Normalizing biproportional methods: new results

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ABSTRACT. Biproportional methods project a matrix **A** to give it the column and row sums of another matrix; the result is **R A S**, where **R** and **S** are diagonal matrices. As **R** and **S** are not identified, it is possible to normalize them. This article starts from the idea developed in de Mesnard (2002) -- any normalization amounts to put constraints on Lagrange multipliers, even when it is based on an economic reasoning, -- to show that it is impossible to calculate the normalized solution at optimum, except by trial and error. Convergence must be proved when normalization is applied at each step on the path to equilibrium. It is also indicated that negativity is not a problem.

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1 Introduction

Biproportional methods project a matrix $\mathbf{A}^0$ to give it the column and row sums of another matrix $\mathbf{A}^1$; the result is $\mathbf{R}\ \mathbf{A}^0\ \mathbf{S}$, where $\mathbf{R}$ and $\mathbf{S}$ are diagonal matrices. As $\mathbf{R}$ and $\mathbf{S}$ are not identified, it is possible to normalize them. This article will start from the idea developed in de Mesnard (2002) -- any normalization amounts to put constraints on Lagrange multipliers, even when it is based on an economic reasoning, -- to show that it is impossible to calculate the normalized solution, except by trial and error, at optimum or along the path to optimum.

2 Remind: Normalizing biproportional methods

The principle of biproportional methods is the following. Assuming that a matrix $\mathbf{A}^0$ is projected on the margins of a matrix $\mathbf{A}^1$ to give a projected matrix denoted $\hat{\mathbf{A}} = K\left(\mathbf{A}^0,\ \mathbf{A}^1\right)$. $\mathbf{R}$ and $\mathbf{S}$ are two diagonal matrices such that $\hat{\mathbf{A}} = \mathbf{R}\,\mathbf{A}^0\,\mathbf{S}$ has the same margins than $\mathbf{A}^1$: $\sum_j \hat{a}_{ij} = a^1_{i\cdot}$ for all i and $\sum_i \hat{a}_{ij} = a^1_{\cdot j}$ for all j. The solutions $\left(r^*_i,\ s^*_j\right)$ for all i, ,j, can be found, for example, by minimizing the quantity of information [1]:

$$\min\ \mathrm{I} = \sum_i \sum_j \hat{a}_{ij} \log \frac{\hat{a}_{ij}}{a^0_{ij}}\ ,$$

s.t. $\sum_j \hat{a}_{ij} = a^1_{i\cdot}$ (multiplier λ_i) and $\sum_i \hat{a}_{ij} = a^1_{\cdot j}$ (multiplier μ_j).

The solution is found after a transformation $r_i = \exp-\left(1+\lambda_i\right)$ for all i and $s_j = \exp-\mu_j$ for all j:

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[1] Other algorithms, and among them the so-called method RAS (Stone and Brown, 1962) or those of (1) (Bachem and Korte, 1979) are possible but it is demonstrated that they all lead to the same solution (de Mesnard, 1994).

(1) $$r_i^* = \frac{z_{i\cdot}^1}{\sum_{j=1}^{m} s_j^* \, z_{ij}^0} \text{ for all } i, \text{ and } s_j^* = \frac{z_{\cdot j}^1}{\sum_{i=1}^{n} r_i^* \, z_{ij}^0} \text{ for all } j$$

that could be changed into:

(1') $$r_i^* = z_{i\cdot}^1 \left(\sum_{j=1}^{m} \frac{z_{\cdot j}^1 \, z_{ij}^0}{\sum_{i=1}^{n} r_i^* \, z_{ij}^0} \right)^{-1} \text{ for all } i$$

It is unique as it is deduced of the minimization of I, a convex and continuously derivable function, on a compact set. After initializing the process by a set of values $r_i^{(0)}$ for all i, for example, **R** and **S** are found iteratively from a mathematical algorithm as follows:

(2) $$r_i^{(k+1)} = \frac{z_{i\cdot}^1}{\sum_{j=1}^{m} s_j^{(k+1)} \, z_{ij}^0} \text{ for all } i, \text{ and } s_j^{(k+1)} = \frac{z_{\cdot j}^1}{\sum_{i=1}^{n} r_i^{(k)} \, z_{ij}^0} \text{ for all } j$$

that could be transformed into:

(2') $$r_i^{(k+1)} = z_{i\cdot}^1 \left(\sum_{j=1}^{m} \frac{z_{\cdot j}^1 \, z_{ij}^0}{\sum_{i=1}^{n} r_i^{(k)} \, z_{ij}^0} \right)^{-1} \text{ for all } i$$

It is demonstrated for RAS that the process is converging [2] (Bacharach, 1970). Biproportional methods are not identified, that is $\mathbf{R}\,\mathbf{A}^0\,\mathbf{S} = (\alpha\,\mathbf{R})\,\mathbf{A}^0\,(\alpha^{-1}\,\mathbf{S})$ for any $\alpha > 0$: it is impossible to give an interpretation to a particular value found for a r_i^* or a s_j^* (even if the products $r_i^*\,s_j^*$ are identified for all i and j). Geometrically, in the space of the r_i^*, the locus of all biproportional solutions is on a hyperplane of dimension $n-1$ passing by the origin. To bypass the difficulty of non identification, it is possible to normalize **R** or **S**, for example by doing arbitrarily $r_1^* = 1$ or $\sum_i r_i = 1$ or following Van der Linden and Dietzenbacher (1995), by

[2] Under some conditions of existence on the matrices $\mathbf{A}^0$ and $\mathbf{A}^1$.

arguing that the global substitution effect is zero for the whole economy, what is less arbitrary (x_j^1 denoting the output of sector j at year 1):

$$(3) \qquad \sum_i \sum_j r_i^* \, a_{ij}^0 \, s_j^* \, \frac{x_j^1}{\sum_j x_j^1} - \sum_i \sum_j a_{ij}^0 \, s_j^* \, \frac{x_j^1}{\sum_j x_j^1} = 0$$

Reporting the expression of s_j^* found from (1), it follows:

$$(4) \qquad \sum_j a_{\cdot j}^1 \, x_j^1 = \sum_j a_{\cdot j}^1 \, x_j^1 \left(\sum_i a_{ij}^0 \right) \left(\sum_i a_{ij}^0 \, r_i^* \right)^{-1}$$

3 Calculability of normalization at optimum

In (de Mesnard 2002), it was stressed that normalizing r_i^* and s_j^* amounts to put a constraint on the multipliers λ_i and μ_j, was is unusual in mathematical optimization. This is not only a question of violating or not the "mathematical law" but this has important consequences. As they are multipliers, r_i and s_j are known after calculating, so, how to know them before calculating, what would be necessary to apply the constraint? Normalization allows to select one set of these parameters, but what set?

Finding a solution amounts to solve the system $\{(1'), (4)\}$. However, as (1) is itself computed by solving (2) iteratively, one has to solve $\{(2'), (4)\}$, that is to compute the intercept between the constraint and the trajectory defined by (2) that ends up on an accumulation point and in fact the trajectory that ends up exactly on the constraint. However, this point is unknown as biproportion is solved after an infinite iterative process; in other words, starting from an initialization, one cannot know by calculation what is this point, only it can be known by numerical computing. Moreover, the initialization point and the trajectory are not unique.

So, one **cannot calculate** the correct set, biproportional solution respecting the normalization constraint (4).

In (de Mesnard, 2002), a 2x2 example was provided: it is recalled here as it could help the reader to understand the new arguments developed in this article. Data are:

$$\mathbf{A}^0 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.1 \end{bmatrix}, \ \mathbf{A}^1 = \begin{array}{cc|c} & & 0.4 \\ & & 0.9 \\ \hline 0.5 & 0.8 & \end{array}, \ x_1^1 = 20, \ x_2^1 = 25 .$$

The solution matrix in all cases is: $\hat{\mathbf{A}} = \begin{bmatrix} 0,0558756 & 0,3441244 \\ 0,4441244 & 0,4558756 \end{bmatrix}$. In the normalization expression, $2\,r_1^*\,s_1^* + 5\,r_1^*\,s_2^* + 6\,r_2^*\,s_1^* + 2.5\,r_2^*\,s_2^* = 8\,s_1^* + 7.5\,s_2^*$, the terms $s_1^* = \frac{0.5}{0.1\,r_1^* + 0.3\,r_2^*}$ and $s_2^* = \frac{0.8}{0.2\,r_1^* + 0.1\,r_2^*}$ are inserted to give the normalization constraint in the space of $(r_1^*,\ r_2^*)$:

(4') $\quad r_2^* = 1.2222 - 1.1667\,r_1^* + 5.5556 \times 10^{-2}\sqrt{484 - 420\,r_1^* + 225\,(r_1^*)^2}$

In the above example, choosing $r_1^{(0)} = 1$ and $r_2^{(0)} = 1$ gives the (unconstrained) solution $r_1^* = 0.553210$ and $r_2^* = 1.465719$ but the normalization formula (4') implies a different value for $r_2^{(*)}$: it must be equal to 1.5714125; one must change the initialization to obtain coincidence. This one is obtained for an initialization by $r_1^{(0)} = 1$ and $r_2^{(0)} = 1.0903506$ to give the solution $r_1^* = 0.579157$ and $r_2^* = 1.534466$ that fully respects (4'); this last value has been **found by trial and error**, not by calculating.

Remark that normalization leads to initialize by not equal values. Also, one notice that the normalized solution must have a valid initialization point, but this one is **not unique**: another initialization is $r_1^{(0)} = 2$ and $r_2^{(0)} = 0.355393$ for the same solution, or also $r_1^{(0)} = 0.5$ and $r_2^{(0)} = 1.6295396$ or $r_1^{(0)} = 0.2$ and $r_2^{(0)} = 2.0229707$, etc.

One can see on the above example that empirically, the initialization points leading to a solution on the normalization constraint (4') are closed to this normalization constraint; the other points are dismissed. A paradox is: it is not because an initialization point is chosen on the normalization constraint that it is a valid constrained solution. For example $r_1^{(0)} = r_2^{(0)} = 1$ is on the constraint (4') but it is not a biproportional solution: $r_1^* = r_2^* = 1$ is false. It is even possible to initialize by negative values to obtain the same constraint solution: $r_1^{(0)} = -0.5$ and $r_2^{(0)} = 3.1115122$ [3].

In this 2-dimensional example, all solutions are aligned on the origin (non identification) what allows to compute exactly the solution by the intersect of the straight line $r_2^* = b\, r_1^*$ and of the line (4'), without passing by (2). However, it is false for $n - 2$ as the intersect of the locus of all biproportional solutions and the surface (4) of dimension $n - 1$ is of dimension $n - 2$ (a line if $n = 3$, a plane surface if $n = 4$, etc.) what leaves an infinite number of possibilities..

Figure 1 about here

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[3] All r_i^* (or s_j^*) can be negative without any problem as they are not identified. In (de Mesnard, 2002), it is stressed that some terms r_i^* could be positive and some negative in equation (4) but this is not a problem also: the solution matrix is always not negative as soon as matrices $\mathbf{A}^0$ and $\mathbf{A}^1$ are not negative.

At the beginning of the iterative process it is possible to have some *r* or *s* terms that are negative and some that are positive, for example if initialization starts with some negative $r_i^{(0)}$. But as they are all positive or zero or all negative or zero at equilibrium, necessarily after a certain moment these terms will turn to have all the same sign. So, the constraints of negativity are not really a problem except if one consider that the solution matrix must be valid (i.e., not negative) at each step *k*.

4 Calculability of normalization on the path to equilibrium

Above, normalization comes at the end of the iterative process when r_i^* (and s_j^*) are calculated at equilibrium. It is not completely satisfactory as (2') is iterative but (4) is not. It could have be preferred to have two iterative equations to form a system so the question is: what happens when normalization comes during the iterative process, that is to say at each step k? It is much more complicated. If it is accepted that normalization must hold not only for equilibrium but also for all steps of the iterative process, one has to calculate the path respecting the normalization at each step k from initialization to equilibrium. And one would demonstrate that the resulting solution converges to the same solution than those given by (4).

As the equilibrium values of r and s terms are found from an iterative process, condition (3), i.e., $\sum_i r_i^* \sum_j a_{ij}^0 s_j^* x_j^1 - \sum_i \sum_j a_{ij}^0 s_j^* x_j^1 = 0$, has to be set not only at equilibrium but also at each step of the iterative calculation of r and s terms, that is:

(5) $$\sum_i r_i^{(k+1)} \sum_j a_{ij}^0 s_j^{(k+1)} x_j^1 = \sum_i \sum_j a_{ij}^0 s_j^{(k+1)} x_j^1$$

Terms $s_j^{(k+1)}$ must be replaced by $s_j^{(k+1)} = a_{\cdot j}^1 \left(\sum_{i=1}^n r_i^{(k)} a_{ij}^0 \right)^{-1}$ found from (2):

(6) $$\sum_j a_{\cdot j}^1 x_j^1 \left(\sum_i a_{ij}^0 r_i^{(k+1)} \right) \left(\sum_{i=1}^n a_{ij}^0 r_i^{(k)} \right)^{-1} = \sum_j a_{\cdot j}^1 x_j^1 \left(\sum_i a_{ij}^0 \right) \left(\sum_{i=1}^n a_{ij}^0 r_i^{(k)} \right)^{-1}$$

One observes that the normalization formula includes both $r_i^{(k)}$ and $r_i^{(k+1)}$ terms: it becomes "intertemporal" (if iterations are compared to time periods). Remark that it becomes impossible to normalize at $k=0$ and one must wait for $k=1$ to do it. Obviously, when $r_i^{(k+1)} \to r_i^{(k)}$ for all i, (6) tends toward (4). Now, one has to solve the system $\{(1'),(6)\}$.

In the example of (de Mesnard, 2002), the normalization expression is:

$$\left(2\, r_1^{(k+1)} + 6\, r_2^{(k+1)}\right) s_1^{(k+1)} + \left(5\, r_1^{(k+1)} + 2.5\, r_2^{(k+1)}\right) s_2^{(k+1)} = 8\, s_1^{(k+1)} + 7.5\, s_2^{(k+1}$$

Replacing $s_1^{(k+1)} = \dfrac{0.5}{0.1\, r_1^{(k)} + 0.3\, r_2^{(k)}}$ and $s_2^{(k+1)} = \dfrac{0.8}{0.2\, r_1^{(k)} + 0.1\, r_2^{(k)}}$ in the normalization expression (5) transforms into:

(6') $$r_2^{(k+1)} = -\frac{6\, r_1^{(k)}\, r_1^{(k+1)} + 13\, r_1^{(k+1)}\, r_2^{(k)} - 14\, r_1^{(k)} - 22\, r_2^{(k)}}{4\, r_1^{(k)} + 7\, r_2^{(k)}}$$

Initializing by $r_1^{(0)} = 1$ and $r_2^{(0)} = 1$, one obtains $r_1^{(1)} = 0.6075949$ and $r_2^{(1)} = 1.4025974$. However normalization formula (6') obliges to associate to $r_1^{(1)} = 0.6075949$ another value for $r_2^{(1)}$, that is $r_2^{(1)} = 2.2232451$: coincidence doesn't hold. One must change initialization, for example, to $r_1^{(0)} = 1$ and $r_2^{(0)} = 1.6287231$ to have coincidence, with $r_1^{(1)} = 0.7606423$ and $r_2^{(1)} = 1.8935493$ as result.

However, at step 2, things are bad as coincidence falls gain, even with these new values: $r_1^{(2)} = 0.7323751$ and $r_2^{(2)} = 1.9265983$ while (6') indicates $r_2^{(2)} = 1.8982382$. So convergence must be demonstrated from $k = 0$ to equilibrium.

5 Conclusion

As the diagonal matrices **R** and **S** of RAS are not identified, it is possible to normalize them. However, in (de Mesnard, 2002) it is stressed that, as **R** and **S** are Lagrange multipliers, normalizing means that Lagrange multipliers are constrained, what is mathematically strange. Starting from the fact that multipliers are known only at the end of an optimization calculation and then by an iterative computation, this article has shown that the normalized solution cannot be calculated but only found by trial and error; convergence must be proved when normalization is applied at each step of the path to equilibrium. However, negativity is not a problem. So, unfortunately, biproportional methods remain unidentified and **R** and **S** must not be interpreted by themselves...

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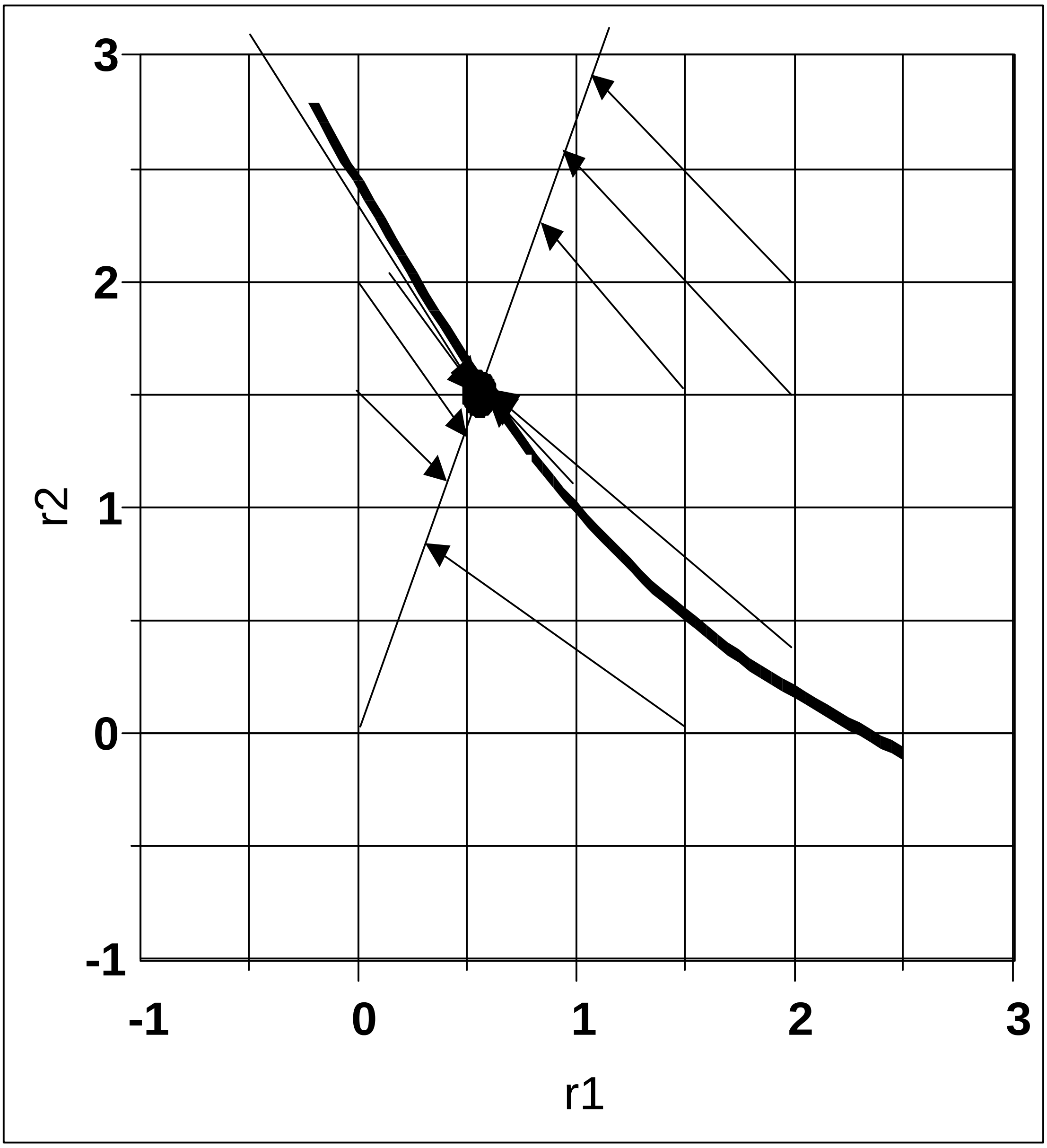

Figure 1. Function $r_2^* = \mathrm{f}(r_1^*)$ of normalization
The paths to equilibrium are indicated by an arrow
The constrained equilibrium is indicated by a bold dot