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Did Ragnar Frisch discover input-output economics?

by

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Abstract

Ragnar Frisch on various occasions claimed that he had invented input-output analysis in a 1934 paper titled "Circulation planning". Although prominently published in Econometrica, Frisch's contribution has hardly been discussed in the input-output literature. Frisch's paper was an attempt to explain the *incapsulating* phenomena of economic depressions and thus arose from a different motivation than Leontief's analysis. This paper aims to clarify what Frisch accomplished relative to Leontief (and implicitly assess his claim). The paper will examine the inventiveness and analytical power of the approach developed by Frisch, and discuss various contexts in which it can be placed in the development of economic thought. The emphasis on distribution rather than on production, which Frisch's contribution suggest, can be traced back to early attempts at outlining general equilibrium relation. The paper will in this context discuss the roots in the history of economic thought both of Leontief's input-output analysis and of Ragnar Frisch's circulation approach.

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1. Introduction

Ragnar Frisch claimed that he had invented input-output analysis in a 1934 paper titled "*Circulation Planning: Proposal for a National Organization of a Commodity and Service Exchange*". This claim, which Frisch mentioned casually on various occasions after World War II, can easily be dismissed. Despite using the open input-output model in his modelling and programming research after 1949, Frisch did not introduce the input-output model in the paper as we know it. The underlying ideas in '*Circulation Planning*' are quite different from those that motivated Wassily Leontief in his development of the input-output analysis. There are, however, striking similarities in the formal structure of the problem Frisch grappled with and what later became known as Leontief's "closed model". Both focused on "circulation" of real flows in the economy and needed similar mathematical prerequisites. This paper aims to clarify what Frisch accomplished relative to Leontief by examining the inventiveness and analytical power of the approach developed by Frisch, and discussing various contexts in which it can be placed in the development of economic thought.

When Frisch first embraced Leontief's work in 1949 and began to teach input-output analysis and build input-output models, he coined a term in Norwegian for the new field, calling it "*kryssløpsanalyse*" (literally: *cross run analysis*), a term he to begin with used also to cover linear programming which was launched at the same time. Also linear programming figured prominently on Frisch's agenda in the 1950s, as it indeed did as a research area in economics at that time.[1] Several future Nobel Laureates engaged deeply in programming problems. Frisch's work in the field can also be traced back to the *Circulation* paper.[2]

Frisch's article appeared in the second year of Econometrica which was edited by Frisch. It was a book length article, inconveniently long for a journal.[3] The article was divided into three parts: "Theory", "Applications" and "Mathematical Appendix". The title pointed to the applications part, which has lost any claim on interest today, how to build a national exchange organization that could deliver what a malfunctioning market system could not. From the title it may not have been obvious to the early econometricians that the article belonged in Econometrica at all. In Frisch's opinion, as indeed stated in the article, it definitely dealt with an *econometric* issue. Frisch had a certain claim on authority here as head coined the term 'econometrics' and also written the constitution of the Econometric Society, which was as close as one could come to a definition of econometrics at that time. Some leading members in the Econometric Society questioned the propriety of publishing the long article written by the editor.[4]

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[1] See e.g. Dorfman, Samuelson and Solow (1958).

[2] The earliest trace of a programming approach in Frisch's work may be Frisch (1932b).

[3] Frisch (1934) is the longest article which ever appeared in Econometrica, apart from Haavelmo's *The Probability Approach in Econometrics* in 1944, a monograph published as a separate supplement to compensate subscribers for the scarcity of articles during the war. Frisch's article, including the mathematical appendix, which was published in the successive issue, came in total to 93 pages!

[4] Frisch's co-founder of the Econometric Society and secretary at the time, Charles Roos, put it like this in a letter to about one year after the article appeared: "I did not think either represented a very vital contribution to economic knowledge. … it does not read like a very important piece of research. I do not mean that is should not have been published, but I do mean that a pair of scissors could have been used to advantage. … There has been some criticism to the effect that you have failed to recognize the difference between mathematical exercises and contributions to economic knowledge by means of mathematics, and that in your paper on "Circulation Planning" … certain elementary mathematical exercises which have no real mathematical interest and are far from economic reality are published under the guise of economic theory." (Roos/Frisch, 13 Feb. 1935).

The *Circulation* paper appeared at the very peak of Frisch's scientific career. Frisch, born 1895, was the jeweller's son destined to take over the family business. As late as 1928-29 he may still have been undecided whether to become a jeweller or an econometrician. An invitation to visit Yale University in 1930, engineered and paid for by Irving Fisher, became decisive. Frisch stayed in USA for a year and half in 1930-31. On his return he became appointed to professor at the University of Oslo and founded his Institute of Economics, which he often referred to, and rightly so, as a laboratory. The research agenda he had worked on in USA resulted in a series of publications in 1932-36, comprising most of those that created his fame as an econometrician. Among these are his key contributions towards developing econometric methods, frequently quoted in works on the history of econometrics, Frisch (1933a, 1934b), Frisch and Waugh (1933), his *Propagation and impulse* explanation of business cycles, Frisch (1933b, 1935a), and his works on marginal utility and price indexes, Frisch (1932a, 1936).[5] The *Circulation* paper is literally squeezed in between Frisch's perhaps two most important contributions, the *Propagation* paper and the *Confluence Analysis* treatise.

This says something of the priority that Frisch gave this effort. The methods and approaches of the *Circulation* paper were related to other works by Frisch. In the opening sections of the paper Frisch argued that consumption behaviour might cause cycles and "violent depressions" through similar mechanisms that in the Propagation analysis had created cycles from production behaviour. The argument was hardly convincing as a business cycle theory and is not further discussed here. The ideas of the *Circulation* paper were also referred to and retrieved by Frisch on a number of future occasions, *inter alia* in papers on multilateral trade in the early post-war years.[6]

Another of Frisch's projects that meant much to him at the time was his production theory or *productivity theory*, as he preferred to call it. It had been virtually completed as a monograph when he left USA in 1931, but for not obvious reasons, it was not published in English until 1965. Frisch's production theory was very thoroughly corroborated mathematically formulated theory of substitution among variable factors in production (although the theory also comprised limitational factors). To Frisch this was a pre-eminent example of "quantitative theory". He drew on it on several occasions for problems outside conventional production, when substitution was a key element in the problem. He thus applied it to the "diet problem", to the question of "optimum population" and he also used in a pioneering work on engineering production function.[7] It is thus not so surprising that it turns up as one his resources in the problem dealt with in the *Circulation* paper.

At the time Frisch wrote *Circulation Planning*, Wassily Leontief was at work at Harvard on pursuing a research path which eventually would give us input-output economics. Leontief (1928) had already begun to develop the foundations of input-output economics in his doctoral thesis that was partially published in 1928. In this thesis he developed a simple 2 sector model that described prices how the surplus product is distributed (Kurz and Salvadori, 2005). These ideas were likely developed further while he was a post doc fellow at the Institute of World Economics in Kiel (later moved to the New School in New York). During this time Alfred Kähler (1933) developed a simple closed input output model (Gehrke, 2000). Leontief (1936, 1937) would later publish two papers in *The Review of Economic Statistics*: One that described the theoretical foundations (1937) and one that used the United States as an empirical example (1936). These ideas were developed further in his famous monograph, published in 1941 and a second edition of the book published in 1951 incorporated some

[5] Frisch (1933c) from the same period figures as pioneering contribution in the history of game theory.

[6] Frisch (1947, 1948), see also Chipman (1998).

[7] Frisch (1941), see Sandmo (1993); Frisch (1940); Frisch (1935c).

articles published in 1944 and afterwards. It may be reasonable say that input-output analysis was established as a field about that time, although the numbers of practitioners were still few. The first Input-Output Conference took place in 1950.

But when and where on this path did the input-output model emerge? For many input-output practitioners a natural answer would be to point to the article by Leontief which opened with the following questions: "How will the cessation of war purchases of planes, guns, tanks, and ships – if not compensated by increased demand for other types of commodities – affect the national level of employment? How many new jobs will be created by the consumers' demand for an additional one million of passenger cars, how many of these jobs can be expected to be located in the automobile industry itself, and how many in other industries such as steel and the chemicals, the coal and the petroleum industries?"[8] The questions were representative of those tackled at an early stage by the new tools of input-output analysis.

Did Frisch and Leontief know each other at that time? They had most likely not met, although they could have been at the same place at the same time. But they still had had a confrontation. Leontief had in Kiel in 1929 published a treatise on the estimation of demand and supply curves in which he claimed to have developed a sophisticated procedure that allowed the estimation of both the demand curve and the supply curve from one set of market observations of price and quantity.[9] Frisch read it in USA in 1930 and pondered over the problem until he published in 1933 a rather sharply formulated criticism of Leontief's claim in the *Pitfalls* essay. Leontief did not yield to the criticism, but defended vigorously his view in a reply. Frisch wrote a rejoinder with an excessively sharp criticism of Leontief.[10]

The next section relates the *Circulation* paper to the historical context. Section 3 sets out the analytic core of Frisch's Circulation Planning scheme and the results he presented. Section 4 gives some details on Frisch's mathematical underpinning of his derivations. Section 5 discusses some aspects of the application Frisch had in mind for his scheme, and section 6 concludes.

2. Depression economics

The opening paragraph of Frisch's article left no doubt about the underlying motivation for his work:

> The most striking paradox of great depressions, and particularly of the present one, is the fact that poverty is imposed on us in the midst of plenty. Many kinds of good are actually present in large quantities, and other kinds could without great difficulty be brought forth in abundance, if only the available enormous productive power was let loose. Yet, in spite of this technical and physical abundance, most of us are forced to cut down consumption. We are compelled to make real sacrifices in order to economize in the use of these very goods and services that *could* easily be produced in abundance if we only could use our resources.[11]

Thus his contribution may be reasonably be considered as belonging to "depression economics", attempts not only to explain but also to propose remedies to counteract the disastrous economic conditions he had observed developing in USA 1930-31 and in Norway after his return.

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[8] Leontief (1944), p.290. The entire article was incorporated in Leontief (1951).

[9] Leontief (1929).

[10] Frisch (1934c), Leontief (1934a, 1934b), see also Hendry and Morgan (1995), 257-270. Schumpeter who was a good friend of Frisch, warned him that he risked ruining the career of Leontief.

[11] Frisch (1934a), p.259.

The crisis was, Frisch asserted, not a "real poverty crisis", the cause was connected with "the present form of organization of industry and trade." A consequence of the disastrous effects of this present organization was the *incapsulating phenomenon* produced during great depressions.[12] Frisch is a little vague about the conceptual content of this phenomenon, it is connected, he asserted, with the fact that "modern economic life has been divided into a number of regions or groups". The economic system will under certain circumstances, create a situation where these groups are forced to "… mutually undermine each other's position. Each group is forced to curtail the use of the goods produced and services rendered by the other groups, which, in turn, will cause a still further contraction of the demand for its own products, and so on. This meaningless vicious circle is what I understand by the incapsulating phenomenon."

Frisch referred to how incapsulation manifested itself internationally through the protectionist tendencies in international trade relations during the current depression. He expressed the opinion that the effect of incapsulation within national borders was even more important, but much less recognized than the international problem. On the other hand the possibilities of doing something to combat the devastating effects of the circulation stagnation were much greater within national borders than internationally. These introductory statements conveys well Frisch's commitment and conviction, but does really illuminate very much the nature of the "incapsulation". We may certainly conclude that he took it for granted that Walras' Law was not operative.

The situation called for "introducing a certain measure of planning in exchange activity." The prime object of the planning would be to utilize more fully the existing productive capacity. This led to an optimum problem of great complexity as increasing the capacity and at the same time conform as much as possible with the particular desires of the individuals and groups involved, were in many cases mutually exclusive: "The weighing of their relative importance is precisely the fundamental problem of the planning."

Frisch was not the only socially conscious economist who devoted attention to this pressing issue of the day. Another was Frisch's co-founder of Econometric Society and its first President , professor at Yale University Irving Fisher. He published the book *Booms and Depressions* which explained the depth of depressions as the result of the effect of deflation upon debt, and wrote a condensed version of his theory for one of the early issues of Econometrica.[13] Frisch made an effort to pay attention to Fisher's ideas in the opening part of the *Circulation* paper.

Fisher's explanation of depressions was that there was overindebtedness for whatever reason, would cause deflation to follow soon after. The liquidation of debts not only caused deflation, but the liquidation could not keep up with the fall in prices. Thus, "the very effort of individuals to lessen their burden of debts increases it, because of the mass effect of the stampede to liquidate in swelling each dollar owed." Thus Fisher's theory also had a strong element of incapsulation. "The more the debtors pay, the more they owe. The more the economic boat tips, the more it tends to tip. It is not tending to tight itself, but is capsizing."[14] That is, unless counteracted by policy. The remedy for debt-deflation was reflation. Fisher blamed in the article the Hoover administration for stopping efforts to prevent the crash in

[12] The dictionary meaning of 'incapsulate' or 'encapsulate' is to 'enclose in or as in a capsule' or 'epitomize, typify'. The meaning Frisch meant to convey was more that of contraction or even imploding. His use of this term, deviating from the dictionary definition, thus, deviating from the dictionary definition, thus adds to the number of the terms he coined.

[13] Fisher (1932, 1933).

[14] Fisher (1933), p.344.

1929 from developing into a depression and praised the new Roosevelt administration for having halted the downswing.[15]

Of course J.M. Keynes was by far the most successful contributor to depression economics. But outside academic circles there were also a great number of amateur economists, laymen and, indeed, crackpots with recipes for economic revival. They were typically monetary reformers with more or less sound proposals for creating more liquidity. They frequently also pursued ideas of barter exchange systems when markets did not function, indeed, to replace the market. Keynes discussed a number of these reformers and their proposals both in *General Theory* and in *Treatise of Money*.[16] These were also represented in Norway, e.g. both Gesellians and adherents of Major Douglas.[17]

Frisch had exerted much activity after his return from USA to Norway in 1931 to influence politicians and the government about reform of the monetary system and a more wisely use of fiscal policy, but largely to no avail. Frisch was eager to exert influence but never seemed - unlike Keynes - to be very successful and perhaps was not very adept as a political operator. Fisher also put much effort into exerting influence on the Roosevelt administration on how to handle the crisis, usually by direct access to Roosevelt.

The involvement in devising complicated schemes for barter trade outside regular market brought Frisch in touch with the fringe and the crackpots of economic policy. It must be understood, however, as a second best choice. His bold efforts to exert influence towards a reform of the monetary system by approaching the government and the general public had run aground. As Frisch wrote to the leadership of the Labour Party whose program he was fairly successful in influencing: "If we are going to stop these encapsulating forces by means of reforming the existing monetary system, many prejudices have to be overcome. To eliminate such prejudices is no easy matter. It is an open question, then, if it is more effective policy to let people retain their old prejudices in peace and to construct new instruments which may fill the function that money has now failed to fill."[18] The new instrument she had in mind were those he promoted in the *Circulation* paper. Frisch had earlier presented macroeconomic policy proposals in a pamphlet, in which he also had explained the saving paradox. Hence, his energetic promotion of the *Circulation* ideas was probably also fed by the frustration experienced.

We may note also in this connection that there was no depression dimension of Leontief's work. His motivation was not related to the problems of the day. He was aiming at providing a detailed empirical underpinning of the equilibrium system drawn up by Walras. It is a kind of paradox here that Frisch's efforts and those of many others at counteracting the depression through new ideas and policy tools came to nothing while the modelling tools that emerged eventually from Leontief's equilibrium oriented project turned out to be useful for very many purposes, not least in economies out of kilter and for counteracting recessions.

[15] Fisher also had personal experience. He is asserted to have lost about $10 mill. of his own and his wife's money in the 1929 crash and its aftermath. At the time of writing he may still have nurtured hope of recovering his losses. See Tobin on "Irving Fisher" in *The New Palgraves*.

[16] Keynes (1930, 1936).

[17] Andvig (1986), chapter 13.

[18] Quoted from Andvig (1986), p.418.

3. Optimal allocation and transaction in Circulation Planning

The theory part of the *Circulation* paper, supported by the mathematical appendix, has a number of inventive and ideas. Some of these ideas can be traced to other work Frisch had conducted. In later work he would draw on the ideas and results he had achieved in the article.

Frisch set out his ideas using the simile of a shoemaker, a tailor and a farmer trading with each other. The setting was a "heavy depression period". The three "partakers" were "mutually waiting for each other's orders, none of them daring to make a purchase, because they do not feel assured that they will be able to sell their own products." The demand was thus latent, rather than manifest. Walras' law did not apply.

Frisch put an "organizer" into the picture, i.e. an organizer working for the *National Exchange Organization*. He went around and elicited from each partaker how much he would buy from each of the others if he was assured of an increase in the sales of his own products equal to the total sum of what he bought from the others. Saving was thus ruled out. The answers were given in a *request table*. The row sum of requests would then by assumption be the production capacity of the partaker, measured in the purchasing power his effort would result in. Suppose the request table looked like table A.

Table A	**Wants to buy from**			
	Shoemaker	**Tailor**	**Farmer**	**Total**
Shoemaker	0	50	100	150
Tailor	40	0	180	220
Farmer	110	170	0	280
Total	150	220	280	650

One cannot avoid finding that this is highly suggestive of an input-output table with rows and columns interchanged. The shoemaker-tailor-farmer seemed to represent commodities and production sectors, but not quite. Frisch's partakers are not very precisely defined, but are indicated to be groups of suppliers, perhaps according to industrial branches, and also subdivided geographically. Some partakers could be trade unions representing uemployed labour power. The table in its entirety would not coincide with the national economy, but would be a voluntarily joined part of an economy where markets by and large had failed to function. The "requests" represented demand, both for consumption, but also for input in the production process.

Table A is, however unrealistic as an observed request table, as it is balanced at the outset. The total value of the goods requested by the shoemaker equals the total request for shoes from the others and similarly for the tailor and the farmer. The organizer then has an easy job in this deadlock economy. He distributes warrants ("commodity notes") according to the requests of each agent. The agent will use the warrants in lack of alternative actions and after the transactions the organizer can collect warrants from each agent in the same amount as was distributed and "burn them". Frisch does not elaborate upon why the market does not function, why the monetary system cannot support the transaction needed etc. This is all left in the background here.

An elicited request table of independent and voluntary requests for goods will in practice never be balanced. In the general case with *n* partakers the request table will be like table B. Row sums are denoted $a_{io} = \Sigma_j a_{ij}$, column sums $a_{0j} = \Sigma_i a_{ij}$ and the grand total $a_{00} = \Sigma_{ij} a_{ij}$. In general, $a_{i0} \neq a_{0i}$.

Table B	**Wants to buy from**						
	1	**2**	**…**	***j***	**…**	***N***	**Total**
1	-	a_{12}	…	a_{1i}	…	a_{1n}	a_{10}
2	a_{21}	-	…	a_{2j}	…	a_{2n}	a_{20}
…	…	…	…	…	…	...	…
i	a_{i1}	a_{i2}	…	a_{ij}	…	a_{in}	a_{i0}
…	…	…	…	...	…	…	…
n	a_{n1}	a_{n2}	…	a_{nj}	…	-	a_{n0}
Total	a_{01}	a_{02}	…	a_{0j}	…	a_{0n}	a_{00}

The purpose of the Circulation Planning is to achieve as large volume of transactions. and thereby employment, as possible. A larger volume would benefit all partakers with excess supply, at least if the request matrix was indecomposable. But to get anywhere the request table must be balanced. Otherwise transactions cannot be fulfilled. Balancing supply and demand is here not via prices and a Walrasian auctioneer. The market mechanism does not function. Balancing will be by engineered quantity adjustment in an optimal way, to be defined.

The request table has n^2-n elements that can be changed to fulfil the n balancing conditions, of which only n-1 are independent. Hence there would be $n^2 - n - (n-1) = (n-1)^2$ degrees of freedom, the question was which constraints ought to apply.

Frisch proceeded by stating principles that could be applied in a rather mechanical way to any given request table. The overriding concern was that the amendments in the request matrix must be such as to "leave the originally requested quantities as far as possible unchanged", That was in a sense the fundamental optimum consideration, accept the partakers' stated preferences whether derived from a utility function or from production needs. As the total request or production capacity for each partaker was voluntarily stated, Frisch took as a basic assumption that the a_{i0}'s could not be exceeded.

Furthermore, Frisch argued that "many – if not most – of the persons or groups" likely to take part in the planned exchange would probably consider it a fair and straight arrangement if the adjustments were made on a strict percentage basis for each person, so that the relative distribution of purchases for any given person is maintained unchanged while the total volume of purchases was adjusted. This led to amending the request matrix by proportional changes of all elements in a row. Frisch calls this the principle of *partakers' percentages*.

To corroborate this principle Frisch came very close to an explicit statement of the key assumption of input-output analysis:

> This principle seems not only fair and straight, but, for many partakers, it will be the only natural solution. We only have to think of an enterprise that requests certain factors of production. To the extent that the coefficients of production are constant for variations in output of the order of magnitude here considered, the principle of partakers' percentages would be the only correct solution.[19]

The question is then, do there exist partakers' percentages, z_i, such that the request table as given in table B corrected according to

(1) $$(1+z_i)a_{ij}$$

[19] Frisch (1934a), p.275.

will become balanced? The answer is yes, but as by assumption $(1+z_i)a_{i0} \le a_{i0}$, the z_i's will be non-positive, it will a balancing downwards.

Taking the total requests of a balanced request table, $\{c_i : i = 1, 2, ..., n\}$, as unknowns instead of the z_i's, i.e. $c_i = (1+z_i)a_{i0}$,we must have fulfilled the equations:[20]

(2) $$\sum_k c_k(\alpha_{ki} - e_{ki}) = 0 \quad i = 1, 2, ..., n$$

where $\left[\alpha_{ij}\right]$ is the request matrix with $\alpha_{ij} = a_{ij} / a_{io}$

and $$e_{ki} = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \ne i \end{cases}$$

This is, of course, nothing else than the equation of the Leontief closed model. The coefficient matrix $\left[\alpha_{ij} - e_{ij}\right]$ of (2) is singular, as each row add to zero, and of rank *n*-1. Frisch showed that the rows of the adjoint of this singular matrix were identical and positive. Let now the elements in any row of the adjoint matrix be divided by the corresponding row sum to get relative numbers which add up to one, say, $P_1, P_2, ..., P_n$. The solution of (2) in terms of c_i can then be written with one degree of freedom as

(3) $$c_i = CP_i \quad \text{where } i = 1, 2, ..., n \quad \text{and} \quad C = \Sigma_i c_i$$

The purpose is to achieve the highest possible volume of total trade, *C*, but with the side condition that no one should be impelled to participate with a larger volume than originally requested, i.e. $c_i \le a_{i0},\ i = 1, 2, ..., n$, which again implies that

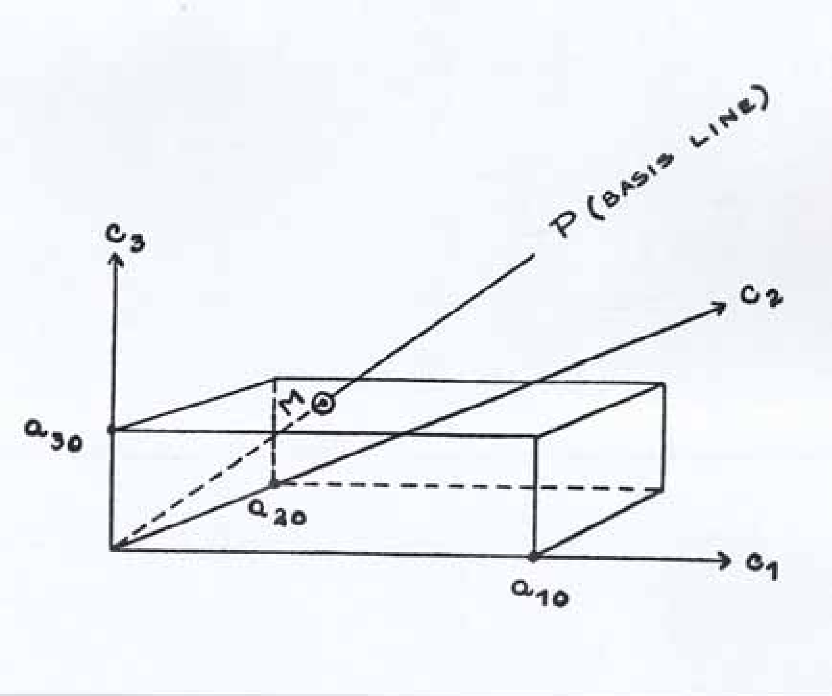
(4) $$C = \min_i \{a_{10} / P_1, a_{20} / P_2, ..., a_{n0} / P_n\}$$

The geometric interpretation in $(c_1,c_2,\ldots,c_n)$-space is that the solution is determined to be where the ray $[CP_1, CP_2, ..., CP_n]$, the direction of which is determined by the structure of the request table, cuts through the square box given by the original total requests $[a_{10},a_{20},\ldots,a_{n0}]$.

The situation can be illustrated in figure 1.

[20] Cf. Frisch (1934a), (4.7) and (4.8), p.276.

Figure 1[21]



Frisch exemplified with the shoemaker-tailor-farmer case. Let us say that the observed request table is table C, which is clearly unbalanced.

Table C	**Wants to buy from**			**Total**
	Shoemaker	**Tailor**	**Farmer**	
Shoemaker	0	75	150	225
Tailor	80	0	360	440
Farmer	330	510	0	840
Total	410	585	510	1505

Cf. Frisch (1934a), (4.15).

Following the procedure outlined above, the uniquely determined partakers' percentages can be worked out to be $w_1 = 0.0$, $w_2 = -0.25$ and $w_3 = -0.5$.

This implied that the limited production capacity of the shoemaker constrained the satisfaction of the tailor to three quarters of his original requests and that of the farmer to only half of his original requests. This again meant that the production of the tailor and the farmer is constrained to three quarters and one half of the stated production capacity, respectively. The corresponding balanced table is table D.

[21] Frisch (1934a), Figure 2, p.279.

Table D	Is allowed to buy from			
	Shoemaker	Tailor	Farmer	Total
Shoemaker	0	75	150	225
Tailor	60	0	270	330
Farmer	165	255	0	420
Total	225	330	420	650

Cf. Frisch (1934a), (4.23).

Frisch thus easily concluded from this preliminary exercise that adaptation according to the principle of partakers' percentages was not enough. More corrections needed to be done in the table to achieve an increase in the total volume of trade. But any further correction would at the same time imply, Frisch argued, "a weighing of the various desires towards each other." He came up against the problem that it was hardly possible to find any "absolute objective and exact scale" for making such comparisons. It is on this point that he let out the following statement which says much about his concern and commitment in this problem:

> We have here one of those cases – so frequent in economic practice – where it can be "proved" by abstract reasoning that a solution is not possible , but where life itself compels us nevertheless to find a away out.[22]

Frisch then inventively introduced additional principles for correcting the request table. While the partakers' percentages modified rows in the request table, the *rationing-percentages*, x_j, modified columns correspondingly. Using both, each element in the request table would thus be corrected to become

(5) $$(1+z_i+x_j)\cdot a_{ij}$$

The rationing-percentages could be of either sign.

One could imagine that only rationing percentages were applied to balance the table and maximize the volume. Varying the rationing-percentages can achieve a similar kind of solution as with partakers' percentages, only with interchanged rows and columns. Thus the question is what can be achieved with both partakers' percentages and rationing-percentages, when the direct effect of the latter is considered as a nuisance. The problem seemed to become an exercise in optimal manipulation of the degrees of freedom.[23]

Frisch goes quite a bit further than (5) in introducing degrees of freedom for correcting the request table. He suggests that there could be a considerable surplus of some commodity which is only a small part of the total production capacity. Perhaps a large partaker did not even request anything of the commodity, thus it would not do to adjust the rationing percentages alone. To relieve the excess supply it could be to some extent forced on all partakers, related to their activity level. It could thus conveniently be distributed by *surplus coefficients*, y_j, among the partakers according to total requests, i.e.

(6a) $$(1+z_i+x_j)\cdot a_{ij}+y_j\cdot a_{i0} \qquad y_j\geq 0$$

[22] Frisch (1934a), p.274.

[23] We may note at this point that the correction of an input-output table by means of the two sets of proportional adjustment by row and column will remind input-output practitioners of problems of updating input-output tables using the RAS-method. This suggests perhaps that a possible application of Frisch's approach, is, or could have been, to the updating of input-output matrices to new gross production totals.

The surplus coefficients are assumed to be non-negative. The distribution of surplus could be made even more elaborate, if the surplus coefficients also depend upon the relative composition of requests of the receiving partaker, i.e.

(6b) $$(1+z_i+x_j)\cdot a_{ij}+\Sigma_k y_{kj}a_{ik} \qquad y_{kj}\geq 0$$

For completeness we mention that Frisch introduces still another correction. Calling it a *service tax*, denoted ρ_i, it is introduced as a wedge between the row sum of the corrected table, c_i, and the column sum, $c_i+\rho_i$. The service tax should add to zero over all partakers. Frisch gives two interpretations. One is, as the name suggests, a service tax levied by the exchange organization, which would enter as one of the partakers. The other interpretation is more interesting, as Frisch on this point leaves the mechanical-numerical approach to consider economic behaviour, albeit somewhat cryptically. "… let us imagine that one of the parties involved would attempt to make an extra profit by raising the price of its product and at the same time trying to force its sale through the exchange organization. … To meet this situation it is desirable to introduce into the planned circulation mechanism some element that may be manipulated so as to produce an effect similar to the one which would in a free market be produced by the adjustment of relative prices." He did not further elaborate on this.

We limit our exposition of Frisch's approach to consider only the rationing-percentages in addition to the partakers' percentages, i.e. corrections according to (5) above. Frisch includes also the other types of corrections in his general scheme.

We shall try to convey how Frisch solves the apparently impossible problem of assuring a "best possible" adaptation of request table to the desires of the partakers. In his approach he draws, as we shall see, on ideas from his production theory.

Taking the sum of over *j* and the sum over *i*, respectively, of (5), we get

(7) $$\begin{aligned}&(1+z_i)\cdot a_{i0}+\Sigma_k x_k a_{ik}=c_i\\&a_{0j}+\Sigma_k z_k a_{kj}+x_j a_{0j}=c_j\end{aligned}$$

Changing *j* to *i* in the second expression and inserting the expression for the z_i's from the first equation, we arrive at

(8) $$a_{0i}+\Sigma_k(c_k-a_{k0}-\Sigma_j x_j a_{kj})\cdot\frac{a_{ki}}{a_{k0}}+x_i a_{0i}=c_i$$

which can be written as

(9) $$\Sigma_k c_k(\alpha_{ki}-e_{ki})=r_i$$

where

(10) $$r_i=\Sigma_k x_k(\gamma_{ik}-e_{ik})a_{0k}$$

where $\gamma_{ij}=\Sigma_k\alpha_{ki}\beta_{kj}$ and $\beta_{ij}=a_{ij}/a_{0j}$

The coefficient matrix of (9) is the same as in (2) and thus singular of rank *n*-1. The elements of the right-hand side vector adds to zero ($\Sigma_j r_j=0$), and (9) thus has a solution with one degree of freedom. Frisch shows that the solution to (9) can be written in line with (3)-(4) as

(11) $$c_i=CP_i-u_iP_i$$

where the P_i's are the same as in (3)-(4) above, and u_i is introduced as an intermediate variable, useful for the ensuing argument. u_i is a linear form in the r_k's,

(12) $$u_i = \Sigma_k r_k Q_{ki}$$

with coefficients Q_{ki} derived from $[\alpha_{ij} - e_{ij}]$, i.e. from the data of the original request table. r_k is again from (10) a linear form in the x_j's. Hence, u_i can be written as a linear form in the x_i's.

(13) $$u_i = \Sigma_j r_j Q_{ji} = \Sigma_j \Sigma_k x_k (\gamma_{jk} - e_{jk}) a_{0j} Q_{ji} = \Sigma_k f_{ik} x_k$$

It can be shown that the u_i's fulfil $\Sigma_j u_j P_j = 0$.

As before none of the c_i's shall be larger than the corresponding a_{i0}'s.

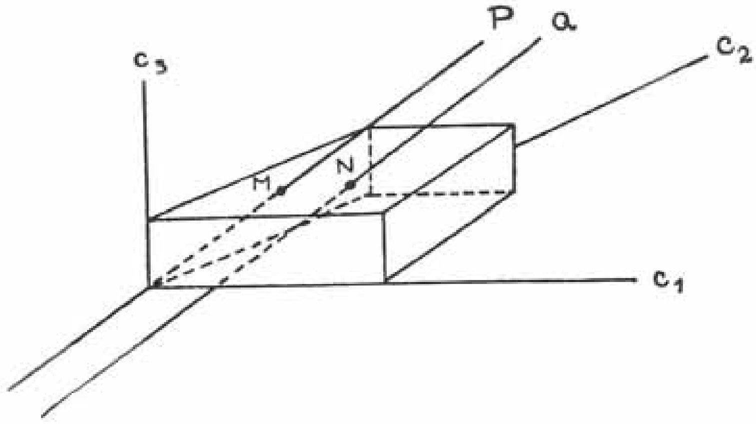
$$(C - u_i)P_i \leq a_{i0} \quad \text{for all } i$$

implying

(14) $$C = \min_i \{ \frac{a_{10}}{P_1} + u_1, \frac{a_{20}}{P_2} + u_2, ..., \frac{a_{n0}}{P_n} + u_n \}$$

The geometric interpretation in $(c_1, c_2, \ldots, c_n)$-space is that the solution point is located at the intersection of the ray [$CP_1, CP_2, ..., CP_n$] translated away from origo by the vector [$u_1 P_1, u_2 P_2, ..., u_n P_n$] and the box given by the original total requests, to reach a possibly higher total volume.

Figure 2[24]



How far towards the maximum volume can one get using this method? It is really a question of the inconvenience of rationing. Frisch at this point introduced ideas from his production theory: "We are indeed confronted with problems very similar to those we meet in productivity theory, only the corresponding problems in circulation planning are more far-reaching and complicated: *how much* should this activity be increased and that other curtailed? All of which means that these problems are essentially *econometric*."

He considered the total volume C as the product, "produced" by means of elementary factors $\{x_i\}$. For any given factor combination, the best possible result would be as given by (14). What needs to be assessed is the "factor cost". We have to assume that there is an inconvenience cost of the x_i's. If we introduce a cost function in the x_i's, the least cost combination corresponding to a given volume can be worked out. This is the *substitumal* curve of Frisch's production theory.[25]

Frisch here reaches a programming formulation with linear constraints. He introduced a quadratic cost function in the x_i's.[26]

(15) $$\Omega = \frac{1}{2}[\frac{x_1^2}{\varepsilon_1} + \frac{x_2^2}{\varepsilon_2} + ... + \frac{x_n^2}{\varepsilon_n}]$$

The optimal way of "producing" a total volume $C^* \leq C^{\max} = \Sigma_i a_{i0}$ is then given by

[24] Frisch (1934a), Figure 2, p.289.

[25] "The *substitumal* is the locus of all points in the factor diagram from which it is *not* possible to undertake an in-every-respect economic substitution." Frisch (1965), Definition (10d.5), p.158.

[26] A quadratic cost function makes sense as an approximation of the cost of deviating from quantities chosen on the basis of a utility function.

$$\text{Min } \Omega \text{ wrt. } x_k \qquad k = 1, 2, ..., n$$

(16) subject to:

$$u_i = \Sigma_k f_{ik} x_k \geq C^* - \frac{a_{i0}}{P_i} \qquad i = 1, ..., n$$

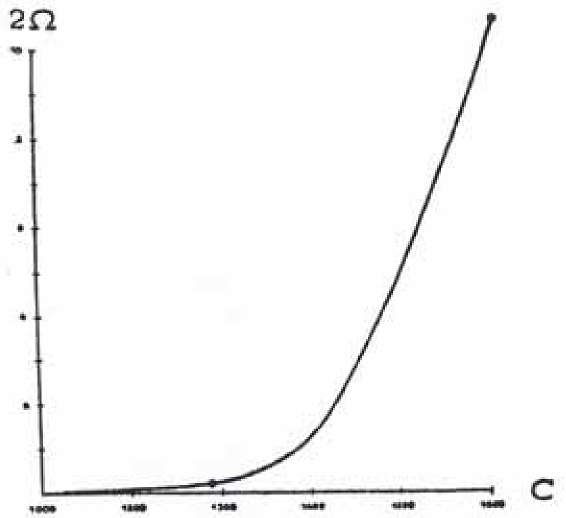
Frisch's attempt to solve this problem is a differential approach of "small tentative steps". Starting from the situation of all x_i's equal to zero, increasing volume targets were set and the corresponding optimal values $\{ x_i : i = 1, 2, ..., n \}$ determined to map out the cost Ω as a function of total volume *C*. For each new value of total volume the starting point in the x_i's was the solution from the previous volume. For this point some of the inequalities of (16) would be fulfilled and could be temporarily ignored. Those which were not fulfilled, say for $i=p,q,\ldots,t$, would be set as equality constraints in the optimization problem.

$$(17) \qquad \frac{\partial(\Omega - \underset{i=p,q,...,t}{\Sigma} \lambda_i \underset{h=1,2,...,n}{\Sigma} f_{ih} x_h)}{\partial x_k} = \frac{x_k}{\varepsilon_k} - \underset{i=p,q,...,t}{\Sigma} \lambda_i f_{ik}$$

Each step would be solved by iteration. For the solution of (17) some of the u_i which at the outset fulfilled the inequality of (16) might as a result of the optimization have become *deficient*, i.e. no longer fulfilling (16) and an iterated procedure with additional constraints would as Frisch presumed, lead towards the *substitumal* point. Repeated for other volumes the entire cost function in terms of *C* could be mapped out.

Although Frisch worked with a somewhat more general case, as indicated above, the numerical illustration he worked out in detail is the problem in (16) for the shoemaker-tailor-farmer case, i.e. starting from table C. This numerical example runs over 14 pages in Econometrics 1934! Frisch first set the coefficients of the "cost function" (15) more or less arbitrarily as follows $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.2$ and $\varepsilon_3 = 0.7$. Then he solved (16) for enough values of C^* to map out the correspondence between volume *C* and the corresponding optimal combination of x_i's. He chose then for illustration to display the cost as a function of volume.

Figure 3[27]



He then showed the solution for the point where full capacity both for the shoemaker and the tailor has been reached. Starting from table C modifying it optimally using both partakers' percentages and rationing-percentages until the capacity of both the shoemaker and the tailor has been reached, results in table E.

Table E	**Is allowed to buy from**			**Total**
	Shoemaker	**Tailor**	**Farmer**	
Shoemaker	0.00	75.83	149.17	225
Tailor	65.93	0.00	374.07	440
Farmer	159.16	364.11	0.00	523.26
Total	225.08	439.94	523.24	1188.29

Cf. Frisch (1934a), (9.27).

The optimal values of partakers' percentages and rationing-percentages for this solution were:

$w_1 = -0.131338$, $w_2 = -0.086695$ and $w_3 = -0.428517$

$x_1 = -0.089193$, $x_2 = 0.142452$ and $x_3 = 0.125775$

The total volume thus increased from 650 to 1188.29, a quite considerable gain, but at the cost of forcing the tailor to buy more from the farmer and less from the shoemaker, and similarly with the farmer.

The Circulation model thus delivered something in terms of allocation improvements, derived in a consistent and stringent setting. Frisch's general can separate from this particular

[27] Frisch (1934a), Figure 1, p.317.

numerical example be said to be a model for coordination and allocation improvements, starting from agents' stated preferences and weighing higher overall transactions to the benefits of all against the (utility) cost of deviation from individual preferences. Hence, another context for Frisch's contribution would be interpersonal utility comparisons.

There are, however, a number of questions that one can raise wit regard to Frisch's model this he does not touch upon. One is the price issue, prices seems to be taken for the outside. There is thus an outside market, exactly how it functions is not known, but clearly there is, or could be some choice of trading within the exchange organization or outside, raising the issue of strategic behaviour, or in general game aspects. Neither is the incentive aspect of stating correct preferences or not, discussed by Frisch.

4. Numerical workability and mathematical requirements

Was the calculation for three partakers undertaken by Frisch as rendered above a workable scheme for a considerable greater number of partakers? Frisch seemed in little doubt about this and added some remarks on the amount of calculation needed. He calculated the number of multiplications needed depending upon the size of the request table and the number of iterations to get the $[Q_{ij}]$ matrix.

Then for the cost curve the key parameters were the number of points on the cost curve, the number of conditional factors needed in the determination of one such point and the average number of steps needed in the convergence process.

For 50 partakers Frisch and reasonable assumption about the other parameters Frisch estimated the calculations with the standard equipment that he had in his laboratory, to require 600 weeks of human effort. "If the work is to be completed in two weeks time, a staff of some 300 computors would thus be needed. … But even assuming that a staff of 300 people is needed, would this be a big item to reckon with? I think it is right to say that it would – even in small country – be nothing but a *quantité negligible* if the system actually was able to secure some of the results aimed at."

With regard to the mathematical results needed to support Frisch's needs, they were to a large extent also results needed for input-output analysis. Frisch proved the results he needed without checking up on the literature. Some results he knew were not new, but he claimed to have derived them "simpler and more systematic than the ones usually found in the literature." Other results like those he called lemmas 1,2 and 3 he claimed not to have seen before.[28]
Frisch's lemma 1 is stated differently, but clearly can be reformulated as a wellknown result today, namely that for an input-output or request matrix $[\boldsymbol{\alpha}]$ with non-negative elements and row sums less than or equal to zero, $[\mathbf{I}-\boldsymbol{\alpha}]$ is invertible and the inverse has only non-negative elements.[29] His lemma 2 showed the result he used in solving (2) above, namely that the adjoint of a request table has all rows identical. Lemma 3 is equivalent to all elements of the adjoint being positive. Frisch stated his results, however, in a way which may have made them less easy to observe for less proficient readers.

To solve (9) and find the coefficient matrix $[Q_{ij}]$ in (12) Frisch uses a power series expansion to calculate $[Q_{ij}]$. Starting from (9) above

$$\Sigma_k c_k(\alpha_{ki} - e_{ki}) = r_i$$

[28] Frisch (1934a), p.422.

[29] The mathematician Karl Goldziher sent Frisch in January 1935 a proof of lemma 1 by Paul Erdös. Erdös drew on theorems by Minkowski and Artin.

rewritten as a matrix equation

$$\mathbf{c}[\boldsymbol{\alpha} - \mathbf{I}] = \mathbf{r}$$

By postmultiplication of $\boldsymbol{\alpha}$ and substitution one can easily derive:[30]

$$\mathbf{c}\boldsymbol{\alpha}^{\mathbf{N}} = \mathbf{c} + \mathbf{r}(\mathbf{I} + \boldsymbol{\alpha} + \boldsymbol{\alpha}^{2} + ... + \boldsymbol{\alpha}^{\mathbf{N}-1})$$

This formulation encompasses the usual power series expansion of the Leontief inverse, but Frisch worked on the closed model and the power expansion on the right does not converge. Neither does $\boldsymbol{\alpha}^{\mathbf{N}}$ converge towards zero. Frisch found that the solution of (9) can be written as stated in (11) and (12) with $\mathbf{Q}$ determined as

$$\mathbf{Q} = \lim_{\mathbf{N}\to\infty}[\mathbf{Q}^{\mathbf{N}}] \quad \text{with}$$

$$\mathbf{Q}^{\mathbf{N}} = \mathbf{1} \times (\boldsymbol{\alpha} - \mathbf{I}) + \mathbf{2} \times (\boldsymbol{\alpha}^{2} - \boldsymbol{\alpha}) +, ..., + \mathbf{N} \times (\boldsymbol{\alpha}^{\mathbf{N}} - \boldsymbol{\alpha}^{\mathbf{N}-1})$$

His mathematic formulation is influenced by his need for finding numerical solution and is less penetrable than they needed to be.[31]

5. The National Exchange Organization

INCOMPLETE

More on Frisch's ideas of a National Exchange Organization from Part B of the *Circulation* paper and other sources.

His involvement with efforts to establish exchange organizations in Norway, see Andvig (1986). Other aspects of Frisch Circulation ideas, interest in monetary reform, Wicksellian roots, etc, see Andvig (1978,1983,1988).

Frisch's interest in National Recovery Administration (NRA). Contact with Fisher in this regard, Fisher's attempt to let Frisch introduce his *Circulation Planning* ideas directly to Frankling Roosevelt in 1934. Visit to NRA in 1934, contact with Charles Roos who worked for NRA.

"If you were running the country, would you rather have Frisch's "Circulation Planning" or Tugwell's "Industrial Discipline" at your elbow." (Roos/Frisch, 13.02.35). Tugwell and NRA, see documents from American Planning Association.

6. Concluding remarks

Ragnar Frisch's *Circulation Planning* article has received relatively little attention, this may very well be due to Frisch's strong emphasis on numerical calculation, his somewhat impenetrable presentation, and also elements of arbitrariness in his assumptions.

The main problem discussed is how to counteract the contraction of production and circulation processes in depressed economies. In dealing with this problem Frisch also

[30] Frisch (1934a), (22.8), p.428.

[31] Even Frisch's pupil and successor Leif Johansen seems to go astray here. He credits Frisch for lacking "only a few comments in order to arrive at the now wellknown power series expansion of the inverse of the Leontief matrix" (Johansen, 1969, p.309). The power series expansion was not unknown at the time and Frisch was hardly unaware of it.

approaches more general problems of planning and coordination when a market mechanism is not an alternative. His general approach of basing the allocation on stated preferences, and balancing the allocation by weighing against each other the interests of the various parties, may be applied to situations widely different from the background which originally motivated Frisch's study.

Frisch had worked on choice situations and preferences in connection with his marginal utility studies around 1930, but it was after World War II that he did most of his work on estimating preferences and using preference functions in connection with modelling. Many of the ideas he then applied can be traced back to the *Circulation* paper.[32]

In connection with his somewhat stylized example of a simple multi-partaker economy in the paper he also treated mathematically a model that is identical to what was later known as the closed input-output model. He deals with this model as a step on the way of more general models structure. Frisch may receive some credit for having pioneered the closed input-output model, but there is no trace of Frisch's work having influenced Leontief or the work of the Kiel School to have influenced Frisch. And unlike Frisch, Leontief's ideas followed more from the ideas of Quesnay and the classical economists and focused directly on the production instead of circulation. This was the direction taken by the Kiel School at the time and what later became the New School in New York and Leontief's institute at New York University.

Clearly, an original contribution by Frisch is his formulation and attempted solution of a programming problem with a quadratic target function and linear constraints.

Among the few assessments of Frisch's *Circulation* paper is Johansen (1969) and Chipman (1998). Chipman views Frisch's scheme as being very close to the idea of the role of a central bank as lender of last resort. This was surely an assessment that would have been appreciated by Frisch, as the whole approach to him was as substitute for a an overhauled and reformed monetary system, not least with regard to the role of the central bank, which at the time was a private joint stock company, something Frisch found quite unsuitable. Some writers have also in recent years found Frisch's contribution an antecedent to modern discussion of banking and coordination.[33]

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[32] See Johansen (1974), also Johansen (1969).

[33] Bryant (1987,1992).

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