On the Solution of Stochastic Input Output-Models

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October 2004

Abstract

The static, open input output-model reads $(I-A)x = d$, where $A \geq 0$ is an $n \times n$ input matrix with $\sum_i a_{ij} < 1$, d and x are $1 \times n$ final demand and output vectors, respectively. In contrast to the familiar model the coefficients of A, d and x are regarded as random variables in this paper, since there are many influences on production and demand, and hence on the resulting output $X = (I - \mathbb{A})^{-1}d = \mathbb{L}d$, where $\mathbb{L} \geq 0$ denotes the Leontief-inverse of $\mathbb{A}$. Correlations within $\mathbb{A}$ or d can be allowed for, but may often be neglected. However, even for independent coefficients within $\mathbb{A}$ and d those within $\mathbb{L}$ and x are correlated.

Empirical input matrices are derived from a database from year 0 such that $(I - A^0)x^0 = d^0$ is fulfilled for this reference system. It is assumed that these values of A^0 and d^0 represent either a) expected values or b) modes of the underlying random variables $\mathbb{A}$ and D. If there is no other information, variances are assumed to be derivable by a 3σ-rule or in case b) by some additional information on unimodal distributions.

Approximations of $\mathrm{E}(\mathbb{L})$ and $Cov(L)$, and $\mathrm{E}(X)$ and $Cov(X)$ are deduced from the Jacobian of the mapping $g\colon \mathbb{A} \mapsto \mathbb{L}$ and the corresponding Hessian. The results simplify considerably if correlations within A and D are negligible, which seems realistic.

*The author thanks Johannes Becker, Christian Conrad, and Berthold Haag for helpful comments, Asmus Löwenhaupt and Arne Nägel for the simulations.

Crude probability regions for $\mathbb{L}$ and the solution x can be given, which may be improved, if knowledge of the distribution types of $\mathbb{A}$ and d is available. Assumption of normality may cause difficulty with respect to the requirement that $(I - \mathbb{A})$ has a non-negative inverse.

In a simulation it is assumed that the column vectors of $\mathbb{A}$ and normed final demand D have independent Dirichlet distributions, or its components Beta distributions $\mathrm{Be}(r, s)$ on $[0, 1]$, which seems reasonable for fractions. With these distributions no problem occurs with invertibility, since it can be ensured that the dominant eigenvalue of $\mathbb{A}$ is less than one. The parameters r and s in the simulation, based on German data, are derived from assumed $\mathrm{E}(Y)$ (or $\mathrm{mode}(Y)$) and $\mathrm{Var}(Y)$. A simple second approximation of $\mathbb{L}$ suggests that the coefficients of $\mathbb{L}$ and x may have Beta distributions of the second kind. This is confirmed by the simulations. Also, the two types of approximations of $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ show a good and satisfactory performance, resp. The theoretically expected approximate, relatively narrow, 2σ-regions contain 90–95 % of the simulation results for most coefficients; the theoretically derived approximate densities mainly accord with the histograms of the simulation.

1 Introduction

Input output-models are usually based on a deterministic input matrix A. However, the observed values of inputs should be seen as realisations of random variables since inputs are affected by random effects, e. g. data errors, compilation, aggregation, prices, factor substitution, technical progress, product and process mix. In the literature, only a small number of contributions start with a stochastic input matrix and draw some conclusions concerning the stochastic Leontief inverse $L(A)$ – for instance, on bounds of its expected values based on Jensen's inequality. However, it is difficult to derive the distributions of the elements of $L(A)$, even if the input coefficients are supposed to be normally distributed as is commonly assumed.

In this paper, the input coefficients are assumed to be beta distributed. The standard beta distribution has the domain of the input coefficients, the interval $[0, 1]$. It depends on two parameters which allow for a high degree of flexibility. In particular, great skewness is admitted. These properties seem to be adequate for modelling the distribution of the large number of very small input coefficients. A is assumed to have a dominant eigenvalue less than one, which is equivalent to existence and non-negativity of $L(A)$.

The derivation of the distributions of $L(\mathbb{A})$ seems out of reach. However, for good lower bounds of $L(\mathbb{A})$ the densities will be given for its diagonal elements, as well as the first two moments for the other elements. The latter result is achieved by applying an approximation of the density of a product of beta random variables proposed by Fan (1991).

A first proxy of the parameters of the beta distributions may even be computed from a single input-output table.

2 Properties of the Beta Distribution

Density of the standard beta distribution $Be(r, s)$:

$$f_X(x) = \frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1} \mathbb{1}_{[0,1]}(x) \quad \text{for } r, s > 0,$$

where $B(r, s)$ denotes the beta function ($B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$).

$$X \sim Be(r,s) \Longrightarrow 1 - X \sim Be(s,r)$$

Moments:

$$\mathrm{E}(X) = \mu_X = \frac{r}{r+s}, \quad \mathrm{E}(X^2) = \frac{r(r+1)}{(r+s)(r+s+1)} = \frac{\mu_X}{1 + s/(r+1)},$$

$$\mathrm{Var}(X) = \sigma_X^2 = \mu_2 = \frac{rs}{(r+s)^2(r+s+1)} = \frac{\mu_X\,(1-\mu_X)}{r+s+1},$$

$$E((X-\mu)^3) = \mu_3 = \mu_2 \frac{2(s-r))}{(r+s)(r+s+2)} = \mu_2 \frac{2(1-2\mu)}{r+s+2},$$

$$\mu_3 > 0 \;\Leftrightarrow\; s > r.$$

If $r, s > 1$, which will be an underlying parameter restriction in the paper, then $\mathrm{Var}(X) < 1/12$ and f_X is unimodal with mode

$$m_X = \frac{r-1}{r+s-2} < \mu_X,$$

$$m_X < \mu_X \;\Leftrightarrow s > r.$$

Distribution of $Y = (1-X)^{-1}$ and moments

The theorem on the density of a function g of a random variable X yields

$$f_Y(y) = \frac{1}{B(r,s)} \left(1 - \frac{1}{y}\right)^{r-1} \left(\frac{1}{y}\right)^{s+1} \mathbb{1}_{[1,\infty)}(y),$$

or with $X = 1 - \frac{1}{Y}$:

$$f_Y(y) = (1-x)^2 f_X(x) \mathbb{1}_{(0,1)}(x).$$

Thus, the beta density of X is dampened by the factor $(1-x)^2$ and transferred to the domain $[1,\infty)$. The factor is decreasing with $x \in [0,1]$. The random variable $Y' := Y - 1 = \frac{X}{1-X}$ has a beta distribution of the second kind $(Be^*(r,s))$(see Härtter 1987, p.162) with density

$$f_{Y'}(y') = (B(r,s))^{-1}(y')^{r-1}(1+y')^{-(r+s)}\mathbb{1}_{[0,\infty)}(y').$$

It is known that for $m, n \in \mathbb{N}$

$$X \sim Be^*\big(\frac{m}{2}, \frac{n}{2}\big) \Rightarrow \frac{n}{m} X \sim F(m,n),$$

so that an F-density results for $Z := \frac{n}{m}Y' = \frac{n}{m}\frac{X}{1-X}$ with $\mathrm{E}(Z) = \frac{n}{n-2}$.

Moments are derived as follows

$$\begin{aligned}
\mathrm{E}(Y) &= \frac{B(r,s-1)}{B(r,s)} = \frac{r+s-1}{s-1} = 1 + \frac{r}{s-1} > \frac{1}{1-\mu_X} = 1 + \frac{r}{s}, \\
\mathrm{E}(Y^2) &= \frac{B(r,s-2)}{B(r,s)} = \frac{(r+s-1)(r+s-2)}{(s-1)(s-2)} = \left(1 + \frac{r}{s-1}\right)\left(1 + \frac{r}{s-2}\right), \\
\mathrm{Var}(Y) &= \frac{r\,(r+s-1)}{(s-1)^2(s-2)} = \mu_Y\,(\mu_Y - 1)\frac{1}{s-2}, \\
\text{and } \ \mathrm{E}(Y') &= \frac{r}{s-1}, \quad \mathrm{Var}(Y') = \mu_{Y'}(\mu_{Y'}+1)\frac{1}{s-2},
\end{aligned}$$

provided that $s > 2$. In the sequel, this assumption for the beta parameter s will be made, whenever $\mathrm{Var}(Y)$ or $\mathrm{E}(Y^2)$ is mentioned. The mode is given with

$$m_Y = \frac{r+s}{s+1} = 1 + \frac{r-1}{s+1} < \frac{1}{1-m_X} = 1 + \frac{r-1}{s-1} < \mu_Y.$$

Densities of X and $Y = 1/(1-X)$

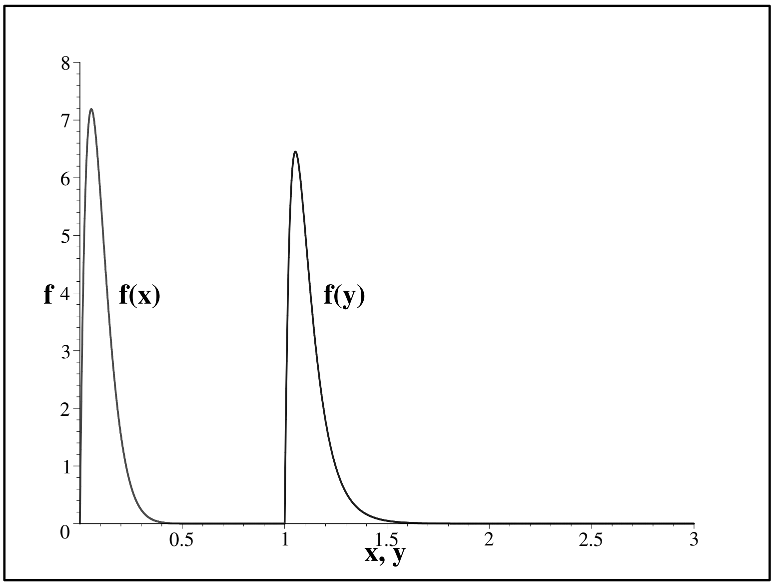

Figure 1: $r = 2, s = 18 \Rightarrow \mu_X = 0.1$

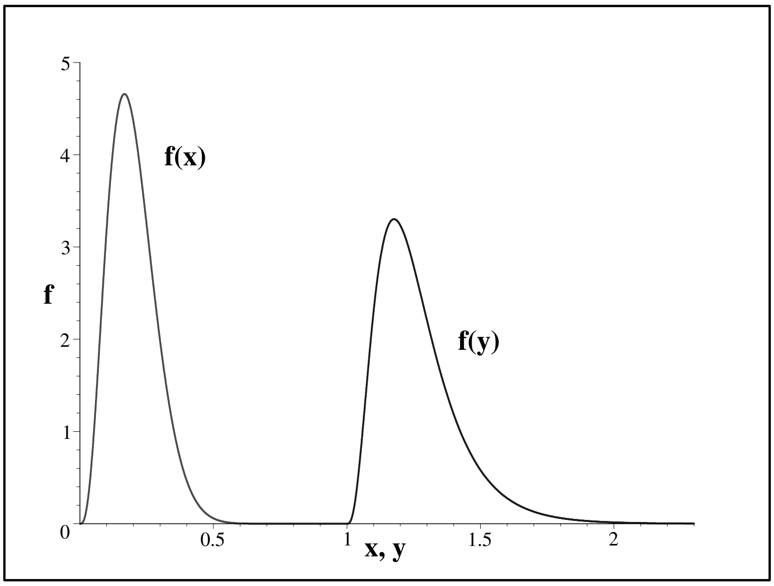

Figure 2: $r = 4, s = 16 \Rightarrow \mu_X = 0.2$

Taylor-approximation for moments of $Y = (1-X)^{-1}$

(second order approximation at μ_X, cf. Dudewicz, Mishra, (1988, p. 264))

$$\begin{aligned}
\mathrm{E}(Y) &\approx g(\mu_X) + g''(\mu_X)\frac{\sigma_X^2}{2} \\
&\le \frac{1}{1-\mu_X}\left(1+\frac{r}{s\,(r+s+1)}\right) \le \frac{1}{1-\mu_X}\left(1+\frac{\mu_X}{s}\right) \\
\mathrm{Var}(Y) &\approx \big(g'(\mu_X)\big)^2\sigma_X^2 = \sigma_X^2\left(1+\frac{r}{s}\right)^4 \ge \frac{r}{s^2}\left(1+\mu_X\right),
\end{aligned}$$

whereby in the first line, the first inequality holds for $\mu_3 > 0$, i. e. for skewness to the right, which is a reasonable assumption for input coefficients. It is equivalent to $s > r$.

The figures show two beta densities X with corresponding transformed distributions of $Y = 1/(1-X)$ which will be relevant for the densities of the diagonal coefficients of the Leontief inverse. Remarkable is the skewness to the right of f_X, which seems adequate for the great number of very small input coefficients. A normal density does not seem to fit this situation, even if truncated, because of its symmetry.

3 First Approximation of the Leontief Inverse and its Moments

If all the elements A_{ij} of the input matrix A have beta distributions, it seems impossible to determine the densities of the elements L_{ij} of the Leontief inverse $L(A)$, since they depend on a ratio of two determinants which are sums of products of the A_{ij}.

Therefore, an approximation of $L(A)$ for a deterministic matrix A is regarded which will allow an approximation of the distributions of its diagonal elements and the moments of the other elements after returning to random input coefficients.

The approximation by minors (Kogelschatz, 1978), which gives good lower bounds for L, reads in its simplest form

$$\tilde{l}_{ii} = \frac{1}{1-a_{ii}} \;\forall\; i \quad \text{and} \quad \tilde{l}_{ij} = \frac{a_{ij}}{(1-a_{ii})(1-a_{jj})} \;\forall\; i,j \text{ with } j \ne i.$$

In the following, this approximation is studied for random variables $A_{ij} \sim Be(r_{ij}, s_{ij})$.

For the diagonal elements of L the densities are given as for $Y = 1/(1-X)$ above:

$$f_{\tilde{L}_{ii}}(\tilde{l}_{ii}) = \frac{1}{\tilde{l}_{ii}^2} \frac{1}{B(r_{ii}, s_{ii})} \left(1 - \frac{1}{\tilde{l}_{ii}}\right)^{r_{ii}-1} \left(\frac{1}{\tilde{l}_{ii}}\right)^{s_{ii}-1} \mathbb{1}_{[1,\infty)}(\tilde{l}_{ii}) \quad \forall\ i.$$

Furthermore,

$$\mathrm{E}(\tilde{L}_{ii}) = 1 + \frac{r_{ii}}{s_{ii}-1} > 1 + \frac{r_{ii}}{s_{ii}} = \frac{1}{1-\mathrm{E}(A_{ii})}$$

and

$$\begin{aligned}\mathrm{Var}(\tilde{L}_{ii}) &= \mathrm{E}(\tilde{L}_{ii}) \frac{\mathrm{E}(\tilde{L}_{ii}) - 1}{s_{ii}-2} = \left(1 + \frac{r_{ii}}{s_{ii}-1}\right) \frac{r_{ii}}{(s_{ii}-1)(s_{ii}-2)} \\ &\approx \mathrm{Var}(A_{ii}) \left(1 + \frac{r_{ii}}{s_{ii}}\right)^4,\end{aligned}$$

according to the Taylor series approximation in section 2.

For the off-diagonal elements of L the situation is complicated. The distribution of $\tilde{L}_{ij}$ depends on a ratio and a product of beta random variables. In the literature, several very complicated formulae can be found, mainly for special cases of products (Johnson, Kotz, Balakrishnan (1995, p. 256f)). A very good approximation of a product of beta random variables, which is usually not beta distributed, was suggested by Fan (1991, p. 4045). By construction, it ensures exact first two moments and, furthermore, performs very well in approximating higher moments as his computations show.

Fan's approximation theorem

If $X_i \sim Be(r_i, s_i)$, X_i are independent random variables and $Z = \prod_{i=1}^k X_i$, then Z has an approximate $Be(R, S)$ distribution with true first two moments, where

$$R := \frac{U\,(U-T)}{T-U^2}, \qquad S := \frac{(1-U)(U-T)}{T-U^2}$$

$$\text{and} \quad U := \prod_{i=1}^k \frac{r_i}{r_i+s_i}, \qquad T := \prod_{i=1}^k \frac{r_i}{r_i+s_i} \cdot \frac{r_i+1}{r_i+s_i+1}$$

By the way, in the book by Johnson, Kotz, Balakrishnan (1995, p. 262), where Fan's result is reported, the formula for p has to be multiplied by $(S-T)$, where p and S correspond to R and U here. The following interpretations result from the independence assumption:

$$U = \prod_{i=1}^{k} \mathrm{E}(X_i) = \mathrm{E}(Z), \qquad T = \prod_{i=1}^{k} \mathrm{E}(X_i^2) = \mathrm{E}(Z^2),$$

$$\text{hence} \quad T - U^2 = \mathrm{Var}(Z).$$

Obviously, $T = UV$ where $V = \prod_{i=1}^{k}(r_i+1)/(r_i+s_i+1)$.

Fan's method will now be applied to the approximation by minors with random input coefficients A_{ij} and $\tilde{L}_{ij}$, where $A_{ij} \sim Be(r_{ij}, s_{ij})$ with $r_{ij}, s_{ij} > 1$. In order to get some information about the distribution of the off-diagonal elements of $\tilde{L}_{ij}$, the denominator $(1-A_{ii})(1-A_{jj}) =: Z_{ij}$ is considered first. It seems realistic to assume the diagonal input coefficients (intrasectoral inputs) to be independent random variables. As Z_{ij} is a product of $(1-A_{ii}) \sim Be(s_{ii}, r_{ii})$ and $(1-A_{jj}) \sim Be(s_{jj}, r_{jj})$ it can be approximated by $\tilde{Z}_{ij} \sim Be(S_{ij}, R_{ij})$ with the same first two moments using the notation from above with additional indices ($i \neq j$ in the following)

$$S_{ij} = \frac{U_{ij}\,(U_{ij}-T_{ij})}{T_{ij}-U_{ij}^2}, \qquad R_{ij} = \frac{(1-U_{ij})\,(U_{ij}-T_{ij})}{T_{ij}-U_{ij}^2}$$

with

$$U_{ij} = \frac{s_{ii}}{r_{ii}+s_{ii}} \cdot \frac{s_{jj}}{r_{jj}+s_{jj}} \quad \text{and} \quad T_{ij} = U_{ij}\frac{s_{ii}+1}{r_{ii}+s_{ii}+1} \cdot \frac{s_{jj}+1}{r_{jj}+s_{jj}+1}$$

In the next step, the distribution of $Y_{ij} := 1/\tilde{Z}_{ij} = 1/(1-\tilde{Z}'_{ij})$, where $\tilde{Z}'_{ij} := 1-\tilde{Z}_{ij}$ with $\tilde{Z}'_{ij} \sim Be(R_{ij}, S_{ij})$, is given as that of $Y := 1/(1-X)$ above

$$f_{Y_{ij}}(y_{ij}) = \frac{1}{B(R_{ij}, S_{ij})}\left(1-\frac{1}{y_{ij}}\right)^{R_{ij}-1}\left(\frac{1}{y_{ij}}\right)^{S_{ij}+1} \mathbb{1}_{[1,\infty)}(y_{ij}).$$

The expected value of $1/\tilde{Z}_{ij}$ is obtained as

$$\mathrm{E}\left(\frac{1}{\tilde{Z}_{ij}}\right) = 1 + \frac{R_{ij}}{S_{ij}-1} = \ldots = \frac{1+U_{ij}-2V_{ij}}{2U_{ij}-V_{ij}(1+U_{ij})} > \frac{1}{\mathrm{E}(Z_{ij})} = 1 + \frac{R_{ij}}{S_{ij}},$$

where $U_{ij} = E(Z_{ij})$ and $V_{ij} \geq U_{ij}$ for r_{kk}, s_{kk} large. In this case, $\mathrm{E}(1/\tilde{Z}_{ij}) \geq (1 - U_{ij})/(U_{ij} - U_{ij}^2) = 1/U_{ij} = 1/\mathrm{E}(Z_{ij})$. Furthermore,

$$\mathrm{E}\left(\frac{1}{\tilde{Z}_{ij}^2}\right) = \left(1 + \frac{R_{ij}}{S_{ij} - 1}\right)\left(1 + \frac{R_{ij}}{S_{ij} - 2}\right).$$

The first two moments of $1/\tilde{Z}_{ij}$ might deviate slightly from those of $1/Z_{ij}$, because the reciprocal is taken of the approximation $\tilde{Z}_{ij}$ instead of Z_{ij}. However, since the latter random variables have equal first two moments, this possibly minute error will be neglected.

In the last step, the numerator A_{ij} of $\tilde{L}_{ij} = A_{ij}/Z_{ij}$ has to be taken into account. The distribution of $\tilde{L}_{ij}$ which is a product of a beta and a transformed approximate beta random variable will not be investigated here. Evidently, no beta density can result since the domain of $\tilde{L}_{ij}$ is $\mathbb{R}_+$.

However, the density of $\tilde{L}_{ij}$, which is mainly concentrated on $[0, 1]$, may also be approximated by a suitable beta distribution (of the second kind) with the same first two moments. For empirical input matrices A the off-diagonal elements of $L(A)$ are smaller than 1.

In the sequel, expected value and variance of $\tilde{L}_{ij}$ will be derived. As before, Z_{ij} is substituted by $\tilde{Z}_{ij}$. Additionally, it is assumed that A_{ij} and A_{ii}, A_{jj} are pairwise independent which may be questionable. The intrasectoral input coefficients A_{ii}, A_{jj} of different sectors have been supposed to fulfil this requirement; A_{ij} and A_{jj} of the same sector (production process) are probably expected to show a slight negative correlation, e.g., if this is neglected, the expected value of $\tilde{L}_{ij}$ turns out to be

$$\begin{aligned}
\mathrm{E}(\tilde{L}_{ij}) &= \mathrm{E}(A_{ij})\ \mathrm{E}\left(\frac{1}{\tilde{Z}_{ij}}\right) = \mathrm{E}(A_{ij})\left(1 + \frac{R_{ij}}{S_{ij} - 1}\right) \\
&= \mathrm{E}(A_{ij})\left(1 + \frac{r_{ii}}{s_{ii} - 1}\right)\left(1 + \frac{r_{jj}}{s_{jj} - 1}\right) =: \mathrm{E}(A_{ij})\, K_{ij} \\
&> \mathrm{E}(A_{ij})\frac{1}{\mathrm{E}(Z_{ij})} = \mathrm{E}(A_{ij})\frac{(r_{ii} + s_{ii})(r_{jj} + s_{jj})}{s_{ii}s_{jj}} \\
&= \mathrm{E}(A_{ij})\left(1 + \frac{r_{ii}}{s_{ii}}\right)\left(1 + \frac{r_{jj}}{s_{jj}}\right) =: \mathrm{E}(A_{ij})\, k_{ij}.
\end{aligned}$$

Obviously, $\tilde{L}_{ij}$ has not only a greater mean but also greater variance than

A_{ij}. This is shown by the usual computation of a variance

$$\mathrm{Var}(\tilde{L}_{ij}) = \mathrm{E}(\tilde{L}_{ij}^2) - \left(\mathrm{E}(\tilde{L}_{ij})\right)^2 .$$

$$\begin{aligned} \mathrm{E}(\tilde{L}_{ij}^2) &= \mathrm{E}\left(A_{ij} \cdot \frac{1}{\tilde{Z}_{ij}}\right)^2 = \mathrm{E}(A_{ij}^2)\ \mathrm{E}\left(\frac{1}{\tilde{Z}_{ij}^2}\right) \\ &= \mathrm{E}(A_{ij}^2)\left(1 + \frac{R_{ij}}{S_{ij}-1}\right)\left(1 + \frac{R_{ij}}{S_{ij}-2}\right) \\ &> \mathrm{E}(A_{ij}^2)\left(1 + \frac{R_{ij}}{S_{ij}-1}\right)^2 . \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(\tilde{L}_{ij}) = \mathrm{E}(A_{ij}^2)\left(1 + \frac{R_{ij}}{S_{ij}-1}\right)\left(1 + \frac{R_{ij}}{S_{ij}-2}\right) \\ - \left(\mathrm{E}(A_{ij})\right)^2 \left(1 + \frac{R_{ij}}{S_{ij}-1}\right)^2 \end{aligned}$$

Since

$$1 + \frac{R_{ij}}{S_{ij}-2} < \left(1 + \frac{R_{ij}}{S_{ij}-1}\right) \cdot \frac{S_{ij}-1}{S_{ij}-2} = K_{ij}\frac{S_{ij}-1}{S_{ij}-2}$$

$$\mathrm{Var}(A_{ij}) K_{ij}^2 \frac{S_{ij}-1}{S_{ij}-2} > \mathrm{Var}(\tilde{L}_{ij})$$

and

$$\begin{aligned} \mathrm{Var}(\tilde{L}_{ij}) &> \mathrm{Var}(A_{ij})\left(1 + \frac{R_{ij}}{S_{ij}-1}\right)^2 = \mathrm{Var}(A_{ij})\, K_{ij}^2 \\ &> \mathrm{Var}(A_{ij})\left[\left(1 + \frac{r_{ii}}{s_{ii}}\right)\left(1 + \frac{r_{jj}}{s_{jj}}\right)\right]^2 = \mathrm{Var}(A_{ij})\, k_{ij}^2 \end{aligned}$$

which follows from the lower bound of $\mathrm{E}(\tilde{L}_{ij})$ mentioned above.

Both lower bounds for $\mathrm{E}(\tilde{L}_{ij})$ and $\mathrm{Var}(\tilde{L}_{ij})$, respectively, differ from the corresponding moments of A_{ij} by augmenting factors which only depend on the ratio of beta parameters r and s of the corresponding diagonal elements. The factor for $\mathrm{Var}(\tilde{L}_{ij})$ is just obtained by squaring the factor for $\mathrm{E}(\tilde{L}_{ij})$. Also the 3σ-region will be extended by the latter factor. For the values of the example in Figure 2, where $\mu = 0.2$ is realistic for average diagonal elements A_{ii} of input matrices with 10 to 15 sectors, one would obtain $E(\tilde{L}_{ij}) > 1.604 \cdot \mathrm{E}(A_{ij})$ and $\mathrm{Var}(\tilde{L}_{ij}) > 2.574 \cdot \mathrm{Var}(A_{ij})$.

4 Proxies of Beta Parameters

After these theoretical considerations the question arises of how to estimate the parameters r and s of the beta distributions within this model for the input coefficients. Estimation from a time series is doubtful since coefficients are changing over time for several reasons. Only input-output tables based on fixed prices should be used. For estimation procedures of the parameters r, s (see Johnson, Kotz, Balakrishnan (1995, p. 221–238)) and with special focus on skewness Moitra (1990).

Even from a single input matrix a first proxy for r, s may be given. A practical proposal made by Bamberg (1976, p. 16) for the moments of an apriori distribution in Bayesian estimation can be applied here. He suggests the mode m as a proxy of μ and asks for the greatest relevant deviation d from μ. According to the 3σ-rule, which says that 99.7 % of the probability of a normal density lies in the 3σ-region and 89 % according to Chebychev's inequality for the least favorable distribution, he suggests to take $d/3$ as a proxy of σ. For unimodal beta densities this probability will be close to that of the normal distribution.

For stochastic input coefficients one may modify this proposal as follows. Take the observed value a_{ij} a) as expected value μ_{ij} of the distribution of A_{ij} and also as deviation d or b) as mode m_{ij}. It is assumed that $a_{ij} \neq 0$ and that the probability that A_{ij} exceeds $2\mu_{ij} = \mu_{ij} + 3\sigma_{ij}$ may be neglected.

When r and s are integers, this probability (or that of a 3σ-region) can be computed by means of the well known relation between the distribution function F of the $Be(r, s)$ and that of the binomial distribution

$$F(x) = \sum_{k=r}^{n} \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{where } n = r + s - 1 \text{ and } x \in [0, 1].$$

For the beta distributions in the figures $P(X > 2\mu) \leq 1.14\,\%$.
The two equations for the moments of A_{ij} read

$$\mu_{ij} := \mathrm{E}(A_{ij}) = \frac{r_{ij}}{r_{ij} + s_{ij}}$$

$$\sigma_{ij}^2 := \mathrm{Var}(A_{ij}) = \frac{\mu_{ij}\,(1-\mu_{ij})}{r_{ij} + s_{ij} + 1}$$

which can be solved for r_{ij} and s_{ij}:

$$r_{ij} = \left(\frac{\mu_{ij}}{\sigma_{ij}}\right)^2 (1-\mu_{ij}) - \mu_{ij}\,, \quad s_{ij} = r_{ij}\left(\frac{1}{\mu_{ij}} - 1\right).$$

Case a): Assume $a_{ij} = \mu_{ij} = 3\sigma_{ij}$. Then

$$r_{ij} = 9 - 10a_{ij} \quad \text{and} \quad s_{ij} = r_{ij}(1/a_{ij} - 1),$$

provided $a_{ij} \neq 0$, else let $a_{ij} = 10^{-3}$, e.g..

With respect to the skewness (to the right) of adequate beta densities it might be preferred to take an assymetric, larger region, say, up to $3a_{ij}$ with $a_{ij} = 2\sigma_{ij}$ in order to capture more of the probability mass. This might fit better for very small a_{ij} and would yield $r_{ij} = 4 - 5a_{ij}$ instead.

In case b), which corresponds to the idea of maximum likelihood estimation, $a_{ij} = m_{ij}$ and again $\mu_{ij} = 3\sigma_{ij}$ are assumed. Then, according to section 2,

$$\mu'_{ij} = m_{ij}\frac{r_{ij}}{r_{ij}-1}\frac{r_{ij}+s_{ij}-2}{r_{ij}+s_{ij}} = m_{ij}\mu_{ij}\left(1 + \frac{s_{ij}-1}{r_{ij}-1}\right)$$

and in the equations of a) μ_{ij} has to be substituted by μ'_{ij}, a third unknown.[1] In order to eliminate μ'_{ij} it is proposed to compute r_{ij} and s_{ij} according to a) and to insert these values into the last equation. With the resulting μ'_{ij} instead of μ_{ij} in the equations of a) solve this system again for r'_{ij} and s'_{ij}.

Example:

$$\begin{aligned} \text{case a)} \quad & a_{ij} = 0.1 \Rightarrow r_{ij} = 8, s_{ij} = 72 \text{ and} \\ & a_{ij} = 0.2 \Rightarrow r_{ij} = 7, s_{ij} = 28 \\ \text{case b)} \quad & a_{ij} = 0.1 \Rightarrow \mu'_{ij} = 0.11 \Rightarrow r'_{ij} = 7.9, s'_{ij} = 62.9 \\ & a_{ij} = 0.2 \Rightarrow \mu'_{ij} = 0.22 \Rightarrow r'_{ij} = 6.8, s'_{ij} = 27.2. \end{aligned}$$

Here, $a_{ij} = \mu_{ij}$ from a) is increased by about 10 % to μ'_{ij} in b), which means that μ'_{ij} exceeds $m_{ij} = a_{ij}$ by this percentage. However, for the densities in figure 1 and 2, μ exceeds m by 80 % and 20 %, respectively.

With the larger region of $3a_{ij}$ one obtains parameter proxies more similar to those in the figures:

$$\begin{aligned} \text{case a)} \quad & a_{ij} = 0.1 \Rightarrow r_{ij} = 3.5, s_{ij} = 31.5 \text{ and} \\ & a_{ij} = 0.2 \Rightarrow r_{ij} = 3, s_{ij} = 12 \\ \text{case b)} \quad & a_{ij} = 0.1 \Rightarrow \mu'_{ij} = 0.132 \Rightarrow r'_{ij} = 3.34, s'_{ij} = 21.96 \\ & a_{ij} = 0.2 \Rightarrow \mu'_{ij} = 0.26 \Rightarrow r'_{ij} = 2.7, s'_{ij} = 7.68. \end{aligned}$$

[1] The prime is used here to distinguish case b) from case a) and does not refer to the beta distribution of the second kind as in section 2.

Now, μ'_{ij} exceeds m_{ij} by about 30 %.

Proxies of the parameters of the density of $\tilde{L}_{ij}$ (for $i \neq j$), if assumed to be approximated by a beta distribution, are derived from its moments

$$\tilde{\mu}_{ij}^L := \mathrm{E}(\tilde{L}_{ij}) = \frac{R_{ij}}{R_{ij} + S_{ij}} = [E(A_{ij})K_{ij} = \mu_{ij}K_{ij}]$$

and

$$\left(\tilde{\sigma}_{ij}^L\right)^2 := \mathrm{Var}(\tilde{L}_{ij}) = \frac{\tilde{\mu}_{ij}^L(1-\tilde{\mu}_{ij}^L)}{R_{ij} + S_{ij} + 1} \gtrapprox \left[\mathrm{Var}(A_{ij})K_{ij}^2 = \sigma_{ij}^2 K_{ij}^2\right]$$

where, in square brackets, the simple approximations from section 3 are given with

$$K_{ij} = \left(1 + \frac{R_{ij}}{S_{ij} - 1}\right) = \left(1 + \frac{r_{ii}}{s_{ii} - 1}\right)\left(1 + \frac{r_{jj}}{s_{jj} - 1}\right).$$

With the same procedure just applied to A we obtain for L

$$\begin{aligned} R_{ij} &\approx \left(\frac{\tilde{\mu}_{ij}^L}{\tilde{\sigma}_{ij}^L}\right)^2 (1-\tilde{\mu}_{ij}^L) - \tilde{\mu}_{ij}^L = \frac{R_{ij}}{S_{ij}}(R_{ij} + S_{ij} + 1)(1-\tilde{\mu}_{ij}^L) - \tilde{\mu}_{ij}^L \\ S_{ij} &\approx R_{ij}\left(\frac{1}{\tilde{\mu}_{ij}^L} - 1\right). \end{aligned}$$

Example for case a)
Assuming $a_{ii} = a_{jj} = 0.2$ yields as above $r_{ii} = r_{jj} = 7,\ s_{ii} = s_{jj} = 28$, hence

$$\begin{array}{llllll} \text{i)} & a_{ij} = \mu_{ij} = 0.1 & \Rightarrow & r_{ij} = 8, & s_{ij} = 72, & \mathrm{Var}(A_{ij}) = 0.00\bar{1} \\ & \quad \Rightarrow \tilde{\mu}_{ij}^L = 0.159, & & R_{ij} \approx 7.41, & S_{ij} \approx 39.33, & \mathrm{Var}(\tilde{L}_{ij}) \approx 0.0028. \\ \text{ii)} & a_{ij} = \mu_{ij} = 0.2 & \Rightarrow & r_{ij} = 7, & s_{ij} = 28, & \mathrm{Var}(A_{ij}) = 0.00\bar{4} \\ & \quad \Rightarrow \tilde{\mu}_{ij}^L = 0.317, & & R_{ij} \approx 5.83, & S_{ij} \approx 12.55, & \mathrm{Var}(\tilde{L}_{ij}) \approx 0.0112. \end{array}$$

When passing from A_{ij} to $\tilde{L}_{ij}$, in the simple approximation the augmenting factors of μ_{ij} and σ_{ij}^2, resp., here are $K_{ij} = 1.586$ and $K_{ij}^2 = 2.515$.

Similarly, a beta distribution of the second kind with the same first two moments can be chosen for approximating the density of $\tilde{L}_{ij}$ with the advantage that both have the same domain. According to section 2 with $Y' = \tilde{L}_{ij}$ for $i \neq j$, for case a) [2]

$$E(\tilde{L}_{ij}) = \frac{R_{ij}}{S_{ij} - 1}$$

and

$$Var(\tilde{L}_{ij}) = \frac{\tilde{\mu}_{ij}^L(\tilde{\mu}_{ij}^L + 1)}{S_{ij} - 2}$$

then have to be solved for the parameters R'_{ij}, S'_{ij} of the beta distribution of the second kind which is supposed to yield a better density proxy of L_{ij}:

$$\begin{aligned} R'_{ij} &= \tilde{\mu}_{ij}^L(S'_{ij} - 1) \approx \tilde{\mu}_{ij}^L \left(1 + \frac{\tilde{\mu}_{ij}^L(1 + \tilde{\mu}_{ij}^L)}{(\tilde{\sigma}_{ij}^L)^2}\right) \\ S'_{ij} &\approx 1 + \frac{R'_{ij}}{\mu_{ij}^L} = 2 + \frac{\tilde{\mu}_{ij}^L(1 + \tilde{\mu}_{ij}^L)}{(\tilde{\sigma}_{ij}^L)^2} > 2. \end{aligned}$$

For the previous example (case a) with $a_{ii} = a_{jj} = 0.2$

$$\begin{aligned} a_{ij} = 0.1 &\Rightarrow R'_{ij} \approx 10.57, S'_{ij} \approx 67.63 \\ a_{ij} = 0.2 &\Rightarrow R'_{ij} \approx 12.15, S'_{ij} \approx 39.31. \end{aligned}$$

5 Distribution of the Leontief Inverse

In this section, the distribution of the Leontief inverse of a stochastic input matrix $\mathbb{A} = (A_{ij})$ will be derived. In order to apply the theorem on the density of functions of random variables, an open set of the domain of $\mathbb{A}$ is needed which will be provided by the requirement $0 < A_{ij} < 1 \,\forall i, j$. The realizations of $\mathbb{A} = (A_{ij})$ and $\mathbb{L} = (L_{ij})$ are denoted by $A = (a_{ij})$ and $L = (l_{ij})$, resp. Again, the dominant eigenvalue of any A has to be assumed to be less than one, which is fulfilled, if all column sums of A do not exceed one and at least one of them is less than one, since A is irreducible. Assuming all of these sums smaller than one gives a suitable open domain.

Thus, transforming $\mathbb{A}$ into $\mathbb{L}(\mathbb{A})$ is described by a mapping

$$g : U := \times_{j=1}^n (S_j^n)^\circ \subset \mathbb{R}_{++}^{n^2} \to V \subset \mathbb{R}_{++}^{n^2},$$

[2]Here, the prime again refers to the beta distribution of the second kind.

where $(S_j^n)^\circ$ is the interior of the standard simplex in $\mathbb{R}^n$ and U, V are open sets in $\mathbb{R}^{n^2}$. Furthermore, $\tilde{g}(\mathbb{A}) := \mathbb{L}(\mathbb{A}) - I$ has the range $\mathbb{R}^{n^2}_{++}$ because $\mathbb{L}(\mathbb{A}) > I$, where I denotes the $n \times n$ identity matrix. $\mathbb{A} > 0$ implies that $\mathbb{L}$ and, furthermore, $L(A)$ is strictly increasing in every coefficient if any element of A is augmented.

Obviously, g is a bijection which is continuously differentiable and g^{-1}, as well. Hence, g is a diffeomorphism. The differentials are given by

$$\begin{aligned} dg &= d(I - \mathbb{A})^{-1} = d\mathbb{L} = \mathbb{L}(d\mathbb{A})\mathbb{L} \\ dg^{-1} &= d(I - \mathbb{L}^{-1}) = d\mathbb{A} = \mathbb{A}(d\mathbb{L})\mathbb{A}, \end{aligned}$$

both with range $\mathbb{R}^{n^2}_{++}$.

Hence, the theorem on the density of functions of random variables can be applied and yields

$$f_{\mathbb{L}}(L) = |\det J(I - L^{-1})|^{-1} f_{\mathbb{A}}(I - L^{-1}) \text{ for } L - I \in \mathbb{R}^{n^2}_{++}$$

and

$$f_{\mathbb{L}}(L) = |\det J(A_L)|^{-1} f_{\mathbb{A}}(A_L), \text{ for } A_L \in U,$$

where J is the Jacobian matrix of the mapping g and A_L denotes the input matrix A with image $L(A)$.

Since g maps matrix A onto a matrix L the functional matrix $J = \frac{\partial L}{\partial A}$ has to be carefully defined. This is done by rearranging an $n \times n$ matrix A into a $n^2 \times 1$ column vector writing the second column of A underneath the first and so forth (e.g., $a_{n+1} := a_{12}, a_{2n+1} := a_{13}, \ldots$). In the same way L is transformed into a column vector l. The $n^2 \times n^2$ functional matrix J consists of n^2 blocks of $n \times n$ submatrices J_{ij} given by $\frac{\partial l^i}{\partial a^j}$ where l^i and a^j are the i-th column of L and the j-th column of A, resp., for $i, j = 1, \ldots, n$. Thus, the diagonal blocks $J_{ii} = \frac{\partial l^i}{\partial a^i}$ capture the direct partial effects within sector i. The coefficients within each block J_{ij} are $j_{hk} = \frac{\partial l^i_h}{\partial a^j_k}$, for $h, k = 1, \ldots, n$.

The Jacobian matrix $(n^2 \times n^2)$ has the following form

$$J(A_L) = \left(\frac{\partial l_{ij}}{\partial a_{hk}} \right) = (l_{ih} l_{kj}) = (l_{ji} L)_{i,j} = L^T \otimes L$$

and the determinant

$$|J| = |L^T|^n |L|^n = |L|^{2n} \Rightarrow |J^{-1}| = |I - A_L|^{2n} = |(I - A_L)^T (I - A_L)|^n = |C|^n .$$

Hadamard's inequality for positive definite matrices yields an upper bound for $|J|$:

$$|C| \leq \prod_{j=1}^{n} c_{jj} = \prod_{j=1}^{n} ||e_j - a^j||_2^2 = \prod_{j=1}^{n} (||\check{a}^j||_2^2 + (1 - a_{jj})^2) = M'(A_L),$$

where $M' \approx \prod_{j=1}^{n}(1-a_{jj})^2 < 1$, if $(I-A_L)$ has a strictly column dominant diagonal, i.e. $1 - a_{jj} > \sum_{i \neq j} a_{ij}$, and if $2a_{ii} \geq a_{ij}\ \forall i$. The symbol $\check{a}^j$ denotes a vector a^j without the j-th component.

The Hadamard-Fischer inequality for (symmetric) positive definite matrices (cf. Horn/Johnson 1985, p. 485) may give an improved bound:

$$|C| \leq c_{nn} \prod_{j=1}^{n-1} \left(c_{jj} - \frac{c_{j,j+1}^2}{c_{j+1,j+1}} \right) = M \leq M' .$$

Hence $|J^{-1}| \leq M^n$, which leads to the following representation of the density of the Leontief inverse for any distribution of $\mathbb{A}$

$$\begin{aligned} f_{\mathbb{L}}(L) &= |L|^{-2n} f_{\mathbb{A}}(I - L^{-1}) = |I - A_L|^{2n} f_{\mathbb{A}}(A_L) \\ &\leq M^n f_{\mathbb{A}}(I - L^{-1}) = M^n f_{\mathbb{A}}(A_L) . \end{aligned}$$

6 Improved Approximation of Moments of $\mathbb{L}$ and X

6.1 Taylor expansion

The moments of a function g of a random variable X can be approximated by a (second order) Taylor expansion around μ_X by (cf. Dudewicz, Mishra (1988, p. 264), see section 2)

$$\mathrm{E}(g(X)) \approx g(\mu_X) + \frac{1}{2} g''(\mu_X)\, \sigma_X^2 ,$$
$$\mathrm{Var}(g(X)) \approx (g'(\mu_X))^2\, \sigma_X^2 .$$

These formulae can be generalized to $n \times n$ random matrices $\mathbb{A}$ and

$g : \mathbb{R}^n \mapsto \mathbb{R}^n$ as follows

$$\begin{aligned}
\mathrm{E}(g_{ij}(\mathbb{A})) &\approx g_{ij}(\mu) + \frac{1}{2}\,\mathrm{E}\left[(vec\Delta\mathbb{A})^T H^{ij}(\mu) vec\Delta\mathbb{A}\right] \\
&= g_{ij}(\mu) + \frac{1}{2} tr\left(H^{ij}(\mu) cov(\mathbb{A})\right) , \\
cov(g(\mathbb{A})) &\approx J(\mu) cov(\mathbb{A}) \left(J(\mu)\right)^T ,
\end{aligned}$$

where $\mu, \Delta\mathbb{A}, J$ and H denote $\mathrm{E}(\mathbb{A}), \mathbb{A} - \mathrm{E}(\mathbb{A})$, the Jacobian and the Hessian of g, respectively. The formula with the trace follows from a result on the expected value of a quadratic form (cf. Magnus, Neudecker, 1988, p. 247) and from $\mathrm{E}(\Delta\mathbb{A}) = 0$.

The Taylor approximations of moments are now applied to the stochastic input output-model with $g : \mathbb{A} \mapsto \mathbb{L}$ with $A^\mu := \mathrm{E}(\mathbb{A})$ and $L^\mu := \mathrm{E}(\mathbb{L})$. Recalling

$$J = L^T \otimes L \quad \text{or} \quad \frac{\partial l_{ij}}{\partial a_{rs}} \,|_\mu = l^\mu_{ir} l^\mu_{sj}$$

the Hessian $H^{ij}(\mu)$ is given with

$$\frac{\partial h^{ij}}{\partial a_{rs} \partial a_{pq}} \,|_\mu = l^\mu_{ir} l^\mu_{qj} l^\mu_{sp} + l^\mu_{ip} l^\mu_{sj} l^\mu_{qr} .$$

Hence, with $Cov(\mathbb{L}) = \left(L^T \otimes L\right) cov(\mathbb{A}) \left(L \otimes L^T\right)$, $\sigma_{rs,pq} := cov(\mathbb{A}_{rs}, \mathbb{A}_{pq})$, $x^\mu := \left(I - A^\mu\right)^{-1} d^\mu$ and $d^\mu = \mathrm{E}(D)$

$$\begin{aligned}
cov(\mathbb{L}_{ij}, \mathbb{L}_{hk}) &\approx \sum_{r,s} l^\mu_{ir} l^\mu_{sj} \sum_{p,q} l^\mu_{hp} l^\mu_{qk} \sigma_{rs,pq} , \\
\mathrm{E}\left(\mathbb{L}_{ij}\right) &\approx l^\mu_{ij} + \frac{1}{2} \sum_{r,s} \sum_{p,q} \left(l^\mu_{ir} l^\mu_{qj} l^\mu_{sp} + l^\mu_{ip} l^\mu_{sj} l^\mu_{qr}\right) \sigma_{rs,pq} , \\
\mathrm{E}(X_i) &\approx x^\mu_i + \frac{1}{2} \sum_{r,s} \sum_{p,q} \left(l^\mu_{ir} l^\mu_{sp} x^\mu_q + l^\mu_{ip} l^\mu_{qr} x^\mu_s\right) \sigma_{rs,pq} .
\end{aligned}$$

The superscript $^\mu$ with l_{ij}, d and x will be suppressed from now on. Assuming $\mathbb{A}$ to be uncorrelated, which is done in the sequel since possible (little negative) correlations within columns seem negligible, e. g. for Dirichlet distributions of the column vectors used in the simulation (see section 7), simplifies the moment proxies as follows

$$cov\left(\mathbb{L}_{ij},\mathbb{L}_{hk}\right)\approx\sum_{r,s}l_{ir}l_{sj}l_{hr}l_{sk}\sigma_{rs}^{2}\geq 0\,,$$

$$\operatorname{Var}\left(\mathbb{L}_{ij}\right)\approx\sum_{r,s}\left(l_{ir}l_{sj}\right)^{2}\sigma_{rs}^{2}\,,$$

$$\operatorname{E}\left(\mathbb{L}_{ij}\right)\approx l_{ij}+\sum_{r,s}l_{ir}l_{sj}l_{sr}\sigma_{rs}^{2}\,.$$

For these approximations, $Cov(\mathbb{L})\geq 0$ and $Cov(\mathbb{L})>0$, if A is irreducible.

Furthermore, terms without diagonal elements of L might be neglected without great loss of precision since $l_{ii}>l_{ij}\;\forall\, i,j$ by a theorem of Metzler given the above assumption that A has column sums less than one. In addition, $\sigma_{ii}^{2}\geq\sigma_{ij}^{2}$ seems to be realistic because $a_{ii}\geq a_{ij}$ is mainly observed in empirical input matrices. Hence, the following reduced smaller sums may be satisfactory proxies:

$$\begin{aligned}cov\left(\mathbb{L}_{ij},\mathbb{L}_{hk}\right)&\approx l_{ii}l_{jj}l_{hi}l_{jk}\sigma_{ij}^{2}+l_{hh}l_{kk}l_{ih}l_{hk}\sigma_{hk}^{2}+l_{ii}l_{hi}\sum_{s}l_{sj}l_{sk}\sigma_{is}^{2}\\&\quad+l_{jj}l_{jk}\sum_{r}l_{ir}l_{hr}\sigma_{rj}^{2}\,,\\\operatorname{Var}\left(\mathbb{L}_{ij}\right)&\approx\left(l_{ii}l_{jj}\right)^{2}\sigma_{ij}^{2}+l_{ii}^{2}\sum_{s\neq j}l_{sj}^{2}\sigma_{is}^{2}+l_{jj}^{2}\sum_{r\neq i}l_{ir}^{2}\sigma_{rj}^{2}\,,\\\operatorname{E}\left(\mathbb{L}_{ij}\right)&\approx l_{ij}+l_{ii}l_{jj}l_{ji}\sigma_{ij}^{2}+l_{ii}\sum_{s\neq j}l_{si}l_{sj}\sigma_{is}^{2}+l_{jj}\sum_{r\neq i}l_{ir}l_{jr}\sigma_{rj}^{2}\,.\end{aligned}$$

These crude proxies need only one row and column of L and $\operatorname{Var}(\mathbb{A})$ saving n^2-2n terms in the Taylor approximation of each of the $2n^2$ moments of $\mathbb{L}$.

These Taylor approximations of $\operatorname{E}(\mathbb{L})$ and $cov(\mathbb{L})$ will now be used to derive proxies for the corresponding moments of the solution X of the stochastic, static input output-model. Final demand D is assumed to be uncorrelated with $\mathbb{A}$ and to be normalized by $\sum d_i=1$, which is no restriction. With $\operatorname{E}(D)=d$ and $\operatorname{E}(X)=x$ for the solution based on $\operatorname{E}(\mathbb{A})$ one obtains

$$\operatorname{E}\left(X_{i}\right)=\sum_{j=1}^{n}\operatorname{E}\left(\mathbb{L}_{ij}\right)\operatorname{E}\left(D_{j}\right)=x_{i}+\sum_{j=1}^{n}\left(\operatorname{E}\left(\mathbb{L}_{ij}\right)-l_{ij}\right)d_{j}$$

and applying the Taylor proxy of $\mathrm{E}\left(\mathbb{L}_{ij}\right)$

$$\mathrm{E}\left(X_i\right) \approx x_i + \sum_r l_{ir} \sum_s l_{sr} x_s \sigma_{rs}^2 = x_i + \sum_s x_s \sum_r l_{ir} l_{sr} \sigma_{rs}^2 \,,$$

where x_s is a d-weighted mean of row s in L and $l_{ir} l_{sr} = \frac{\partial l_{ir}}{\partial a_{rs}}$.

The double sum provides a proxy for the underestimation (negative bias), if $\mathrm{E}(X_i)$ is estimated by x_i, the solution of the system based on $\mathrm{E}(\mathbb{A})$. The bias is influenced by all elements of $\mathbb{L}$ and all variances of $\mathbb{A}$, which are weighted by the vector x.

The derivation of a proxy for $\mathrm{Var}(X_i)$ is more complicated because of the correlation within $\mathbb{L}$.

$$\mathrm{Var}(X_i) = \sum_{j,k} cov\left(\mathbb{L}_{ij} D_j, \mathbb{L}_{ik} D_k\right) \quad \text{with}$$

$$\begin{aligned} cov\left(\mathbb{L}_{ij} D_j, \mathbb{L}_{ik} D_k\right) &= cov\left(\mathbb{L}_{ij}, \mathbb{L}_{ik}\right)\left[cov\left(D_j, D_k\right) + \mathrm{E}\left(D_j\right)\mathrm{E}\left(D_k\right)\right] \\ &\quad + \mathrm{E}\left(\mathbb{L}_{ij}\right)\mathrm{E}\left(\mathbb{L}_{ik}\right) cov\left(D_j, D_k\right) \end{aligned}$$

Assuming D to be uncorrelated yields a simple Taylor approximation

$$cov\left(\mathbb{L}_{ij} D_j, \mathbb{L}_{ik} D_k\right) \approx \sum_r l_{ir}^2 \sum_s l_{sj} l_{sk} \left(d_j d_k + \delta_{jk}\, \sigma^2(D_j)\right) \sigma_{rs}^2 + \delta_{jk} l_{ij} l_{ik}\, \sigma^2(D_j) \,,$$

and finally,

$$\mathrm{Var}\left(X_i\right) \approx \sum_r l_{ir}^2 \sum_s \left(x_s^2 + \sum_j l_{sj}^2\, \sigma^2(D_j) \right) \sigma_{rs}^2 + \sum_j l_{ij}^2\, \sigma^2(D_j) \,.$$

This proxy is now regarded for the special variances of $\mathbb{A}$ and D

$$\sigma_{ij}^2 = \left(\frac{a_{ij}}{k}\right)^2 \quad \text{and} \quad \sigma^2(D_j) = \left(\frac{d_j}{k}\right)^2 \quad \text{for } k = 2 \text{ or } 3$$

which are used in the simulation for the case with observations as expected values. Then

$$\begin{aligned} \mathrm{Var}(\tilde{X}_i) &\approx \sum_r l_{ir}^2 \sum_s \left(x_s^2 + \sum_j l_{sj}^2 \left(\frac{d_j}{k}\right)^2 \right) \left(\frac{a_{rs}}{k}\right)^2 + \sum_j l_{ij}^2 \left(\frac{d_j}{k}\right)^2 \\ &= \frac{1}{k^2} \left[\sum_r l_{ir}^2 \sum_s \left(x_s^2 + \sum_j \left(l_{sj} \frac{d_j}{k} \right)^2 \right) a_{rs}^2 + \sum_j \left(l_i j d_j\right)^2 \right] \end{aligned}$$

$$\mathrm{Var}(X_i) \approx \frac{1}{k^2} \left[\sum_s \sum_r \left(l_{ir} a_{rs}\right)^2 \left(x_s^2 + \frac{1}{k^2} \sum_j \left(l_{sj} d_j\right)^2 \right) + \sum_j \left(l_{ij} d_j\right)^2 \right]$$

Upper bounds are derived as follows because of $LA = L - I$ and $\sum y_i^2 \leq (\sum y_i)^2$

$$\begin{aligned}
\mathrm{Var}(X_i) &\leq \frac{1}{k^2}\left[\sum_s (l_{is} - \delta_{is})^2 \left(x_s^2 + \frac{1}{k^2}\sum_j (l_{sj}d_j)^2\right) + \sum_j (l_{ij}d_j)^2\right] \\
&\leq \frac{1}{k^2}\left[\sum_j \left((l_{ij}d_j)^2 + \frac{k^2+1}{k^2}(l_{ij} - \delta_{ij})^2 x_j^2\right)\right] \\
&= \frac{1}{k^2}\left[\frac{k^2+1}{k^2}(1 - 2l_{ii})\, x_i^2 + \sum_j l_{ij}^2 \left(d_j^2 + \frac{k^2+1}{k^2}x_j^2\right)\right]
\end{aligned}$$

and

$$\mathrm{Var}(X_i) < \frac{1}{k^2}\left[\frac{k^2+1}{k^2}(1 - 2l_{ii})\, x_i^2 + x_i^2 + \frac{k^2+1}{k^2}\sum_j (l_{ij}x_j)^2\right] ,$$

and finally,

$$\mathrm{Var}(X_i) < \frac{k^2+1}{k^4}\left(\sum_j (l_{ij}x_j) + (l_{ii} - 1)^2 x_i^2\right) .$$

The upper bounds only depend on row i of $\mathbb{L}$, d, x and k, but no longer (directly) on $\mathbb{A}$.

A simple but crude lower bound for $\mathrm{Var}(\tilde{X}_i)$ is given with

$$\mathrm{Var}(\tilde{X}_i) \geq \frac{1}{k^2}\left[\sum_s (l_{ii}a_{is})^2 \left(x_s^2 + \frac{1}{k^2}(l_{ss}d_s)^2\right) + (l_{ii}d_i)^2\right]$$

For the proxy of $\mathrm{E}(X_i)$ an upper bound is found analogously:

$$\begin{aligned}
\mathrm{E}(X_i) &\approx x_i + \frac{1}{k^2}\sum_s x_s \sum_r l_{ir}l_{sr}a_{rs}^2 \\
&\leq x_i + \frac{1}{k^2}\sum_s x_s \left(\sum_r l_{ir}a_{rs}\right)\left(\sum_r l_{sr}a_{rs}\right) \\
&= x_i + \frac{1}{k^2}\sum_s x_s (l_{is} - \delta_{is})(l_{ss} - 1) \\
&= x_i\left(1 + \frac{1}{k^2}(l_{ii} - 1)^2\right) + \frac{1}{k^2}\sum_{s\neq i} x_s (l_{ss} - 1)\, l_{is} .
\end{aligned}$$

A crude lower bound is given with

$$\mathrm{E}(X_i) \geq x_i + \frac{1}{k^2}\left(\sum_r x_r l_{ir} l_{rr} a_{rr}^2 + \sum_s x_s l_{ii} l_{si} a_{is}^2\right) .$$

6.2 Inversion-approximation

A further approximation improves the simple approximation by minors, which was introduced in section 3. It starts from the identity $L(I-A) = I$ which implies

$$l_{ii} = \frac{1}{1-a_{ii}}\left(1 + \sum_{k\neq i} l_{ik} a_{ki}\right) \quad \forall\, i .$$

Substituting l_{ik} with the minor-proxy yields

$$\tilde{l}_{ii} \approx \frac{1}{1-a_{ii}}\left(1 + \frac{1}{(1-a_{ii})}\sum_{k\neq i}\frac{a_{ik}a_{ki}}{1-a_{kk}}\right)$$

as a proxy from below. Similarly, it can be seen that

$$l_{ij} = \frac{1}{1-a_{jj}}\left(l_{ii}a_{ij} + \sum_{h\neq i,j} l_{hh} a_{ih} a_{hj}\right)$$

where proxies for l_{ii} and l_{hh} may be inserted to get $\tilde{l}_{ij}$. Or, less precise,

$$\tilde{l}'_{ij} = \frac{1}{(1-a_{ii})(1-a_{jj})}\left(a_{ij} + \sum_{h\neq i,j}\frac{a_{ih}a_{hj}}{1-a_{hh}}\right) .$$

Another good proxy is given with

$$\tilde{l}_{ij} \approx \tilde{l}_{ii}\tilde{l}_{jj}a_{ij} .$$

These approximations based on only one row and column of A and its diagonal are considerably better than the minor-proxies based on a 2×2 submatrix.

If $\mathbb{A}$ is uncorrelated, proxies of the moments are given with

$$\mathrm{E}\left(\tilde{\mathbb{L}}_{ii}\right) \approx R_{ii}\left(1 + R_{ii}(1 + R_{ii}^2\sigma_{ii}^2)\sum_{k\neq i} R_{kk}\mu_{ik}\mu_{ki}\right)$$

$$\mathrm{E}\left(\tilde{\mathbb{L}}_{ij}\right) \approx R_{jj}\left(\mathrm{E}\left(\tilde{\mathbb{L}}_{ii}\right)\mu_{ij} + \sum_{h\neq i,j}\mathrm{E}\left(\tilde{\mathbb{L}}_{hh}\right)\mu_{ih}\mu_{hj}\right) ,$$

where $R_{ii} = (1 - \mu_{ii})^{-1}$. Improvement can be achieved if R_{ii} is increased by $(1 - \mu_{ii})^{-3}\sigma_{ii}^2$ due to Taylor.

In contrast to Taylor's method no knowledge of $L(E(\mathbb{A}))$ at the expansion point is needed.

Since proxies for the variances of $\mathbb{L}$ are complicated, in the simulation it is assumed that $\mathbb{A}_{ij}$ has a Beta distribution $\mathbb{A}_{ij} \sim Be\,(r_{ij}, s_{ij})$. Then, according to section 3, simple approximating Beta distributions of the second kind have the following variances:

$$\mathrm{Var}\left(\tilde{\mathbb{L}}_{ii}\right) = \frac{\mathrm{E}\left(\tilde{\mathbb{L}}_{ii}\right)\left(\mathrm{E}\left(\tilde{\mathbb{L}}_{ii}\right) - 1\right)}{(s_{ii} - 2)},$$

and

$$\mathrm{Var}\left(\tilde{\mathbb{L}}_{ij}\right) = \frac{\mathrm{E}\left(\tilde{\mathbb{L}}_{ij}\right)\left(\mathrm{E}\left(\tilde{\mathbb{L}}_{ij}\right) + 1\right)}{(s_{ij} - 2)}.$$

These proxies are used in the simulation with the inversion-approximation (denoted there M2).

Furthermore, R_{ii} is then replaced by $\mathrm{E}\,(1/(1 - \mathbb{A}_{ii})) = 1 + r_{ii}/(s_{ii} - 1)$. The moments of X then follow in a similar way as described above with Taylor.

$$\mathrm{E}\left(\tilde{X}_i\right) = \sum_j \mathrm{E}\left(\tilde{\mathbb{L}}_{ij}\right) d_j\,,$$

and, if D is uncorrelated,

$$\mathrm{Var}\left(\tilde{X}_i\right) = \sum_{j,k} cov\left(\tilde{\mathbb{L}}_{ij}, \tilde{\mathbb{L}}_{ik}\right)\left(d_j d_k + \delta_{jk}\,\sigma^2(D_j)\right) + \sum_j l_{ij}^2\,\sigma^2(D_j)\,,$$

where, since $l_{ij} \approx l_{ii} l_{jj} a_{ij}$,

$$cov\left(\tilde{\mathbb{L}}_{ij}, \tilde{\mathbb{L}}_{ik}\right) \approx \mathrm{E}\left(\tilde{\mathbb{L}}_{jj}\right)\mathrm{E}\left(\tilde{\mathbb{L}}_{kk}\right)\mu_{ij}\mu_{ik} Var\left(\tilde{\mathbb{L}}_{ii}\right)\,,$$

if positive correlations within $diag\,\mathbb{L}$ as well as between $diag\,\mathbb{L}$ and $\mathbb{A}$ are neglected, which underestimates $Var(\tilde{X})$.

7 Simulation

The simulation is based on German input output-data of the Federal Statistical Office for 1998. Empirical input matrices are derived from a database from year 0 such that $(I - A^0)x^0 = d^0$ is fulfilled for this reference system. It is assumed that these values of A^0 and d^0 represent either a) expected values or b) modes of the underlying random variables $\mathbb{A}$ and D which corresponds to maximum likelihood estimation. If there is no other information, variances are assumed to be derivable by a 3σ-rule. The input coefficients a_{ij} are taken i) as $3\sigma_{ij}$ or, alternatively, as ii) $2\sigma_{ij}$. In both versions it is assumed that an interval of length $6\sigma_{ij}$ (ending with i) $2a_{ij}$ or ii) $3a_{ij}$) captures nearly the whole propability mass. Version ii) seems to fit better to the asymmetric beta distributions, skewed to the right, of the bulk of very small input coefficients.

Two types of approximation of $\mathbb{L}$ are applied: α) Taylor series and β)inversion-approximation of each coefficient of L by only one corresponding row and column of A. Approximations of $\mathrm{E}(\mathbb{L})$ and $cov(\mathbb{L})$, and $\mathrm{E}(X)$ and $cov(X)$ are deduced from the Jacobian and Hessian of the mapping g. The results simplify considerably if correlations within A and D are negligible, which seems realistic.

Crude probability regions for $\mathbb{L}$ and the solution x can be given, which may be improved, if knowledge of the distribution types of $\mathbb{A}$ and D is available. Assumption of normality may cause difficulty with respect to the requirement that $(I - \mathbb{A})$ has a nonnegative inverse.

In a simulation it is assumed that the column vectors of A and normed final demand D have independent Dirichlet distributions, or its components Beta distributions $\mathrm{Be}(r, s)$ on $[0, 1]$. It seems reasonable to start with a multivariate standard beta distribution (Dirichlet-distribution (cf. Johnson, Kotz, Balakrishnan, 2000) for each sector j since the input coefficients and the value added coefficient v_j add up to one (columnwise). Thus, this restriction may be taken into account. In particular, this would be important for estimation procedures. If all v_j are positive, then existence and non-negativity of $L(A)$ are ensured according to Brauer/Solow. This is a realistic assumption for all possible realizations of the random matrix A.

Then, the marginal distributions of $Y_i = A_{ij_0}$ are standard beta distributions $Be(r_i, s_i = R - r_i)$, where $R := \sum_{k=0}^{n} r_k$ and r_0 is the first beta parameter of v_{j_0}. Hence, μ and σ^2 are given as before. Therefore, only

$$cov(Y_i, Y_j) = -\frac{r_i r_j}{R^2(R+1)} = -\frac{\mu_i \mu_j}{R+1}$$

has to be taken into account. Because of this negative correlation $E(\tilde{\mathbb{L}}_{ij})$ is smaller than computed above under the assumption of pairwise uncorrelatedness. However, $cov(Y_i, Y_j)$ is very small on the average. For n sectors the average absolute covariance approximately amounts to $1/n^2(R+1) < 1/n^3$, which might be neglected in comparison to the average $E(\tilde{L}_{ij}) \approx 1/n$. This is done here.

Beta random variables are obtained with the BB-algorithm proposed by Chen (cf. Johnson, Kotz, Balakrishnan, 1995, p. 216, with a misprint (γ^V instead of γV)). With these distributions no problem occurs with invertibility, since it can be ensured that the dominant eigenvalue of A is less than one. The density of $\mathbb{A}$ transformed by g is theoretically derived for any distribution and, in particular, applied to the beta distribution. The parameters r and s in the simulation are computed from German data assumed to be $\mathrm{E}(Y)$ (or mode(Y)), and $\mathrm{Var}(Y)$ derived by a 3σ- (or 2σ)-rule. A simple approximation of $\mathbb{L}$ by minors suggests that the coefficients of $\mathbb{L}$ may have Beta distributions (of the second kind). Theoretical considerations show that this distribution type then should also be appropriate for the solution X. This is confirmed by the simulations. Furthermore, the two types of approximations of $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ show a good performance. The theoretically expected approximate, relatively narrow, 2σ-regions contain $90 - 95\,\%$ of the simulation results for most coefficients. For version 2, taking into account the skewness to the right of the distribution of A, here, it is assumed that the relation of $1:2$ for left and right parts (with respect to $\mathrm{E}(Y)$) of the probability region for A may be transferred to L as a proxy because its distributions are also skewed to the right. The theoretically derived approximate densities of L and x mainly accord with the histograms of the simulation. As a result, it may be possible to do without simulations within this model framework.

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A Appendix: Diagonal elements of the Leontief-inverse

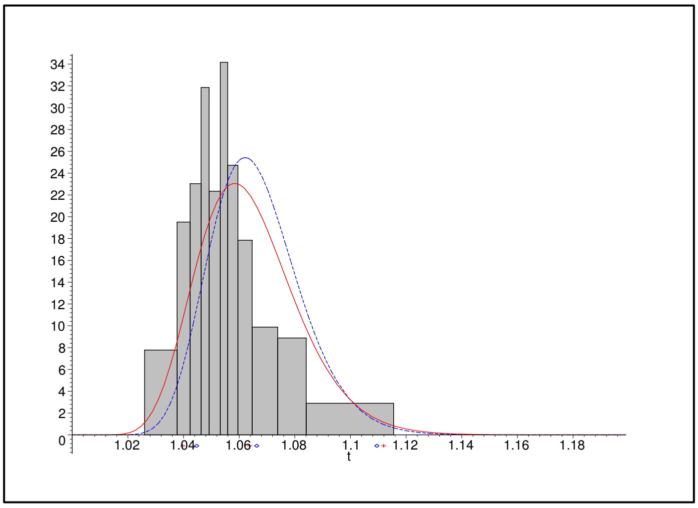

Figure 3: (i,i)=(1,1), exact=1.06316697, arithmetic mean=1.05769370, $S^2_{(n-1)}$=.00030499. M2 (inversion-approximation, blue line): estimated expected value =1.06629872, estimated variance=.00026219, 75.00% in interval. Taylor (red line): estimated expected value =1.06377250, estimated variance=.00032571, 87.00% in interval. Var.coeff.(simulation)=.016470, Var.coeff.(M2)=.015185, Var.coeff(Taylor)=.016965

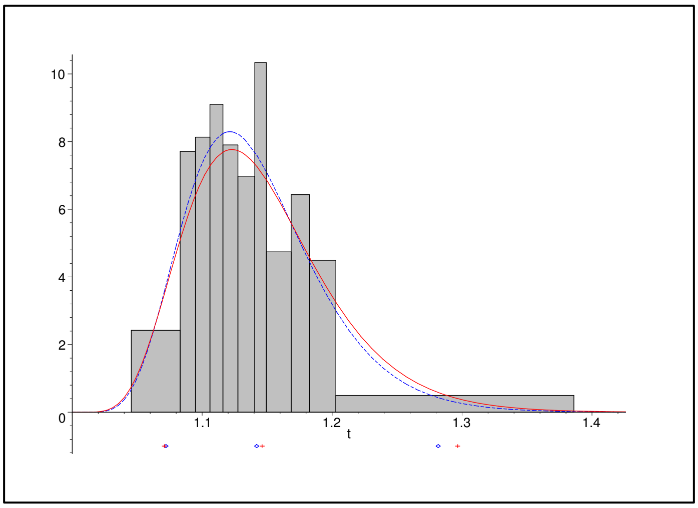

Figure 4: (i,i)=(2,2), exact=1.14310441, arithmetic mean=1.13967796, $S^2_{(n-1)}$=.00249099 M2: estimated expected value =1.14197183, estimated variance=.00274387, 93.00% in interval. Taylor: estimated expected value =1.14616170, estimated variance=.00318553, 94.50% in interval. Var.coeff.(simulation)=.043683, Var.coeff.(M2)=.045870, Var.coeff(Taylor)=.049243

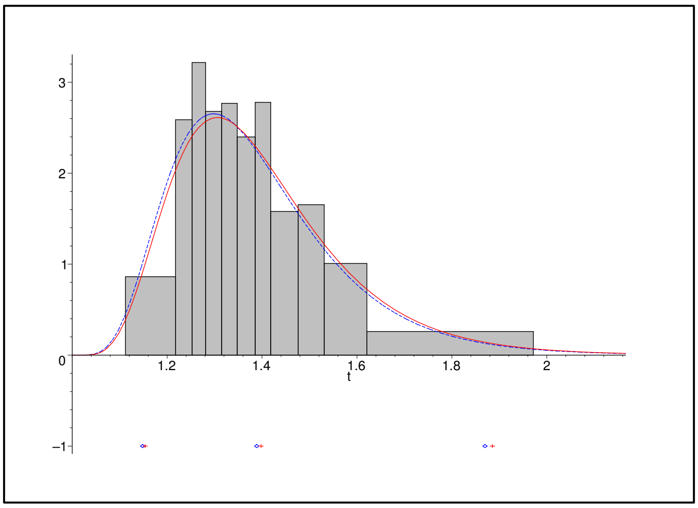

Figure 5: (i,i)=(3,3), exact=1.37450546, arithmetic mean=1.39105082, $S^2_{(n-1)}$=.02403603 . M2: estimated expected value =1.38874883, estimated variance=.03254323, 97.00% in interval. Taylor: estimated expected value =1.39771263, estimated variance=.03347502, 97.00% in interval. Var.coeff.(simulation)=.111173, Var.coeff.(M2)=.129899, Var.coeff(Taylor)=.130901

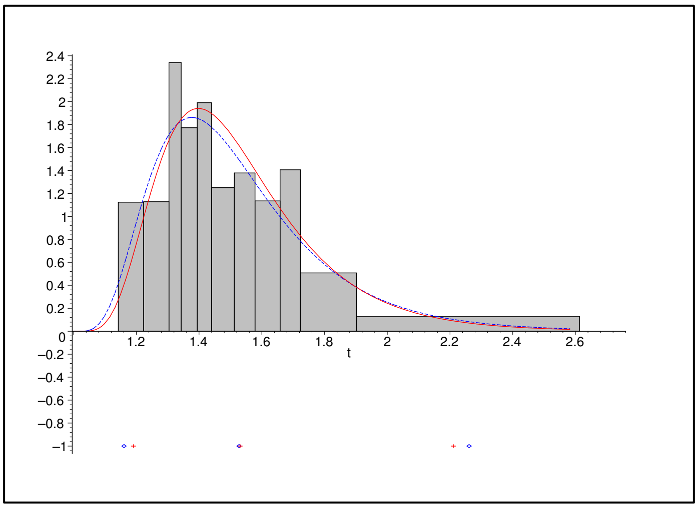

Figure 6: (i,i)=(4,4), exact=1.48791139, arithmetic mean=1.52147896, $S^2_{(n-1)}$=.06236211 . M2: estimated expected value =1.52729593, estimated variance=.07568236, 98.00% in interval. Taylor: estimated expected value =1.53082818, estimated variance=.06501317, 95.00% in interval. Var.coeff.(simulation)=.163722, Var.coeff.(M2)=.180125, Var.coeff(Taylor)=.166561

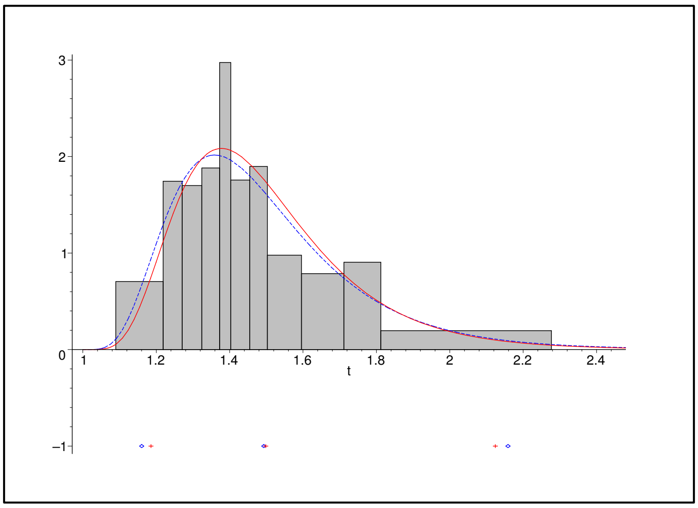

Figure 7: (i,i)=(5,5), exact=1.46092907, arithmetic mean=1.47999187, $S^2_{(n-1)}$=.05527118 . M2: estimated expected value =1.49312223, estimated variance=.06228505, 95.50% in interval. Taylor: estimated expected value =1.49845880, estimated variance=.05504847, 93.50% in interval. Var.coeff.(simulation)=.158453, Var.coeff.(M2)=.167146, Var.coeff(Taylor)=.156577

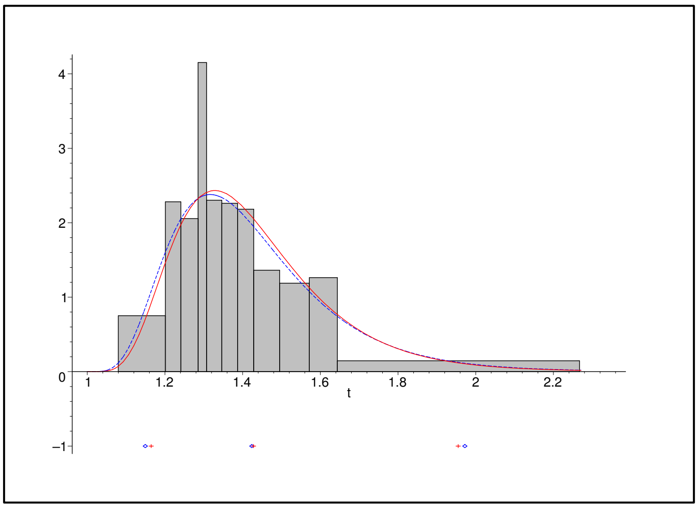

Figure 8: (i,i)=(6,6), exact=1.40000966, arithmetic mean=1.41107149, $S^2_{(n-1)}$=.04302302 . M2: estimated expected value =1.42364494, estimated variance=.04229164, 93.50% in interval. Taylor: estimated expected value =1.42789508, estimated variance=.03902955, 91.50% in interval. Var.coeff.(simulation)=.146627, Var.coeff.(M2)=.144453, Var.coeff(Taylor)=.138357

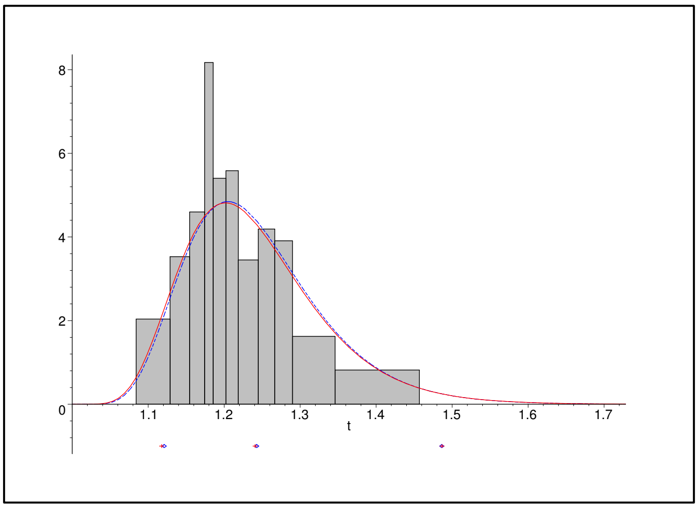

Figure 9: (i,i)=(7,7), exact=1.23369370, arithmetic mean=1.22373533, $S^2_{(n-1)}$=.00605759 . M2: estimated expected value =1.24279952, estimated variance=.00834130, 94.00% in interval. Taylor: estimated expected value =1.24076088, estimated variance=.00850822, 94.50% in interval. Var.coeff.(simulation)=.063442, Var.coeff.(M2)=.073488, Var.coeff(Taylor)=.074341

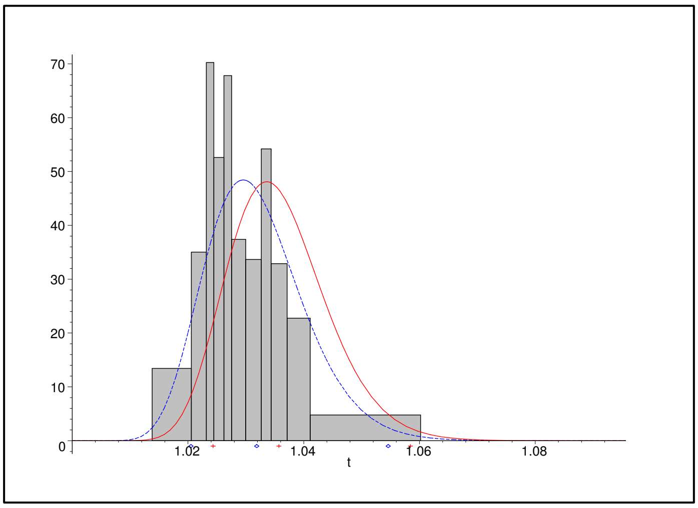

Figure 10: (i,i)=(8,8), exact=1.03518597, arithmetic mean=1.03023664, $S^2_{(n-1)}$=.00006946 . M2: estimated expected value =1.03187433, estimated variance=.00007263, 90.00% in interval. Taylor: estimated expected value =1.03570382, estimated variance=.00007266, 73.50% in interval. Var.coeff.(simulation)=.008069, Var.coeff.(M2)=.008259, Var.coeff(Taylor)=.008230

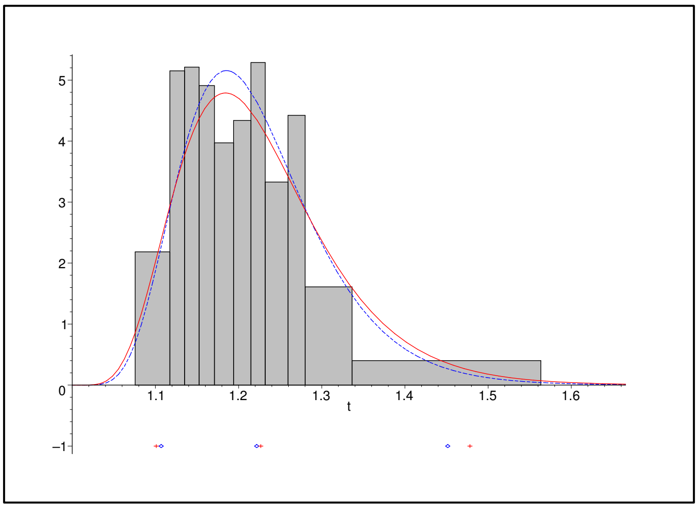

Figure 11: (i,i)=(9,9), exact=1.21973733, arithmetic mean=1.21417974, $S^2_{(n-1)}$=.00769641 . M2: estimated expected value =1.22179241, estimated variance=.00742207, 91.00% in interval. Taylor: estimated expected value =1.22674757, estimated variance=.00889194, 92.00% in interval. Var.coeff.(simulation)=.072073, Var.coeff.(M2)=.070512, Var.coeff(Taylor)=.076868

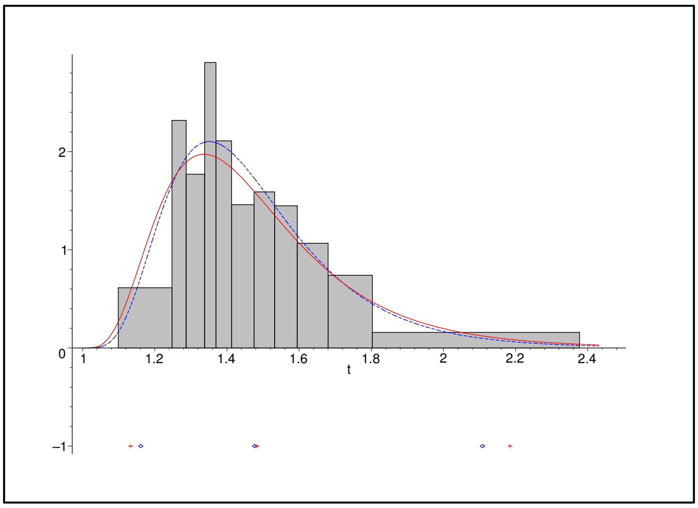

Figure 12: (i,i)=(10,10), exact=1.44888249, arithmetic mean=1.48359548, $S^2_{(n-1)}$=.04976908 . M2: estimated expected value =1.47668964, estimated variance=.05607942, 95.50% in interval. Taylor: estimated expected value =1.48345906, estimated variance=.06911659, 97.00% in interval. Var.coeff.(simulation)=.149995, Var.coeff.(M2)=.160366, Var.coeff(Taylor)=.177221

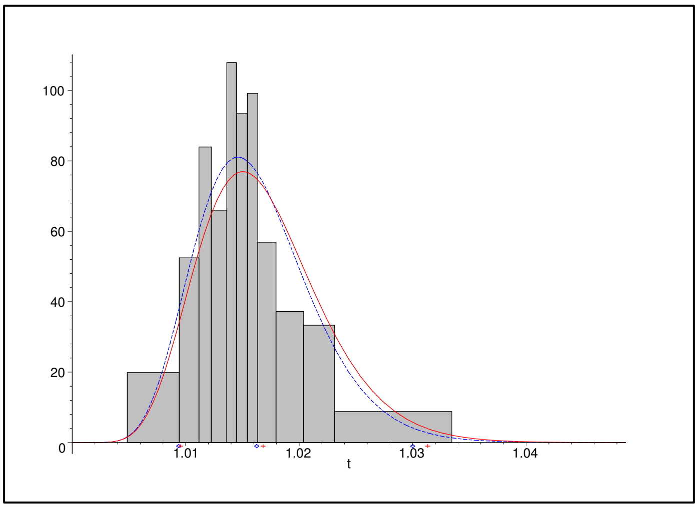

Figure 13: (i,i)=(11,11), exact=1.01674161, arithmetic mean=1.01564277, $S^2_{(n-1)}$=.00002589 . M2: estimated expected value =1.01625626, estimated variance=.00002656, 90.00% in interval. Taylor: estimated expected value =1.01682036, estimated variance=.00002960, 88.50% in interval. Var.coeff.(simulation)=.004997, Var.coeff.(M2)=.005071, Var.coeff(Taylor)=.005351

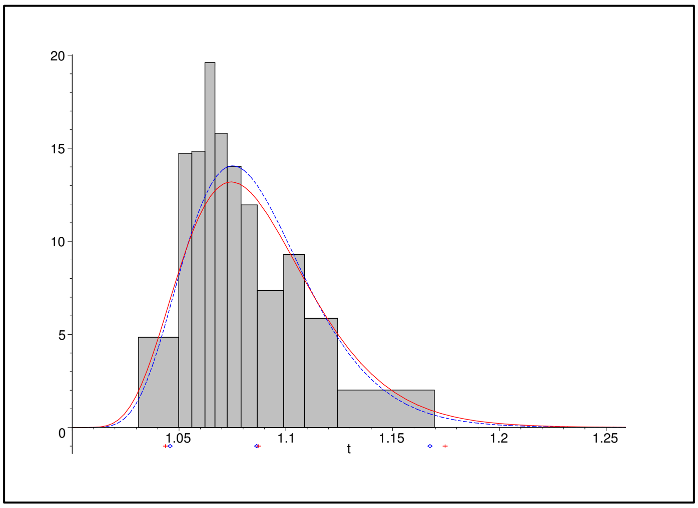

Figure 14: (i,i)=(12,12), exact=1.08619042, arithmetic mean=1.08194653, $S^2_{(n-1)}$=.00081617 . M2: estimated expected value =1.08636468, estimated variance=.00092563, 93.00% in interval. Taylor: estimated expected value =1.08728893, estimated variance=.00107131, 95.00% in interval. Var.coeff.(simulation)=.026339, Var.coeff.(M2)=.028006, Var.coeff(Taylor)=.030103

B Appendix: Off-diagonal elements of the Leontief-inverse

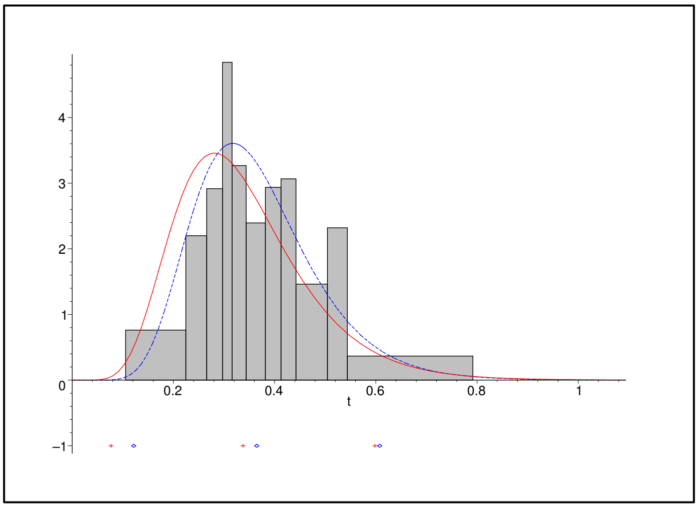

Figure 15: (i,j)=(1,7), exact=.33423555, arithmetic mean=.37341845, $S^2_{(n-1)}$=.01541727 . M2: estimated expected value =.36459644, estimated variance=.01477006, 96.00% in interval. Taylor: estimated expected value =.33750631, estimated variance=.01699031, 95.50% in interval. Var.coeff.(simulation)=.331680, Var.coeff.(M2) =.333333, Var.coeff.(Taylor) =.386206

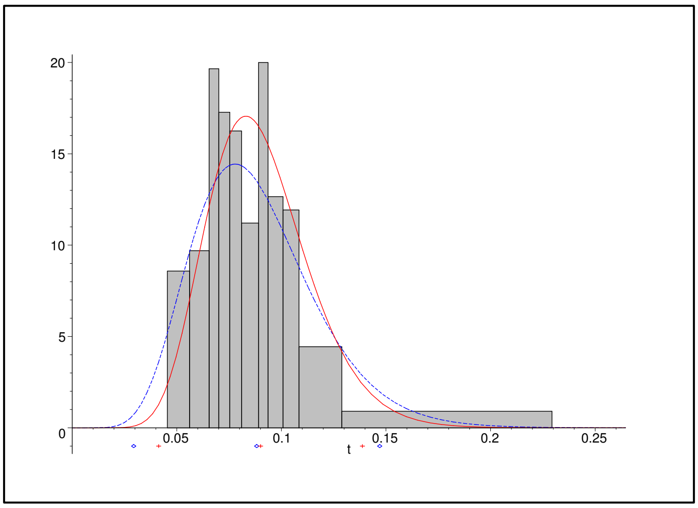

Figure 16: (i,j)=(9,2), exact=.08855437, arithmetic mean=.08874559, $S^2_{(n-1)}$=.00079986 . M2: estimated expected value =.08820121, estimated variance=.00086438, 95.50% in interval. Taylor: estimated expected value =.09004951, estimated variance=.00059328, 94.50% in interval. Var.coeff.(simulation)=.317886, Var.coeff.(M2) =.333333, Var.coeff.(Taylor) =.270488

C Appendix: Solution components of the stochastic input output-model

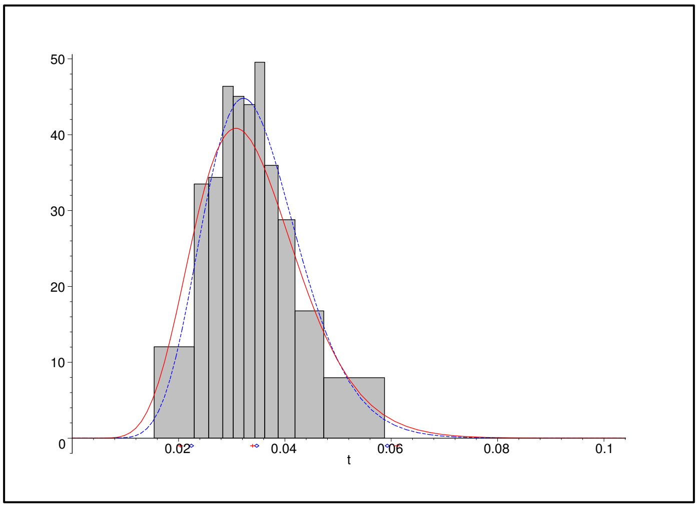

Figure 17: i=1, exact=.03361670, arithmetic mean=.03404627, $S^2_{(n-1)}$=.00008133 . M2: estimated expected value =.03469478, estimated variance=.00008486, 91.00% in interval. Taylor: estimated expected value =.03393318, estimated variance=.00010412, 94.50% in interval. Var.coeff.(simulation)=.264215, Var.coeff.(M2) =.265512, Var.coeff.(Taylor) =.300704

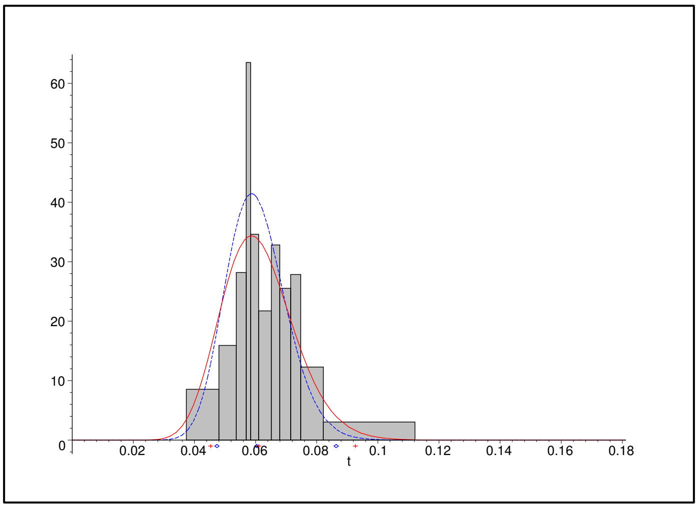

Figure 18: i=2, exact=.05990588, arithmetic mean=.06464508, $S^2_{(n-1)}$=.00017029 . M2: estimated expected value =.06036574, estimated variance=.00009513, 87.00% in interval. Taylor: estimated expected value =.06110863, estimated variance=.00014022, 92.00% in interval. Var.coeff.(simulation)=.201359, Var.coeff.(M2) =.161576, Var.coeff.(Taylor) =.193776

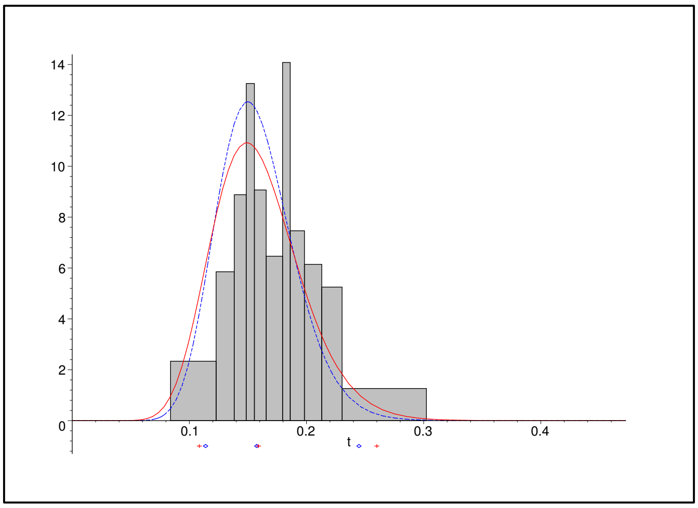

Figure 19: i=3, exact=.15492811, arithmetic mean=.17557625, $S^2_{(n-1)}$=.00173286 . M2: estimated expected value =.15747426, estimated variance=.00107107, 89.50% in interval. Taylor: estimated expected value =.15901284, estimated variance=.00143497, 94.50% in interval. Var.coeff.(simulation)=.236498, Var.coeff.(M2) =.207825, Var.coeff.(Taylor) =.238226

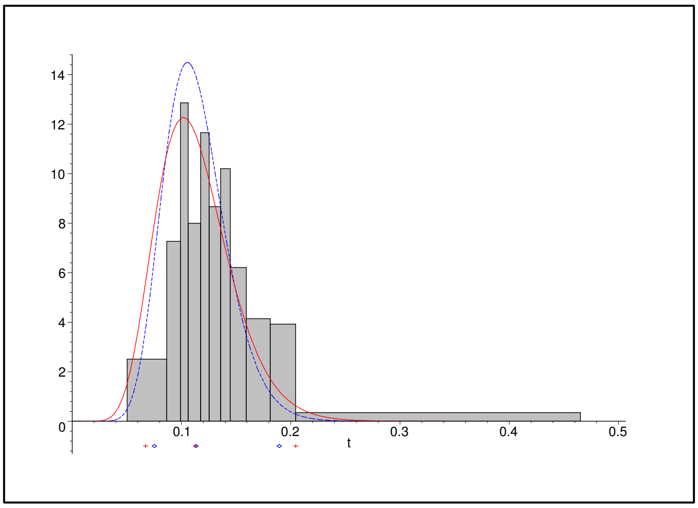

Figure 20: i=4, exact=.10824516, arithmetic mean=.14006088, $S^2_{(n-1)}$=.00270309 . M2: estimated expected value =.11327215, estimated variance=.00081649, 82.00% in interval. Taylor: estimated expected value =.11293384, estimated variance=.00117877, 89.50% in interval. Var.coeff.(simulation)=.370275, Var.coeff.(M2) =.252263, Var.coeff.(Taylor) =.304012

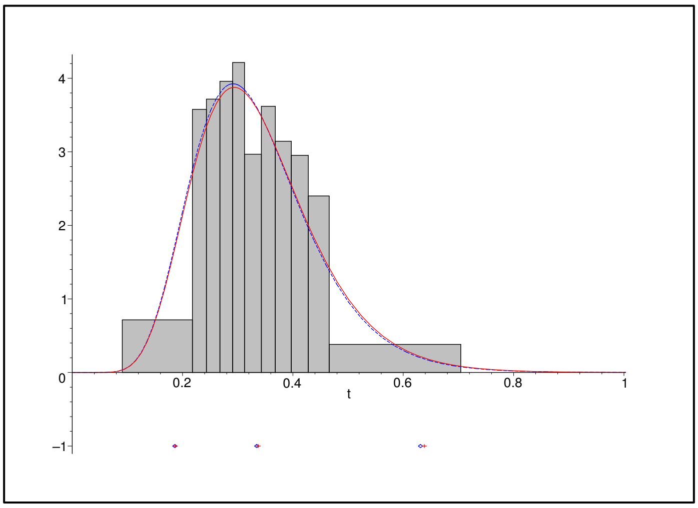

Figure 21: i=5, exact=.32824753, arithmetic mean=.33939668, $S^2_{(n-1)}$=.01139405 . M2: estimated expected value =.33438189, estimated variance=.01239037, 94.00% in interval. Taylor: estimated expected value =.33715145, estimated variance=.01273639, 94.00% in interval. Var.coeff.(simulation)=.313721, Var.coeff.(M2) =.332889, Var.coeff.(Taylor) =.334733

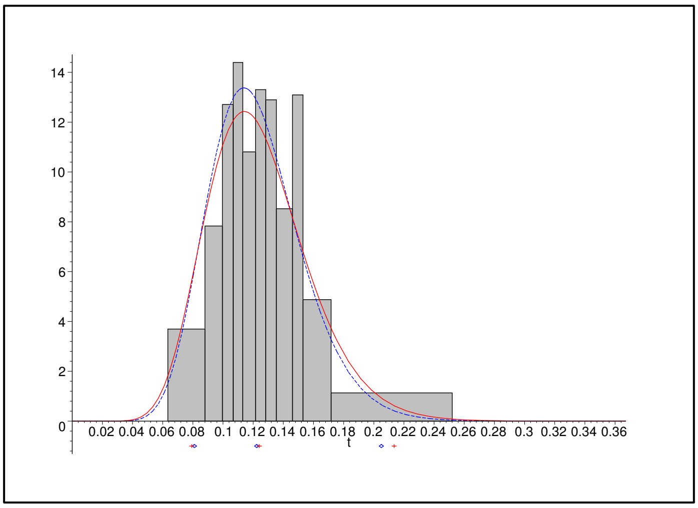

Figure 22: i=6, exact=.12090179, arithmetic mean=.12779481, $S^2_{(n-1)}$=.00101556 . M2: estimated expected value =.12234544, estimated variance=.00096047, 93.50% in interval. Taylor: estimated expected value =.12402966, estimated variance=.00112529, 93.50% in interval. Var.coeff.(simulation)=.248743, Var.coeff.(M2) =.253311, Var.coeff.(Taylor) =.270462

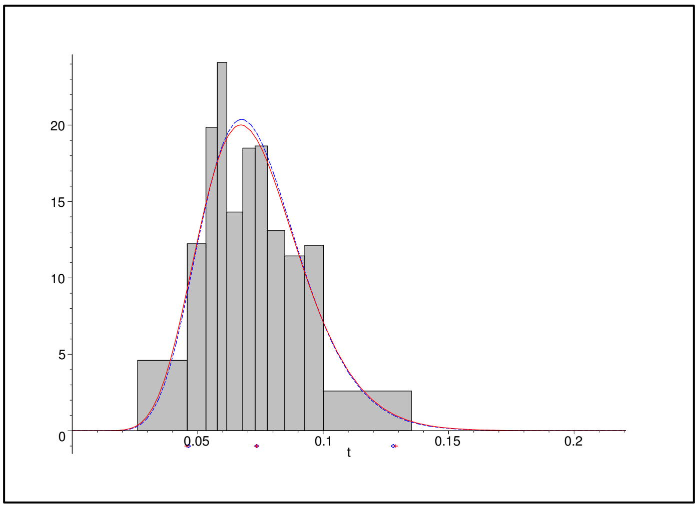

Figure 23: i=7, exact=.07283257, arithmetic mean=.07228684, $S^2_{(n-1)}$=.00043234 . M2: estimated expected value =.07351924, estimated variance=.00041642, 90.00% in interval. Taylor: estimated expected value =.07343453, estimated variance=.00043335, 90.50% in interval. Var.coeff.(simulation)=.286924, Var.coeff.(M2) =.277565, Var.coeff.(Taylor) =.283478

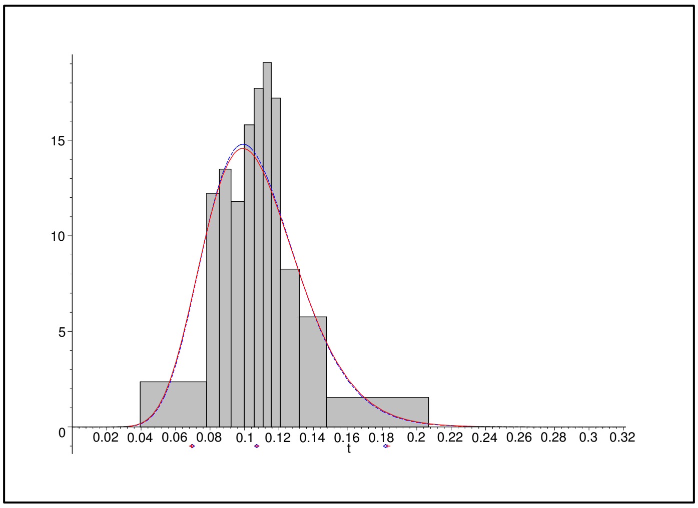

Figure 24: i=8, exact=.10659005, arithmetic mean=.10979957, $S^2_{(n-1)}$=.00078298 . M2: estimated expected value =.10709090, estimated variance=.00078737, 93.00% in interval. Taylor: estimated expected value =.10720800, estimated variance=.00081302, 93.00% in interval. Var.coeff.(simulation)=.254206, Var.coeff.(M2) =.262022, Var.coeff.(Taylor) =.265964

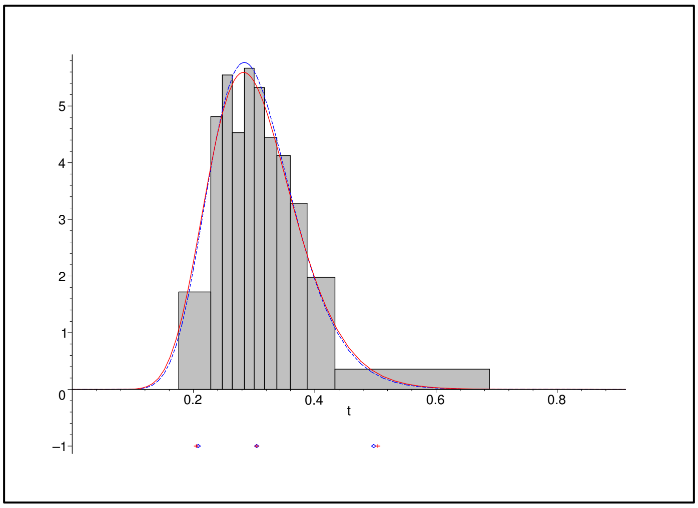

Figure 25: i=9, exact=.30055556, arithmetic mean=.31937791, $S^2_{(n-1)}$=.00636517 . M2: estimated expected value =.30424774, estimated variance=.00523755, 93.00% in interval. Taylor: estimated expected value =.30438346, estimated variance=.00560275, 94.00% in interval. Var.coeff.(simulation)=.249179, Var.coeff.(M2) =.237868, Var.coeff.(Taylor) =.245912

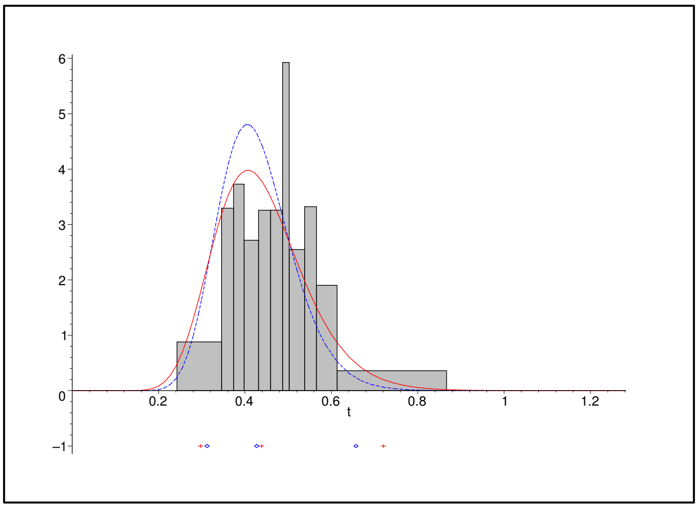

Figure 26: i=10, exact=.42445949, arithmetic mean=.47700948, $S^2_{(n-1)}$=.01171086 . M2: estimated expected value =.42719243, estimated variance=.00742581, 90.50% in interval. Taylor: estimated expected value =.43845616, estimated variance=.01119914, 97.00% in interval. Var.coeff.(simulation)=.226297, Var.coeff.(M2) =.201720, Var.coeff.(Taylor) =.241360

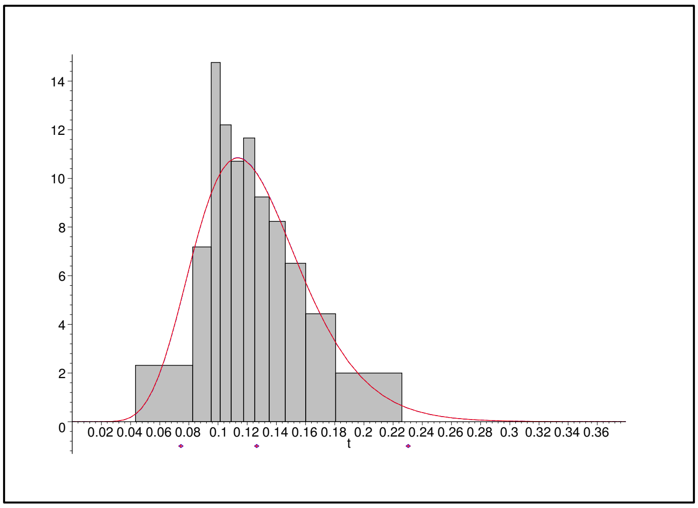

Figure 27: i=11, exact=.12621878, arithmetic mean=.12635430, $S^2_{(n-1)}$=.00132576 . M2: estimated expected value =.12648105, estimated variance=.00151695, 93.50% in interval. Taylor: estimated expected value =.12641081, estimated variance=.00151963, 94.50% in interval. Var.coeff.(simulation)=.287444, Var.coeff.(M2) =.307936, Var.coeff.(Taylor) =.308379

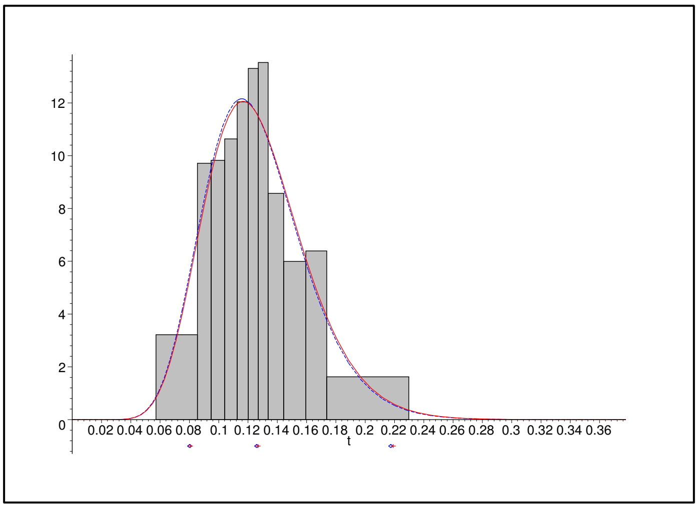

Figure 28: i=12, exact=.12617985, arithmetic mean=.12710113, $S^2_{(n-1)}$=.00121609 . M2: estimated expected value =.12593797, estimated variance=.00117813, 92.00% in interval. Taylor: estimated expected value =.12686433, estimated variance=.00119775, 91.50% in interval. Var.coeff.(simulation)=.273681, Var.coeff.(M2) =.272546, Var.coeff.(Taylor) =.272799