Sraffa's System and Mark-up Pricing

Professor Takashi Yagi

Gunma University,Faculty of Social and Information Stuidies
4-2 Aramaki-Machi, Maebashi-shi, Gunma,371-8510,JAPAN
E.mail: yagi@si.gunma-u.ac.jp

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1 Introduction

The first aim of this paper is a theoretical examination of the mark-up pricing in the light of the Sraffa System in order to interpret the prices of Input-Output Table in the case that each industry has its own rates of profit and wage which are different from those of other industries. The second aim of this paper is a reconsideration of the mark-up equation of Macroeconomic meaning.

In the first section, we will explain our interpretation of the Sraffa system. In order to interpret the Sraffa sytem, we will introduce an enlarged Sraffian price system, which we call the Evaluation system. We will explain the properties of the Sraffian price system in our Evaluation system. In the Sraffa system, it is assumed that the equal rate of profit and the equal wage rate are prevailing throughout industries. If we denote the distributional ratio measured in terms of the standard income by θ_{sp}, the price vector expressed in terms of the labour commanded by $\mathbf{p}_w$ and the price vector expressed in terms of the labour embodied in the Standard net product by $\mathbf{p}_v$, we can find the relationship $\mathbf{p}_w$=$(1+\theta_{sp})$ $\mathbf{p}_v$. This is a theoretical relation obtained when the Leontief input coefficient matrix, the labour coefficient vector, the total labour and the equal rate of profit are given. We will also use the notion of cost price $\mathbf{p}_c$ which is defined as $\mathbf{p}_v$=$(1+\theta_{sp})\mathbf{p}_c$. In the second section, we will try to explain the deviation of the actual price from the Sraffian cost price. By doing so, we will interpret the connection between the Sraffian price system and the mark-up pricing theory.

In the third section, we will examine the general price level, by constructing index numbers based on the Sraffa system. From our analysis, it comes to that the mark-up equation used in the Macroeconomics should be modified by considering the distribution effect to the general price level.

2 The Sraffian Price System

2.1 The Actual Total Labour and the Sraffian Price System

In this section, we will explain the simple Sraffa system, the case of single product industries. There is no joint production. Land and fixed capital are excluded. Each industry produces a single commodity by using a certain quantity of labour and certain quantities of commodities as a means of production. The number of industries and thus the number of products is considered to be n.

We will express the Sraffa system with matrix algebra. If we denote the row vector of output by $\mathbf{x}$, the transposed matrix of the Leontief input coefficient matrix by $\mathbf{A}$, and the row vector of the actual net product by $\mathbf{y}$, then the quantity equation can be represented by

$$\mathbf{x} = \mathbf{y} + \mathbf{x}\mathbf{A} \tag{1}$$

This means that the quantity produced is divided into two parts; the surplus produced in the economy and the replacement of commodities used up in the production processes. $\mathbf{x}\mathbf{A}$ denotes the commodities that should be replaced. For simplicity, we will use the notation

$$\mathbf{k}_y = \mathbf{x}\mathbf{A} \tag{2}$$

This is the row vector of the produced means of production or capital goods. The equation (1) can be rewritten as

$$\mathbf{x} = \mathbf{y} + \mathbf{k}_y \tag{3}$$

The vector $\mathbf{y}$ denotes the surplus. From (1)(3), we also obtain

$$\mathbf{y} = \mathbf{x}(\mathbf{I} - \mathbf{A}) = \mathbf{x} - \mathbf{k}_y \tag{4}$$

where $\mathbf{I}$ is the unit matrix. The equation (4) gives the definition of the row vector of the actual net product that makes up the actual national income. The j th component of $\mathbf{y}$ represents the net product of industry j. If each industry produces a surplus, $\mathbf{y}$ will be strictly positive: $\mathbf{y} > 0$, and if there is no surplus in some industries, $\mathbf{y}$ will be semi-positive: $\mathbf{y} \geq 0$.

[**Assumption 1**] $\quad \mathbf{x} > 0, \quad \mathbf{y} \geq 0, \quad \mathbf{A} > 0$

We assume that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{A}$ are given exogenously. We also assume, for simplicity, that all products are basic commodities, and there is no non-basic commodity. Then, $\mathbf{A}$ is an indecomposable matrix.

[**Assumption 2**] *Matrix* $\mathbf{A}$ *is indecomposable*

The trivial case of the uniform capital intensity for all sectors will not be considered in this paper.

Let us turn to the labour vector. We denote the column vector of the Leontief type labour input coefficient vector by $\mathbf{l}_A$.

[**Assumption 3**] $\quad \mathbf{l}_A > 0$

It enables us to define the actual total labour (L_A) as $\mathbf{x}\mathbf{l}_A$, i.e.

$$L_A = \mathbf{x}\mathbf{l}_A \tag{5}$$

Let us call $\mathbf{l}_A$ the actual labour coefficient vector, which is corresponding to $\mathbf{x}\mathbf{l}_A$. We assume that $\mathbf{l}_A$ is given exogenously.

Now we proceed to define the price equation system given by the technique [$\mathbf{A}$, $\mathbf{l}_A$]. We call the exchange-ratios, which enable the system to be economically viable, the prices of commodities or simply the prices. Let us denote the column vector of the commodity prices by $\mathbf{p}_A$. And let us denote the rate of profits by r, which is assumed to be uniform all over the economic system. Moreover let us denote the maximum rate of profits by R. Then the rate of profits will take real numbers ranging from 0 to R ($0 \leq r \leq R$). Similarly, a uniform rate of wage (post factum) is assumed to be prevailing in the economy. It is indicated by w_A. From above, the price system can be written as

$$\mathbf{p}_A = (1+r)\mathbf{A}\mathbf{p}_A + w_A\mathbf{l}_A \tag{6}$$

In (6), there are (n+2) unknowns and n equations, so that the degree of freedom is two. *Assumptions 1,2* and the assumption of $\mathbf{l}_A > 0$ assure a positive solution of $\mathbf{p}_A$ for all r ($0 \leq r \leq R$).

2.2 Normalization of Total Labour and the Sraffian Price System

Sraffa assumes that the actual total labour is equal to unity. Let us formulate this assumption. The actual total labour is represented as $\mathbf{x}\mathbf{l}_A$, which may not be necessarily equal to unity. In order to distinguish it from the actual labour coefficient vector $\mathbf{l}_A$, we denote it by $\mathbf{l}_S$, and define it as

$$\mathbf{l}_S = (1/\mathbf{x}\mathbf{l}_A)\mathbf{l}_A \tag{7}$$

We will call this normalized vector the standard labour coefficient vector. Pre-multiplying $\mathbf{l}_S$ by $\mathbf{x}$, the normalized total labour of system can be represented as $\mathbf{x}\mathbf{l}_S$. By definition, $\mathbf{x}\mathbf{l}_S$ is equal to unity. We will call this normalized total labour the Standard total labour or in short Standard labour. Let us denote the standard labour by L_S. The standard labour will be represented as

$$L_S = (1/L_A)\mathbf{x}\mathbf{l}_A = \mathbf{x}\mathbf{l}_S = 1 \tag{8}$$

Moreover, since $\mathbf{l}_A > 0$ from *Assumption 3*, the standard labour coefficient vector becomes strictly positive. Therefore, we have

$$\mathbf{l}_S > 0 \tag{9}$$

Since we changed the labour coefficient vector from $\mathbf{l}_A$ to $\mathbf{l}_S$, we must alter the notation of the wage variable. In place of w_A corresponding to $\mathbf{x}\mathbf{l}_A$, we denote the wage corresponding to $\mathbf{x}\mathbf{l}_S$ by w_S. Let us make clear the implication of the wage variable w_S corresponding to $\mathbf{x}\mathbf{l}_S$. Since we have two types of labour coefficient vector (i.e. $\mathbf{l}_A$ and $\mathbf{l}_S$), the total wage will be represented either by $w_A\mathbf{x}\mathbf{l}_A$ or by $w_S\mathbf{x}\mathbf{l}_S$. Although $w_A \neq w_S$ and $\mathbf{x}\mathbf{l}_A \neq \mathbf{x}\mathbf{l}_S$, $w_A\mathbf{x}\mathbf{l}_A$ is equal to $w_S\mathbf{x}\mathbf{l}_S$ in the aggregates. Therefore we have

$$w_A\mathbf{x}\mathbf{l}_A = w_S\mathbf{x}\mathbf{l}_S \tag{10}$$

This is the relationship between w_A and w_S.

The price vector should also be redefined because we altered the definition of the labour input coefficient vector from $\mathbf{l}_A$ to $\mathbf{l}_S$. Let us denote the price vector corresponding to the standard labour coefficient vector $\mathbf{l}_S$ by $\mathbf{p}_S$. Then we can reformulate the price vector given by the technique $[\mathbf{A}, \mathbf{l}_S]$ as

$$\mathbf{p}_S = (1+r)\mathbf{A}\mathbf{p}_S + w_S\mathbf{l}_S \tag{11}$$

Comparing (6) and (11), the wage variable is altered from w_A to w_S, and the price vector is altered from $\mathbf{p}_A$ to $\mathbf{p}_S$. Then the relationship between $\mathbf{p}_A$ and $\mathbf{p}_S$ can be represented as

$$\mathbf{p}_S = (1/\mathbf{x}\mathbf{l}_A)\mathbf{p}_A \tag{12}$$

Assumptions 1-3 assure a positive solution of $\mathbf{p}_S$ for all r ($0 \leq r \leq R$). In the following part, we will take the equation (11) as that of the Sraffa System.

2.3 Sraffa's Standard System

The standard system is defined as a virtual system whose quantity vector corresponds to the eigenvector of the input coefficient matrix $\mathbf{A}$ and which has a uniform rate of surplus in physical terms throughout industries. If we denote the physical rate of surplus by Π, and the row vector of output of the standard system by $\mathbf{h}$, the standard system can be represented as

$$\mathbf{h} = (1+\Pi)\mathbf{h}\mathbf{A} \tag{13}$$

If we denote the standard commodity vector by $\mathbf{u}$, it can be defined from (13) as

$$\mathbf{u} = \mathbf{h}(\mathbf{I}-\mathbf{A}) = \Pi\mathbf{h}\mathbf{A} \tag{14}$$

The components of this vector correspond to the commodities constituting the standard commodity.

It is indeed true that Sraffa introduced the standard commodity for the analysis of price changes and referred to the term frequently, but what Sraffa adopted in his analysis was not the standard commodity in general but the standard net product. The standard net product is the standard commodity when the total labour of the standard system is equal to the Standrad total labour (the normanilzed actual total labour). If we denote the row vector of the total output produced in the standard system when the total labour of the standard system is equal to the total labour of the actual system by $\mathbf{q}$ in place of $\mathbf{h}$, the relationship between $\mathbf{q}$ and $\mathbf{h}$ will be represented by

$$\mathbf{q} = t\mathbf{h} \tag{15}$$

where t is a positive scalar. Like (13), the vector $\mathbf{q}$ is defined as the vector that is given by

$$\mathbf{q} = (1+\Pi)\mathbf{q}\mathbf{A} \tag{16}$$

and, in addition, that satisfies the following assumption

$$\mathbf{q}\mathbf{l}_S = \mathbf{x}\mathbf{l}_S \tag{17}$$

From (16), we can obtain, like (14), the following equation

$$\mathbf{s} = \mathbf{q}(\mathbf{I}-\mathbf{A}) = \Pi\mathbf{q}\mathbf{A} \tag{18}$$

This is called the standard net product or standard national income. The vector $\mathbf{s}$ is uniquely determined if the matrix $\mathbf{A}$ and the total labour $\mathbf{x}\mathbf{l}_S$ are given exogenously.

The standard net product vector $\mathbf{s}$ is a bundle of commodities potentially producible with the given technique $[\mathbf{A},\mathbf{l}_S]$ and the given total labour $\mathbf{x}\mathbf{l}_S$.

When the wage is equal to zero, the price equation (11) will reduce to

$$\mathbf{p}_S = (1 + R)\mathbf{A}\mathbf{p}_S \tag{19}$$

This implies that the price vector $\mathbf{p}_S$ is the right-hand eigenvector of the matrix $\mathbf{A}$ and $1\ /(1{+}R)$ is its eigenvalue. On the other hand, in the equation (18), the vector $\mathbf{q}$ is the left-hand eigenvector of the matrix $\mathbf{A}$ and $1/(1{+}\Pi)$ is its eigenvalue. From the Perron-Frobenius theorem, we have

$$\Pi = R \tag{20}$$

Substituting (20) into (18), we obtain

$$\mathbf{s} = \mathbf{q}(\mathbf{I} - \mathbf{A}) = R\mathbf{q}\mathbf{A} \tag{21}$$

This is a useful expression of the standard net product.

The aggregate capital of the standard system should be calculated as follows. Let us show the calculation of the standard capital by aggregating the produced means of production. Let us denote the vector of the produced means of production of the standard system by $\mathbf{k}_S$, then we can define it by

$$\mathbf{k}_S = \mathbf{q}\mathbf{A} \tag{22}$$

The standard net product is defined by (21). Post-multiplying (21) by price vector $\mathbf{p}_S$, we have

$$\mathbf{s}\mathbf{p}_S = R\mathbf{q}\mathbf{A}\mathbf{p}_S \tag{23}$$

From (22)(23), we can obtain

$$\mathbf{s}\mathbf{p}_S = R\mathbf{k}_S\mathbf{p}_S \tag{24}$$

Then, if we adopt the following condition

$$\mathbf{s}\mathbf{p}_S = 1 \tag{25}$$

and if we denote the price vector expressed in terms of the standard net product by

$$\mathbf{p}_{sp} = \mathbf{p}_S/\mathbf{s}\mathbf{p}_S \tag{26}$$

then we have the value of the standard capital as follows

$$K_S = \mathbf{k}_S\mathbf{p}_{sp} = 1/R \tag{27}$$

This is the aggregate value of the standard capital. It is important to understand that the value of standard capital is independent of the variation of the prices and distribution. It is given simply as the inverse of R.

2.4 Distribution measured in terms of the Standard Net Product

Under the condition (25), w_S has three different implications. The wage expressed in terms of the standard net product can be defined as

$$w_{sp} = w_S/\mathbf{sp}_S \tag{28}$$

Under the condition (25), we have

$$w_{sp} = w_S \tag{29}$$

Under the condition (25), the total wage measured in terms of the standard net product can be represented as

$$W_{sp} = w_S\mathbf{xl}_S/\mathbf{sp}_S = w_S \tag{30}$$

Also the wage-standard income ratio (the wage share to the standard income) can be represented as

$$\omega_{sp} = W_{sp}/\mathbf{sp}_{sp} = w_S \tag{31}$$

where ω_{sp} is a pure number.

Then let us explain the wage curve under the condition (25). From (11), we can obtain

$$\mathbf{q}(\mathbf{I}-\mathbf{A})\mathbf{p}_S = (r/R)R\mathbf{qAp}_S + w_S\mathbf{ql}_S \tag{32}$$

Substituting (17)(21) into (32), we can obtain

$$(1-r/R)\mathbf{sp}_S = w_S\mathbf{xl}_S \tag{33}$$

From(8), we can obtain

$$(1-r/R)\mathbf{sp}_S = w_S \tag{34}$$

Under the condition (25), we can obtain

$$r = R(1-w_{sp}) \tag{35}$$

where $0 < w_{sp} \leq 1$ or alternatively $0 \leq r < R$. This means that the wage curve is independent of the price changes. We have got, at this stage, to the well-known w-r relationship of the Sraffian framework.

The relationship between the wage share and the profit share to the standard national income π_{sp} will become

$$1 = \omega_{sp} + \pi_{sp} \tag{36}$$

The profit share to the standard national income π_{sp} can be represented as

$$\pi_{sp} = r\mathbf{k}_S\mathbf{p}_S/\mathbf{sp}_S \tag{37}$$

Then, from (27)(37), we have

$$\pi_{sp} = r/R \tag{38}$$

From (36)(38), we have

$$\omega_{sp} = 1 - r/R \tag{39}$$

The standard distributional ratio θ_{sp} can be represented as

$$\theta_{sp} = \Pi_S/W \tag{40}$$

or we have

$$\theta_{sp} = \pi_{sp}/\omega_{sp} \tag{41}$$

Therefore, from (38)(39), we can obtain

$$\theta_{sp} = r/(R - r) \tag{42}$$

The distribution ratio changes in a simple rule of (42), and it is independent of the changes in the prices.

3 Adding-up or Mark-up Interpretation of Prices

3.1 The Evaluation System for $\mathbf{p}_v$

From (33), under the condition of (25), we can derive the equality between the standard income $\mathbf{sp}_S$ and the standard labour $\mathbf{xl}_S$. This equality is represented as

$$\mathbf{sp}_S = \mathbf{xl}_S \tag{43}$$

Since the right member of this equation is measured in terms of unit of labour, the left member should also be measured in terms of unit of labour. In order to explain reasonably the equality of (43), which is measured in terms of unit of labour, we will introduce the value of labour v_L into the equation (43) as follows.

$$\mathbf{sp}_v = v_L \mathbf{xl}_S \tag{44}$$

If we use the following notation

$$\mathbf{p}_v = \mathbf{p}_S/\ v_L \tag{45}$$

then from (44)(45) we have

$$\mathbf{sp}_v = \mathbf{xl}_S \tag{46}$$

The left member of this equation means that the aggregate value of the standard net product is obtained by post-multiplying the vector $\mathbf{s}$ by $\mathbf{p}_v$. The right member is the standard total labour. Then the standard national income in terms of the quantity of labour is equal to the standard total labour. Since the standard labour is set equal to unity, we have

$$\mathbf{sp}_v = 1 \tag{47}$$

The standard condition (normalization condition) of the Sraffian price system is

$$v_L = 1 \tag{48}$$

Under this condition, the wage will be

$$w_v = w_S / v_L \tag{49}$$

In order to show the relationship between the standard condition (25) and the Sraffian price system, we will introduce an enlarged price system, which we call *Evaluation System* (see Yagi [1999]). When a set of data ($\mathbf{x}$, $\mathbf{A}$, $\mathbf{l}_S$, $\mathbf{s}$, $R = \Pi$) is given, we have the following Evaluation System

$$[\textit{Evaluation System } 1] \quad \begin{aligned} v_L &= 1 && (50)\\ \mathbf{s}\mathbf{p}_S &= v_L \mathbf{x}\mathbf{l}_S && (51)\\ w_v &= w_S / v_L && (52)\\ \mathbf{p}_v &= \mathbf{p}_S / v_L && (53)\\ \mathbf{p}_S &= (1+r)\mathbf{A}\mathbf{p}_S + w_S \mathbf{l}_S && (54) \end{aligned}$$

where $0 \leq r < R$. In Evaluation System 1, there are $(2n+3)$ independent equations and $(2n+4)$ unknowns (i.e. v_L, w_S, w_v, $\mathbf{p}_S$, $\mathbf{p}_v$, r). The degree of freedom is one. The exogenous rate of profits will make the system determinate. Under the condition of $v_L = 1$, we can obtain for all r ($0 \leq r < R$),

$$r = R(1 - w_S) \tag{55}$$

When the standard condition is given by (48), we have

$$\mathbf{p}_v = (1 - r/R)[\mathbf{I} - (1-r)\mathbf{A}]^{-1}\mathbf{l}_S \tag{56}$$

This is an interpretation of the Reduction equation of Sraffa [1960].

3.2 The Commanded Labour $\mathbf{p}_w$

The prices of commodities can be divided into two parts: the part of wage and the profit as surplus. From (36)(56), we have

$$\mathbf{p}_v = \omega_{sp}\mathbf{p}_v + \pi_{sp}\mathbf{p}_v \tag{57}$$

If we rewrite (57) as

$$\mathbf{p}_v / \omega_{sp} = \mathbf{p}_v + (\pi_{sp}/\omega_{sp})\mathbf{p}_v \tag{58}$$

and let us denote

$$\mathbf{p}_w = \mathbf{p}_v / \omega_{sp} \tag{59}$$

then we have

$$\mathbf{p}_w = \mathbf{p}_v + \theta_{sp}\mathbf{p}_v \tag{60}$$

or

$$\mathbf{p}_w = (1 + \theta_{sp})\mathbf{p}_v \tag{61}$$

The standard condition for price vector $\mathbf{p}_w$ can be considered as the quantity of labour which can be purchased by the standard net product. The standard for $\mathbf{p}_w$ is represented as

$$\mathbf{s}\mathbf{p}_w = R/(R-r) \tag{62}$$

From The reduced form in respect of $\mathbf{p}_w$ can be represented as

$$\mathbf{p}_w = [\mathbf{I} - (1-r)\mathbf{A}]^{-1}\mathbf{l}_S \tag{63}$$

Comparing (63) with (56), we can obtain the equation (61). The interepretation of (61) is very interesting because it explains the resolution of prices into two different parts. The above explanation crucially depends on the choice of standard conditions. If we consider the j th price of the equation (61), we will have

$$p_w^{(j)} = (1+\theta_{sp})p_v^{(j)} \tag{64}$$

3.3 The Cost Price System $\mathbf{p}_c$

From (57), we have

$$\mathbf{p}_v = \omega_{sp}\mathbf{p}_v + (\pi_{sp}/\omega_{sp})\omega_{sp}\mathbf{p}_v \tag{65}$$

In first term of the right member, π_{sp} / ω_{sp} means the distributional ratio between the profit and the wage, or in other words the profit rate over the one unit of wage. In the second term of the right member, $\omega_{sp}\mathbf{p}_v$ means the wage part of the price and it can be interpreted as the part of the wage cost of one unit of a commodity. Now let us use the notation

$$\mathbf{p}_c = \omega_{sp}\mathbf{p}_v \tag{66}$$

then from (65) we have

$$\mathbf{p}_v = \mathbf{p}_c + \theta_{sp}\mathbf{p}_c \tag{67}$$

The first term of the right member means the wage cost in a price, and the second means the profit per unit of wage in a price. The price $\mathbf{p}_v$ is divided into the wage part and the profit part. From (67) we have

$$\mathbf{p}_v = (1+\theta_{sp})\mathbf{p}_c \tag{68}$$

This is a similar relationship with (61).

Now let us interpret the price vector $\mathbf{p}_c$ in our Evaluation system. If we denote the wage basket vector by $\mathbf{b}_S$ given by the standard net product, then it will be defined as

$$\mathbf{b}_S = ((R-r)/R)\mathbf{s} \tag{69}$$

On the other hand, corresponding to the vector $\mathbf{b}_S$, we can define the following new labour coefficient vector

$$\mathbf{l}_B = ((R-r)/R)\mathbf{l}_S \tag{70}$$

This is the vector corresponding to the labour amount to produce the wage paid in terms of the standard income. By using the j th component of the vector $\mathbf{l}_B$, we can obtain the labour amount to produce the j th component of the standard net product vector $\mathbf{s}$. Let us express it by $L_b^{(j)}(r)$, then it will be defined as follows

$$L_B^{(j)}(r) = b^{(j)}l_S^{(j)} = q^{(j)}l_B^{(j)}(r) = ((R-r)/R)q^{(j)}l_S^{(j)} \tag{71}$$

The column vector whose components are given by the equation (71) can be represented as follows

$$\mathbf{L}_B = [L_B^{(1)}(r), L_B^{(2)}(r), \cdots, L_B^{(n)}(r)] \tag{72}$$

The total amount of labour to produce the wage part of the standard net product will be given as

$$L_B(r) = \mathbf{x}\mathbf{l}_B(r) = \mathbf{q}\mathbf{l}_B(r) = \mathbf{e}\mathbf{L}_B \tag{73}$$

Corresponding to the equation (44), let us consider the equality between the value of wage basket and the labour amount of the wage basket as follows

$$\mathbf{b}_S\mathbf{p}_B = v_L\mathbf{x}\mathbf{l}_B \tag{74}$$

The value relationship of this equation is measured in terms of the anonymous unit. Let us define the price vector

$$\mathbf{p}_c = \mathbf{p}_B/v_L \tag{75}$$

Then under the condition of v_L =1, from (74)(75), we obtain

$$\mathbf{b}_S\mathbf{p}_c = \mathbf{x}\mathbf{l}_B \tag{76}$$

When a set of data ($\mathbf{x}$, $\mathbf{A}$, $\mathbf{l}_S$, $\mathbf{s}$, $R = \Pi$) is given, we have the following Evaluation System

[**Evaluation System 2**]

$$\begin{aligned}
v_L &= 1 && (77)\\
\mathbf{b}_S &= ((R-r)/R)\mathbf{s} && (78)\\
\mathbf{l}_B &= ((R-r)/R)\mathbf{l}_S && (79)\\
\mathbf{b}_S\mathbf{p}_B &= v_L\mathbf{x}\mathbf{l}_B && (80)\\
\mathbf{p}_c &= \mathbf{p}_B/v_L && (81)\\
\mathbf{p}_B &= (1+r)\mathbf{A}\mathbf{p}_B + w_S\mathbf{l}_B && (82)
\end{aligned}$$

where $0 \leq r < R$. In *Evaluation System 2*, there are ($4n$+2) independent equations and ($4n$+3) unknowns (i.e. v_L, w_S, $\mathbf{p}_c$, $\mathbf{p}_S$, r, $\mathbf{b}_S$). The degree of freedom is one. The exogenous rate of profits will make the system determinate. The reduced form in respect of $\mathbf{p}_c$ becomes

$$\mathbf{p}_c = (1 - r/R)[\mathbf{I} - (1-r)\mathbf{A}]^{-1}\mathbf{l}_B \tag{83}$$

Comparing with (56), we have

$$\mathbf{p}_c = (1 - r/R)\mathbf{p}_v \tag{84}$$

or we have

$$\mathbf{p}_v = R/(R-r)\mathbf{p}_c \tag{85}$$

Therefore we have

$$\mathbf{p}_v = (1 + \theta_{sp})\mathbf{p}_c \tag{86}$$

This interepretation is very interesting because it explains the resolution of prices into two different parts. The above explanation crucially depends on the choice of standard conditions. If we consider the j th price of the equation (86), we will have

$$p_v^{(j)} = (1 + \theta_{sp})p_c^{(j)} \tag{87}$$

3.4 The Actual Price and Mark-up Interpretation

The Sraffian price system adopts the assumption that the rate of profit is equal throughout industries and the wage rate is also equal throughout industries. On the contrary, if we look at the I-O table, the actual distributional ratio varies through industries. Therefore let us consider the case that an industry has its own distributional ratio and there are as many distributional ratios as the number of exsisting industries.

Let us denote the distributional ratio of industry j by $\theta^{(j)}$. We have a set of distributional ratios as follows

$$[\theta^{(1)}, \theta^{(2)}, \cdots, \theta^{(n)}] \tag{88}$$

Moreover, if we denote the actual wage share of industry j by $\omega^{(j)}$, then we can show a set of them as follows

$$[\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(n)}] \tag{89}$$

If we denote the actual price of commodity j by $p_{actual}^{(j)}$, then we can show a set of them as follows

$$[p_{actal.}^{(1)}, p_{actal.}^{(2)}, \cdots, p_{actal.}^{(n)}] \tag{90}$$

In order to compare $p_{actal.}^{(1)}$ with $p_v^{(j)}$, let us normalize the actual prices of (90) as

$$[p_{S\ (actal).}^{(1)}, p_{S\ (actal).}^{(2)}, \cdots, p_{S\ (actal).}^{(n)}] \tag{91}$$

which satisfies the following condition:

$$\mathbf{s}\mathbf{p}_{S\ (actual)} = 1 \tag{92}$$

where $\mathbf{p}_{S\ (actual)}$ is the price vector whose elements are the prices of (91). If we denote the actual cost price by $p_{c\ (actal)}^{(j)}$, let us asuume that the following relation will hold between the actual price and the actual cost price

$$p_{S\ (actal)}^{(j)} = (1 + \theta^{(j)})p_{c\ (actal)}^{(j)} \tag{93}$$

On the other hand, in the Sraffian price system, the equation (87) holds between $p_v^{(j)}$ and $p_c^{(j)}$. Therefore, let us define the ratio between $p_{S\ (actal)}^{(j)}$ and $p_v^{(j)}$ as follows

$$\frac{p_{S\ (actal)}^{(j)}}{p_v^{(j)}} = 1 + m \tag{94}$$

Then, from (94), we have

$$p_{S\ (actal)}^{(j)} = (1+m)p_v^{(j)} \tag{95}$$

or, from (94)(87), we have

$$p_{S\ (actal)}^{(j)} = (1+m)(1+\theta_{sp})p_c^{(j)} \tag{96}$$

We are considering that the equation (96) expresses the deviation from the cost price $p_c^{(j)}$as reference . The equation (96) can be considered as a kind of mark-up pricing equation.

4 Reconsideration of Mark-up Equation in Macroeconomics

4.1 A Price System with No Surplus and the Standard Productivity Index

Now let us proceed to the case of $r = 0$ and $\mathbf{A} \neq 0$. In this case, the Sraffian price system (11) reduces to the following equation

$$\mathbf{p}_S = \mathbf{A}\mathbf{p}_S + v_L \mathbf{l}_S \tag{97}$$

where $v_L = w$, because $r = 0$. From (97), we can obtain

$$\mathbf{p}_S = v_L[\mathbf{I} - \mathbf{A}]^{-1}\mathbf{l}_S \tag{98}$$

Let us denote the vertically integrated labour coefficient vector by v_S, it can be defined as (see Pasinetti [1973][1977])

$$\mathbf{v}_S = [\mathbf{I} - \mathbf{A}]^{-1}\mathbf{l}_S \tag{99}$$

This vector is measured in terms of physical unit of labour. Therefore from (98)(99), we can obtain

$$\mathbf{p}_S = v_L \mathbf{v}_S \tag{100}$$

In this equation, v_L plays the role of connecting the physical labour coefficients to the prices of commodities. From (45)(100), we have

$$\mathbf{p}_v = \mathbf{v}_S \tag{101}$$

In equation (101), the price vector in terms of labour becomes equal to the vertically integrated labour coefficient vector.

The Standard labour coefficient vector of *Period 1* can be defined as

$$\mathbf{l}_S^1 = (1/\mathbf{x}^1\mathbf{l}_A^1)\mathbf{l}_A^1 \tag{102}$$

The Standard labour coefficient vector of *Period 2* can be defined as

$$\mathbf{l}_S^2 = (1/\mathbf{x}^1\mathbf{l}_A^1)\mathbf{l}_A^2 \tag{103}$$

The superscript 1 indicates *Period 1* and the superscript 2 indicates *Period 2*. Let us consider the intertermporal comparisons of prices when the rate of profit is equal to zero. A set of given data is ($\mathbf{x}^1$, $\mathbf{A}^1$, $\mathbf{l}^1_A$, L^1_A, $\mathbf{x}^2$, $\mathbf{A}^2$, $\mathbf{l}^2_S$,L^2_A).

[Two Period Price System 1]

$$\mathbf{p}^1_S = \mathbf{A}\mathbf{p}^1_S + v^1_L \mathbf{l}^1_S \quad (104)$$
$$\mathbf{p}^2_S = \mathbf{A}\mathbf{p}^2_S + v^2_L \mathbf{l}^2_S \quad (105)$$

This system has $2n$ independent equations and $(2n+2)$ variables. In this case, we can also set the values of labour of each period equal to unity, i.e.

$$v^1_L = v^2_L = 1 \quad (106)$$

Let $\mathbf{v}^1_S$ be the vertically integrated labour coefficient of *Period 1* and $\mathbf{v}^2_S$ be that of *Period 2*, then it can be defined as

$$\mathbf{v}^1_S = [\mathbf{I} - \mathbf{A}^1][\mathbf{I} - \mathbf{A}]^{-1}\mathbf{l}^1_S \quad (107)$$
$$\mathbf{v}^2_S = [\mathbf{I} - \mathbf{A}^2][\mathbf{I} - \mathbf{A}]^{-1}\mathbf{l}^2_S \quad (108)$$

These vectors are measured in terms of physical unit of labour. When the rate of profit is equal to zero, the equations of the above system become

$$\mathbf{p}^1_v = \mathbf{v}^1_S \quad (109)$$
$$\mathbf{p}^2_v = \mathbf{v}^2_S \quad (110)$$

Since the vertically integrated labour coefficient vectors can be compared with each other, the price vector of *Period 1* can be compared with that of *Period 2*. The equations (107)(108) express only the physical quantity of labour required to produce one unit of commodities.

Now let us construct index numbers with the price vectors (107)(108) and the standard net product vectors. The vector of the standard net product of each period will be given as

$$\mathbf{s^1} = \mathbf{q^1}(\mathbf{I} - \mathbf{A^1}) \quad (111)$$
$$\mathbf{s^2} = \mathbf{q^2}(\mathbf{I} - \mathbf{A^2}) \quad (112)$$

If we denote the input cost index by C_{sv}, it can be represented as

$$C_{sv} = \frac{\mathbf{s}^2\mathbf{p}^2_v}{\mathbf{s}^1\mathbf{p}^1_v} = \frac{\mathbf{s}^2\mathbf{v}^2_S}{\mathbf{s}^1\mathbf{v}^1_S} \quad (113)$$

The labour input index can be defined as

$$L^2_S = L^2_A / L^1_A \quad (114)$$

Since $\mathbf{q}^1\mathbf{l}^1_S = \mathbf{s}^1\mathbf{v}^1_S$ and $\mathbf{q}^2\mathbf{l}^2_S = \mathbf{s}^2\mathbf{v}^2_S$, we have

$$L^1_S = \mathbf{s}^1\mathbf{v}^1_S \quad (115)$$
$$L^2_S = \mathbf{s}^2\mathbf{v}^2_S \quad (116)$$

Therefore, the labour input index can be represented by

$$L_S^2 = \mathbf{s}^2\mathbf{v}_S^2/\mathbf{s}^1\mathbf{v}_S^1 \tag{117}$$

From (113)(117) , we have

$$C_{sv} = L_S^2 \tag{118}$$

Let us denote the price index of Fisher type by P_{sv}, the output index of Fisher type by Q_{sv}. Then The price index and output index defined can be represented as

$$P_{sv} = \sqrt{\frac{\mathbf{s}^1\mathbf{v}_S^2}{\mathbf{s}^1\mathbf{v}_S^1} \cdot \frac{\mathbf{s}^2\mathbf{v}_S^2}{\mathbf{s}^2\mathbf{v}_S^1}} \qquad Q_{sv} = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_S^2}{\mathbf{s}^1\mathbf{v}_S^2} \cdot \frac{\mathbf{s}^2\mathbf{v}_S^1}{\mathbf{s}^1\mathbf{v}_S^1}} \tag{119}$$

From (117)(119), these indexes can be reduced to a simpler form as follows

$$P_{sv} = \sqrt{\frac{\mathbf{s}^1\mathbf{v}_S^2}{\mathbf{s}^2\mathbf{v}_S^1} \cdot L_S^2} \qquad Q_{sv} = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_S^1}{\mathbf{s}^1\mathbf{v}_S^2} \cdot L_S^2} \tag{120}$$

From the theory of index number, the relationship between C_{sv}, P_{sv}, Q_{sv} becomes

$$C_{sv} = P_{sv} \cdot Q_{sv} \tag{121}$$

In our model, however, from (118)(120)(121), we can obtain the following relationship

$$1/P_{sv} = Q_{sv}/L_S^2 = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_S^1}{\mathbf{s}^1\mathbf{v}_S^2} \cdot \frac{1}{L_S^2}} \tag{122}$$

This is a quite interesting result because the inverse of the price index is equal to the output index divided by L_S^2. In (121), the productivity change is calculated both from the cost side by $1/P_{sv}$ and from the quantity side by Q_{sv}/L_S^2. In Hicks[1981], Hicks called the inverse of price index the productivity index. Our productivity index can be defined by

$$\Lambda_{sv} = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_S^1}{\mathbf{s}^1\mathbf{v}_S^2} \cdot \frac{1}{L_S^2}} \tag{123}$$

It should be stressed that Λ_{sv} is obtained by the given production techniques [$\mathbf{A}^1$, $\mathbf{l}_A^1$; $\mathbf{A}^2$, $\mathbf{l}_A^2$] and the total labour of each period, i.e. L_A^1 and L_A^2. Therefore if we want to measure the technological productivity change rate of the economy, Λ_{sv} is useful because Λ_{sv} is independent of the changes in compositions of the net product. Λ_{sv} is considered to reflect the changes in production techniques, the labour growth and the output growth.

4.2 A Sraffian Evaluation System for Price Comparison

Now let us construct our Two-Periods Model for price comparison when the rate of profit is positive. A set of given data is ($\mathbf{x}^1$, $\mathbf{A}^1$, $\mathbf{l}_S^1$, $\mathbf{q}^1$, $\mathbf{s}^1$, r^1,R^1, $\mathbf{x}^2$, $\mathbf{A}^2$, $\mathbf{l}_S^2$, $\mathbf{q}^2$, $\mathbf{s}^2$, r^2,R^2). Then we have the following two-period system of evaluation.

$$
\textbf{[Two Period Evaluation System 2]} \quad
\begin{array}{rcll}
\mathbf{s}^1\mathbf{p}_S^1 &=& v_L^1\mathbf{x}^1\mathbf{l}_S^1 & (124)\\
\mathbf{s}^2\mathbf{p}_S^2 &=& v_L^2\mathbf{x}^2\mathbf{l}_S^2 & (125)\\
\mathbf{p}_S^1 &=& (1+r^1)\mathbf{A}^1\mathbf{p}_S^1 + w_S^1\mathbf{l}_S^1 & (126)\\
\mathbf{p}_S^2 &=& (1+r^2)\mathbf{A}^2\mathbf{p}_S^2 + w_S^2\mathbf{l}_S^2 & (127)
\end{array}
$$

In this system, there are ($2n$+2) independent equations and ($2n$+4) unknowns (w_S^1, $\mathbf{p}_S^1$, v_L^1, w_S^2, $\mathbf{p}_S^2$, v_L^2). If the condition of standard for each period is given, the above system will become determinate. In this system, the following proposition will hold (see the revised version of Yagi[2000b]).

[Theorem] In the above Two-Period Evaluation System, the value of labour of *Period 1* is equal to the value of labour of *Period 2* if and only if

$$r^1 = R^1(1-w_S^1) \quad \textit{and} \quad r^2 = R^2(1-w_S^2) \tag{128}$$

where $0 \leq r^1 < R^1$ and $0 \leq r^2 < R^2$.

[Proof] In the above Two-Period Evaluation System, we have the following for all $r^1$of $0 \leq r^1 < R^1$ and for all $r^2$of $0 \leq r^2 < R^2$

$$v_L^1 = 1 \iff r^1 = R^1(1-w_S^1) \tag{129}$$
$$v_L^2 = 1 \iff r^2 = R^2(1-w_S^2) \tag{130}$$

Then we have

$$v_L^1 = v_L^2 = 1 \iff r^1 = R^1(1-w_S^1) \quad \textit{and} \quad r^2 = R^2(1-w_S^2) \tag{131}$$

From this, Theorem is verified.

Q.E.D.

This theorem states that the prices of different periods are measured in terms of a common unit, unit of labour. Therefore, under the condition of (128), the prices of different periods can be compared with each other. Theorem states that the equations of (128) can be replaced by the condition

$$v_L^1 = v_L^2 = 1 \tag{132}$$

Either (128) or (132) can make the Two-Period Evaluation System determinate. It is more important to understand that, in the Two-Period Evaluation System, the labour can be considered as the intertemporal standard of value by the condition (128). Under the condition (128) or (132), the prices and wages of both periods are measured in terms of the same unit of labour. The Reduction equations, or the reduced forms of $\mathbf{p}_v^1$, $\mathbf{p}_v^2$, are represented as

$$\mathbf{p}_v^1 = \mathbf{p}_S^1/v_L^1 = (1-r^1/R^1)[\mathbf{I}-(1+r^1)\mathbf{A}^1]^{-1}\mathbf{l}_S^1 \tag{133}$$
$$\mathbf{p}_v^2 = \mathbf{p}_S^2/v_L^2 = (1-r^2/R^2)[\mathbf{I}-(1+r^2)\mathbf{A}^2]^{-1}\mathbf{l}_S^2 \tag{134}$$

These price vectors are measured in terms of the same unit of labour when the condition (128) or (132) is adopted. The total labour of the economy can be represented by

$$L_S^1 = \mathbf{s}^1\mathbf{p}_v^1 \tag{135}$$
$$L_S^2 = \mathbf{s}^2\mathbf{p}_v^2 \tag{136}$$

4.3 Positive Profit and Reconsideration of the Mark-up Equation of Macroeconomics

When the rate of profit is positive, the input cost index can be represented as

$$C_{sp(r^1,r^2)} = \frac{\mathbf{s}^2\mathbf{p}_v^2}{\mathbf{s}^1\mathbf{p}_v^1} \tag{137}$$

Let us denote the price index of Fisher type by $P_{sp(r^1,r^2)}$, the output index of Fisher type by $\Omega_{sp(r^1,r^2)}$. Then the price index and output index defined can be represented as

$$P_{sp(r^1,r^2)} = \sqrt{\frac{\mathbf{s}^1\mathbf{p}_v^2}{\mathbf{s}^1\mathbf{p}_v^1} \cdot \frac{\mathbf{s}^2\mathbf{p}_v^2}{\mathbf{s}^2\mathbf{p}_v^1}} \qquad \Omega_{sp(r^1,r^2)} = \sqrt{\frac{\mathbf{s}^2\mathbf{p}_v^2}{\mathbf{s}^1\mathbf{p}_v^2} \cdot \frac{\mathbf{s}^2\mathbf{p}_v^1}{\mathbf{s}^1\mathbf{p}_v^1}} \tag{138}$$

From (137)(138), these indexes can be reduced to a simpler form as follows

$$P_{sp(r^1,r^2)} = \sqrt{\frac{\mathbf{s}^1\mathbf{p}_v^2}{\mathbf{s}^2\mathbf{p}_v^1} \cdot L_S^2} \qquad \Omega_{sp(r^1,r^2)} = \sqrt{\frac{\mathbf{s}^2\mathbf{p}_v^1}{\mathbf{s}^1\mathbf{p}_v^2} \cdot L_S^2} \tag{139}$$

From the theory of index number, the relationship between $C_{sp(r^1,r^2)}$, $P_{sp(r^1,r^2)}$, $\Omega_{sp(r^1,r^2)}$ becomes

$$C_{sp(r^1,r^2)} = P_{sp(r^1,r^2)} \cdot \Omega_{sp(r^1,r^2)} \tag{140}$$

From (115)(116)(117)(135)(136)(137)(139)(140), we can obtain the following relationship

$$1/P_{sp(r^1,r^2)} = \Omega_{sp(r^1,r^2)}/L_S^2 = \sqrt{\frac{\mathbf{s}^2\mathbf{p}_v^1}{\mathbf{s}^1\mathbf{p}_v^2} \cdot \frac{1}{L_S^2}} \tag{141}$$

This is also a quite interesting result. Let us define the value productivity index as follows

$$\Lambda_{sp(r^1,r^2)} = \sqrt{\frac{\mathbf{s}^2\mathbf{p}_A^1}{\mathbf{s}^1\mathbf{p}_A^2} \cdot \frac{1}{L_S^2}} \tag{142}$$

It should be stressed that $\Lambda_{sp(r^1,r^2)}$ is obtained by the given data $[\mathbf{A}^1, \mathbf{l}_A^1, r^1; \mathbf{A}^2, \mathbf{l}_A^2, r^2]$ and the total labour of each period, i.e. L_A^1 and L_A^2. I calls $\Lambda_{sp(r^1,r^2)}$ the Standard Value Productivity index, which is obtained by synthesizing Hicks's idea of the Opportunity cost approach and Sraffa's idea of the Standard net product. $\Lambda_{sp(r^1,r^2)}$ is also one of the indicators of social productivity changes. It should be noticed that $\Lambda_{sp(r^1,r^2)}$ is obtained, under the condition of (128), by the production techniques, the rate of profit of each period, and the total labour of each period, or in other words by a set of data data

$[\mathbf{A}^1, \mathbf{l}^1_A, r^1, L^1_A;\ \mathbf{A}^2, \mathbf{l}^2_A, r^2, L^2_A]$. $\Lambda_{sp(r^1,r^2)}$ is independent of the composition of demand. From(141) (142), we have the price index

$$P_{sp(r^1,r^2)} = 1/\Lambda_{sp(r^1,r^2)} \tag{143}$$

Let us rewrite this as

$$P_{sp(r^1,r^2)} = \frac{1}{\Lambda_{sv}} \frac{\Lambda_{sv}}{\Lambda_{sp(r^1,r^2)}} \tag{144}$$

Now let us consider the mark-up pricing equation of macroeconomic meaning by using the price index of (139). The relationship of profit share π_{sp} and the wage share ω_{sp} is represented as

$$\pi_{sp} + \omega_{sp} = 1 \tag{145}$$

Then from the price index of (139) we have

$$P_{sp(r^1,r^2)} = \omega_{sp} P_{sp(r^1,r^2)} + \pi_{sp} P_{sp(r^1,r^2)} \tag{146}$$

From this we have

$$P_{sp(r^1,r^2)} = (1 + \theta_{sp}) \omega_{sp} P_{sp(r^1,r^2)} \tag{147}$$

Since the equality of $\omega_{sp} = w_S$ holds in the Sraffian price system, we have

$$P_{sp(r^1,r^2)} = (1 + \theta_{sp}) w_S P_{sp(r^1,r^2)} \tag{148}$$

From this we have

$$P_{sp(r^1,r^2)} = (1 + \theta_{sp}) \frac{w_S}{\Lambda_{sv}} \frac{\Lambda_{sv}}{\Lambda_{sp(r^1,r^2)}} \tag{149}$$

Let us denote the ratio of $\Lambda_{sv}\ /\ \Lambda_{sp(r^1,r^2)}$ by $\xi(r^1, r^2)$, i.e.

$$\xi(r^1, r^2) = \frac{\Lambda_{sv}}{\Lambda_{sp(r^1,r^2)}} = \frac{P_{sp(r^1,r^2)}}{P_{sv}} \tag{150}$$

From (149)(150) we have

$$P_{sp(r^1,r^2)} = (1 + \theta_{sp}) \frac{w_S}{\Lambda_{sv}} \xi(r^1, r^2) \tag{151}$$

This is our formulation of Mark-up equation of macroeconomic meaning. It is important to understand that the equation (151) has the term $\xi(r^1, r^2)$. In usual formulation of Mark-up pricing equation, the effect of the distribution to the general price ($\xi(r^1, r^2)$) is nelected. However, as is seen in (151), even in the case when the equal rate of profit and the equal wage rate are privailing through industries, we should consider the effect of $\xi(r^1, r^2)$.

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