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Approximate Surrogate Production Functions[1]

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Summary:

The Cambridge debate showed that an aggregation of capital is not possible in general. A recent investigation has found one example for reswitching and several for reverse capital deepening, but the paradoxes seem not to be frequent. The paper provides a theoretical justification of this result and shows how wage curves of input-output matrices with small non-dominant eigenvalues become quasi-linear with some numéraires. Large random systems lead to the genesis of such states. Approximate surrogate production functions then seem possible. A family of economic systems with constant capital composition allows to construct a surrogate production function.

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[1] I should like to thank Christian Bidard, Sergio Parrinello, Anwar Shaikh, Lefteris Tsoulfidis and the participants of seminars held at Duisburg, Paris, Rome, New York and Graz for valuable advice. Special thanks are due to Christian Schmidt for the empirical investigation reported in section 8. The responsibility for errors remains my own.

1. Production functions and the paradoxes of capital

The debate about capital theory, beginning with Robinson (1953-54), but really focussed on Samuelson's surrogate production function (Samuelson 1962), showed that a rigorous aggregation of capital and hence a logically stringent construction of the production function were impossible (Garegnani 1970, Harcourt 1972, Pasinetti 1966, Sen 1974). The theoretical problem turned out to differ from other aggregation problems in economics, because it concerned produced means of production. Simple criteria to rule out the paradoxes failed (Gallaway and Shukla 1976). The debate has remained open (Cohen and Harcourt 2003) and is topical because production functions are ubiquitous in endogenous growth theory. Implications of the paradoxes of capital theory for intertemporal general equilibrium theory have also been found (Garegnani 2003, Schefold 1997, 2005, 2008) but are not considered here. Anwar Shaikh (1987) extended the critique to empirical methods of estimating production functions and was answered by Solow (1987). However, the opponents found no solid ground on which they could have reached common conclusions. One side insists on the use of production functions for theoretical and pragmatic reasons, the other denies the legitimacy of the approach.

Recently, an empirical inquiry (Han and Schefold 2006) showed that empirical examples for reverse capital deepening (see below) exist, but are not frequent. If this result proves to be robust, it might help to settle the debate by defining a compromise. At any rate, it is our purpose here to provide theoretical arguments justifying this result.

The name of the surrogate production function already suggested that its originator Samuelson (1962) had something less than perfect in mind. We return to the old debate in order to find out to what extent the criteria for a rigorous construction may be relaxed without falling into arbitrariness[2]. The assumptions made for the construction of the surrogate production function are as usual: one deals with a closed economy, with a linear technology, constant returns to scale, single product industries, and labour of uniform quality. There is no reason to generalise at this stage, since the introduction of heterogeneous labour, of fixed capital and joint production and of variable returns to scale does not render the existence of the surrogate production function more likely. No special form of the production function will be postulated[3]. The assumption of perfect competition should be retained[4].

Hence we assume a finite number of methods of production, available for the production in each industry in the form of a book of blueprints. Competition will then ensure that, at any given rate of profit, a certain combination of methods will be chosen, one in each industry, such that positive normal prices and a positive wage rate result, expressed in terms of a numéraire. The wage rate can then be drawn in function of the

2 For a first incomplete enunciation of some of the results of this paper (sections 1-4 only) see Schefold 2008a.

3 The likelihood of the existence of C.E.S. production functions is discussed in Schefold 2006.

4 A set-theoretical description of technological alternatives does not eliminate the possibility of paradoxes of capital theory, as long as strict convexity is not postulated, and strict convexity is an extremely problematic assumption (see Schefold 1976).

rate of profit for this combination of methods between a rate of profit equal to zero and a maximum rate of profit, and the 'individual' wage curve for this technique will be monotonically falling (see Han and Schefold 2006 for a more detailed description). If the choice of technique is repeated at each rate of profit, starting from zero, different individual wage curves will appear on the envelope of all possible wage curves, and the envelope will also be monotonically falling. Technical change is '*piecemeal*' in that only one individual wage curve will be optimal in entire intervals, except at a finite number of switch points where generically only two wage curves intersect and where a change of technique generically takes place only in one industry, so that the two wage curves to the left and to the right of the switch point will have all other methods in all other industries in common. The intensity of capital and output per head change discontinuously at the switch points (they can be represented geometrically for a given individual wage curve $w(r)$, if the numéraire consists of the vector of output per head in the stationary state): output per head equals $w(0)$ and capital per head $k = (w(0) - w(r))/r$ (see diagram 1).

If many individual wage curves appear successively on the envelope, this envelope may be replaced by a smooth approximation, and each point on this modified envelope can be thought to represent one individual technique, represented by an individual wage curve. The surrogate production function then is defined by taking the tangent to this modified envelope (supposed to be convex to the origin): the slope of the tangent is equal to capital per head and the intersection of the tangent with the abscissa is equal to output per head, as in diagram 1. One thus obtains capital per head $k(r)$ and output per head $y(r)$ as functions of the rate of profit, and one can show from this parameter representation that a production function $y = f(k)$ must exist (Schefold 1989, p. 297-298). If and only if the individual wage curves are linear, the construction is rigorous in that output per head and capital per head of techniques individually employed will be equal to those which we have just defined, and the paradoxes of capital theory (to be discussed presently) will then be absent.

However, the critique of the surrogate production functions starts from the observation that individual wage curves will in general not be linear and the envelope will not be necessarily convex to the origin; envelope $\hat{w}(r)$ in diagram 1 provides an example. Output per head at $\tilde{r}$ is given by $\tilde{w}(0)$, where $\tilde{w}(r)$ is the individual wage curve tangent to $\hat{w}(r)$ at $\tilde{r}$. With non-linear wage curves, there is likely to be a divergence between output per head and capital per head in the individual industry and the corresponding values which follow from the definition of the surrogate production function; this divergence is called *declination* and it is illustrated in diagram 1: output per head would be $\hat{y}$ and $k = tg\alpha$, if the individual wage curve $\hat{w}(r)$ was linear, but since this is not the case, there is the declination $\tilde{w}(0) - \hat{y}$. Output per head equals $\hat{y}$ according to the definition of the surrogate production function, but actual output per head really is $\tilde{w}(0)$, if $r = 0$.

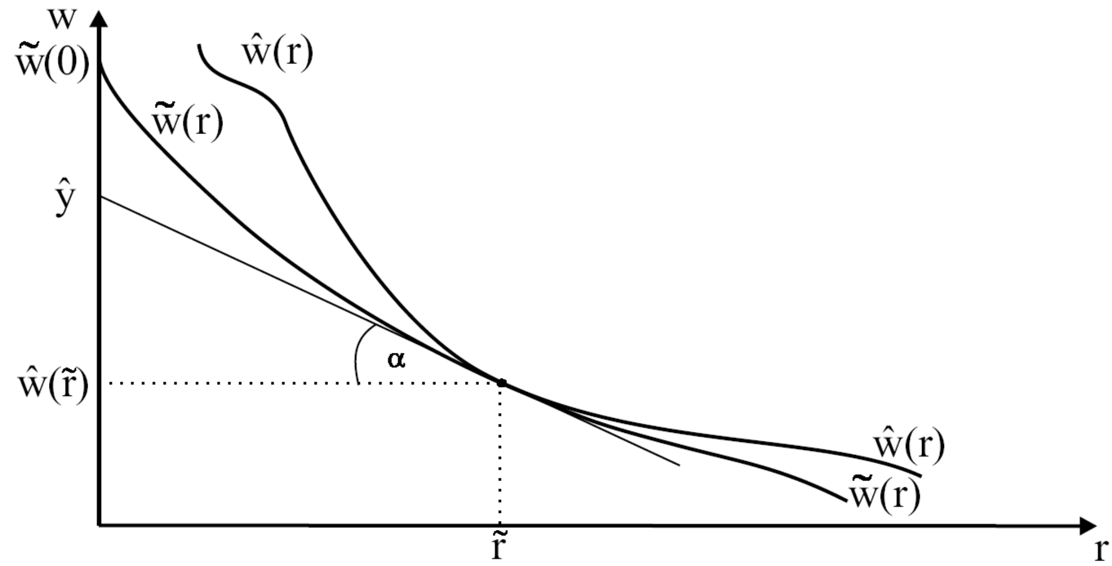


Diagram 1: Declination $\tilde{w}(0) - \hat{y}$. The surrogate production function yields output per head $\hat{y} = \hat{w}(\tilde{r}) + rtg\alpha,\ tg\alpha = -\hat{w}'(\tilde{r}) = K/L$, since $tg\alpha = (\hat{y} - \hat{w})/\tilde{r}$, but actual output per head equals $\tilde{w}(0)$ in the stationary state.

Declination entails an inaccuracy of the procedure of aggregation. The paradoxes of capital concern the change of techniques, engendered by the change of distribution and visible in the piecemeal change of wage curves on the envelope. The phenomenon which has attracted most attention is that of reswitching and reverse capital deepening: there may be switch points on the original envelope such that the intensity of capital does not fall with the rate of profit (*reverse capital deepening*), and the individual wage curve may have appeared on the envelope already at a lower rate of profit (*reswitching*). It is also possible that capital per head rises with the rate of profit in the industry where the switch of methods of production takes place (*reverse substitution of labour*) and, surprisingly, reverse capital deepening (the perverse change of aggregate capital per head) and reverse substitution of labour (a perverse change of capital per head at the industry level) need not go together[5] in systems with more than two industries (Han and Schefold 2006). *Returns of processes* seem to be frequent: a process which is used in

5 The main case in which this *paradox of paradoxes* occurs is given, if reverse capital deepening at r_3 is associated with three switchpoints $r_1,\ r_2,\ r_3$ between two wage curves $w^*,\ w^{**}$, with $-1 < r_1 < 0 < r_2 < r_3 < Min(R^*, R^{**})$, with r_3 on the envelope, with w^* above w^{**} at $r = 0$ and for $r > r_3$, and with the intersection at r_2 dominated by some third wage curve $w^+,\ R^+ < r_3$ (the reader is advised to draw the diagram). As r rises from $r_3 - \varepsilon$ to $r_3 + \varepsilon,\ \varepsilon > 0$ in a stationary state, the aggregate intensity of capital rises paradoxically, since $w^*(0) > w^{**}(0)$. But the constellation also implies that, as the switch takes place in one sector, say $1,\ l_1$ increases. For we have $w^*(-1) < w^{**}(-1)$ with $1 = \mathbf{dp}^*(-1) = \mathbf{dp}^{**}(-1) = w^*(-1)\mathbf{dl}^* = w^{**}(-1)\mathbf{dl}^{**}$, hence $\mathbf{dl}^* > \mathbf{dl}^{**}$ and, since $l_i^* = l_i^{**};\ i = 2,...,n;\ l_1^* > l_1^{**}$. By definition of a switchpoint $(1+r)\mathbf{a}_1^*\mathbf{p}^* + w^* l_1^* = (1+r)\mathbf{a}_1^{**}\mathbf{p}^{**} + w^{**} l_1^{**}$ and $w^* = w^{**}$ at r_3. We conclude that raising r from $r_3 - \varepsilon$ to $r_3 + \varepsilon$ leads to a fall in the intensity of capital in the sector, where the switch in the method of production occurs, while the intensity of capital in the aggregate rises. Reverse capital deepening, a rise of the intensity of capital as the rate of profit rises, is curious enough, but it is even more curious that this can happen, while the intensity of capital falls, as usually expected, in the sectors where the change of technique takes place. The phenomenon should not be confused with a Wicksell effect.

one industry in one interval of the rate of profit is used again in another interval, but not in between. This is a generalisation of reverse capital deepening. It can be shown to imply large changes of relative prices and capital values and it demonstrates that processes cannot be classed as being inherently more or less capital-intensive, prior to their use in specific systems and at specific levels of distribution.

Some thought (e.g. myself, Schefold 1989 [1971], p. 298) that reverse capital deepening might be about just as likely (frequent) as 'normal' switches and that one would encounter 'many' individual wage curves succeeding each other on the envelope in a piecemeal fashion (it was conceded that 'reswitching' might be 'rather unlikely' in Schefold 1997, p. 480). A different picture emerges in Han and Schefold (2006), where it is assumed that techniques used in the past, as represented in corresponding input-output tables, could be used again, and that similarly the technique used in another country could be used at home. Comparing only two input-output tables in this manner results in a multitude of wage curves, since two methods (the foreign method or that of the past) are available as alternatives to the actual method employed in each industry so that 2^n alternative systems result, if both input-output tables are composed of n sectors.

Han and Schefold (2006) analysed envelopes derived from nearly 500 pairs of input-output tables for economies different in space or time (32 tables with 36 sectors). It was not possible to compute all the 2^{36} wage curves for each of 496 pairs, but the envelopes were obtained by means of linear programming. Contrary to our expectations, reverse capital deepening and reverse substitution of labour are obtained only in a little less than 4 % of all switch points on the envelopes. Technical change is confirmed as piecemeal, but, also surprisingly, only about 10 wage curves out of 2^{36} = 68 719 476 736 appear on average on each envelope. (This number will increase, if more than two - say m - input-output tables of n sectors are combined to define a book of blueprints as we shall see in more detail in section 7.)

Joan Robinson used to tell me that if one technique was really better than another, their wage curves would not cross at all - I replied that the stylised facts of growth theory (constant capital-output ratio, weak dependence of the capital-labour ratio of a given technique on the rate of profit) implied near linear wage curves turning around the maximum rate of profit (Schefold 1997, p. 277) - paper first published 1979), hence, with perturbed techniques, one would have to expect some switchpoints near the maximum rate of profit. At the other extreme, neoclassical theory and Sraffa share the expectation that, as one moves down the envelope, there will be a 'rapid succession of switches' (Sraffa 1960, p. 85).

To look at the effect of all '*combinations*' of methods on the envelope was the starting point of the analysis of Han and Schefold (2006). Similar empirical investigations would be welcome to confirm or question their results. There are considerable methodological problems; they are discussed in the paper itself. Meanwhile, theoretical reflections on this peculiar outcome may be useful. The critics of neoclassical theory can point out that, for the first time, an empirical case of reswitching and many of reverse capital deepening have been found. But the frequency is not sufficient to destroy neoclassical hopes that the production function might survive as an approximation, similar perhaps not to the more rigorous laws of physics but to the empirical generalisations, supported by some theoretical considerations, which one finds in biology. The discussion then moves on a plane lower than that of the critique of pure theory in which approximations are not permitted. There must be theories also for approximations in the measurement of capital. It once was appropriate to confront the measurement 'in which the statisticians were mainly interested' (only 'approximate') and 'theoretical measures' which 'required absolute precision' and corresponded to 'pure definitions of capital', as 'required' by the theories (statement by Piero Sraffa of 1958, as quoted by Sen 1974, p. 331). We now want to create a theory for the approximations.

We know the characteristics individual wage curves would have to have for a rigorous construction: they would have to be linear. The envelope would then become convex to the origin, declination would vanish and the intensity of capital would fall with any increase of the rate of profit.

The open question thus is whether the surrogate production function can be defined under assumptions which are sufficiently general to take the relevant aspects of real modern economies into account and sufficiently specific to rule out the paradoxes of the capital theory in a form which would render meaningless the theoretical analysis or its application. This construct - if it exists - could be called an '*approximate surrogate production function*'.

2. Foundations of the approximation

The original surrogate production function had linear wage curves, and strictly linear wage curves imply that prices are equal to labour values (unless the numéraire is very special). Prices and values can differ substantially, as Ian Steedman and Judith Tomkins (1998) assert. It would not only be ironic to fall back on a primitive form of the labour theory of value (Marx had prices of production as transformed labour values), but there is also a specific inconsistency implied by the assumption of prices equal to values: it can be shown that two techniques with linear wage curves, due to uniform organic compositions of capital, can not coexist at a switch point; the switch would violate the principle of combination. For if their linear wage curves cross, a combination of the methods of the techniques will exist, with a wage curve dominating this point of intersection (Salvadori and Steedman 1988). The reason is that technical change on the envelope must be piecemeal. If we have a wage curve of a technique with uniform composition of capital on the envelope, more than one method must change in order to get to another technique which is also characterised by a uniform composition of capital.

A linear wage curve also results if the basket of goods defining the numéraire happens to be equal to Sraffa's standard commodity. The deeper reason why wage curves otherwise are not straight derives from a property of the so-called '*regular*' Sraffa systems introduced by Schefold (1989 [1971]): A system $(\mathbf{A},\mathbf{l})$ is regular, if the eigenvalues of $\mathbf{A}$ are semi-simple and if $\mathbf{l}$ is not orthogonal to any of the left-hand eigenvectors of $\mathbf{A}$. This property is generic and equivalent to the linear independence of the vectors $\mathbf{l}, \mathbf{Al}, \ldots, \mathbf{A}^{n-1}\mathbf{l}$. The point here is that it is also equivalent to the linear indepence of the price vector $\mathbf{p}(r)$, taken at n different rates of profit, i.e. to the linear independence of $\mathbf{p}(r_1),\ldots,\mathbf{p}(r_n);\ r_1 < \ldots < r_n$. The implied movement of relative prices entails the curvature of $w(r)$, unless the numéraire happens to be an eigenvector.

The two constellations mentioned above, which lead to linear wage curves, both concern the eigenvectors of the input matrix. If the labour theory of value holds and relative prices are constant, they must be equal to the relative prices formally obtained at a rate of profit equal to -1. They will then be equal to relative direct labour inputs. Hence, the labour vector must be the Frobenius eigenvector of the input matrix, if the labour theory of value holds. The standard commodity, on the other hand, is known to be the dual positive eigenvector. In the former case, the linear wage curve is possible because the system is not regular, in the second, because the numéraire is an eigenvector, which also implies an irregularity according to the extended definition in Schefold (1997, p. 116). Schefold (1989 [1971]) further considered the other eigenvalues of the input matrix. A transformation, which will be used again here, showed that relative prices as functions of the rate of profit took a very simple form, related to the properties of Sraffa's standard system, if the eigenvalues other than the Frobenius eigenvalue were zero. Thirty years later, Christian Bidard proved a hypothesis by Bródy and showed in a seminal paper together with Tom Schatteman (Bidard and Schatteman 2001) that the eigenvalues other than the dominant eigenvalue will tend to zero for larger and larger random matrices, and their result has been generalised and proved independently by mathematicians since (see below). On this basis, one can show (see section 3) that large 'random' systems will exhibit wage curves of even curvature.

We thus have three properties on which the construction of approximate surrogate production functions might perhaps be based, because they lead to even, more linear wage curves and they thus reduce both the risk of the paradoxes and declination: they would be based on systems with prices not differing much from labour values, with numéraire vectors not differing much from the standard commodity and with matrices having small eigenvalues (except for the dominant one).

However, there are three additional supporting properties. One can observe that the magnitudes on which the paradoxes of capital depend are locally continuous functions of elements of the input matrix, of the labour vector and of the numéraire, so that each single small change of methods of production in different industries can only exert a small effect on the aggregates, and if the system is large and the changes are many, rare paradoxical changes will, as it were, disappear in the noise of frequent transitions (the numerical results in Han and Schefold 2006 had this character[6]). The argument fails, if the paradoxes are frequent. That the paradoxes are rare would have to be shown by means of more empirical studies and will here theoretically be supported by means of the first three arguments.

The fifth argument concerns declination only and is discussed in Schefold (2006): One can prove that declination will diminish, if a positive rate of growth, g, is introduced, and declination disappears in the golden rule case $r = g$. Reswitching, by contrast, exists also in the golden rule case and independently of the choice of the numéraire: a

6 See table 2 in Han and Schefold (2006), where reverse capital deepening is of the order of magnitude of one percent.

technique, which had been in use at low rates of profit, reappears at high rates. But capital per head falls at both switch points, since declination disappears with $r = g$. If the first switchpoint is dominated by a third technique, capital reversing will thus not be observed on the envelope and two paradoxes (the hidden return of a technique and the change in the value of capital) compensate each other. This golden rule case is only of theoretical interest, however, since the important applications of the production function concern problems of employment and distribution which typically occur at low or zero rates of growth (in particular: there is unemployment in a stagnant economy and the question is whether lowering wages and raising the rate of profit will induce a choice of technique which eliminates unemployment). Declination increases with the curvature of one individual wage curve which appears on the envelope. Multiple switches become more likely with changes of the curvature of two individual wage curves which have at least one switchpoint in common on the envelope.

The last argument is randomness which leads not only to small eigenvalues for large matrices (argument three) but which also (and much more generally) lends stability to aggregates, as was observed by many and also, relatively early, by Marx. He based his assertion that total profits could be represented as a redistribution of surplus value partly on the mistaken mathematical argument of the algebraic 'transformation' of values into prices, partly on the hypothesis that the deviations of prices from values were random and would cancel on average. Here we can state in a like way that changes of distribution may have large effects on the relative prices of certain capital goods, but only a smaller effect on the aggregate price of all capital goods.

Our construction thus will be an attempt to render arguments often heard in oral discussions about the justifiability of production functions more precise: they are supposed to summarise what is taken to hold on average in a convenient and elegant form. The question becomes that of whether (or what extent) actual systems can be said to have random properties. It is surprising how little theoretical work exists on this question.

The reader should note that we are here not talking about the uncertainty of the measurement of individual coefficients of the system. The error involved in the measurement of individual elements of the matrix can be quite different according to the industry and the input concerned and must be reflected in distributions of the likely magnitude of the input which are specific to this element. The uncertainty about the inputs of random matrices, by contrast, consists in an uncertainty regarding the methods of production; only a specific mean is assumed to be given for the distribution of coefficients in each industry. The deterministic counterparts of the systems so obtained are irregular. Schefold had shown in 1971 (see Schefold 1989 [1971]) that the neoclassical construction is based on irregular systems and that irregular systems are of measure zero in the set of all systems and thus not generic. Surrogate production functions are therefore definitely not rigorous, if the systems are regular, as here defined. But it now seems possible that large random systems are not far from an 'irregular' state, and irregular systems, though not generic, might turn out to represent valid approximations to reality. We might go further and say that they play a role similar that played by *attractors* in chaos theory, since we shall provide an argument in section 7 according to which techniques with relatively straight wage curves have an evolutionary advantage.

We start from the first three arguments in this paper which concern the forms of the individual wage curves, hence they concern both the paradoxes and declination. Randomness will be considered in section 3 and continuity will be brought in in section 6, but we shall not use the golden rule assumption to eliminate declination. Preliminary investigations have led me to the conviction that no single of the three algebraic properties can serve to justify the construction of an approximate surrogate production function. Whether combinations of them (or of all six effects) can do that is again our open question in a more developed form.

In a preliminary attempt to solve it (Schefold 2008a), I proposed to discuss 'families' of wage curves defined by some common properties of the techniques involved. The families were called 'closed', if combinations of two techniques and their wage curves lead to a combined optimal technique and wage curve which still belonged to the same family.

Three such 'families' were discussed. One, based on '*circular*' systems as extensions of '*Austrian*' (Schefold 1999) models[7], was used to show that wage curves with extreme curvature are possible. The second, on the contrary, is taken up again here, using a more general notion of randomness than in Schefold (2008a), in order to demonstrate how near-linearity may be obtained. Since this family is associated with random 'large' input-output systems, the result justifies the use of approximate surrogate production functions to the extent that real systems are random. A third family will eventually be found, the deterministic counterpart of the second, now also generalised, which exhibits strictly linear wage curves and which thus permits the construction of a surrogate production function, but under a further restrictive condition.

3. Systems with small non-dominant eigenvalues

The techniques can be represented by Sraffa systems (Sraffa 1960) of the usual form:

$$(1+r)\mathbf{Ap}+w\mathbf{l}=\mathbf{p},$$

where $\mathbf{A}=(a_{ij})$ is the input matrix, $\mathbf{l}=(l_i)$ is the (positive) labour vector (column), $\mathbf{p}=(p_i)$ is the vector of prices; $i,j=1,...,n$; w is the wage rate, r is the rate of profit and $\mathbf{d}=(d_1,...,d_n)$ is the numéraire vector (row); prices are normalised so that $\mathbf{dp}=1$ for all r. The systems are assumed to be semi-positive, basic (indecomposable) and productive. Productivity can be ensured by assuming that there is a surplus with $\mathbf{eA}\le\mathbf{e}$ ($\mathbf{e}$ is the summation vector). The prices expressed in this numéraire and the wage rate will then be positive for $0\le r\le R$.

7 Circular systems allow to focus on the contrast between the ease with which examples of reswitching of the wine-and-oak-chest type could be constructed (Sraffa 1960), and the difficulty of finding reswitching in interdependent systems. It was at the origin of the false hypothesis advanced by Levhari (1965) that reswitching would not occur in an interdependent basic system (the possibility of a continuous transition from non-basic to basic systems was noted only afterwards by Levhari and Samuelson 1966, p. 519).

We now assume that the non-dominant eigenvalues of the input-output systems are small. As Bidard and Schatteman (2001) have shown, in the article already quoted, the non-dominant eigenvalues of so-called *random matrices* (with a random distribution of positive coefficients) have the property that the non-dominant eigenvalues all tend to zero as the number of sectors increases. It will help the understanding of what follows, if intuitive reasons for their result are given (more on the actual mathematical proof below).

Let $a_{ij}^{(n)}$; $i,j = 1,\ldots,n$; denote the elements of the semipositive indecomposable matrices $\mathbf{A}^{(n)}$; $n = 1,2,\ldots$. The $a_{ij}^{(n)}$ are random variables, i.i.d., with mean μ. The averages of each row and each column of $\mathbf{A}^{(n)}$ will tend to μ because of the strong law of large numbers. We therefore get (omitting superscript n) $\left|\mathbf{ea}^j/n - \mu\right| < \varepsilon > 0$ ($\mathbf{e}$ is the summation vector, $\mathbf{a}^j$ the column of $\mathbf{A}$), so that all column averages differ from μ only by a given ε, if n is large enough. This implies that the column sums approach $n\mu$ and tend thus to be equal, if n increases so slowly that $n\varepsilon$ can be made to go to zero. Hence $dom\mathbf{A}$ tends to $n\mu$ and $\mathbf{e}$ is the Frobenius eigenvector of $\mathbf{A}$. Given the distribution of the elements, $dom\mathbf{A}$ increases with n for given μ, but it is more instructive to assume $\mu = \lambda / n$, with $dom\mathbf{A}$ tending to λ.

To get an idea why all other eigenvalues will tend zero, we define $\mathbf{q}_i = \mathbf{e}_1 - \mathbf{e}_i$, $\mathbf{e}_i$; $i = 1,\ldots,n$; being the unit (row) vectors, and we obtain $\mathbf{q}_i\mathbf{A} = \mathbf{a}_1 - \mathbf{a}_i$; $i = 2,\ldots,n$. The elements of $\mathbf{q}_i\mathbf{A}$ will nearly be normally distributed with mean zero because of the central limit theorem, and $\left|\mathbf{q}_i\mathbf{A}\right|$ will be small, if the variance of the elements of $\mathbf{a}_1 - \mathbf{a}_i$ is small. $\mathbf{A}$ will then, for large n, be close to matrix $\mu\mathbf{E}$ ($\mathbf{E}$ is the matrix with all elements equal to one); the $\mathbf{q}_i$ are eigenvectors of $\mu\mathbf{E}$ with eigenvalues equal to zero; $i = 2,\ldots,n$. The proof of Bidard-Schatteman (2001) does not require the small variance argument and ensures convergence by having recourse to higher moments of the distribution.

A rigorous mathematical statement had independently been given by Goldberg et al. (2000), and this has been generalised significantly by Goldberg and Neumann (2003). The latter theorem is as follows (Goldberg and Neumann 2006, p. 749): The elements of $\mathbf{A}$ are random with mean $1/n$ and the rows of $\mathbf{A}$ independent. The variance of $b_{ij} = a_{ij} - (1/n)$ is bounded by c/n^2, and the absolute value of the covariance of any row b_i is bounded by c/n^3, c constant. For $0 < \delta < 1$, $0 < p < 1$ there is $N(\delta,p)$ such that for any $n > N(\delta,p)$ and for any γ with $1 > \gamma > \delta$, at least $n-1$ of the eigenvalues of $\mathbf{A}$ are in an open disc of radius γ around the origin.

Given the specification of the theorem, $dom\mathbf{A}$ tends to one and $\mathbf{A}$ tends to be stochastic (*i.e.* $\mathbf{A}$ tends to fulfil $\mathbf{eA} = \mathbf{e}$). It turns out that the subdominant eigenvalues tend to zero not only for random matrices with a common mean for all elements of the matrix, but it suffices - given the other assumptions - that each row has its own mean. Intuitive argument: if the rows of $\mathbf{A}$ have mean c_i/n, $\overline{\mathbf{A}} = (a_{ij}/c_i)$ has mean $1/n$. Note that we should *reduce* the generality of our analysis, if we postulated that both the rows *and* the columns of the input matrix were i.i.d. (cf. Section 5).

Of course, random matrices are not the only matrices with small non-dominant eigenvalues. Other such matrices will be discussed in sections 5, 6 and 7. It is clear that the elements of input-output tables are not strictly random: they are not independent, in that, if, e.g., a_{ij} is a chemical used in the production of a pharmaceutical product i, the quantity a_{ik} may denote another chemical required in a precise amount. We assume an identical distribution only out of *a priori* ignorance. But the assumption that the a_{ij} are i.i.d. in each row perhaps is not so bad in the large; we shall end with a generalisation and admit a determinate trend in section 6.

We start with $\mathbf{A} \geq \mathbf{0}$ basic, with eigenvalues $(1+R_i)^{-1}$; $i = 1,...,n$; where $R_2,...,R_n$ are different 'large' maximum rates of profit (except for the 'true' maximum rate of profit R_1 which corresponds to the Frobenius eigenvalue)[8]. We have $(1+R_i)\mathbf{q}_i\mathbf{A} = \mathbf{q}_i$, $\mathbf{l} \geq \mathbf{0}$, $\mathbf{d} \geq \mathbf{0}$. With any of the associated eigenvectors we get (proof by inversion of the matrix)

$$\mathbf{q}_i\left(\mathbf{I} - (1+r)\mathbf{A}\right)^{-1} = \frac{1+R_1}{R_i - r}\mathbf{q}_i.$$

This is a generalisation of Sraffa's standard system where $\mathbf{d} = \mathbf{q}_1 = \mathbf{q}(\mathbf{I} - \mathbf{A})$, $R_1 = R$ is the maximum rate of profit, with normalisation $\mathbf{ql} = 1$, $\mathbf{el} = 1$; this $\mathbf{d}$, taken as the numéraire, yields Sraffa's familiar linear wage curve in terms of standard prices $\bar{\mathbf{p}}$:

$$1 = \mathbf{q}(\mathbf{I} - \mathbf{A})\bar{\mathbf{p}} = r\mathbf{qA}\bar{\mathbf{p}} + \bar{w}\mathbf{ql} = (r/R)\mathbf{q}(\mathbf{I} - \mathbf{A})\bar{\mathbf{p}} + \bar{w}\mathbf{ql} = (r/R) + \bar{w}.$$

One thus has the wage curve in terms of the standard commodity

$$\bar{w} = 1 - \frac{r}{R}.$$

We generalise Sraffa's normalisation by putting $\mathbf{q}_i\mathbf{l} = \dfrac{R_i}{1+R_i}$ (assuming $\mathbf{q}_i\mathbf{l} \neq 0$, which means that $\mathbf{l}$ is not an eigenvector of $\mathbf{A}$ and the labour theory of value does not hold, for if it does hold, the wage curve is straight anyway). We now choose any arbitrary numéraire $\mathbf{d} > \mathbf{0}$ and represent the numériare as a linear combination of the eigenvectors: $\mathbf{d} = \lambda_1\mathbf{q}_1 + ... + \lambda_n\mathbf{q}_n$.

We thus obtain a simplified formula for the inverse of the wage rate

$$\frac{1}{w} = \mathbf{d}\left(\mathbf{I} - (1+r)\mathbf{A}\right)^{-1}\mathbf{l} = \sum \lambda_i\mathbf{q}_i\left(\mathbf{I} - (1+r)\mathbf{A}\right)^{-1}\mathbf{l} = \sum \lambda_i \frac{1+R_i}{R_i - r}\mathbf{q}_i\mathbf{l} = \sum \lambda_i \frac{R_i}{R_i - r} = \sum \frac{\lambda_i}{1 - \dfrac{r}{R_i}}.$$

8 We make the generic assumption that the eigenvalues are different.

The numéraire $\mathbf{d}$ here can be chosen so that $w(0)=1$ which is equivalent to $\sum \lambda_i = 1$. We can show (Schefold 2008a) that $\lambda_1 > 0$ and that the vector

$$\hat{\mathbf{q}} = \sum_{i=2}^{n} \lambda_i \mathbf{q}_i$$

is real. Obviously, the standard commodity represents the special case where $\lambda_1 = 1$, $\lambda_2 = ... = \lambda_n = 0$ so that $\mathbf{d} = \mathbf{q}_1 = \mathbf{q}(\mathbf{I}-\mathbf{A})$; then we have again

$$\overline{w} = 1 - \frac{r}{R}.$$

But the general formula for the wage is

$$w = \frac{1}{\dfrac{\lambda_1}{1-\dfrac{r}{R}} + \displaystyle\sum_{i=2}^{n} \dfrac{\lambda_i}{1-\dfrac{r}{R_i}}}.$$

Since $\lambda_1 > 0$, and since w is real if r is real (so that $\lambda_1/(1-r/R)$ is real), the second term in the denominator must also be real, as a sum of possibly complex terms. The wage curves w and $\overline{w}$ intersect at the maximum wage rate and at the maximum rate of profit, for $w(0)=\overline{w}(0)=1$ and $w(R)=\overline{w}(R)=0$; both curves fall monotonically. However, we have $w(r) \equiv \overline{w}(r)$ only for $\lambda_2 = ... = \lambda_n = 0$.

We are now interested in a family of wage curves for which the absolute values of $R_2,...,R_n$ are large enough so that each r/R_i can be ignored for $0 \le r < R$. This will be the case in particular for random matrices, for $1/(1+R_i)$ will then tend to zero almost surely with $c(p)/\sqrt{n}$, p probability, $0 \le p < 1$, c constant, according to Goldberg et al. (2000, p. 150). We further suppose that $\mathbf{d}$ is defined in such a way that $\sum \lambda_i$ remains bounded. One obtains an approximate wage curve $\tilde{w}(r)$, putting $z = \lambda_2 + ... + \lambda_n$:

$$\tilde{w}(r) = \frac{1}{\dfrac{\lambda_1}{1-\dfrac{r}{R}} + \lambda_2 + ... + \lambda_n} = \frac{R-r}{R\lambda_1 + (R-r)z} = \frac{R-r}{R-zr},$$

where $\lambda_1 + z = 1$. z must be real in the limit. It can be positive; we then must have $z < 1$, since $\lambda_1 + z = 1$. Or z can be negative, with $\lambda_1 > 1$. Two cases result, represented by two hyperbolas, one concave, one convex (for diagrams see Schefold (2008a).

4. First conclusions and discussion of the main assumption

The actual wage curve $w(r)$ will be very close to the hyperbola $\tilde{w}(r)$ and may cross it several times. And it is clear that thy hyperbola given by $\tilde{w}$ will approximate the linear wage curve of the standard wage $\overline{w}$ the better, the closer z is to zero, for the asymptotes of the two hyperbolas will then move to infinity and the wage curve $\tilde{w}$ will become linear. The case favourable for the construction of the surrogate production function and for neoclassical theory is obtained with $z < 0$, for the hyperbola will then be convex, and it will be relatively straight, if $|z|$ is small. A positive z implies $0 < z < 1$, since the wage curve cannot diverge to infinity for $0 \le r \le R$.

We thus identify two properties of the systems which lead together to almost linear wage curves:

1. If the non-dominant eigenvalues of the matrix are small enough, a simple hyperbolic form of the wage curve results; it is, as it were, very smooth. The wage curve, which in general is given by the ratio of two polynomials in r, of degree $n-1$ and n respectively, reduces here to a ratio of two polynomials of the first degree; *we conclude that any wage curve which is more complicated than a simple hyperbola owes these complications to non-dominant eigenvalues which are not equal to zero*.
2. If we have a hyperbola and want to obtain a nearly linear wage curve, it is important that z be close to zero so that the hyperbola is 'stretched'. This happens, if λ_1 is close to 1, *which means that the numéraire is close to the Frobenius eigenvector of the system*.

It is at once plausible that systems with relatively simple wage curve a will have prices that are relatively simple as functions of the rate of profit. Since we are mainly interested in the production function, we here only show that standard prices are linear functions of the rate of profit, if the non-dominant eigenvalues are small. Let the labour vector be represented as a linear combination $\mathbf{l} = \gamma_1\mathbf{x}_1 + ... + \gamma_n\mathbf{x}_n$ of the right-hand side eigenvectors $\mathbf{x}_i$, $(1+R_i)\mathbf{A}\mathbf{x}_i = \mathbf{x}_i$. Then we obtain for standard prices by a transformation analogous to that of section 3, with $R = R_1$,

$$\overline{\mathbf{p}} = \left(1 - \frac{r}{R}\right)\left(\mathbf{I} - (1+r)\mathbf{A}\right)^{-1}\sum \gamma_i \mathbf{x}_i = \frac{R_1 - r}{R_1}\sum_{i=1}^{n}\frac{1+R_i}{R_i - r}\gamma_i\mathbf{x}_i,$$

hence, if $R_2,...,R_n$ tend to infinity,

$$\overline{\mathbf{p}}(r) = \frac{1+R}{R}\gamma_1\mathbf{x}_1 + \sum_{i=2}^{n}\left(1 - \frac{r}{R}\right)\gamma_i\mathbf{x}_i.$$

This formula allows us to interpret $\overline{\mathbf{p}}(r)$ as a linear function of the extreme values $\overline{\mathbf{p}}(R) = \left[(1+R)/R\right]\gamma_1\mathbf{x}_1$ and labour values $\mathbf{u}$, $\mathbf{u} = \overline{\mathbf{p}}(0) = \overline{\mathbf{p}}(R) + \gamma_2\mathbf{x}_2 + ... + \gamma_n\mathbf{x}_n$:

$$\overline{\mathbf{p}}(r) = \overline{\mathbf{p}}(R) + \left(1 - \frac{r}{R}\right)\left(\overline{\mathbf{p}}(0) - \overline{\mathbf{p}}(R)\right),$$

therefore

$$\overline{\mathbf{p}}(r) = \mathbf{u} + \frac{r}{R}\left(\overline{\mathbf{p}}(R) - \mathbf{u}\right).$$

Such linear deviations of prices from values were empirically observed and discussed by A. Shaikh (1998), and by Mariolis and Tsoulfidis (2009) who note that $rk\mathbf{A} = 1$ represents an interesting case.

The observation that prices are near-linear functions of the rate of profit, as found by Shaikh, and by Mariolis and Tsoulfidis, and the near-linear wage curves which have appeared frequently in the empirical literature since Krelle (1976), could be explained by the assumption of small non-dominant eigenvalues, coupled with a numéraire not much different from the standard commodity. The latter assumption is roughly fulfilled, since net output or consumption are usually taken as price indices, but the near-linearity of the wage curve usually is preserved, if other price standards are chosen, and this seems to indicate that the former hypothesis also is important, because small non-dominant eigenvalues imply linear standard prices.

It is not sufficient only to postulate a numéraire close to the standard, also this property, if fulfilled exactly yields a linear wage curve. For if the standard is only approximate, "wiggles" of the wage curve may remain, even if, possibly only of small amplitude, leading to reverse capital deepening or even the paradox of paradoxes.

But why should we expect non-dominant eigenvalues to be small in a large class of systems? A complete mathematical answer to this question would presuppose a satisfactory solution to the inverse eigenvalue problem, applied to the whole spectrum of eigenvalues of a semipositive matrix (Minc 1988, p. 183). I offer some heuristic considerations.

It is easy to see that it suffices to analyse the case $\mathbf{eA} = \mathbf{e}$ (so-called stochastic matrices - 'stochastic' is not to be confused with 'random') - Gantmacher 1966, p. 74) so that $dom\mathbf{A} = 1$. The other eigenvalues must then be interior points of the unit circle or they are complex numbers z on the unit circle with $z = e^{2\pi ip/q}$; p, q natural numbers (the case of imprimitive matrices, Gantmacher 1966, p. 70). This suggests that the unit circle would gradually be filled by the eigenvalues of non-negative matrices picked out at random, but the theorems cited show that the subdominant eigenvalues of random matrices tend to concentrate at the centre of the circle as the order n of the matrices increases.

One might think that the concentration of the non-dominant eigenvalues at the centre of the circle holds not only for random matrices but for some larger class of matrices which would not necessarily be random. To identify such a class would be of interest for economists. However, an analysis based on the theory of matrix rings in Schefold (2008a) shows by means of a counterexample that the non-dominant eigenvalues do not generally tend more rapidly to zero for larger systems than for smaller ones, if we are concerned with primitive input matrices. Another special example in Schefold

(2008a) shows that the moduli of the eigenvalues can even be distributed uniformly between zero and the dominant eigenvalue[9]. The theorems mentioned in section 3 in fact prove the convergence of the non-dominant eigenvalues to zero only for random matrices. Hence we seem compelled to conclude that the decisive property leading to non-dominant eigenvalue is randomness, not so much the dimension of the matrix or any other structural property.

Now it is important to realise that the focus on randomness is more than a sophisticated return to one-commodity systems which have linear wage curves and for which surrogate productions functions exist. The presentation in Schefold (2008a) could create this mistaken impression, since it was assumed there that all elements of the random matrix had the same mean, while we here allow the means of the rows to differ. As a matter of fact, individual coefficients can vary considerably, even if there is only one mean, according to the assumptions of the theorem by Goldberg and Neumann referred to above. If the means of the rows differ, the levels of the use of capital changes from sector to sector, as can be seen when we complete our analysis by turning to determinate matrices with non-dominant eigenvalues strictly equal to zero - the matrices of section 5 represent, so to speak, certainty equivalents of the random matrices.

5. Wage curves of matrices with vanishing non-dominant eigenvalues

Consider indecomposable semipositive matrices with (without loss of generality) $\mathbf{eA} = \mathbf{e}$ (so-called 'stochastic' matrix) and with zero being a root of multiplicity $n-1$ of the characteristic polynomial, *i.e.* $\mu_2 = ... = \mu_n = 0$, where $\mu_2,...,\mu_n$ are the non-dominant eigenvalues. The associated $\mathbf{q}_2,...,\mathbf{q}_n$ are the associated eigenvectors which form a basis of $\mathbb{R}^n$, together with the Frobenius eigenvector $\mathbf{q}_1$. For any $\mathbf{x}\ \varepsilon\ \mathbb{R}^n$, $\mathbf{x} = \xi_1\mathbf{q}_1 + ... + \xi_n\mathbf{q}_n$, we have $\mathbf{xA} = \Sigma\xi_i\mathbf{q}_i\mathbf{A} = \xi_1\mathbf{q}_1$; hence $\mathbf{A}$ maps $\mathbb{R}^n$ on $\mathbb{R}$ and $rk\mathbf{A} = 1$. The rows $\mathbf{a}_i$ of $\mathbf{A}$ therefore are proportional and there is a column vector $\mathbf{c} = (c_1,...,c_n)^{\mathrm{T}} > 0$ such that $\mathbf{a}_i = c_i\mathbf{e}$. In fact, we can write $\mathbf{A} = \mathbf{ce}$, with $\mathbf{ec} = 1$. This implies $\mathbf{A} > \mathbf{0}$, for if any $a_{ij} = 0$, we should get $c_i = 0$ and $\mathbf{a}_i = 0$ so that $\mathbf{A}$ would not be indecomposable, contrary to the assumption.

Conversely, the only large random semipositive indecomposable and stochastic matrix with all non-dominant eigenvalues *strictly* equal to zero can be written as $\mathbf{A} = \mathbf{ce}$ (using the normalisation of the dominant eigenvalue introduced above). The elements of $\mathbf{a}_i$ must be equal to their mean c_i; and the analogous argument could be made with regard to the columns of $\mathbf{A}$. But it is natural from the economic point of view to regard the rows as representations of methods of production, the coefficients of which are random numbers taken from a distribution with mean c_i. In the deterministic case, these coefficients are equal to c_i. The family of systems with non-dominant eigenvalues

9 Let $\mathbf{D} = diag\{1/n,...,(n-1)/n,1\}$ and $\mathbf{A} = \mathbf{D} + \varepsilon\mathbf{E}$; $e_{ij} = 1$ for all i, ji; ε small. The eigenvalue μ_i of $\mathbf{A}$ will be close to $(n-i+1)/n$.

close to zero can be represented by the matrices towards which the members of the family converge. These are now the objects of our analysis.

Although they are positive, the matrices with zero non-dominant eigenvalues appear as matrices which could be called *one-industry systems of even capital composition* (even capital composition because all elements of each row are equal). The even capital composition is the deterministic counterpart of the assumption that the elements o each row are i.i.d. with a mean specific for the row c_i. We only add the special assumption $\mathbf{ec} = 1$ so that there is no surplus (this assumption will be dropped in the next section). We speak of one-industry systems, as opposed to one-commodity systems. The latter have linear wage curves, as is well known. That we are here dealing in essence with only one industry follows from the proportionality of the rows of $\mathbf{A}$. It can be made more evident by means of the following transformation. Define

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & . & . & . & 0 \\ -c_2/c_1 & 1 & . & . & . & 0 \\ & . & . & . & & \\ -c_n/c_1 & 0 & . & . & . & 1 \end{pmatrix}.$$

We consider prices in the interval $0 \le \rho \le 1$, as in section 2, since $R = 0$. The system of price equations

$$\rho\mathbf{Ap} + w\mathbf{l} = \mathbf{p}$$

then is equivalent to

$$\rho\overline{\mathbf{A}}\mathbf{p} + w\bar{\mathbf{l}} = \mathbf{Bp}$$

where

$$\overline{\mathbf{A}} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} = \mathbf{QA},\ \bar{\mathbf{l}} = \begin{pmatrix} l_1 & & \\ l_2 - (c_2/c_1)l_1 & & \\ . & . & . \\ l_n - (c_n/c_1)l_1 & & \end{pmatrix} = \mathbf{Ql},\ \mathbf{B} = \mathbf{QI} = \mathbf{Q}.$$

The transformed matrix $\overline{\mathbf{A}}$ is not positive. The transformation yields simplified equations in which industries $2,...,n$ seem to use no inputs other than labour, but they are not non-basic industries in the sense of Schefold (1989 [1971]). There is a similarity between the transformation using matrix $\mathbf{Q}$ here and that used to eliminate non-basics of the type of land in Sraffa in the case of differential rent of the second kind, but the procedures are not identical and no commodity here is a non-basic in the sense of the definition used in Schefold (1989 [1971]), p. 58. A comparison can also be made with the construction of the centre of a fixed capital system (Schefold 1989 [1971], p. 147), but again the transformation is not identical. A kind of joint production results. These one-industry systems with their peculiar wage curves are a novelty in the Sraffa literature.

The price equations (after the transformation) may now also be written as

$$p_1 = \mathbf{a}_1\mathbf{p} + wl_1$$
$$p_i - (c_i / c_1)p_1 = w(l_i - (c_i / c_1)l_1).$$

They could be solved by elimination, but it is more convenient to return to the original system with $\mathbf{A} > \mathbf{0}$, which is a single product system, and to solve by means of the standard commodity, here obviously given by the proportions of $\mathbf{e}$. We define standard prices $\overline{\mathbf{p}}$ by $\mathbf{e}\overline{\mathbf{p}} = 1$. In fact, normalising $\mathbf{el} = 1$:

$$1 = \mathbf{e}\overline{\mathbf{p}} = \rho\mathbf{eA}\overline{\mathbf{p}} + \overline{\mathrm{w}}\mathbf{el} = \rho + \overline{\mathrm{w}};$$

the standard wage $\overline{\mathrm{w}}$ is

$$\overline{\mathrm{w}} = 1 - \rho .$$

Standard prices thus fulfil

$$\overline{\mathbf{p}}(0) = \mathbf{l},$$

since $\mathbf{A}\overline{\mathbf{p}} = \mathbf{ce}\overline{\mathbf{p}} = \mathbf{c}$. More generally,

$$\overline{\mathbf{p}}(1) = \mathbf{c},$$

and, using $\mathbf{ce} = \mathbf{A}$,

$$\overline{\mathbf{p}} = \rho\mathbf{A}\overline{\mathbf{p}} + \overline{\mathrm{w}}\mathbf{l} = \rho\mathbf{ce}\overline{\mathbf{p}} + \overline{\mathrm{w}}\mathbf{l} = \rho\mathbf{c} + \overline{\mathrm{w}}\mathbf{l}.$$

The formula is even simpler than that derived for standard prices of random matrices in section 4. The irregular character of these matrices now comes to the fore. Regular systems, as defined in Schefold (1989 [1971]) and explained in section 1 above, have the property that the vector of prices of a system $\mathbf{p}(r)$, taken at n different rates of profit $r_1,...,r_n$, turns into n linearly independent vectors $\mathbf{p}(r_1),...,\mathbf{p}(r_n)$. This is here not the case. For instance, $\overline{\mathbf{p}}(\rho)$ is a linear combination $\overline{\mathbf{p}}(\rho) = \rho\overline{\mathbf{p}}(1) + (1-\rho)\overline{\mathbf{p}}(0)$. Note that the dimension of the space in which $\overline{\mathbf{p}}(\rho)$ moves therefore is lower than n as soon as $n \geq 3$. Two-dimensional examples often cannot reflect the full complexity of capital theory. The linearity of the price function will be shown to have important consequences for the effect of the choice of the numéraire on the form of the wage curve.

Prices in terms of the wage rate result at once. They rise in parallel up to the maximum rate of profit.

$$\hat{\mathbf{p}} = \overline{\mathbf{p}} / \overline{\mathrm{w}} = \frac{1}{1-\rho}(\mathbf{l} + \rho(\mathbf{c} - \mathbf{l})).$$

If we now use a more general numéraire $\mathbf{d} \geq \mathbf{0}$, the wage, expressed in this standard, will be, with $1 = \mathbf{dp} = \mathbf{d\hat{p}} \cdot w$,

$$w = \frac{1}{\mathbf{d\hat{p}}} = \frac{1 - \rho}{\mathbf{dl} + \rho\mathbf{d}(\mathbf{c} - \mathbf{l})}.$$

We thus find that the possibilities for obtaining a linear wage curve have considerably been enlarged.

a) There remains the familiar possibility of the wage being measured in terms of Sraffa's standard, here with $\mathbf{d} = \mathbf{e}$. Since $\mathbf{ec} = \mathbf{el}$, we already obtained $\overline{w} = 1 - \rho$.

b) The linear solution based on prices being equal to values here follows for $\mathbf{c} = \mathbf{l}$. This implies labour values, since we here get, with $\mathbf{c} = \mathbf{l}$, $\overline{\mathbf{p}} = \rho\mathbf{c} + \overline{w}\mathbf{l} = (\rho + \overline{w})\mathbf{l} = \mathbf{l}$. In general, it is necessary and sufficient for prices being equal to values that $\mathbf{l}$ is an eigenvector of $\mathbf{A}$. In fact, we have $\mathbf{Al} = \mathbf{l}$, since $\mathbf{Ac} = \mathbf{c}$.

c) The new and main result is that linear wage curves result, if $\mathbf{d}(\mathbf{c} - \mathbf{l}) = 0$. Since $\mathbf{e}(\mathbf{c} - \mathbf{l}) = 0$, $\mathbf{c} - \mathbf{l}$ has both positive and negative components for $\mathbf{c} \neq \mathbf{l}$, and the set D of all numéraires which result in linear wage curves $D = \{\mathbf{d} \geq \mathbf{0} | \mathbf{d}(\mathbf{c} - \mathbf{l}) = 0\}$ is an $(n-1)$-dimensional hyperplane containing $\mathbf{e}$. The scope of the systems generating linear (or almost linear) wage curves thus is considerably enlarged.

d) The main result can be rendered in a different form. One now easily sees how the effects of the numéraire being close to the standard and of the prices being near values may re-enforce each other. Suppose that $\mathbf{d} = \mathbf{e} + \mathbf{m}$, where the elements of row vector $\mathbf{m}$ represent small deviations of the numéraire from the standard, and $\mathbf{l} = \mathbf{c} + \mathbf{n}$, where $\mathbf{n}$ is a column vector of small deviations of the labour inputs from that labour vector which would cause prices to be equal to values, and suppose that the n_i have mean zero. The expression $\mathbf{d}(\mathbf{c} - \mathbf{l})$, which causes the deviation of the wage curve from linearity, becomes $\mathbf{d}(\mathbf{c} - \mathbf{l}) = (\mathbf{e} + \mathbf{m})(\mathbf{l} - \mathbf{n} - \mathbf{l}) = -\mathbf{en} - \mathbf{mn} = -\mathbf{mn}$, where $\mathbf{mn}$ is a sum of magnitudes of the second order of smallness.

We repeat: These one-industry systems of even capital composition represent the deterministic counterpart of the systems with random matrices of section 3. The price vectors and wage curves of random systems are approximations of the price vectors and wage curves encountered here. It remains to analyse technical choice and technical change. This must be based on a generalisation; we abandon the even composition of capital and the assumption that the matrices are stochastic, i.e. we cease to assume that $dom\mathbf{A} = 1$.

6. One-industry systems with strictly linear wage curves

So far, we have analysed prices and wage curves of individual systems, but the construction of an approximate surrogate production function requires the comparison of wage curves of systems. Systems differ in their processes, and the totals of commodities employed in production are different fractions of their gross output. The general form of semipositive indecomposable input-output matrices $\mathbf{A}$ with vanishing non-dominant eigenvalues is

$$\mathbf{A} = \mathbf{cf},$$

where $\mathbf{c}$ is a positive column vector, $\mathbf{f}$ a positive row vector, as can easily be shown, using the same argument as at the beginning of the last section. These are one-industry systems, but not of even capital composition, since the comparison of systems now requires the representation of a determinate structure of industries and of the surplus of the economy. Hence the maximum rate of profit must change with the change of systems. The generalisation seems small, but is important because random matrices were defined by the assumption that the elements of each row $\mathbf{a}_i$ are i.i.d. with mean c_i; it then follows that the non-dominant eigenvalues tend to zero for large matrices. The deterministic counterpart of these random matrices therefore are one-industry systems of even capital composition: $\mathbf{A} = \mathbf{ce}$ (where $\mathbf{ec} = dom\mathbf{A} < 1$, if $\mathbf{A}$ is not stochastic).

Since the consideration of technical change compels us to consider one-industry systems of general capital composition, with $\mathbf{A} = \mathbf{cf}$, we could here introduce random processes by assuming that we start from a given system with $\mathbf{A} = \mathbf{cf}$, hence with non-dominant eigenvalues equal to zero, and we could regard technical change as a perturbation of the elements of $\mathbf{A}$ (and as changes of $\mathbf{l}$) such that the non-dominant eigenvalues remain small. This would represent a true generalisation, compared to the random matrices considered in section 3, but we can only hint at it, since a rigorous mathematical theory determining the admissible extent of the perturbations, e.g. in terms of the admissible variance of the elements of a perturbed input matrix, does not seem to be available (the non-dominant eigenvalues obviously cease to be small if no conditions are imposed on the perturbations). The procedure will be justified further below; it means that we experiment with a compromise between randomisation for the representation of large systems (the statistical view) and the determination of the individual structure of production. Physics is sometimes said to be most difficult where quantum mechanics and classical mechanics meet, but our approach in the following model is very simple: we assume that the perturbations are small enough.

If $\mathbf{A}$ is productive with $R > 0$, we have $\mathbf{fA} = \mathbf{fcf}$, hence $\mathbf{f}$ is the Frobenius eigenvector, $1 + R = 1/\mathbf{fc}$ and $\mathbf{fc} = dom\mathbf{A} < 1$. The price equations

$$\mathbf{p} = (1+r)\mathbf{cfp} + w\mathbf{l}$$

correspond to those of a one-industry system, generalised from that used in the previous section, and the price vector $\overline{\mathbf{p}}$ in terms of $\mathbf{f}$ is irregular as a function of r. With $\mathbf{fl} = L$, we obtain

$$1 = \mathbf{f}\overline{\mathbf{p}} = (1+r)\mathbf{fcf}\overline{\mathbf{p}} + \overline{w}\mathbf{fl},$$

therefore linear functions for wage curve and standard prices:

$$\overline{w} = \frac{1}{\mathbf{fl}}\left(1-(1+r)\mathbf{fc}\right) = \frac{R-r}{(1+R)L},$$

$$\overline{\mathbf{p}} = (1+r)\mathbf{c} + \frac{R-r}{(1+R)L}\mathbf{l},$$

hence

$$\hat{\mathbf{p}} = \frac{\overline{\mathbf{p}}}{\overline{w}} = \mathbf{l} + L(1+R)\frac{1+r}{R-r}\mathbf{c},$$

and the wage curve for any numéraire $\mathbf{d} \geq \mathbf{0}$ results:

$$w = \frac{1}{\mathbf{d}\hat{\mathbf{p}}} = \frac{R-r}{(R-r)\mathbf{dl} + L(1+R)(1+r)\mathbf{dc}} = \frac{R-r}{R\mathbf{dl} + L(1+R)\mathbf{dc} + r\left(L(1+R)\mathbf{dc} - \mathbf{dl}\right)}.$$

This is the familiar hyperbola which differs from the corresponding formula in section 5 because of the re-introduction of the surplus and of a level of employment not normalised to unity. The hyperbola becomes linear for (1) $\mathbf{d} = \mathbf{f}$ (standard numéraire) or (2) for $(1+R)\mathbf{c} = \mathbf{c}/\mathbf{fc} = \mathbf{l}/L$ (labour theory) or, more generally, (3) for $\mathbf{d}(L(1+R)\mathbf{c} - \mathbf{l})$ - this condition can be analysed as in section 5: $\mathbf{d}$ can be any point on the intersection of an $(n-1)$-dimensional hyperplane and the positive orthant - or

$$(1+R)\mathbf{dc} = \mathbf{dc}/\mathbf{fc} = \mathbf{dl}/L = \mathbf{dl}/\mathbf{fl}.$$

We are now interested in the conditions under which technical changes within the same family of systems will leave the wage curves straight. The last condition opens up new possibilities. The change can affect the labour vector and/or the input-output matrix, and if the latter, it can in principle affect $\mathbf{c}$ or $\mathbf{f}$. We interpret $\mathbf{f}$ as the *composition of capital* which remains the same for all activity levels $\mathbf{q}$, given $\mathbf{A}$, since $\mathbf{qA} = \mathbf{q}(\mathbf{cf}) = (\mathbf{qc})\mathbf{f}$ varies only with the total volume $\mathbf{qc}$. We interpret $\mathbf{c}$ as the *distribution of capital* over industries, since the total volume $\mathbf{qc}$ of the capital goods of composition $\mathbf{f}$ is distributed in proportion c_i to the inputs $\mathbf{a}_i$ of industry i. Note that c_i can also be interpreted as an index of productivity (the smaller c_i, the smaller the proportion of $\mathbf{f}$ required to produce one unit of commodity i), and c_i can, with $\mathbf{A}$ considered as random, still be interpreted as a mean pertaining to industry inputs $\mathbf{a}_i$, if $\mathbf{a}_i = c_i\mathbf{e}$, but c_i is not to be confused with an activity level: it characterises the inputs relative to the output and is not, as an activity level would be, a common multiplier for both.

With a given capital *composition*, all industries are thought to be somewhat alike (equal apart from random perturbations) - around the year 1900, steel is important in each, and electronics around 2000. On the other hand, some idea of a physical capital-labour ratio is associated with every method, hence - with labour not random - the idea of a given *distribution* of capital. We now assume that technical change affects the methods employed in different industries, say in industry i, by a change of the distribution of capital c_i or the labour input l_i, but that the composition of capital does not change.

This assumption defines a family of systems. It is restrictive, but not arbitrary, for the following reason.

The family of systems under consideration shall represent the states to which an economy tends under the influence of technical change, understood as a perturbation of the initial state of a one-industry system $\mathbf{A} = \mathbf{cf}$. If a small number of industries switch the methods employed, they affect primarily the distribution of capital and the labour inputs. Although the individual industries change their individual compositions of capital as well, the average composition $\mathbf{f}$ cannot change much, and the non-dominant eigenvalues, although they will start to deviate from zero, can do so only by small amounts. For the eigenvalues are roots of the polynomial $|\lambda\mathbf{I} - \mathbf{A}| = 0$. The roots of the polynomial in function of its coefficients form a complex Riemannian manifold, and each root locally is a continuous function of these coefficients according to well-known theorems of higher function theory. Hence it seems appropriate to represent the effect of a small number of technical changes on a one-industry matrix by the assumption that they affect not so much $\mathbf{f}$ but $\mathbf{c}$ and $\mathbf{l}$. We shall speak of a family of one-industry matrices of *stable capital composition*.

We start from a given system in this family, assuming that the wage curve happens to be linear. If this is the case because of the most general condition (3) above, i.e. because $\mathbf{dc/fc} = \mathbf{dl/fl}$, without assuming that the numéraire is proportional to the Frobenius vector $\mathbf{f}$, technical change will leave the wage curve straight only if a proportionate change of $\mathbf{c}$ and / or $\mathbf{l}$ takes place. E.g. all components of $\mathbf{c}$ rise and $\mathbf{c}$ is replaced by $\bar{\mathbf{c}} = \alpha\mathbf{c}$, $\alpha > 1$, and $\mathbf{l}$ is reduced to $\bar{\mathbf{l}} = \beta\mathbf{l}$, $0 < \beta < 1$. This could be a process of technical change as mechanisation, taking place in time at a given rate of profit: a process of accumulation with technical progress as in classical theory. Or we could have different techniques for different levels of α and β, available at the same time, as in neoclassical theory, and in this case the different straight wage curves would seem to correspond to a surrogate production function reflecting the possibility of substitution, hence of choosing between different degrees of mechanisation which would be optimal at different levels of distribution (the wage curves of less mechanised techniques would appear on the envelope at higher rates of profit).

However, the construction would not be generally valid for the reason encountered in section 2: even if only the techniques represented by $\mathbf{c}$ and $\bar{\mathbf{c}}$, $\mathbf{l}$ and $\bar{\mathbf{l}}$ and hence apparently only two wage curves were given, it would be possible to combine the methods of both. These combinations would give rise to many more wage curves, and these would in general not be straight. A similar argument could be made, if the wage curve of the system from which we start would be straight because of condition (2).

But matters are different for matrices of the family of one-industry matrices of stable capital composition, if the numéraire is chosen according to condition (1). We then have $\mathbf{d} = \mathbf{f}$ and the term $\mathrm{L}(1+\mathrm{R})\mathbf{dc} - \mathbf{dl}$, which causes the hyperbolic form of the wage curve, becomes $(\mathbf{fl/fc})\mathbf{fc} - \mathbf{fl}$. The latter expression vanishes for all $\mathbf{c}$ and $\mathbf{l}$. The wage curve therefore is linear and may be written as

$$\overline{w} = \frac{1}{\mathbf{fl}}\big(1 - (1+r)\mathbf{fc}\big),$$

with $\overline{w}(0) = (1 - \mathbf{fc})/\mathbf{fl}$, $\overline{w}(\mathrm{R}) = 0$, $\mathrm{R} = (1/\mathbf{fc}) - 1$. The wage curve remains straight for any changes of $\mathbf{c}$ and $\mathbf{l}$ and for their combinations, and the position of each wage curve can be defined by calculating the corresponding $\overline{w}(0)$ and R. This family thus gives rise to a *surrogate production function*, if the analysis is limited to one-industry systems and if $\mathbf{f}$ remains strictly constant. We are dealing with an *approximate surrogate production* function, if changes of $\mathbf{f}$ are small because of corresponding random perturbations of the methods of production and if they are slow because of a slow determinate trend of $\mathbf{f}$. Our argument of the stable capital composition then is justified by the slow movements of averages of large systems, in spite of the obvious heterogeneity of the compositions of capital in any small number of industries picked out at random from the empirical input-output table of an actual economy. Because of the deterministic trend admitted for $\mathbf{f}$, the capital composition, we might speak of *near-random systems*.

7. The form of an approximate surrogate production function

It is not enough to enumerate the conditions under which approximate surrogate production function exists; we need to know more about randomness, and about the form of such production functions, for many applications do not only presuppose the most general property, diminishing returns, but also an appropriate constancy of the elasticity of substitution both as distribution varies in a given period and for a given distribution over time. These problems require empirical studies, but some heuristic considerations, based on the narrow empirical basis available, are possible and relevant. We begin with the form. The form of the production function - if it exists - depends on the form of the envelope of wage curves.

Techniques are thought to change continuously along a neoclassical production function, but we mentioned the empirical finding (section 1) that only a few out of many individual wage curves appear on the envelope of the individual wage curves, if the methods of production are combined from two input-output tables as in Han and Schefold (2006) - we also mentioned the stark contrast between Joan Robinson's and Piero Sraffa's views on this matter.

It appears that there will in fact be 'many' switches along the envelope of individual wage curves resulting from a book of blueprints of m input-output tables, representing m countries, if m is sufficiently large, and if the techniques employed in each country are sufficiently different, according to the following heuristic consideration. Assume that the wage curves are nearly linear in what we call the relevant range $0 \le r \le R_m$, where R_m is the maximum of the maximum rates of profit of each of the m^n techniques. We assume $R_m > 0$ and, for simplicity, reserving the discussion of complications for later elaborations, that all these techniques are indecomposable, which means that we can number the techniques $i = 1,2,...,m^n$, and their maximum rates of profit accordingly, and $R_i > 0$ for all techniques which are viable. Techniques will not necessarily all be viable, however. It is possible that the combination of less efficient methods from different countries lead to wage curves which are only defined in the extended range $\{-1 \le r \le 0\}$. It is clear that $w_i(-1) > 0$ for all i, since we assume the labour vectors to be positive.

The wage curves of inefficient techniques i thus are positive in part of the extended range $\{-1 \le (1+r_1) \le 1+R_i\}$, with $-1 < R_i \le 0$.

Each wage curve can be characterised fully by $w_i(0)$ and R_i[10], since we have so far assumed them to be linear. If $R_i > 0$, R_i may be interpreted as output-capital ratio, and $w_i(0)$ as output per head or aggregate labour productivity (if $w_i(0) > 0$). Denote the minimum (maximum) of $w_i(0)$ and R_i by w_m, w_M, R_m, R_M respectively, with $w_m \le w_M < \infty$, $-1 < R_m \le R_M < \infty$, and the intervals $\{w | w_m \le w < w_m\}$, $\{r | R_m \le r \le R_m\}$ by I_w and I_r. As we saw, $-1 < R_m < 0$ is possible.

If the m countries all employ the same technique, all wage curves coincide and I_w and I_r shrink to a point. Always retaining the assumption of linear wage curves, we can interpret I_w and I_r as measures of the *diversity* of the techniques available to these countries. This technical diversity is large in particular, if the book of blueprints contains (possibly many) techniques which are not viable.

If any two points w^* in I_w and r^* in I_r are given, one could imagine that there had to be some wage curve w_i nearly connecting them, in that $|w^* - w_i(0)|$ and $|r^* - R_i|$ would be small, since the number of wage curves is so large (say 10^{100}, with $m = 10$ and $n = 100$), and some kind of even distribution of their endpoints could be imagined. The wage curves would then fill the quadrangle spanned by (w_m, w_M, R_m, R_M) fairly evenly. The upper envelope (the upper side of the quadrangle) would consist of one wage curve, representing one technique dominantly throughout the relevant range, and Joan Robinson (see section 1) would have been right.

The mistake in this consideration is obvious. Suppose there was a planner who had to choose techniques represented by wage curves $w_i(r)$ with a view either to maximise $w_i(0)$ or R_i. The combinations to be chosen would be different, and if $w_i(0)$ was maximal, this would imply that R_i was not, and conversely[11]. The same would hold for the lower envelope, with the planner trying to minimise either the productivity of labour or the output-capital ratio. Hence the wage curves must be contained in a concave lens spanned between I_w and I_r; the sides I_w and I_r will be the longer and the lens the more concave, the more the countries are different[12]. If the countries are very different,

10 We disregard that $w_i(0) = 0$ and $R_i = 0$ by coincidence for some technique which is not viable.

11 The likelihood of $w_i(0) = w_M$ and $R_i = R_M$ for one i must be very small.

12 The envelopes are given by the following linear programs, first discussed extensively in Schefold (1997, chapter 5, first published 1978), the second to be analysed analogously $(\mathbf{B} - (1+r)\mathbf{A} = \mathbf{C})$:

upper envelope, primal:	Max	$\mathbf{dp}$,	S.T.	$\mathbf{Cp} \le \mathbf{l}$,	$\mathbf{p} \ge \mathbf{0}$,
upper envelope, dual:	Min	$\mathbf{ql}$,	S.T.	$\mathbf{qC} \ge \mathbf{d}$,	$\mathbf{q} \ge \mathbf{0}$.
upper envelope, primal:	Min	$\mathbf{dp}$,	S.T.	$\mathbf{Cp} \ge \mathbf{l}$,	$\mathbf{p} \ge \mathbf{0}$,
lower envelope, dual:	Max	$\mathbf{ql}$,	S.T.	$\mathbf{qC} \le \mathbf{d}$,	$\mathbf{q} \ge \mathbf{0}$.

As usual, r has to be interpreted as the growth rate in the dual upper envelope: The program has a feasible solution in the primal $\mathbf{p} = \mathbf{0}$ and in the dual, because there are viable techniques, with one in fact expects Sraffa's 'rapid succession of switches', mentioned above, especially in the present case based on linear wage curves. The low number of wage curves observed on the envelope in Han and Schefold (2006) then must be explained by the fact that $m = 2$ (only the input-output tables of pairs of countries defined the book of blueprints) and that the countries were similar (all belonged to the OECD).

So far, we have considered a book of blueprints represented by the collection of the input-output tables of a group of m countries. In order to assess the form of the envelope, we consider two cases.

1. If the tables are taken from the same year (*book of contemporary tables*), technical diversity has been said to be small by definition, if I_w and I_r are comparatively short intervals so that the 'lens' is 'thin'. It is indeed plausible that the upper and the lower envelope will not be far apart, if the countries are similar. The envelopes are far apart, if some combinations are not viable. How could this be? If just one country employs relatively wasteful methods of production in most industries, the country is not similar to the others, and if an inefficient technique can be combined from the techniques used by different countries, the corresponding inefficient methods are used by different countries in different industries so that the countries are, insofar, not similar[13] either.

The comparison here is made at a given rate of profit, as results from the international mobility of capital, but it is supposed not to be fully effective: the tables are not equalised completely - an assumption made to mirror a competitive process which leads to full homogeneity within but not across countries. It is instructive to remember what would happen according to the theory of normal prices, if capital and labour were perfectly and quickly mobile, so that real wages would get equalised, and this while distribution varied slowly over time. It was pointed out in Schefold (1997, p. 126) that, if the techniques represented regular systems, all techniques would then have to be equal. For pure competition in international trade without specialisation forces relative prices to be equal. The price vectors of any country assume n linearly independent values at n different rates of profit. Equality of relative prices then is possible only if all countries use the same technique. International trade does not only equalise factor prices - variable factor prices (barring specialisation) actually equalise countries!

$R_m \geq \mathbf{0}$ by assumption. Lower envelope: $\mathbf{q} = \mathbf{0}$ is feasible in the dual and there is a solution to the primal in a neighbourhood of $r = -1$ since each row of $\mathbf{C}$ then is positive.

13 Example: Two countries with tables $\mathbf{A}^{\mathrm{I}}$, $\mathbf{A}^{\mathrm{II}}$, table for upper envelope $\mathbf{A}^{+}$ and for lower envelope $\mathbf{A}^{-}$:

$$\mathbf{A}^{\mathrm{I}} = \begin{pmatrix} 0 & \frac{5}{4} \\ \frac{1}{2} & 0 \end{pmatrix},\ \mathbf{A}^{\mathrm{II}} = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{4}{3} & 0 \end{pmatrix},\ \mathbf{A}^{+} = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix},\ \mathbf{A}^{-} = \begin{pmatrix} 0 & \frac{5}{4} \\ \frac{4}{3} & 0 \end{pmatrix},\ \text{hence } R^{\mathrm{I}} = \sqrt{8/5} - 1,$$

$R^{\mathrm{II}} = 1/2$, $R^{+} = \sqrt{6} - 1$, $R^{-} = \sqrt{3/5} - 1$ and $R^{+} > R^{\mathrm{II}} > R^{\mathrm{I}} > 0$, $R^{-} < 0$. If $\mathbf{l}^{\mathrm{I}} < \mathbf{l}^{\mathrm{II}}$, we can have $w^{\mathrm{I}}(0) > w^{\mathrm{II}}(0)$, so that the wage curves of the two countries have a switchpoint, with $w^{+}(0) > w^{\mathrm{I}}(0) > w^{\mathrm{II}}(0)$ and $w^{-}(0) < 0$. The countries are dissimilar: each has one 'large' input coefficient, but each in a different sector, and upper and lower envelopes are above and below the wage curves of the countries.

This does not happen in reality - for (apart from specialisation which does happen) three principal reasons, I suggest: 1. Factor prices move only little, and not necessarily in concert. 2. But to some extent, they do, and through this and other mechanisms, technical diversity between countries is reduced but not eliminated. 3. The pressure to reduce technical diversity is diminished, if the techniques adopted are not regular - hence if wage curves are more or less linear. In this sense, irregular systems are like attractors, and a new reason has been given why relatively straight wage curves predominate[14].

2. We now turn to the consideration of successive tables of one country over several years $t = 1,2,...,T$ (*retrospective comparison of tables*). According to the stylised facts of economic growth, I_r must shrink to a point (constant capital output ratio) and the difference between w_m and w_M reflects the growth of the productivity of labour between the first and the last year in the comparison (cf. Schefold 1997, p. 277, with diagram). The lens becomes a triangle. The wage curve of the table of year t must here in fact, at the actual rate of profit, be on the envelope of the wage curves to be formed from the book of blue prints of the tables pertaining to the years t, $t-1$, $t-2$, …, 1; the wage curve of each year is at the upper envelope of the wage curves generated by the technology known in that year n by virtue of the maximisation of profits, and the wage curve of the table (technique) of year T is at the given rate of profit on the upper envelope of all. And here we can even argue that the wage curve of the first year will tend to be on the lower envelope at the given rate of profit r - no assumption about strict linearity is needed:

Let $\left(\mathbf{A}^{\mathrm{I}},\mathbf{l}^{\mathrm{I}}\right)$ be the technique realised in t and $\left(\mathbf{A}^{\mathrm{II}},\mathbf{l}^{\mathrm{II}}\right)$ in $t+1$. Technique I is still remembered in $t+1$ so that $w^{\mathrm{II}}(r)$ is on the envelope of the wage curves resulting from I and II. Now $w^{\mathrm{I}}(r)$ will tend to be on the lower envelope. To show it, we need to assume $\mathbf{p}^{\mathrm{II}} < \mathbf{p}^{\mathrm{I}}$, with both price vectors being roughly proportional. This is likely to be the case, if - as is typical for the growth scenarios corresponding to the 'stylised facts'- the growth of labour productivity is even and predominates over other forms of technical progress. Now let any mixed technique $\left(\mathbf{A}^{*},\mathbf{l}^{*}\right)$ be given consisting of processes of I and II. If any of the processes of II used in $\left(\mathbf{A}^{*},\mathbf{l}^{*}\right)$, say $\left(\mathbf{c}_i^{*},l_i^{*}\right)$, $\mathbf{c}_i = \mathbf{e}_i - (1+r)\mathbf{a}_i$, made losses, if evaluated in terms of prices $\mathbf{p}^{\mathrm{I}}$, it would not be used in II, for $\mathbf{c}_i\mathbf{p}^{\mathrm{I}} < l_i$ contradicts $\mathbf{c}_i\mathbf{p}^{\mathrm{II}} = l_i$ if $\mathbf{p}^{\mathrm{II}} = \lambda\mathbf{p}^{\mathrm{I}}$, $0 < \lambda < 1$. Hence

$$\left(\mathbf{I} - (1+r)\mathbf{A}^{*}\right)\mathbf{p}^{\mathrm{I}} \geq \mathbf{l}^{*}.$$

Since $\mathbf{p}^{\mathrm{I}} > 0$, $l^{*} > 0$ and $\mathbf{A}^{*}$ is indecomposable by assumption, $\left(\mathbf{I} - (1+r)\mathbf{A}^{*}\right)^{-1} > 0$ and

14 Intuitively: Assume that silver and gold exchange at 12 : 1 everywhere. Suppose that the rate of profit falls by three percentage points and that this exchange rate stays constant everywhere except in capital island, where it changes to 10 : 1. What will happen? - Clearly, countries less affected by such shocks of relative prices will have an evolutionary advantage.

$$\mathbf{p}^{\mathrm{I}} > \left(\mathbf{I} - (1+r)\mathbf{A}\right)^{-1}\mathbf{l}^{*} = \mathbf{p}^{*},$$

where all prices are in terms of labour commanded; we get $w^{\mathrm{I}}(r) < w^{*}(r)$ for all numéraires.

The retrospective comparison of tables, with the growth of the productivity of labour dominating, thus yields a lens of wage curves, with the latest on the upper, the earliest on the lower envelope. If we add that the wage curves are straight and that R does not move according to the stylised facts, the lens becomes the triangle we referred to.

To look at a limited number of tables, as we did in both cases (book of blueprints of contemporary tables and retrospective book) is a step towards realism, since an entrepreneur will hardly ever have knowledge of all methods of production used in his industry anywhere in the world or at any time in history. The boundary of the outlook is different for different entrepreneurs and includes projects not yet realised by any firm. The spectrum of such potential techniques can be imagined to be very rich. But the problem of the surrogate production function concerns the substitution to be realised in direct response to changes in factor prices, and that implies first to look for methods of production of proven efficacy - imitation is difficult enough.

No inconsistency arises, if the reader wishes to enlarge the book o blueprints by projects, but the conclusions drawn about the form of the envelope are then not valid, whereas, if we concentrate on country comparisons, we found that the number of wage curves on the envelope increases with the number of countries and its convexity with technical diversity. This is consistent with the idea of an approximate surrogate production function. The stylised facts, by contrast, lead back to the ideas of one best technique. This is due to the assumption that R is given. If the assumption of a given output-capital ratio is dropped, since it is an *explanandum*, not an *explanans* of growth theory, the resulting individual wage curves will form a lens resembling that of the contemporary book of blueprints, except that, if accumulation takes place at a given rate of profit r (which also must be explained), $w_1(r)$ will be in principle on the lower, $w_T(r)$ on the upper envelope. Then we get again an approximate surrogate production function.

8. The realism of the assumptions and the postulates of the theory

We know that wage curves are not strictly linear. Our construction of the approximate surrogate production function rests on the assumption that the input-output tables are perturbations of input-matrices $\mathbf{A} = \mathbf{cf}$, with $\mathbf{f}$ roughly fixed, $\mathbf{c}$ changing, with $\mathbf{l}$ such that prices differ not much from values and with perturbations which keep the non-dominant eigenvalues small (a sufficient condition for this is that $\mathbf{A}$ stays random). Hence a dilemma for neoclassical theory arises: substitution effects will become interesting (and will dominate other influences on distribution), if they are large and noticeable, but then the production function as the main tool for their analysis breaks down. In particular, wage curves will become less linear, because $\mathbf{f}$, the composition of capital, will change and deviate from the numéraire which had been chosen, hence declination becomes a problem.

Moreover, the larger the book of blueprints, the more likely the occurrence of non-dominant eigenvalues, which are not small (the absolute number of such techniques certainly increases, while their relative frequency may remain the same). This means that the wage curves are more complicated than hyperbolas according to section 3, so that two individual wage curves can intersect more than twice. The paradox of paradoxes, mentioned in section 1, for which empirical examples were found in Han and Schefold (2006), provides evidence that this can happen, for it involves three switchpoints between two wage curves.

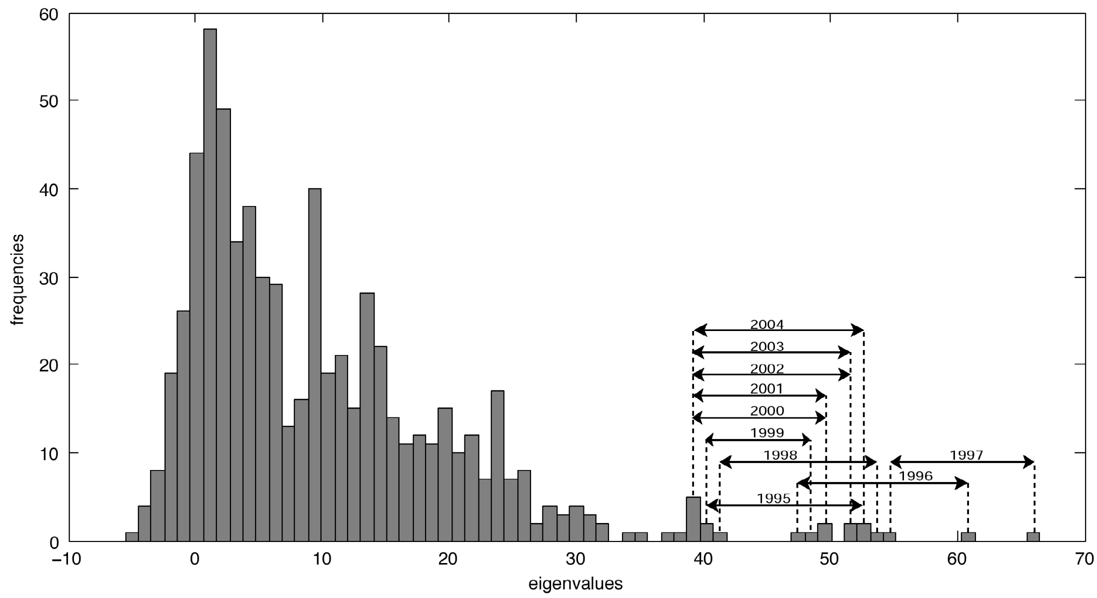
The degree of deviation between prices and values has been discussed since Ricardo. The difference between usual price indices and the standard basket is relatively easy to assess intuitively and its consequences can be calculated. The problem of assessing the likelihood for non-dominant eigenvalues of the input-matrices to be small is new and less tractable.

There is no room in this theoretical paper for an exhaustive empirical analysis of books of blueprints and in particular of whether input matrices are random, either in the specific sense of this concept as defined in section 3, with the rows of the input coefficients being i.i.d., or with some other distribution. A priori, large input coefficients can be expected to occur more rarely than smaller ones, but very small input coefficients are economically meaningless. The distribution of input coefficients thus seems likely to be skewed. We know, on the other hand, that it is not necessary that the rows be i.i.d., given that the non-dominant eigenvalues are strictly zero also if $\mathbf{A} = \mathbf{cf}$ (section 6).

The coefficients of the labour vector follow a definite trend over time: they diminish per unit of output, while this fairly steady rise of labour productivity is due to intensification or it is made possible by changing the material means of production, some of which go up as in the case of mechanisation, while others are reduced because of the saving of raw materials (Schefold 1997, chapter 11). There is no reason to expect a definite trend, to this extent, there is an a priori expectation that randomness prevails and that therefore most eigenvalues of input matrices are small.

In order to make the step from the retrospective consideration of the evolution of the techniques used by one country to the comparison of the techniques of several in a given period (the contemporary book of blueprints), I assume here that the countries constantly exchange methods of production through competition, re-enforcing the trend of labour productivity and introducing more variety (randomness), as far as the material means of production are concerned. Hence there is randomness both in the technique actually employed and in the alternatives, looking across countries and back to the past. There are certain trends, on the other hand, in the path-dependence of the evolution of individual methods of production - tankers, which simply get larger, are a trivial example. I suspect -but I cannot prove it - that such trends do not mean that all non-dominant eigenvalues become more or less equally larger than those of random matrices, but that their distribution becomes more skewed (concentration of non-dominant eigenvalues near zero, few non-dominant eigenvalues near in modulus to the dominant eigenvalue).

I can report on a provisional empirical investigation by Christian Schmidt, to whom I owe thanks. He has analysed 10 input-output tables for Germany, 1995-2004, with 68 sectors. Diagram 2 shows a histogram for the eigenvalues in function of their real parts. It can be seen that each of the largest (in terms of their real parts) non-dominant eigenvalues is separated by about one quarter from the corresponding dominant eigenvalue and that most other eigenvalues cluster near zero[15]. The temporal variation of the dominant roots is in part due to changes in capacity utilisation. The distribution of eigenvalues is skewed in that positive real parts occur more frequently than negative ones. Hence the clustering looks more pronounced, if the moduli, not the real parts, are taken as independent variables (diagram not shown).

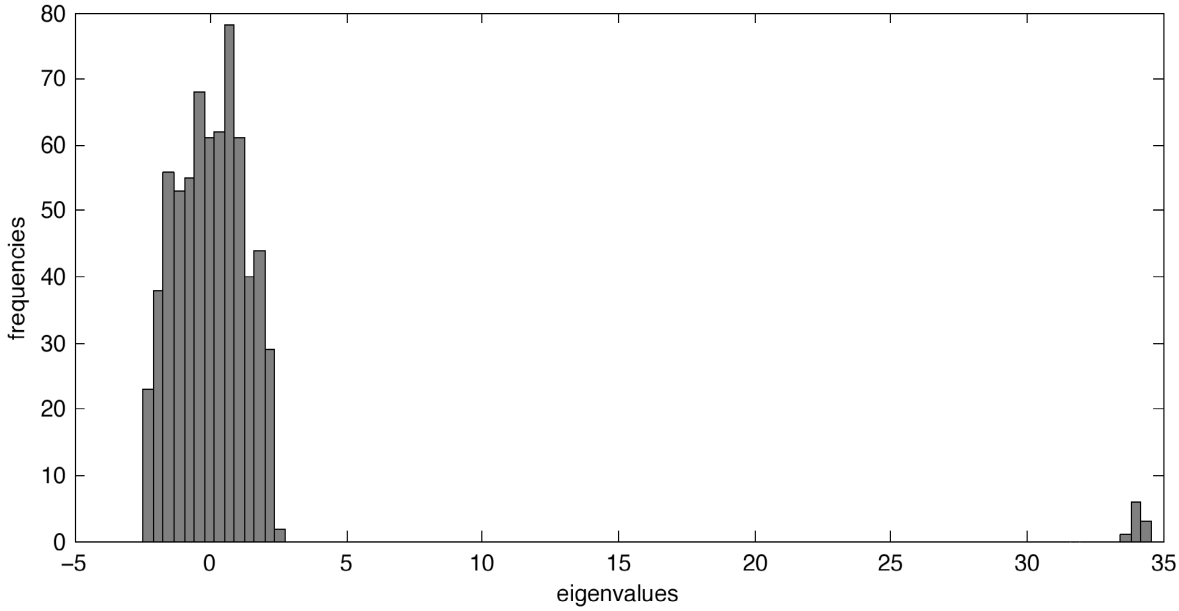


Real parts of eigenvalues

Diagram 2: The empirical distribution of the eigenvalues of 10 input-output tables for Germany, 1995-2004. The arrows indicate the distance between the dominant eigenvalue and the largest (in terms of its real part) non-dominant eigenvalue of the corresponding table.

10 matrices, 68 by 68, with elements between zero and one, were randomly generated for comparison. One therefore expects an average dominant eigenvalue of 0,5. The distribution of all eigenvalues is shown in diagram 3 in function of the real parts (the unit interval on the abscissa has been divided by 70 to facilitate the comparison with diagram 2).

15 Source: Statistisches Bundesamt (2007): Fachserie 18, Volkswirtschaftliche Gesamtrechnungen, Reihe 2, Input-Output-Rechnung 1995-2004, Wiesbaden. All tables are in current prices. Input coefficients then are used (percent of gross sectoral output). Classification of goods according to the European Standard of 2002. Three sectors were omitted, for which all coefficients were zero in the case of Germany. The 71 by 71 tables therefore were reduced to 68 by 68. All matrices have rank 68 (test by means of singular value decomposition).



Real parts of eigenvalues

Diagram 3: The distribution of eigenvalues of 10 positive matrices, randomly generated, 68 by 68, with elements between 0 and 1. All matrices have full rank.

The non-dominant eigenvalues cluster more tightly around zero than in the empirical case of diagram 4. Yet, the eigenvalues of the empirical input-output tables are also concentrated near zero, for 50 % of their eigenvalues are smaller than 7.1 *in modulus* (the dominant eigenvalues vary between 48.1 and 66.5). Theoretically, the non-dominant eigenvalues could all be equal in modulus to the dominant one or they could be uniformly distributed between zero and the dominant eigenvalue, as we saw at the end of section 4. Hence there seems to be a strong element of randomness in input-output tables, but they are not fully random. It must be left to future research to characterise this property more exactly, taking also other factors into account, in particular fixed capital.

Meanwhile, we conclude that randomness is only approximated in reality, much in the same way as we found - and as has been established in the literature - that prices are not equal to values, but not very distant from them either, and that the numéraire-vector yielding a linear wage curve, Sraffa's standard (the activity levels corresponding to balanced growth at a maximal rate) are represented by price standards in actual use, such as the vector of net output, only approximately.

We thus get back to the postulate from which we started: not one of the three properties conducive to straight or hyperbolic wage curves taken in isolation, but only all three taken together explain the empirical finding of the rarity the paradoxes of capital and of the near linearity of empirical wage curves.

The form of the approximate surrogate production function now follows. It is defined if the wage curves are obtained from a book of blueprints which is large, but not too large and of limited technical diversity - a book of blueprints such as that represented by the input-output tables of similar but not identical countries. For the wage curves will then be linear, since combinations will not destroy the properties of prices being fairly close to values, of the standards being close to each other (so that a common numéraire helps linearising the wage curves) and of near randomness reducing the number of wiggles of wage curves (so that capital reversing becomes rare and insignificant in its effect on averages). The envelope of these wage curves is convex to the origin and declination remains small. The approximate surrogate production function therefore exists, but, as it were, only *locally*: if technical diversity increases because the book of blueprints widens (more countries being compared in space or more techniques being considered over time) or because countries which are more different come in or if large variations in distribution are considered, declination is likely to increase. And no argument has been found as to why the elasticity of substitution should assume a specific value and be constant with variations of r or over time.

9. Conclusions

This paper has been written with the intention of taking up the challenge posed by the contrast between the claim of the Cambridge critique to have successfully undermined neoclassical theory by means of the discovery of the paradoxes of capital theory and the empirical finding that these paradoxes appear to be rare - an appearance which seems to confirm Joan Robinson's treatment of them as mere 'curiosa' (Robinson 1956, p. 109). We found, after the exposition of the problem (section 1):

1. The paradoxes are easy to generate, if only non-basics or 'Austrian' processes are involved. The analysis of the closest analogon of 'Austrian' processes among basic systems, the family of circular systems, revealed that the curvature of wage curves and hence the magnitude of declination may still become arbitrarily large, but the direct confrontation of the wage curves so obtained is not licit because of combinations of processes. The paradoxes do not disappear in consequence, but their likelihood is diminished (section 2 and Schefold 2008a).

2. Large random systems, the second family considered, approximate hyperbolic or linear wage curves, because the non-dominant eigenvalues tend to zero. This property, re-enforced by numéraires close to the standard, leads to near-linear wage curves (sections 3 and 4).

3. But the results do not extend directly to large deterministic systems and a random system is not the stochastic analogue of a one-commodity world (section 4).

4. The deterministic counterpart of random systems consists of the family of one-industry systems, with even composition of capital. They lead to hyperbolic wage curves, and it is here easy to show how the properties of prices being equal to values and of the numéraire being close to the standard re-enforce each other so as to generate almost linear wage curves (section 5). Irregular Sraffa Systems, though not generic, are relevant as approximations.

5. The one-industry systems of stable capital composition form a family for which rigorous surrogate production functions exist. They may also be used to represent the

classical process of accumulation with mechanisation. If technical change takes the form of perturbations of one-industry systems with a slow change of the capital composition, the wage curves remain approximately straight and an approximate surrogate production function exists (section 6).

6. The results provide a theoretical explanation for the empirical finding in Han and Schefold (2006) that reverse capital deepening and reverse substitutions of labour can exist but must be rare. For if real systems are approximately random but not strictly random, and if the numéraire is near but not equal to the standard commodity, wage curves are nearly linear (section 7). However, the existence of a small number of non-dominant eigenvalues which are not small can lead to 'wiggles' of the wage curves such that two individual wage curves may occasionally intersect not only more than once but more than twice (the paradox of paradoxes of section 1).

7. If technical diversity is sufficiently large to generate Sraffa's 'rapid succession of switch points', it becomes worthwhile to establish a relationship between the intensities of capital of the techniques adopted at various levels of the rate of profit and the levels of output per head and thereby to construct a production function, but declination will be sufficiently small only, if technical diversity remains limited. The approximate surrogate production function so obtained therefore was characterised as *local*. No reason was found to postulate a specific level of the elasticity of substitution and to expect its constancy with variations of distribution and changes of the book of blueprints over time, except by a kind of inertia (sections 7 and 8).

Yet the vision is not devoid of content: capital can be aggregated, although the means of production are heterogeneous and themselves produced in an interdependent system, and the aggregate capital, combined with labour, yields an aggregate output such that diminishing partial returns obtain. The construction is possible by a kind of statistical smoothing, although the individual techniques are linear, of the fixed coefficients variety, with constant returns to scale. The problem formulated by Hicks (1932) - how is marginal productivity to be reconciled with fixed coefficients of production? -, for which Samuelson found his ingenious but incomplete solution, can be approached successfully in a stochastic setting. The construction seems sufficiently robust to support contentions such as that which I associate with Böhm-Bawerk (1914): suppose both factors are fully employed, suppose that trade unions enforce a rise of real wages. Mere profit maximisation will then lead to an increase in the intensity of capital and hence to unemployment. Can it be cured by Keynesian means, either because of a demand effect resulting from an increase in wages or the by the state expenditure, while real wages stay at their elevated level? The answer is trivial: no, since full employment of 'capital' was assumed. The idea of rigidly given levels of capacity and other, hidden assumptions[16] of this story may be discussed, but that leads into a different territory. The point is that the argument can no longer be simply dismissed on the basis of the critique of capital.

16 For instance: the converse case - immigration leads from full employment to unemployment to be cured by a fall in real wages and a lower intensity of capital - is less plausible because one only rarely returns to old, more labour-intensive techniques..

We have confirmed, on the other hand, that the production function is not based on foundations which are both rigorous and general, and that our less rigorous construction, with its introduction of a statistical notion, randomness, does not support the whole edifice built on the production function. Constant and stable elasticities of substitution found no support in this investigation. The theory of normal prices, with the wage curve as its main tool, emerges as the fundamental concept, and the aggregate production function is a derived concept of limited applicability. For instance, if the above problem of Böhm-Bawerk's is posed, its solution may be sought directly by visualising the rise in real wages in the diagram of wage curves and by determining the more capital-intensive technique directly; with the advantage of rendering the problem of the transition to the new technique more explicit. The more fundamental point which has been established in this paper therefore is the inverse relationship between the rate of profit and the intensity of capital which holds in most cases, while the paradoxes of capital are rare.

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