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The Average Propagation Length: An Extended Analysis

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Abstract

The Average Propagation Length (APL) is a powerful tool in production chain analysis. It measures the inter-industry linkages from the dimension of length. The APL and its variant have been applied to many areas, such as important production chains identification, upstreamness measure and fragmentation measure. This study extends the APL from a double counting perspective. It shows that the APL is equivalent to the time of double counting of a sector's gross output in the production of another sector's final product. From the double counting perspective, we show that the APL can be easily extended to answer many other interesting questions in a clear manner. For instance, the APL can be decomposed to separately measure the average times of each sector visited by the gross output of a specific sector in the production of a final product. The APL can be extended to group-wise APL, which gives the APL between any two groups of sectors. Both the APL and the upstreamness measure are special cases of the group-wise APL. As an empirical study, the position of each country in the global production chain of all world final products is identified by means of group-wise APL.

1. Introduction

In the economic system, the production sectors are closely connected to each other via intermediate transactions. The element of the Leontief inverse is used to measure the inter-sector linkages in the input-output analysis(Miller and Blair, 2009). The size of the element in Leontief inverse indicates the strength of the inter-sector linkage. Another dimension of the inter-sector linkage is the length of the linkage. Dietzenbacher *et al*. (2005) propose the average propagation length (APL) index in an input-output framework to measure the length of the linkage between two sectors. It is defined as the average number of steps taken by the final product of one sector to affect the gross output of another sector.

The APL has attracted many researchers' interests. The relationship between APLs and direct input coefficients is investigated by Lu and Xu (2013) by using the Sherman-Morrison formula. Its concept and applicable scope become clearer with a series of discussions from Oosterhaven and Bouwmeester (2013). The APL has been widely used in many areas. For instance, Dietzenbacher and Romero (2007) identify the important production chains in an interregional framework of Europe by means of APL. Inomata (2008) develops a new measurement of fragmentation based on the APL approach. Antràs et al. (2012) propose an "upstreamness" measure of production based on the input-output methodology, which can be regarded as an invariant of the APL. The "upstreamness" measure can be used to identify the position of each industry in the national production chain. Yu et al. (2014) use APL as one of the components for their vulnerability index to identify the post-disaster key sector prioritization.

"Made in the world" has become a distinctive feature of the world economy. The production of fragmentation impels many international trade insights to be re-considered in the context of global production network and in terms of value added (Timmer *et al*., 2013; Foster and Stehrer, 2013; Koopman, *et al*., 2014). The global value chain and global production chain therefore have become important concerns of current international trade studies (Timmer *et al*., 2012; Baldwin and Lopez Gonzalez, 2013). This study is relevant to the global production chain analysis, attempting to extend the applications of APL in global production chain analysis based on the world input-output table. We find that answering many interesting questions in global production chain requires developing a group-wise APL based on the world input-output table. For instance, where does China as a whole locate at the global production chain of all world products, upstream or downstream? Where does Korea as a whole locate at the global production chain of all world electrical products? Since a specific country includes a group of industries and a specific industry spreads over a group of countries in the world input-output tables the current sector-wise APL cannot be used to answer such questions. Instead, a group-wise APL, namely the APL between two groups, should be used.

This study first revises the APL formula by taking into account the initial effect which is neglected in the current APL. This revision can add some good properties to the APL. On the one hand, under this revision it makes no difference which number is selected as the initial step if only the ordinal position of the APL is concerned[1]; on the other hand, the revised APL is equivalent to the time of double counting of a sector's gross output in the production of another sector's final product. Hence, calculating the APL can be replaced by calculating the time of double counting in certain complicated situations. Many extensions of the APL can be derived from the double counting perspective. The APL can be decomposed into a series of sub-APLs, which indicate the average times of each sector visited by the gross output of a specific sector in the production of a certain final product. The APL can be further extended to the group-wise APL, which gives the APL between any two groups of sectors. We show that both the APL and the upstreamness measure are special cases of the group-wise APL. The group-wise APL therefore can be regarded as the generalized APL.

The remaining content of this paper is organized as follows. Section 2 introduces the concept of APL. Section 3 investigates the equivalence between double counting and APL. Section 4 shows two useful extensions of the APL, the APL decomposition

[1] The initial step in the APL starts from 0 and the initial step in the upstreamness measure starts from 1.

and the group-wise APL. Section 5 gives an application of the group-wise APL in global production chain. Section 6 concludes.

2. The Average Propagation Length (APL)

The average propagation length (APL) is first proposed by Dietzenbacher *et al*. (2005) based on the input-output framework. In a closed economic system, for instance, in the global economic system, the gross output of each sector satisfies

$$x_i = \sum_{j=1}^{n} z_{ij} + f_i \tag{1}$$

Where x_i is the gross output of sector *i*; z_{ij} is the gross output of sector *i* consumed by sector j as intermediate input; f_i is the gross output of sector *i* consumed as final product. Define $a_{ij} = \dfrac{z_{ij}}{x_j}$ as a series of input coefficients of sector *j*. Formula (1) can be rewritten as

$$x_i = \sum_{j=1}^{n} a_{ij} x_j + f_i \tag{2}$$

In matrix form, Formula (2) can be expressed as

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f} \tag{3}$$

Where $\mathbf{x}$ is the gross output vector; $\mathbf{A}$ is the input coefficient matrix; $\mathbf{f}$ is the final demand vector. The solution of Formula (3) is

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} = \mathbf{L}\mathbf{f} \tag{4}$$

$\mathbf{L} \equiv (\mathbf{I} - \mathbf{A})^{-1}$ is the so-called Leontief inverse. Its expansion is

$$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^{2} + \mathbf{A}^{3} + \dots \tag{5}$$

The element of the Lentief inverse indicates that one unit increase in the final demand of sector *j* will directly and indirectly cause l_{ij} unit increase in sector *i*'s gross output. The expression of l_{ij} is

$$l_{ij}=\begin{cases} a_{ij}+\sum_k a_{ik}a_{kj}+\sum_k\sum_s a_{ik}a_{ks}a_{sj}+\dots & i\neq j\\ 1+a_{ij}+\sum_k a_{ik}a_{kj}+\sum_k\sum_s a_{ik}a_{ks}a_{sj}+\dots & i=j\end{cases} \tag{6}$$

Formula (6) indicates that the total effect l_{ij} can be decomposed into a direct effect and a series of indirect effects. When *i*=*j*, the initial effect "1" is also taken into account. The effect a_{ij} needs to visit one sector or take one step before it arrives at sector *i*. In the same sense, the effect $a_{ik}a_{kj}$ needs to visit two sectors or take two steps before it arrives at sector *i*, etc. The visited sectors can be identical. For instance, when $k\neq i$, the effect $a_{ik}a_{kj}$ needs to visit sector *k* and sector *i* before it arrives at sector *i*; when $k=i$, the effect $a_{ik}a_{kj}$ also needs to visit two sectors, but the sectors are the same, namely sector *i*.

According to Dietzenbacher and Romero's formula (2007), the APL between sector *i* and sector *j* is defined as the average number of steps taken by the final product of sector *j* to affect the gross output of sector *i*. the APL between sector *i* and sector *j* is

$$APL_{ij}=\begin{cases} 1\times\dfrac{a_{ij}}{l_{ij}}+2\times\dfrac{\sum_k a_{ik}a_{kj}}{l_{ij}}+3\times\dfrac{\sum_k\sum_s a_{ik}a_{ks}a_{sj}}{l_{ij}}+\dots & i\neq j\\ 1\times\dfrac{a_{ij}}{l_{ij}-1}+2\times\dfrac{\sum_k a_{ik}a_{kj}}{l_{ij}-1}+3\times\dfrac{\sum_k\sum_s a_{ik}a_{ks}a_{sj}}{l_{ij}-1}+\dots & i=j\end{cases} \tag{7}$$

The APLs can be expressed in a matrix form[2]

$$\frac{1\times\mathbf{A}+2\times\mathbf{A}^2+3\times\mathbf{A}^3+\dots}{\mathbf{L}-\mathbf{I}}=\frac{\mathbf{L}(\mathbf{L}-\mathbf{I})}{\mathbf{L}-\mathbf{I}} \tag{8}$$

Antràs *et al*. (2012) propose an "upstreamness" measure of production based on the input-output methodology. This measure presents the average distance of a sector's gross output from the final products of all sectors. It thus can be used to describe the upstreamness of each sector in the production chain of the whole economy.

Based on Formula (4) and (6), the gross output of sector *i* can be expressed as

[2] The matrix division in this study is defined as element-wise division.

$$x_i = f_i + \sum_j a_{ij} f_j + \sum_j \sum_k a_{ik} a_{kj} f_j + \sum_j \sum_k \sum_s a_{ik} a_{ks} a_{sj} f_j + \dots \tag{9}$$

The upstreamness of sector *i* is defined as the weighted average distance of its gross output to the final products. The weight is generated from Formula (9). According to Antràs *et al*. (2012), the upstreamness of sector *i* is defined as

$$u_i = 1 \times \frac{f_i}{x_i} + 2 \times \frac{\sum_j a_{ij} f_j}{x_i} + 3 \times \frac{\sum_j \sum_k a_{ik} a_{kj} f_j}{x_i} + 4 \times \frac{\sum_j \sum_k \sum_s a_{ik} a_{ks} a_{sj} f_j}{x_i} + \dots \tag{10}$$

In Formula (10), each weight is multiplied by its distance from final products plus one. As pointed by Antràs *et al*. (2012), this is somewhat *ad hoc*. Furthermore, Formula (10) can be expressed as the following matrix form.

$$\mathbf{u} = \frac{(1 \times \mathbf{I} + 2 \times \mathbf{A} + 3 \times \mathbf{A}^2 + 4 \times \mathbf{A}^3 + \dots)\mathbf{f}}{\mathbf{x}} = \frac{\mathbf{LLf}}{\mathbf{Lf}} \tag{11}$$

The upstreamness measure can be regarded as an invariant of the APL. If only consider the average distance of a sector to a specific final product, the upstreamness measure turns into the following matrix form

$$\mathbf{U} = \frac{\mathbf{LL}}{\mathbf{L}} \tag{12}$$

The detailed derivation is illustrated in Appendix 1. For the final product of a specific sector, the upstreamness measure shown by Formula (12) is very similar to the APL shown by Formula (8). The difference comes from two aspects. First, the initial distance of a sector’s gross output to final product or the initial step taken by the final product to affect a sector’s gross output is different. The initial distance or step in upstreamness measure starts from 1, while that in APL starts from 0. Second, the initial effect is taken into account to calculate the weight in the upstreamness measure, while the initial effect is neglected in the APL.

If the initial distance or step starts from 0 and the initial effect is taken into account, easy to verify that the formula for APL is turned into the following expression.

$$\mathbf{APL} = \frac{0 \times \mathbf{I} + 1 \times \mathbf{A} + 2 \times \mathbf{A}^2 + 3 \times \mathbf{A}^3 + \dots}{\mathbf{L}} = \frac{\mathbf{L(L-I)}}{\mathbf{L}} \tag{13}$$

The element APL_{ij} in the $\mathbf{APL}$ matrix satisfies

$$APL_{ij} = \frac{\sum_{k \neq j} l_{ik} l_{kj} + l_{ij}(l_{jj} - 1)}{l_{ij}} \tag{14}$$

We suggest using Formula (13) to calculate the APLs or its variants in practice. First, the initial effect should be taken into account. On the one hand, the initial effect is also an effect, although it takes 0 steps to propagate to itself. A sector benefits significantly from an exogenous increase in its final demand. It therefore should play a role in calculating the weight. On the other hand, if the initial effect is taken into account, the ordinal position of each sector in the production chain of a specific sector's final product does not change regardless of which number is selected as the initial step or distance (see the proof in Appendix 2). Second, Formula (13) can be intuitively interpreted as the "time of double counting" of a sector's gross output in the production of a specific sector's final product. This will be further investigated in the next section.

If applying the same rule to the upstreamness measure, namely the initial step or distance starts from 0 and the initial effect is taken into account, the upstreamness measure can be revised as

$$\mathbf{u} = \frac{0 \times \mathbf{f} + 1 \times \mathbf{Af} + 2 \times \mathbf{A}^2\mathbf{f} + 3 \times \mathbf{A}^3\mathbf{f} + \ldots}{\mathbf{Lf}} = \frac{\mathbf{L(L-I)f}}{\mathbf{Lf}} \tag{15}$$

The element u_i in the $\mathbf{u}$ matrix satisfies

$$u_i = \frac{\sum_j \sum_{k \neq j} l_{ik} l_{kj} f_j + \sum_j l_{ij}(l_{jj} - 1) f_j}{\sum_j l_{ij} f_j} \tag{16}$$

3. Double counting and APL

The production sectors in the economic system are connected together via the production network. The completion of a sector's final product requires inputs from other sectors. The total requirement of each involved sector in the production of a specific final product can be calculated by using the Leontief inverse matrix. For instance, l_{ij} is the total gross output of sector *i* required to produce 1 unit final product of sector *j*. The total requirement of a sector can be further decomposed into a series of flows which reach the specific final product via a corresponding series of paths.

$l_{ij}(i \neq j)$ can be decomposed as

$$l_{ij} = a_{ij} + \sum_k a_{ik}a_{kj} + \sum_k \sum_s a_{ik}a_{ks}a_{sj} + \ldots$$

The flow a_{ij} is the gross output of sector i directly consumed by 1 unit final product of sector j. It therefore reaches and is embedded in the final product of sector j via Path 1 in Figure 1. The flow $a_{ik}a_{kj}$ is the gross output of sector i indirectly consumed by 1 unit final product of sector j. The 1 unit final product of sector j first consumes a_{kj} unit gross output of sector k; this a_{kj} unit gross output of sector k further consumes $a_{ik}a_{kj}$ unit gross output of sector i. The flow $a_{ik}a_{kj}$ therefore reaches and is embedded in the final product of sector j via Path 2 in Figure 1. In the same sense, the flow $a_{ik}a_{ks}a_{sj}$ reaches and is embedded in the final product of sector j via Path 3 in Figure 1. Analogically，corresponding paths can be matched to the other flows in l_{ij}.

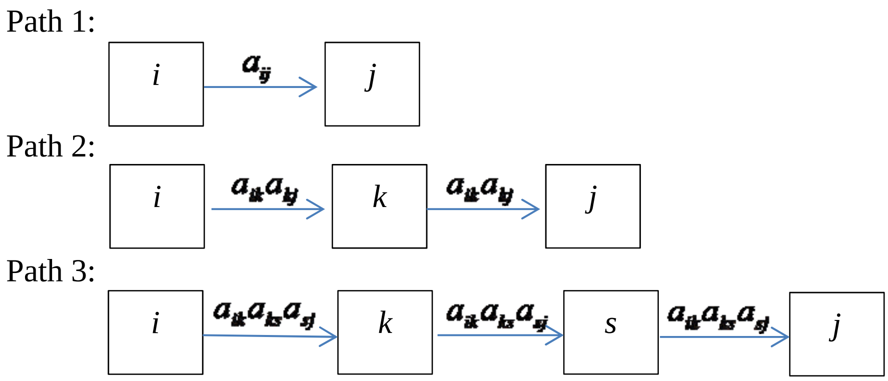

Figure 1 Paths of flows

On each path, the corresponding flow needs to visit a series of sectors before it ends up in a final product. For instance, the flow a_{ij} needs to visits one sector, namely sector j, before it ends up in the final product of sector j; the flow $a_{ik}a_{kj}$ needs to visit two sectors, namely sector k and sector j and the flow $a_{ik}a_{ks}a_{sj}$ needs to visit three sectors, namely sector *k*, sector *s* and sector *j*, before they end up in the final product of sector *j*. Meanwhile, the flow is embedded in the gross output of the sector visited by it, because it is consumed by this sector. For instance, the flow $a_{ik}a_{ks}a_{sj}$ is embedded in the gross output of sector *k*, when it firstly visits this sector; this flow is further embedded in the gross output of sector *s*, when it secondly visits sector *s* carried by sector *k*. Finally, it is embedded in the final product of sector *j* when it visits sector *j* carried by sector *s*. Obviously, each type of flow is double counted by the gross outputs of sectors involved in the corresponding flow path. It is self-evident that the time of double counting of each type flow equals the number of sectors visited by this flow. For instance, the flow $a_{ik}a_{ks}a_{sj}$ is double counted three times, by sector *k*, sector *s* and sector *j* respectively. Meanwhile, the flow $a_{ik}a_{ks}a_{sj}$ visits three sectors, also sector *k*, sector *s* and sector *j* respectively. Hence, the following Axiom exists.

Axiom 1 *In the production of any final product, for a specific flow from the gross output of an involved sector to this final product, the time of double counting of the specific flow along the path equals the number of sectors visited by this specific flow.*

As shown in Figure 1, the total gross output of each sector required to produce a specific final product is realized through a series of paths with different flows. Exactly as the definition of APL, the number of sectors visited by the total required gross output of a sector can be defined as the weighted average number of sectors visited by each flow, with the share of each flow in total required gross output as weight. In the same sense, the time of double counting of the total required gross output of a sector can also be defined as the weighted average time of double counting of each flow. According to Axiom 1, in the production of any final product, the number of sectors visited by the total required gross output of an involved sector equals the time of double counting of the total required gross output of this sector.

Hence, in the production of a specific final product, calculating the number of sectors visited by any involved sector's gross output and the time of double counting of this sector's gross output are two sides of a coin. It therefore can be expected that the formula for the time of double counting of sector *i's* gross output in the production of sector *j's* final product and the formula for APL_{ij} are identical. Before verifying this proposition, we first introduce the formula for the time of double counting of a sector's gross output in the production of another sector's final product.

The gross output of sector *i* embedded in the gross output of sector *j* is

if $i \neq j$

$$\begin{aligned} & a_{ij}x_j + \sum_k a_{ik}a_{kj}x_j + \sum_k \sum_s a_{ik}a_{ks}a_{sj}x_j + \cdots \\ & = (a_{ij} + \sum_k a_{ik}a_{kj} + \sum_k \sum_s a_{ik}a_{ks}a_{sj} + \cdots)x_j \\ & = l_{ij}x_j \end{aligned} \tag{17}$$

if $i = j$

$$\begin{aligned} & x_j + a_{ij}x_j + \sum_k a_{ik}a_{kj}x_j + \sum_k \sum_s a_{ik}a_{ks}a_{sj}x_j + \cdots \\ & = (1 + a_{ij} + \sum_k a_{ik}a_{kj} + \sum_k \sum_s a_{ik}a_{ks}a_{sj} + \cdots)x_j \\ & = l_{ij}x_j \end{aligned} \tag{18}$$

In matrix form, the gross output of a sector embedded in the gross output of another sector can be expressed as $\mathbf{L}\hat{\mathbf{X}}$, where $\hat{\mathbf{X}}$ is the diagonal matrix generated by the gross output vector $\mathbf{x}$. Its elements in the *j*th column gives the gross output of each sector embedded in the gross output of sector *j*.

Next, we investigate the time of double counting of each sector's gross output in the production of a specific final good. First, the gross output of each sector $\mathbf{x}^*$ required to produce the final good of sector *j* satisfies

$$\mathbf{x}^* = \mathbf{L}\mathbf{f}_{(j)}$$

Where $\mathbf{f}_{(j)}$ is a vector with the j^{th} element equaling f_j and the other elements equaling 0. The gross output of each sector embedded in the gross output of all sectors required to produce f_j is

$$\mathbf{x}^{**} = \mathbf{L}\mathbf{x}^* = \mathbf{L}\mathbf{L}\mathbf{f}_{(j)}$$

The time of double counting of each sector's gross output in the production of f_j is

$$\frac{\mathbf{x}^{**}-\mathbf{x}^{*}}{\mathbf{x}^{*}}=\frac{\mathbf{LLf}_{(j)}-\mathbf{Lf}_{(j)}}{\mathbf{Lf}_{(j)}}=\frac{\mathbf{L(L-I)e}_{(j)}}{\mathbf{Le}_{(j)}}$$

Where $\mathbf{e}_{(j)}$ is a vector with the j^{th} element equaling 1 and the other elements equaling 0. The time of double counting of each sector's gross output in the production of any sector's final good can be summarized in the following matrix form.

$$\frac{\mathbf{X}^{**}-\mathbf{X}^{*}}{\mathbf{X}^{*}}=\frac{\mathbf{LL\hat{F}}-\mathbf{L\hat{F}}}{\mathbf{L\hat{F}}}=\frac{\mathbf{L(L-I)}}{\mathbf{L}} \tag{19}$$

Where $\mathbf{\hat{F}}$ is the diagonal matrix generated from the final demand vector. It can be seen that Formula (19) is exactly the same as the revised APL matrix shown by Formula (13). Hence, the following theorem is proved.

Theorem 1 *The formula for the time of double counting of a sector's gross output in the production of another sector's final product and the formula for the APL between these two sectors are identical.*

4. Extended APLs

Axiom 1 and Theorem 1 show that calculating the APL and calculating the time of double counting are the same. Hence, calculating the APL can also be replaced by calculating the time of double counting. The calculation procedure of the time of double counting is simpler than that of APL, especially when extending the current APL to answer other questions discussed later on. This section shows that from the time of double counting perspective, the APL on the one hand can be easily decomposed to a series of sub-APLs. Each sub-APL indicates the number of times of each sector visited by the gross output of an involved sector in the production of a specific final product. On the other hand, the APL can be easily extended to group-wise APL. The current APL and the upstreamness measure proposed by Antràs *et al.* (2012) are both special cases of the group-wise APL.

4.1 Decompose the APL

APL_{ij} measures the number of sectors visited by sector *i*'s gross output in the production of sector *j*'s final product. As we point out before, each sector can be visited several times by the same sector's gross output. Therefore, the sum of the times of each sector visited by sector *i*'s gross output equals the number of sectors visited by sector *i*'s gross output in the production of sector *j*'s final product. The other way around, APL_{ij} can be decomposed into a series of sub-APLs; each sub-APL indicates the times of each sector visited by sector *i*'s gross output in the production of sector *j*'s final product.

According to Axiom 1, in the production of sector *j*'s final product, if sector *i*'s gross output visits sector *k* *m* times, sector *i*'s gross output will be double counted *m* times by sector *k*'s gross output. The gross output of sector *i* and sector *k* required to produce 1 unit sector *j*'s final product are l_{ij} and l_{kj}. In the production of sector *j*'s final product, according to Formula (17)-(18), the gross output of sector *i* embedded in the gross output of sector *k* is $l_{ik}l_{kj}$. Hence, the time of sector *i*'s gross output double counted by sector *k* is $\frac{l_{ik}l_{kj}}{l_{ij}}$. It also indicates that in the production of sector *j*'s final product the times of sector *k* visited by sector *i*'s gross output is $\frac{l_{ik}l_{kj}}{l_{ij}}$. In addition, as only double counted time is relevant, when *k*=*j* the gross output of sector *i* embedded in 1 unit final product of sector *j* should be deducted. In this situation, the time of sector *i*'s gross output double counted by sector *j* or the times of sector *j* visited by sector *i*'s gross output in the production of sector *j*'s final product is $\frac{l_{ij}(l_{jj}-1)}{l_{ij}}$.

Formula (20) demonstrates that the sum of the times of each sector visited by sector *i*'s gross output in the production of sector *j*'s final product equals APL_{ij}.

$$
\begin{aligned}
&\frac{l_{i1}l_{1j}}{l_{ij}}+\frac{l_{i2}l_{2j}}{l_{ij}}+\ldots+\frac{l_{ij}(l_{jj}-1)}{l_{ij}}+\ldots+\frac{l_{in}l_{nj}}{l_{ij}}\\
&=\frac{\sum_{k\neq j} l_{ik}l_{kj}+l_{ij}(l_{jj}-1)}{l_{ij}}\\
&=APL_{ij}
\end{aligned}
\tag{20}
$$

Taking the illustration from the other way around, the APL can be decomposed into a series of sub-APLs. For instance, APL_{ij} can be decomposed into the sum of APL_{ij}^{k}, where $APL_{ij}^{k} = \frac{l_{ik}l_{kj}}{l_{ij}}(k \neq j)$ or $APL_{ij}^{k} = \frac{l_{ik}(l_{kj}-1)}{l_{ij}}(k = j)$. The sub-APL " APL_{ij}^{k} " indicates the times of sector k visited by sector i's gross output in the production of sector j's final product.

4.2 Group-wise APL

The current APL is defined in a sector to sector manner. This part shows that the APL can be simply extended to group-wise APL (GAPL) which is defined in a group to group manner. This extension can show us the number of sectors visited by the gross output of a specific group of sectors in the production of another group of sectors' final products or the number of steps taken by a change in a group of final products to affect the gross output of another group of sectors. In the group-wise case, the flow of a specific group of sectors to the final products of another group of sectors is realized via many more paths than the sector to sector case. This causes complexities in the calculation of each path's weight. Nevertheless, we find that the GAPL can be simply calculated from the perspective of double counting.

Suppose P and G are two groups consisting of a series of sectors, respectively. The gross output of each sector in group P required to produce the final products of sector group G is $\mathbf{i}'_P\mathbf{L}\mathbf{f}_G$, where $\mathbf{i}'_P$ is a summation vector with ones in corresponding cells of $\mathbf{i}'_P$ for the sectors in group P and the remaining cells are zeros; $\mathbf{f}_G$ is a final product vector with final product values of sectors in group G in corresponding cells of $\mathbf{f}_G$ and the remaining cells are zeros. In the production of sector group G's final products, the gross output of sectors in group P embedded in the gross output of all involved sectors is $\mathbf{i}'_P\mathbf{L}\mathbf{L}\mathbf{f}_G$. The time of double counting of sector group P's gross output in the production of $\mathbf{f}_G$ is

$$\frac{\mathbf{i}'_P\mathbf{LLf}_G-\mathbf{i}'_P\mathbf{Lf}_G}{\mathbf{i}'_P\mathbf{Lf}_G}=\frac{\mathbf{i}'_P\mathbf{L}(\mathbf{L}-\mathbf{I})\mathbf{f}_G}{\mathbf{i}'_P\mathbf{Lf}_G}$$

Therefore, the group-wise APL between sector group P and sector group G can be calculated as

$$APL_{PG}=\frac{\mathbf{i}'_P\mathbf{L}(\mathbf{L}-\mathbf{I})\mathbf{f}_G}{\mathbf{i}'_P\mathbf{Lf}_G}=\frac{\mathbf{i}'_P\mathbf{L}(\mathbf{L}-\mathbf{I})\overline{\mathbf{f}}_G}{\mathbf{i}'_P\mathbf{L}\overline{\mathbf{f}}_G} \tag{21}$$

Where $\overline{\mathbf{f}}_G$ is the normalized vector of $\mathbf{f}_G$ with the share of each sector in the total final products of group G.

When group *P* includes only one sector, say sector *i*, and group G also includes only one sector, say sector *j*, it can be verified that APL_{PG} equals to the APL_{ij} in Formula (14)

$$APL_{ij}=\frac{\sum_{k\neq j} l_{ik}l_{kj}+l_{ij}(l_{jj}-1)}{l_{ij}}$$

When group *P* includes only one sector, say sector *i*, and group G includes all sectors of the economy, it can be verified that APL_{PG} turns into the upstreamness measure in Formula (16).

$$u_i=\frac{\sum_j\sum_{k\neq j} l_{ik}l_{kj}f_j+\sum_j l_{ij}(l_{jj}-1)f_j}{\sum_j l_{ij}f_j}$$

Therefore, the GAPL can be regarded as a generalized APL. The APL and the upstreamness measure are all special cases of the GAPL.

The GAPL is especially useful in the context of global production chains. For instance, based on the world input-output table, the GAPL can be used to measure the position of each country in the whole global production chain, the position of each country in the global production chain of a specific industry, etc.. As each country has a group of industries and each industry spreads over a group of countries in the world input-output table, the GAPL should be used.

5. The position identification in global production chain

As an example of the application of GAPL, this section attempts to identify the position of each country in the global production chain. If the gross outputs of country *A* visit more industries in the production of world final products than the gross outputs of country *B*, we say that country *A* is closer to the upstream of the global production chain than country *B*. This requires calculating the GAPL between the gross outputs of all industries in a country and the world final products.

The world input-output database (WIOD) provides a series of world input-output tables from 1995 to 2011. These world input-output tables are industry by industry type tables. Each table includes 40 countries and a RoW (rest of the world) as well as 35 industries for each country. The detailed information on the table construction procedure can be found in Timmer (2012) and Dietzenbacher *et al*. (2013).

Let the 35 industries of a specific country constitute group *P* and the 35*41 industries constitute group *G*. Based on Formula (21), the GAPLs between each country's gross outputs and the world final products from 1995 to 2011 are calculated. These GAPLs indicate the average number of industries visited by a country's gross outputs in the global production chain of a specific year.

It can be seen from Figure 2 that almost all countries' GAPLs increased during 1995-2011. The countries with largest increase in GAPL are Korea and Taiwan, increasing from 1.16 in 1995 to 1.63 in 2011 and from 1.08 in 1995 to 1.54 in 2011, respectively. Other countries such as Turkey, Austria, Germany and Denmark also experienced a relatively large increase in the GAPL. The increased GAPL means that the average number of industries visited by almost every country's gross outputs increased during 1995-2011. Both the lengthened intra-country production chain and the lengthened inter-country production chain could cause this increase. Undoubtedly, the prevailing phenomenon of production of fragmentation which lengthens the inter-country production chain is an important reason for the significantly increased GAPLs shown in Figure 2.

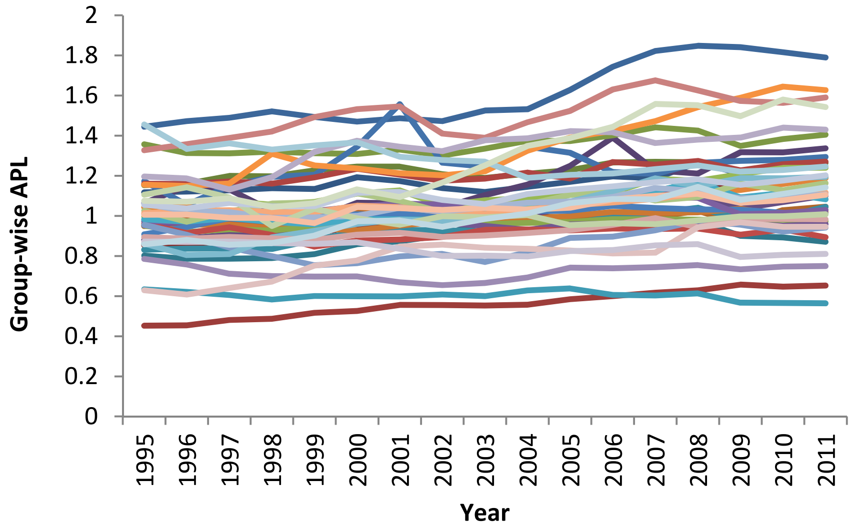

Figure 2 the GAPL for 40 countries and RoW (1995-2011)

Note: each line corresponds to a country

We further sort the GAPLs of all countries. Figure 3 shows the sorting results for 1995 and 2011. It can be seen that during 1995-2011 Korea and Taiwan obviously moved up the global production chains. Korea moved from the eighth place in 1995 to the second place in 2011 and Taiwan moved from the twelfth place in 1995 to the forth place in 2011. Identifying from the current position, Korea and Taiwan are very close to the upstream. Germany also moved up, from the position close to the downstream to the current position close to the midstream. China and Russia basically locate at the upstream of the global production chain all the time, ranked first and fifth in 2011 respectively. US, Brazil and Mexico locate at the downstream of the global production chain; Japan, Italy and Canada basically locate at the midstream of the global production chain.

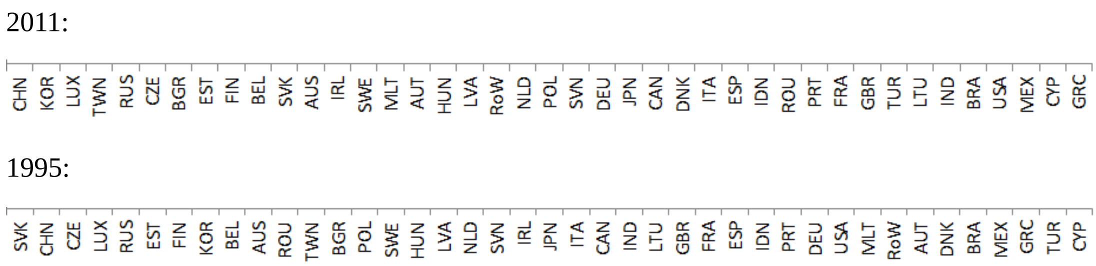

Figure 3 the position of each country in the global production chain (1995, 2011)

One of the implications for each country's position in the global production chain is probably that if the world final demand decreases proportionally across industries and countries for an exogenous shock, the countries located at the downstream such as US, Brazil and Mexico will be affected in relatively short time than the countries located at the upstream, such as China, Korea and Taiwan. In the same sense, along with the rising of the world final demand, the countries located at the downstream will recover first.

6. Conclusion

The APL measures inter-industry linkages from the dimension of length. It is an important complement to the Leontief inverse which measures the inter-industry linkages from the dimension of strength. Related applications have proved that the APL and its invariants are very useful tools in production chain analysis.

This study first gives a revised APL by taking the initial effect into the weight calculation. This revised APL is proved to be equivalent to the time of double counting of a sector's gross output in the production of a specific final product. The time of double counting is relatively easy to calculate based on the input-output analysis. Hence, calculating the APL can be replaced by calculating the time of double counting in certain complicated situations.

From the double counting perspective, on the one hand the APL can be clearly decomposed into a series of sub-APLs which indicate the average times of each sector visited by the gross output of a specific sector in the production of a certain final product. The sub-APLs show the contribution of each involved sector to the APL between two specific sectors. On the other hand, the APL can be extended to group-wise APL (GAPL) which gives the APL between two groups of sectors. The APL and the upstreamness measure are special cases of GAPL. The GAPL thus can be regarded as the generalized APL.

The GAPL is especially useful in the global production chain analysis. We calculate the GAPLs between each country and the whole world final products based on the world input-output tables from 1995 to 2011. The results indicate that almost all countries' GAPLs increased during 1995-2011. The prevailing phenomenon of production of fragmentation, which lengthens the inter-country production chains, is one of the important factors pulling up the GAPLs. We also find that Korea and Taiwan obviously moved up the global production chains in recent years. These two countries with China and Russia locate the upstream of the global production chain; Germany also moved up in recent years, from a downstream position to a midstream position. US, Brazil and Mexico locate at the downstream of the global production chain; Japan, Italy and Canada locate at the midstream of the global production chain.

Appendix 1

The gross output of sector *i* required in the production of the final good of sector *j*, f_j, is

$$x_i^* = l_{ij} f_j = \begin{cases} a_{ij} f_j + \sum_k a_{ik} a_{kj} f_j + \sum_k \sum_s a_{ik} a_{ks} a_{sj} f_j + \dots & i \neq j \\ f_j + a_{ij} f_j + \sum_k a_{ik} a_{kj} f_j + \sum_k \sum_s a_{ik} a_{ks} a_{sj} f_j + \dots & i = j \end{cases}$$

According to the definition of Antràs et al., the average distance of sector *i* to the final product of sector *j* is

$$u_{ij} = \begin{cases} 2 \times \frac{a_{ij} f_j}{x_i^*} + 3 \times \frac{\sum_k a_{ik} a_{kj} f_j}{x_i^*} + 4 \times \frac{\sum_k \sum_s a_{ik} a_{ks} a_{sj} f_j}{x_i^*} + \dots & i \neq j \\ 1 \times \frac{f_j}{x_i^*} + 2 \times \frac{a_{ij} f_j}{x_i^*} + 3 \times \frac{\sum_k a_{ik} a_{kj} f_j}{x_i^*} + 4 \times \frac{\sum_k \sum_s a_{ik} a_{ks} a_{sj} f_j}{x_i^*} + \dots & i = j \end{cases}$$

Substituting x_i^* with $l_{ij} f_j$ yields

$$u_{ij} = \begin{cases} 2 \times \frac{a_{ij}}{l_{ij}} + 3 \times \frac{\sum_k a_{ik} a_{kj}}{l_{ij}} + 4 \times \frac{\sum_k \sum_s a_{ik} a_{ks} a_{sj}}{l_{ij}} + \dots & i \neq j \\ 1 \times \frac{1}{l_{ij}} + 2 \times \frac{a_{ij}}{l_{ij}} + 3 \times \frac{\sum_k a_{ik} a_{kj}}{l_{ij}} + 4 \times \frac{\sum_k \sum_s a_{ik} a_{ks} a_{sj}}{l_{ij}} + \dots & i = j \end{cases}$$

In matrix form:

$$\mathbf{U} = \frac{\mathbf{I} + 2\mathbf{A} + 3\mathbf{A}^2 + 4\mathbf{A}^3 + \dots}{\mathbf{L}} = \frac{(\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A})^{-1}}{\mathbf{L}} = \frac{\mathbf{LL}}{\mathbf{L}}$$

Appendix 2

Suppose that the initial step or distance starts from *k* in one scenario and starts from *s* in another scenario.

If the initial steps or distance starts from *k*, the formula for APL is

$$\frac{k\times\mathbf{I}+(k+1)\times\mathbf{A}+(k+2)\times\mathbf{A}^{2}+(k+3)\times\mathbf{A}^{3}+\ldots}{\mathbf{L}}$$

Let $\mathbf{H}=k\times\mathbf{I}+(k+1)\times\mathbf{A}+(k+2)\times\mathbf{A}^{2}+(k+3)\times\mathbf{A}^{3}+\ldots$, then

$$\mathbf{AH}=k\times\mathbf{A}+(k+1)\times\mathbf{A}^{2}+(k+2)\times\mathbf{A}^{3}+(k+3)\times\mathbf{A}^{4}+\ldots$$

$$\mathbf{H}-\mathbf{AH}=k\times\mathbf{I}+\mathbf{A}+\mathbf{A}^{2}+\mathbf{A}^{3}+\mathbf{A}^{4}+\ldots=\mathbf{L}+(k-1)\mathbf{I}$$

Therefore,

$$\mathbf{H}=\mathbf{L}[\mathbf{L}+(k-1)\mathbf{I}]$$

Then, under scenario one the formula for APL is simplified as

$$\frac{\mathbf{L}[\mathbf{L}+(k-1)\mathbf{I}]}{\mathbf{L}}$$

In the same sense, if the initial steps or distance starts from s, the formula for APL is

$$\frac{\mathbf{L}[\mathbf{L}+(s-1)\mathbf{I}]}{\mathbf{L}}$$

The difference between the APLs under two scenarios is

$$\frac{\mathbf{L}[\mathbf{L}+(k-1)\mathbf{I}]}{\mathbf{L}}-\frac{\mathbf{L}[\mathbf{L}+(s-1)\mathbf{I}]}{\mathbf{L}}=\frac{(k-s)\mathbf{L}}{\mathbf{L}}=(k-s)\mathbf{E}$$

Where $\mathbf{E}$ is a matrix with all elements being ones. The above formula indicates that all APLs have an identical difference under different initial step or distance scenarios. Therefore, the ordinal position of each sector in the production chain of a specific sector does not change regardless of which number is selected as the initial step or distance.

Reference

Antràs, P, Chor, D., Fally, T., Hillberry, R. 2012. Measuring the Upstreamness of Production and Trade Flows. *American Economic Review Papers and Proceedings* 102, no. 3: 412-416.

Baldwin, R.E., Lopez Gonzalez, J..2013. Supply-Chain Trade: A Portrait of Global Patterns and Several Testable Hypotheses. *NBER Working Paper* 18957.

Dietzenbacher, E., Los, B., Stehrer, R., Timmer, M.P., de Vries G.J. 2013. The Construction of World Input-Output Tables in the WIOD Project. *Economic Systems Research*, 25: 71-98.

Dietzenbacher, E., Romero, I., Bosma, N.S. 2005. Using Average Propagation Lengths to Identify Production Chains in the Andalusian . *Estudios de Economia Aplicada* 23: 405-422.

Dietzenbacher, E., Romero, I. 2007. Production Chain in an Interregional Framework: Identification by Means of Average Propagation Lengths. *International Regional Science Review* 30: 362-383.

Fally, T., Hillberry, R. 2013. Quantifying the upstreamness in East Asia: Insights from a Coasian Model of Production Staging. Available at: http://apts.em.seu.edu.cn/APTS2013_Program/Paper2013/Russell%20Hillberry.pdf.

Foster-Mc Gregor, N., Stehrer, R. 2013. Value Added Content of Trade: A Comprehensive Approach. *Economics Letters*, 120: 354-357.

Inomata, S. 2008. A New Measurement for International Fragmentation of the Production Process: An International Input-Output Approach. IDE Discussion Paper NO. 175.

Koopman, R., Wang, Z., Wei, S. 2014. Tracing Value-Added and Double Counting in Gross Exports. *American Economic Review*, (forthcoming).

Lu, X., Xu, J. 2013. Explaining the Changes in ALPs by Identifying ALPs-Important Coefficients: An Empirical Application of China. *Journal of Systems Science and Complexity* 26: 383-406.

Miller, R. E, Blair, P. D. 2009. Input-Output Analysis: Foundation and Extension, Cambridge University Press, New York, U.S.A.

Oosterhaven, J, Bouwmeester, M. 2013. The Average Propagation Length: Conflicting Macro, Intra-industry, and Interindustry Conclusion. *International Regional Science Review* 36: 481-491.

Timmer, M.P., Los, B., Stehrer, R., de Vries, G.J. 2013. Fragmentation, Incomes and Jobs: An Analysis of European Competitiveness. *Economic Policy*, 28: 613-661.

Timmer, M.P., Erumban, A.A., Los, B., Stehrer, R., de Vries, G.J. 2012. Slicing up Global Value Chains. *WIOD Working Paper* Nr.12.

Timmer, M.P. 2012. The World Input-Output Database (WIOD): Contents, Sources and Methods. *WIOD Working Paper* Nr.10.

Yu, K.D.S., Tan, R.R., Aviso, K.B., Promentilla, M.A.B., Santos, J.R. 2014. A Vulnerability Index for Post-Disaster Key Sector Prioritization. *Economic Systems Research* 26:81-97.