Denton PFD and GRP benchmarking are friends. An empirical evaluation on Dutch Supply and Use Tables

Jacco Daalmans*
Dutch Central Bureau of Statistics
Statistics Netherlands
j.daalmans@cbs.nl

Tommaso Di Fonzo†
Istat – Advanced School for Statistics and Socio-Economic Analyses
tdifonzo@istat.it

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Abstract

Temporal benchmarking according to Denton (1971) is widely used in the production process of statistical offices. Statistics Netherlands has been using a multivariate Denton method for the compilation of large, fully consistent, quarterly and annual supply and use tables. The purpose of Denton methods is to achieve consistency between high and low frequency data (e.g. quarterly with annual data). The high frequency data are adjusted to align with the low frequency data, while preserving as much as possible the short-term movements of the preliminary high frequency data. It is often claimed that the Proportionate First Differences (*PFD*) variant of Denton's benchmarking, which is the most used in practice, is a close approximation of the Growth Rates Preservation (*GRP*) benchmarking, which is considered as an 'ideal' benchmarking procedure to preserve short term movements of the indicator series. In addition, the *PFD* criterion is more often applied, because the resulting mathematical problem is easier to solve. In this paper we will search for empirical examples, from Dutch Supply and Use Tables, in which *PFD* does not work as expected. Examples in which the dynamics of the indicator series is not preserved well by Denton *PFD* benchmarking, whereas *GRP* benchmarking works better, are shown. A second aim of the paper is to present simple heuristic procedures that approximate the *GRP* criterion in the multivariate case, whose implementation involves the solution of a standard quadratic-linear problem instead of a linearly constrained non-linear one. The heuristics will be empirically compared with *PFD* and *GRP* in order to evaluate their possible ability to preserve the preliminary growth rates better than the *PFD* procedure.

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1. Introduction

Benchmarking monthly and quarterly series to annual data is a common practice in many National Statistical Institutes. The benchmarking problem arises when time series data for the same target variable are measured at different frequencies with different level of accuracy and there is the need to remove discrepancies between annual benchmarks and corresponding sums of the sub-annual values.

Usually, the low frequency (or annual) data sources describe levels and long-term trends better than the high frequency (sub-annual) sources. The latter, on the other side, provide the only information on the short-term movements. Therefore in benchmarking, the low-frequency benchmarks are fixed, while preserving as much as possible the short-term movements of the sub annual sources.

Several benchmarking methods are available in the literature. Multiplicative methods try to preserve the relative changes of the preliminary high-frequency time series, while additive methods are aimed to preserve the changes in absolute terms. In this paper the focus will be solely on multiplicative variants of benchmarking.

One of the most popular multiplicative benchmarking procedures is the Proportional First Differences (*PFD*) Denton method (Denton, 1971), which attempts to keep the relative differences between the benchmarked and the preliminary data as constant as possible over time. Mathematically, the Denton method deals with a linearly constrained quadratic optimization problem, for which many efficient solution techniques exist.

A second method, the Growth Rates Preservation (*GRP*) benchmarking procedure by Causey and Trager (1981; see also Trager, 1982, and Bozik and Otto, 1988) minimizes the sum of the squared differences between the growth rates of the target and of the preliminary series. Bloem *et al* (2001, p.100) claim that this objective function is grounded on an ``ideal” movement preservation principle, “formulated as an explicit preservation of the period-to-period rate of change'' of the preliminary series. This benchmarking procedure basically looks for a solution to a linearly constrained non linear problem. For large scale applications, specialised optimization software is needed in order to find the optimal solution.

Both methods, *PFD* and *GRP*, can be applied in univariate and multivariate situations.

In the univariate case, there can only be *temporal constraints*, coming from the needed consistency between subannual time series and annual benchmarks. In the multivariate case there are also constraints between different time-series, usually valid at each time, the so-called *contemporaneous constraints*.

Because of the technical difficulty of applying *GRP*, Denton *PFD* is often used, despite its weaker theoretical foundation. For example, the Denton approach is used for reconciling Dutch National Accounts (see Bikker *et al.*, 2013). The aim of this paper is to present two new heuristics that better preserve the initial growth rates than Denton *PFD* and that are at the same time much easier to implement than standard *GRP* benchmarking and reconciliation procedures.

The results of this papers are useful to practitioners in the field, who will be able to obtain accurate results by using simple, understandable and easy to implement procedures.

First, in Section 2, we will show an example from Dutch Supply and Use Tables, that illustrates that *Denton PFD* leads to suboptimal results. In Section 3 a formal description of Denton *PFD* and *GRP* benchmarking will be given. The multivariate case is the topic of Section 4. Thereafter, in Section 5 two new heuristics will be presented. In Section 6 several empirical results will be given. Finally, Section 7 concludes this paper.

2. An example from Dutch SUT

In the literature (Cholette, 1984, Bloem *et al.*, 2001, Dagum and Cholette, 2006) it is often claimed that the Denton *PFD* procedure produces results very close to the *GRP* benchmarking. Di Fonzo and Marini (2010) explain that the approximation worsens if the preliminary series have a large variability and/or a large bias with respect to the annual benchmarks. Some empirical examples are shown in Harvill Hood (2005) and Titova *et al.* (2010). In this section we will add one example obtained from a data set of Dutch Supply and Use Tables (SUT)[3].

In the example twelve quarters are benchmarked to three years. The example is of some case with high variability. Figure 1 compares the results of *GRP* with Denton *PFD*.

The most remarkable difference occurs in Quarter 9. At first sight the differences in results may seem minor, but these are more apparent, if we express the period-to-period changes as a percentage, which is done in Table 1. For 8 out of 11 cases, *GRP* better preserves the initial period to period changes than Denton *PFD*.

We refer to Section 6 for a more extensive comparison between Denton *PFD* and *GRP*, based on several data sets.

Figure1. Case 1. Source data, GRP and Denton PFD results

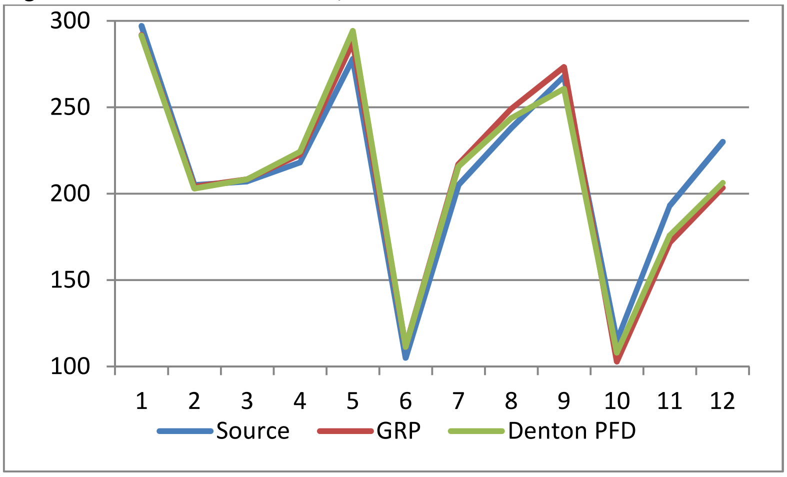

Table 1. Case 1. Quarter-to-quarter growth rates (%)

Period	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10	10-11	11-12
Source	-31,0	1,0	5,3	27,5	-62,2	95,2	16,1	12,6	-57,1	67,8	19,2
GRP	-30,0	2,1	6,8	29,2	-61,3	95,1	14,7	9,8	-62,4	66,9	18,5
PFD	-30,3	2,6	7,7	31,2	-62,2	93,8	13,0	7,0	-58,6	63,0	17,3

[3] For confidentiality reasons, The Supply and Use tables have been aggregated to the so-called 'publication level'.

3. Formal description of univariate benchmarking methods

Because the temporal aggregation constraints are the same for Denton *PFD* and *GRP*, these are described first. Thereafter Denton *PFD* and *GRP* benchmarking procedures are explained.

3.1 General notation and temporal constraints

Let $b_T, T = 1, \dots, N$, and p_t, t = 1,..,n, be, respectively, the temporal benchmarks and the high-frequency preliminary values of an unknown target variable x_t.

Let s be the aggregation order (e.g., s=4 for quarterly-to-annual aggregation, s=12 for monthly-to-annual aggregation, s=3 for monthly-to-quarterly aggregation), and let **A** be a $(N \times n)$ temporal aggregation matrix, converting n high-frequency values into N low-frequency ones (we assume n=sN). If we denote with **x** the (n ×1) vector of high-frequency values, and with **b** the (N×1) vector of low-frequency values, the aggregation constraints can be expressed as **Ax** =**b**.

Depending on the nature of the involved variables (e.g., flows, averages, stocks), the temporal aggregation matrix **A** is usually written as

$$\mathbf{A} = \mathbf{I}_N \otimes \mathbf{a}^T, \qquad (1)$$

where the (s ×1)vector **a** may assume one of the following forms:

1. Flows: $\mathbf{a} = \mathbf{1}_s = (1, \dots, 1)^T$,
2. Averages: $\mathbf{a} = \frac{1}{s}\mathbf{1}_s$,
3. Stocks (end-of-the-period): $\mathbf{a} = (0, \dots, 0, 1)^T$,
4. Stocks (beginning-of-the-period): $\mathbf{a} = (1, 0, \dots, 0)^T$.

For flow variables the sum of s subannual values need to be in line with the annual benchmark. That is:$\sum_{t \in T} x_t^* = b_T,\ T = 1, \dots, N$.

Denoting by **p** the vector of preliminary values (in general it is **Ap≠b**, otherwise no adjustment would be needed), we look for a vector of benchmarked estimates **x*** which should be \`as close as possible' to the preliminary values, and such that **Ax*=b**.

To this end, some characteristics of the original series p should be kept into consideration. For example, in an economic time series framework, the preservation of the temporal dynamics (however defined) of the preliminary series is often a major interest of the practitioner.

3.2 Growth Rates Preservation (GRP)

For flows series, Causey and Trager (1981; see also Monsour and Trager, 1979, and Trager, 1982) consider a criterion to be minimized explicitly related to the growth rate, which is a natural measure of the movement of a time series:

$$f(\mathbf{x}) = \sum_{t=2}^{n} \left(\frac{x_t}{x_{t-1}} - \frac{p_t}{p_{t-1}} \right)^2, \qquad (2)$$

and search for values x_t^*, t = 1,..., n, which minimize the criterion (2) subject to the aggregation constraints $\sum_{t \in T} x_t^* = b_T,\ T = 1, \dots, N$. In other words, the benchmarked series is estimated in such a way that its temporal dynamics, as expressed by the growth rates $\frac{x_t^*}{x_{t-1}^*}$, be as close as possible to the temporal dynamics of the preliminary series, where the \`distance' from the preliminary growth rates $\frac{p_t}{p_{t-1}}$ is given by the sum of the squared differences.

In this paper we consider a more general formulation of the *GRP* benchmarking problem, valid not only for flows variables, that is:

$$\min_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } \mathbf{Ax} = \mathbf{b} \quad (3)$$

where **A** is the temporal aggregation matrix (1). The criterion (2) is clearly a non-linear and also a non-convex function. The constrained minimization problem (3) has not linear first-order conditions for a stationary point, and thus it is not possible to find an explicit, analytic expression for the solution. On the other hand, provided that both p_t and x_t, t= 2,…,n-1, be different from zero, $f(\mathbf{x})$ is a twice continuously differentiable function, which makes it possible to use several iterative minimization algorithms (Nocedal and Wright, 2006).

3.3 Modified Denton *PFD*

Denton (1971) proposed a benchmarking procedure grounded on the *Proportionate First Differences* (*PFD*) between the target and the original series. Cholette (1984) slightly modified the result of Denton, in order to correctly deal with the starting conditions of the problem. The *PFD* benchmarked estimates are thus obtained as the solution to the constrained quadratic minimization problem

$$\min_{x_t} f(\mathbf{x}) = \sum_{t=2}^{n} \left(\frac{x_t}{p_t} - \frac{x_{t-1}}{p_{t-1}}\right)^2, \quad \text{subject to } \mathbf{Ax} = \mathbf{b}. \quad (4)$$

In matrix notation, the *PFD* benchmarked series is contained in the ($n \times 1$) vector $\mathbf{x}^{PFD}$ solution to the linear system (Di Fonzo and Marini, 2010)

$$\begin{bmatrix} \mathbf{M} & \mathbf{A}^{\mathrm{T}} \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{PFD} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix}, \quad (5)$$

where λ is a ($N \times 1$) vector of Lagrange multipliers, $\mathbf{M} = \mathbf{P}^{-1}\boldsymbol{\Delta}_n^T\boldsymbol{\Delta}_n\mathbf{P}^{-1}$,$\mathbf{P} = \mathrm{diag}(\mathbf{p})$and $\boldsymbol{\Delta}_n$ is the ((n-1) ×n) first differences matrix

$$\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}. \quad (6)$$

Notice that $\boldsymbol{\Delta}_n^T\boldsymbol{\Delta}_n$ has rank n-1 (Cohen *et al.*, 1971, p. 122), so **M** is singular. However, given that matrix **A** has full row rank, and provided no preliminary value is equal to zero[4], the coefficient matrix of system (5) has full rank (Di Fonzo and Marini, 2010). Then, $\mathbf{x}^{PFD}$can be obtained from

$$\begin{bmatrix} \mathbf{x}^{PFD} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{M} & \mathbf{A}^{\mathrm{T}} \\ \mathbf{A} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix}. \quad (7)$$

Alternatively, a solution can be obtained from the so-called *augmented form* of the problem, see Appendix A.1. This approach requires the computation of the inverse of one ($n \times n$) and one ($N \times N$) matrix. For comparison, $\begin{bmatrix} \mathbf{M} & \mathbf{A}^{\mathrm{T}} \\ \mathbf{A} & \mathbf{0} \end{bmatrix}$ in (7) is a (N+n) square matrix. To the best of our knowledge,

[4] When some p_t is null, the growth rates of the preliminary data are not defined. As mentioned in Bikker *et al.* (2013) a multivariate benchmarking model is not appropriate. A possible solution (see, for example, Cholette and Chhab, 1991, p. 413) consists in setting the originally null preliminary data at a very small value, e.g. p_t=0.001. This may however result in unstable results.

the expression of Denton's PFD benchmarking through the augmented form has not been previously mentioned in the literature.

4. The multivariate case

4.1 General notation and temporal constraints

Let us consider *m* vectors of (*n*×1) high frequency preliminary values $\mathbf{p}_\mathrm{j}$ and of (*N*×1) benchmark (low frequency) data. In a typical multivariate reconciliation problem, the researcher is looking for reconciled values $\mathbf{x}_j$, *j* = 1,…,*m* which are in line with *temporal* and *contemporaneous* constraints. The vectors $\mathbf{x}_j$and $\mathbf{p}_j$are combined in the (*mn*×1) vectors **x** and **p**. That is:

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_m \end{pmatrix}, \mathbf{p} = \begin{pmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_m \end{pmatrix}$$

For each time-series there the number of temporal constraints is *N*. Therefore, the total number of these constraints is *Nm*. Suppose that at each period, *k* contemporaneous constraints apply. Thus, the total number of these constraints is *nk*.
As shown in Di Fonzo and Marini (2011), all constraints can be expressed in matrix form as follows:

$$\mathbf{Hx} = \mathbf{y}_\mathrm{a}. \qquad (8)$$

where $\mathbf{y}_\mathrm{a}$ is a ((*nk* + *mN*)×1) vector of contemporaneous and temporally aggregated values: $\mathbf{y}_\mathrm{a} = [\mathbf{z}^T\mathbf{y}_0^T]^T$, with **z** a (*nk*×1)vector of high frequency contemporaneously aggregated values, and $\mathbf{y}_0 = [\boldsymbol{y}_{01}^T \ldots \boldsymbol{y}_{0m}^T]^T$ is a (*mN*×1) vector of *m* temporally aggregated vectors. In general, it holds that **Hp** ≠ $\mathbf{y}_\mathrm{a}$, i.e. the provisional data do not satisfy all constraints.

4.2 Multivariate GRP and multivariate Denton PFD

The multivariate extensions of *GRP* and Denton *PFD* are given by (Di Fonzo and Marini, 2012):

$$\min_{x_{jt}} f(\mathbf{x}) = \sum_{j=1}^{m}\sum_{t=2}^{n}\left(\frac{x_{jt}}{x_{jt-1}} - \frac{p_{jt}}{p_{jt-1}}\right)^2, \quad \text{subject to } \mathbf{Hx} = \mathbf{y}_\mathrm{a} \qquad (9)$$

$$\min_{x_{jt}} f(\mathbf{x}) = \sum_{j=1}^{m}\sum_{t=2}^{n}\left(\frac{x_{jt}}{p_{jt}} - \frac{x_{jt-1}}{p_{jt-1}}\right)^2, \quad \text{subject to } \mathbf{Hx} = \mathbf{y}_\mathrm{a} \qquad (10)$$

respectively, where $p_{jt}$and x_{jt} denote the preliminary and reconciled subannual values of time-series *j* at period *t*.
Analoguous to the univariate case, it is not possible to derive an analytical expression for the solution of *GRP*.
The Denton *PFD* benchmarked series $\mathbf{x}^{PFD}$ is contained in the (*mn×1*) vector solution to the linear system (Di Fonzo and Marini, 2010)

$$\begin{bmatrix} \mathbf{M} & \mathbf{H}^T \\ \mathbf{H} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{x}^{PFD} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y}_\mathrm{a} \end{bmatrix}, \qquad (11)$$

where *λ* is a ((*nk* + *mN*)×1) vector of Lagrange multipliers and

$$\mathbf{M} = \mathbf{P}^{-1}\boldsymbol{\Delta}^T\boldsymbol{\Delta}\mathbf{P}^{-1}, \qquad (12)$$

$\mathbf{P} = \mathrm{diag}(\mathbf{p})$ and $\mathbf{\Delta}$ is a (m(n-1) × mn) matrix, defined by $\mathbf{\Delta}= \mathrm{diag}(\mathbf{\Delta}_j)$, in which all m matrices $\mathbf{\Delta}_j$ are the ((n-1) × n) first difference matrix $\mathbf{\Delta}_n$, as given in (6).

It should be noted that, unlike the univariate case, this coefficient matrix is not necessarily invertible. For example, the coefficient matrix is not invertible if some of the rows of $\mathbf{H}$ are lineary dependent. This happens if one of the constraints can be expressed as a linear combination of the other constraints.

The linear system has a solution only provided that the vector $\begin{bmatrix}\mathbf{0}\\ \mathbf{y}_\mathrm{a}\end{bmatrix}$ lies in the range space of the coefficient matrix $\begin{bmatrix}\mathbf{M} & \mathbf{H}^T\\ \mathbf{H} & \mathbf{0}\end{bmatrix}$. The system can be solved in a direct way through an appropriate factorization of the coefficient matrix, a facility offered by most of the mathematical software used by the practitioners (we used Matlab, but Gauss, SAS, R could work as well).

Alternatively, the approach of the augmented problem can be followed (see Appendix A.2). This approach requires the computation of one ((nk + mN)×1) matrix and m times the inverse of a *(n x n)* matrix. For comparison, the coefficient matrix $\begin{bmatrix}\mathbf{M} & \mathbf{H}^T\\ \mathbf{H} & \mathbf{0}\end{bmatrix}$ in (11) is a square matrix of dimension(mn + nk + mN), where mn is the number of variables and nk + mN the number of constraints. In most large-scale practical situation, we have that mn>>nk + mN, from which it follows that using the augmented problem, instead of the standard approach, may lead to a computational gain.

5. Two new heuristics

In this section the two new iterative heuristics are described for the multivariate case. The univariate problem is a special case in which *m,* the number of series, is one.

5.1 Heuristic Growth Rates Preservation (*HGRP*)

The first heuristic is based on a relationship between the GRP function in (9) and the PFD criterion in expression (10):

$$\sum_{j=1}^{m}\sum_{t=2}^{n}\left(\frac{x_{jt}}{x_{jt-1}}-\frac{p_{jt}}{p_{jt-1}}\right)^2 = \sum_{j=1}^{m}\sum_{t=2}^{n}\left[\frac{p_{jt}}{x_{jt-1}}\left(\frac{x_{jt}}{p_{jt}}-\frac{x_{jt-1}}{p_{jt-1}}\right)\right]^2 \qquad (13)$$

The idea is to exploit relationship (13) by considering at a first stage a feasible estimate of the series of interest, say $\tilde{\boldsymbol{x}}$, and (ii) by using it in a Weighted Denton *PFD* procedure (*WPFD*) aimed at minimizing the weighted *PFD* criterion

$$\min_{x_{jt}}\sum_{j=1}^{m}\sum_{t=2}^{n}\left[w_{jt}\left(\frac{x_{jt}}{p_{jt}}-\frac{x_{jt-1}}{p_{jt-1}}\right)\right]^2, \quad \text{subject to } \mathbf{Hx}=\mathbf{y}_\mathrm{a}, \qquad (14)$$

where the weights w_{jt} are defined by:

$$w_{jt}=\frac{p_{jt}}{\tilde{x}_{jt-1}}, \quad j=1,\dots,m;\ t=2,\dots,n. \qquad (15)$$

The solution of this problem will be denoted by $\mathbf{x}^{WPFD[1]}$.
The objective function in (14) approximates the *GRP* function in (13). The more $\tilde{\boldsymbol{x}}$ resembles the *GRP* results, the better this approximation. Consequently, the solution of (14) can be considered as an approximation of the optimal *GRP* solution.
Provided that the two steps shown so far (the estimation of $\tilde{\mathbf{x}}$ and then of $\mathbf{x}^{WPFD[1]}$) give an improvement in the *GRP* criterion, i.e.

$$f\left(\mathbf{x}^{WPFD[1]}\right) < f(\tilde{x}),$$

where function *f*(.) is the global *GRP* criterion defined in (9), we can iterate by updating the weights in (15). That is, by using $\mathbf{x}^{WPFD[1]}$ in (15) instead of $\tilde{\boldsymbol{x}}$. Thereafter, the minimization problem (14) is solved again, using the updated weights, from which we obtain a new solution $\mathbf{x}^{WPFD[2]}$.
The process is repeated until no improvement in the global *GRP* criterion (9) is found. As a result a sequence of solutions is obtained, that are hoped to finally converge to the *GRP* result.
The iterative procedure starts with a feasible solution $\tilde{\boldsymbol{x}}$. In order to find such a solution we can apply a multivariate Denton *PFD* procedure, i.e. the solution to problem in (14) with all weights set to one.
The new estimate, say $\mathbf{x}^{WPFD}$, can be obtained from the linear system

$$\begin{bmatrix}\mathbf{Q} & \mathbf{H}^T\\ \mathbf{H} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}^{WPFD}\\ \lambda\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{y}_\mathrm{a}\end{bmatrix},$$

where $\mathbf{Q} = \mathbf{P}^{-1}\boldsymbol{\Delta}^T\mathbf{W}\mathbf{W}\boldsymbol{\Delta}\mathbf{P}^{-1}$, and $\mathbf{W}$ = diag($\mathbf{w}$) with dimensions *m(n -1)*× *m(n - 1*).
A more formal description of the stopping criterion will be based on the measure α_k

$$\alpha_k = \frac{f(\mathbf{x}^{k-1}) - f(\mathbf{x}^{\boldsymbol{k}})}{f(\mathbf{x}^{k-1})}, \quad k = 1, \dots \qquad (16)$$

where *k* denotes the iteration and *f* stands for the *GRP* criterion in (9).The iterations proceed for $\alpha_k > \varepsilon$, where ε is a small tolerance value, usually set at 1*e*-6.
When the iterations terminate, the `Heuristic *GRP*' (*HGRP*) benchmarked series are given by

$$\boldsymbol{x}^{HGRP} = \begin{cases} \boldsymbol{x}^k & \text{if } 0 < \alpha_k < \epsilon \\ \boldsymbol{x}^{k-1} & \text{if } \alpha_k < 0 \end{cases}$$

The heuristic can be summarised as follows:
Step 1: Apply Denton *PFD*, call its solution $\boldsymbol{x}^{[0]}$;
Step*k*: with k >1:

- Update the weights in (15), by using $\boldsymbol{x}^{[k-1]}$instead of $\widetilde{\boldsymbol{x}}$.
- Apply Weighted Denton *PFD* (14),using the updated weights, to obtain the solution $\boldsymbol{x}^{[k]}$.
- Stop, if the stopping criterion, based on (16), is fulfilled.

We will show in Section 6 that the iterated procedure can be considered as an `heuristic' approximation of the `true' *GRP* benchmarking procedure. It provides good quality results both in terms of precision of the estimates, the degree of movement preservation cannot be less than that of Denton *PFD,* and of simplicity of the implied mathematics, the procedure is just a small extension to Denton *PFD*.
In practical situations, we have found that very few iterations are needed to converge at the final estimates, and that performing only one iteration leads to a considerable improvement over Denton *PFD* already.

Example 1

We consider a simple example to illustrate the procedure. In this example 6 monthly preliminary values are benchmarked to 2 quarterly totals. The monthly values are (80, 100, 80) for the three months within both quarters. The two quarterly benchmarks are 300 and 200, respectively. For both quarters the sum of the three benchmarked monthly values has to be the same as the corresponding quarterly benchmark. The result of Denton *PFD* and the heuristic procedure are shown in Table 2. The *GRP* criterion in the last column refers to the measure in formula (2). The weights are defined in (15).
The first iteration is the best iteration, it leads to the lowest *GRP* criterion value. The iterations are stopped after the second iteration, because α_2 has a negative value: $\alpha_2 = (0.0609 - 0.0688)/0.0609$.

Table 2. Results of Example 1.

	M1	**M2**	**M3**	**M4**	**M5**	**M6**	**GRP-criterion**
Preliminary	80.00	100.00	80.00	80.00	100.00	80.00	
Denton PFD	98.41	117.50	84.09	69.76	74.80	55.44	0.0743
- weights(1)		1.02	0.68	0.95	1.43	1.07	
HGRP 1st it.	100.35	120.94	78.71	65.36	76.61	58.04	0.0609
- weights(2)		1.00	0.66	1.02	1.53	1.04	
HGRP 2nd it.	102.77	125.34	71.89	63.00	77.25	59.75	0.0688

5.2 Taylor linearization of the GRP objective function (TLGRP)

The second heuristic is based on Taylor linearization of the ratios

$$\frac{x_{jt}}{x_{jt-1}}, \qquad j = 1, \dots, m; \;\; t = 2, \dots, n$$

within the *GRP* objective function in (9).
First-order Taylor linearization around the points $(\tilde{x}_{jt-1,}\tilde{x}_{jt})$ leads to:

$$\frac{x_{jt}}{x_{jt-1}} \simeq \frac{\tilde{x}_{jt}}{\tilde{x}_{jt-1}} + \frac{1}{\tilde{x}_{jt-1}}\left(x_{jt} - \frac{\tilde{x}_{jt}}{\tilde{x}_{jt-1}} x_{jt-1}\right). \tag{17}$$

After substitution of (17), we obtain the following *GRP* objective function

$$\min_{x_{jt}} \sum_{j=1}^{m}\sum_{t=2}^{n}\left[\frac{1}{\tilde{x}_{jt-1}}\left(x_{jt} - \frac{\tilde{x}_{jt}}{\tilde{x}_{jt-1}} x_{jt-1}\right) + \frac{\tilde{x}_{jt}}{\tilde{x}_{jt-1}} - \frac{p_{jt}}{p_{jt-1}}\right]^2, \tag{18}$$

i.e. a sum of squared linear terms. The optimization problem can also be written as

$$\min_{x_{jt}} \sum_{j=1}^{m}\sum_{t=2}^{n}\left(a_{jt}x_{jt-1} + b_{jt}x_{jt} \; + q_{jt}\right)^2, \text{ subject to } \mathbf{Hx} = \mathbf{y}_{\mathrm{a}}, \tag{19}$$

where

$$a_{j,t} = -\frac{\tilde{x}_{jt}}{\left(\tilde{x}_{jt-1}\right)^2}, \qquad b_{jt} = \frac{1}{\tilde{x}_{jt-1}}, \qquad q_{jt} = \frac{\tilde{x}_{jt}}{\tilde{x}_{jt-1}} - \frac{p_{jt}}{p_{jt-1}}.$$

Because of the linear terms, the objective function in (19) is technically easier to solve than the *GRP* objective function in (9).
It can be expected that the solution of (19) closer approximates optimal '*GRP*' than the vector of input data **x**. We can exploit this result in the development of a ' Newton-like' iterative procedure. In this procedure Taylor approximation is applied repeatedly, each time around different points, that closer and closer approximate the *GRP*-result.
In the first iteration the preliminary values are used, leading to a first approximation of *GRP*. In the second iteration we apply Taylor iteration again around the outcomes of the first iteration. Since the linearization is performed around a closer to optimal point, it can be expected that the result of the second iteration will also be closer to optimal GRP. If this is actually the case, a third iteration is performed. The process stops if no improvement in the GRP criterion is found.
We now proceed to explain how the problem in (19) can be solved. For this purpose, we consider the band-diagonal matrices $\mathbf{M}_j$, j =1,…,m, each of dimensions (n-1 ×n), whose non-zero items are given by

$$m_{j(t-1,t-1)} = a_{jt}, \qquad m_{j(t-1,t)} = b_{jt}, \qquad j = 1,\dots,m, \;\; t = 2,\dots,n$$

meaning that:

$$\mathbf{M}_j = \begin{bmatrix} a_{j,2} & b_{j,2} & 0 & \cdots & 0 & 0 \\ 0 & a_{j,3} & b_{j,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{j,n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{j,n} & b_{j,n} \end{bmatrix}.$$

Given the (n– 1×1) vectors

$$\mathbf{q}_j = [q_{j,2} q_{j,3} \cdots q_{j,n}]', \qquad j = 1, \dots m,$$

it is easy to check that the objective function in (19) can be written as

$$\sum_{j=1}^{m} (\mathbf{M}_j \mathbf{x}_j + \mathbf{q}_j)' (\mathbf{M}_j \mathbf{x}_j + \mathbf{q}_j). \tag{20}$$

or in matrix form

$$(\mathbf{Mx} + \mathbf{q})' (\mathbf{Mx} + \mathbf{q}). \tag{21}$$

where **M** = *diag*($\mathbf{M}_j$) and $\mathbf{q} = [\mathbf{q}_1^T \cdots \mathbf{q}_m^T]'$.

The minimization of function in(20) subject to the constraints **Hx** = $\mathbf{y}_a$ is rather straightforward: given a ((nk + mN)×1) vector of Lagrange multiplier λ, the optimal solution is obtained by solving the linear system

$$\begin{bmatrix} \mathbf{M}'\mathbf{M} & \mathbf{H}' \\ \mathbf{H} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\mathbf{Mq} \\ \mathbf{y}_a \end{bmatrix}, \tag{22}$$

Notice that matrix $\mathbf{M}'\mathbf{M}$ in the top-left part of the coefficient matrix of system (22) has not full rank, and thus the formula for the inverse of a partitioned matrix cannot be applied. This means that the benchmarked estimates according to the objective function (20) are given by vector $\hat{\mathbf{x}}$ in the following expression:

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{M}'\mathbf{M} & \mathbf{H}' \\ \mathbf{H} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{Mq} \\ \mathbf{y}_a \end{bmatrix}. \tag{23}$$

The heuristic can be summarised as follows:

Step 1: Apply Taylor linearization about the preliminary values[5], i.e. consider function (17) with $\tilde{x}_t = p_t$. The solution of system (19) is labelled as $\hat{\mathbf{x}}^1$.

Step*k*: (with k> 1), Apply Taylor linearization about $\hat{\boldsymbol{x}}^{k-1}$, i.e. $\tilde{\boldsymbol{x}} = \hat{\boldsymbol{x}}^{k-1}$. The new optimal vector is denoted $\hat{\boldsymbol{x}}^k$.

Convergence condition: Apply the same convergence condition considered for the *HGRP* estimates (see Section 5.1).

We will show in Section 6 that the iterated procedure provides good results that closely approximate 'true' *GRP*. The results will be more accurate than that of the *HGRP* procedure explained in Section 5.1.

[5] From our experiments on many time series it turned out that it is not necessary to use a benchmarked series (e.g., Denton PFD) as starting values. Using the preliminary series does not worsen the final solution nor results in more iterations.

There are some clear differences between the two heuristics in this paper. The *HGRP* in Section 5.1 uses Denton *PFD* as point of departure. This is not the case in the Taylor linearization approach, which is applied directly on the preliminary values.

The *HGRP* approach leads to an optimization problem that can be seen as an extension of Denton *PFD*. The Taylor linearization algorithm results in an optimization problem that is rather different from Denton *PFD*.

Example 2

We continue Example 1 and show the results of the first three iterations of the Taylor linearization approach.

Table 3. Results of Example 2.

	M1	**M2**	**M3**	**M4**	**M5**	**M6**	**GRP-criterion**
Preliminary	80.00	100.00	80.00	80.00	100.00	80.00	
Denton PFD	98.41	117.50	84.09	69.76	74.80	55.44	0.0743
TLGRP1st it.	98.68	120.01	81.31	67.72	77.12	55.16	0.0636
TLGRP2nd it.	99.95	121.19	78.86	65.71	76.89	57.40	0.0607
TLGRP3th it.	100.14	121.43	78.43	65.61	76.89	57.49	0.0607

Here, the third iteration gives the best results in terms of GRP-criterion. Notice that the criterion values is better than 0.00609, the optimum value that has been obtained in Table 2.

The TLGRP heuristic can be extended to an algorithm that produces optimal results in all the experiments we performed. The extension consists of a so-called line-search procedure, by analogy with the line search procedure in Newton's optimization algorithm. This means that when it happens that in iteration *k* there is no improvement in the GRP-criterion,

$$f(\boldsymbol{x}) = \sum_{j=1}^{m}\sum_{t=2}^{n}\left(\frac{x_{jt}}{x_{jt-1}} - \frac{p_{jt}}{p_{jt-1}}\right)^2 \quad , \tag{24}$$

we do not stop the iterations, but we search for some α, 0 <α<1, that minimizes $f((1-\alpha)\mathbf{x}_{k-1} + \alpha\mathbf{x}_k)$. The *k*th iteration is performed with $(1-\hat{\alpha})\mathbf{x}_{k-1} + \hat{\alpha}\mathbf{x}_k$ instead of $\mathbf{x}_k$, where $\hat{\alpha}$ denotes the optimum value of α and $\mathbf{x}_k$ is the solution obtained after iteration *k*. In practise, it is not even necessary to find the 'best' value of α, rather it suffices to use some value that is 'close' to it.

We will not consider this extension in the remainder of this paper, because it is not a heuristic. The further development of the approach would be an interesting topic for further research.

6. Empirical results

In this section we apply the Heuristic *GRP* (*HGRP*) and the Taylor linearization approach (*TLGRP*) to real-life test data. In the Subsections 6.1 – 6.4 we consider univariate benchmarking problems. These are data sets with many time-series, without any constraints between them. Thus, each time-series can be benchmarked in isolation. Thereafter, in Subsection 6.5 we consider a multivariate problem, with temporal and contemporaneous constraints.

6.1 Datasets

The first two data sets are obtained from quarterly and annual Dutch Supply and Use Tables. The second data set contains a small selection of time-series from the original, non-aggregated Dutch Supply and Use Tables.[6]

The third and fourth data set come from the EU Quarterly Sector Accounts (EUQSA) and the Canadian Monthly Retail Trade Survey (MRTS); both have been described extensively in Di Fonzo and Marini (2011). All data sets contain flow variables only. Table 4 gives the main characteristics of these data sets.

The last row of Table 4 shows a measure of the temporal discrepancy. This measure compares the low frequency period-to-period changes of the aggregated high-frequency series with the low-frequency benchmarks. Table 4 gives the average of the absolute values of these differences, measured in percent point. The larger this difference, the more the movement of the provisional data needs to be adjusted in order to achieve consistency with the benchmarks.

From Table 4 it follows that this “average discrepancy” is the largest for Data set 2.

Table 4. Characteristics of the test data.

	Data set 1	Data set 2	Data set 3	Data set 4
Number of Low-freq. benchmarks	3 years	3 years	7 years	13 years
Number of High-freq. periods	12 quar-ters	12 quar-ters	28 quar-ters	156 months
Number of time-series	2,252	372	61	227*
Average discrepancy (in% point)	1,84	19,33	2,56	0,72

*= the original data set contains 236 series. Nine series, for which data are not available for all 13 years have been left out.

6.2 Methods

The following methods have been compared:

1. GRP
2. Modified Denton PFD (as explained in Subsection 3.3)
3. HGRP (as explained in Subsection 5.1)
4. TLGRP (as explained in Subsection 5.2)
5. HGRP1: the result after one iteration of HGRP
6. TLGRP1: the result after one iteration of TLGRP

6.3 Criteria

In order to evaluate the performance of the heuristics we use two criteria.

The first is based on the so-called *GRP* gap. The *GRP* gap compares the *GRP* criterion value (as defined in formula (2)) of the *GRP* method with the criterion value of some other method. The

[6] Due to confidentiality we are not allowed to provide any more information on these data sets.

GRP method will by definition lead to the smallest *GRP* criterion value. The *GRP* gap is an indication of how much worse some alternative method preserves the initial growth rates.
The first performance criterion shows the relative reduction in *GRP*-gap, when some heuristic is used instead of Denton *PFD*. That is:

$$\text{Relative Reduction in GRP gap} = \frac{\text{Heuristic}-GRP}{PFD-GRP},$$

where Heuristic, *GRP* and *PFD* stand for the *GRP* criterion value of those methods.
The relative reduction of the *GRP* gap has a clear intuitive interpretation: it is the percentage of the potential improvement over Denton *PFD* that is actually achieved by using a heuristic. A relative reduction of zero means that there is no improvement over Denton *PFD*. On the contrary, a relative reduction of one means that some heuristic preserves the growth rates as good as the optimal *GRP*-method.
The second performance criterion is the relative difference of the *GRP*-criterion of some heuristic, compared to the optimum value:

$$\text{Relative Difference GRP criterion (RD)} = \frac{\text{Heuristic}-\text{GRP}}{\text{GRP}}$$

We adopt the same standards as in Di Fonzo and Marini (2012). According to this standard, a heuristic result is considered:

- Best: if 0 ≤ RD ≤ 0,0001 (i.e. within 0.01% of the optimal solution)
- Very accurate: if 0< RD ≤ 0,001 (i.e. within 0.1% of the optimal solution)
- Accurate: if 0< RD ≤ 0,01 (i.e. within 1% of the optimal solution)
- Acceptable: if 0 < RD ≤ 0.1 (i.e. within 10 % of the optimal solution)
- Bad: if RD > 0.1 (i.e. not within 10% of the optimal solution).

Note that the outcome of the heuristic is either “acceptable” or “bad”. All “accurate” results are “acceptable” as well, but the opposite is not necessarily true.

6.4 Results

The Tables 5a-b show the minimum, the 10 percent percentile and the median of the relative reduction of the GRP gap, taken over all time-series in the data set.

Table 5a. Minimum and median of the relative reduction GRP-gap, over all time-series

	Minimum				Median			
	HGRP	**TLGRP**	**HGRP1**	**TLGRP1**	**HGRP**	**TLGRP**	**HGRP1**	**TLGRP1**
Data set 1	61.50	99.45	61.50	-49.37	99.97	100.00	99.96	99.92
Data set 2	84.07	99.62	79.10	-69.83	99.90	100.00	99.87	99.67
Data set 3	0.00	8.85	0.00	-55.29	99.99	100.00	99.98	99.91
Data set 4	0.00	-176.93	0.00	-312.99	99.95	100.00	99.95	99.79

Table 5b. 10%-Percentile of the relative reduction GRP-gap, over all time-series

	10%-percentile			
	HGRP	**TLGRP**	**HGRP1**	**TLGRP1**
Data set 1	99.10	100.00	98.99	98.12
Data set 2	98.14	100.00	97.63	91.72
Data set 3	97.61	98.73	95.18	92.51
Data set 4	99.43	99.29	99.39	98.74

It can be concluded that *TLGRP* and *HGRP* perform very well; in more than 90 percent of the cases the distance with respect to the optimal solution is more than 90 percent smaller when compared with Denton *PFD*. Even after one iteration the initial growth-rates are much better preserved (following from *HGRP1* and *TLGRP1* in Table 5a-b). From this it follows that the improvement in the first iteration is much larger than the improvement in all further iterations.
It can also be seen in Table 5a-b that the Taylor linearization approach (*TLGRP*) performs better than *HGRP*. In fact, *TLGRP* leads to the optimal solution for the majority of case.
In some few cases, in Data sets 3 and 4, *HGRP* does not lead to a better *GRP* criterion than modified Denton *PFD*. *TLGRP* can even lead to a worse result than Denton *PFD*, which can be seen from the negative *GRP* gap reduction.
The results of our second performance indicator are shown in Table 6. This indicator is based on the relative difference of the Heuristic's result from the optimal growth rate preservation. The relative differences are classified into five categories, from "best" to "bad", as explained in Subsection 6.3. Table 6 shows the percentages of the time-series in each category.

Table 6.Percentage of time-series in 5 categories of accuracy

Data set 1	**Best**	**Very Accurate**	**Accurate**	**Acceptable**	**Bad**
Denton PFD	0.10	0.34	8.84	57.83	42.17
HGRP	45.61	91.31	99.85	100.00	0.00
TLGRP	99.80	100.00	100.00	100.00	0.00
HGRP1	44.62	89.64	99.66	100.00	0.00
TLGRP1	56.31	83.85	98.77	99.95	0.05
Data set 2					
Denton PFD	0.00	0.00	8.87	52.15	47.85
HGRP	38.17	91.40	99.46	100.00	0.00
TLGRP	100.00	100.00	100.00	100.00	0.00
HGRP1	33.33	85.48	98.12	99.46	0.54
TLGRP1	30.65	70.97	95.70	99.46	0.54
Data set 3					
Denton PFD	0.00	4.92	24.59	57.38	42.62
HGRP	68.85	77.05	96.72	100.00	0.00
TLGRP	86.89	93.44	98.36	100.00	0.00
HGRP1	63.93	75.41	88.52	100.00	0.00
TLGRP1	55.74	72.13	86.89	98.36	1.64
Data set 4					
Denton PFD	0.00	0.00	39.65	87.22	12.78
HGRP	87.67	97.36	99.56	100.00	0.00
TLGRP	91.63	95.15	98.24	100.00	0.00
HGRP1	85.90	97.36	99.56	99.56	0.44
TLGRP1	75.77	93.39	98.24	100.00	0.00

The percentage of the cases for which Denton *PFD* yields "acceptable" growth rate preservation varies from 52 percent in dataset 2 to 87 percent in dataset 4. Considering the "Dutch" Supply and Use tables, it follows that Denton *PFD* leads to "acceptable" Growth rates Preservation for about 50-60 percent of the cases, but "Very accurate" results hardly occur. These findings are more or less in line with other studies.

Table 6 shows that the initial discrepancies are the smallest for dataset 4. This is also the data set for which Denton performs best. In general, the results show that Denton PFD performs best when the initial temporal discrepancies are small.

HGRP clearly leads to better growth rate preservation than Denton *PFD*. In at least 98% of all time-series in all data sets "acceptable" growth rate preservation has been achieved. TLGRP in turn outperforms *HGRP*: in more than 98% of the cases "accurate" results have been obtained in all data sets.

Finally, Table 7a and 7b show the distribution of the number of iterations for the HGRP and TLGRP heuristics. For most of the cases one or two iterations are enough.

On average, *TLGRP* requires more iterations than *HGRP*. However, note that *HGRP* starts with a Denton *PFD* reconciliation, which is not counted as an iteration, while *TLGRP* does not start with reconciled data.

Table 7a. Distribution of the number of iterations required for HGRP (in%)

	Data set			
	1	2	3	4
1 iteration	37.02	25.27	40.98	56.39
2 iterations	61.71	72.58	52.46	43.61
3 iterations	1.23	1.61	6.56	0.00
4 iterations	0.05	0.54	0.00	0.00
5 iterations	0.00	0.00	0.00	0.00
>5 iterations	0.00	0.00	0.00	0.00

Table 7b. Distribution of the number of iterations required for TLGRP (in%)

	Data set			
	1	2	3	4
1 iteration	24.35	11.83	11.48	21.15
2 iterations	70.30	68.01	77.05	78.85
3 iterations	5.20	19.35	11.48	0.00
4 iterations	0.15	0.27	0.00	0.00
5 iterations	0.00	0.54	0.00	0.00
>5 iterations	0.00	0.00	0.00	0.00

6.5 Multivariate problem

In this subsection we present the results of one large, multivariate benchmarking problem.

This problem is derived from the aggregated Dutch Supply and Use Tables. The data set contains 4,273 time-series of which data are available for 12 quarters and 3 years.

Each quarter there are 140 contemporaneous constraints. An example of such a constraint is that the total use of some good has to be the same as the total supply. Each time-series is involved with at least one constraint, most time-series are included in two constraints.

As mentioned in Bikker *et al.* (2013) an additive benchmarking model has to be preferred for time-series with positive as well as negative values and time-series with values close to zero. For this reason, a multiplicative benchmarking method can only be applied properly to a selection of 2,252 time-series: the time-series that are included in dataset 1 in Section 6.4.

A multivariate benchmarking method has been applied, as described by Bikker *et al.* (2013). In this model additive and multiplicative model types are combined. That is: for some of the time-series a multiplicative model is applied, i.e. Denton *PFD*, while for other time-series an additive model is chosen, i.e. Denton Additive First Differences. This all happens within one mathematical model.

The two heuristics *HGRP* and *TLGRP* can only be used for time-series to which a multiplicative benchmarking model can be properly applied. This means that we compare two kind of multivariate models: (*i*) a multivariate model that combines additive Denton and proportional Denton; (*ii*) a multivariate model that combines additive Denton with one of the heuristic procedures.

Table 8 shows the results of our first performance indicator: the relative reduction of the *GRP* gap. This measure is based on the aggregated *GRP* gap of the 2,252 series to which a multiplicative model has to be applied.

Table 8. Percentage reduction of the total GRP-gap

	HGRP	***TLGRP***	***HGRP1***	***TLGRP1***
Reduction of GRP-gap	98.41	100.00	98.41	82.49

It can be seen that the Taylor linearisation approach (*TLGRP*) leads to the optimal growth rate preservation in this case, while *HGRP* is close to optimal.

The second performance indicator is the relative difference of the *GRP*-criterion. The results of this indicator are displayed in Table 9.

Table 9. Relative difference from optimal GRP

	Denton				
	PFD	***TLGRP***	***HGRP***	***HGRP1***	***TLGRP1***
Relative difference from optimal GRP-criterion (in %)	33,58%	0,00%	0,53%	0,53%	5.88%
Judgement	Bad	Best	Accurate	Accurate	Acceptable

Finally, the total number of iterations is 1 for *HGRP* and 4 for *TLGRP*.

The most important result of this subsection is that it shows that the heuristics can be applied in large-scale multivariate benchmarking problems that occur in practise.

7. Conclusions

Two well-known multiplicative benchmarking methods are Denton Proportionate First Differences (*PFD*) and Growth Rates Preservation (*GRP*). The first method is technically easier to apply, while the latter is often considered the ideal method.
In the literature, it is known that Denton PFD often closely approximates GRP, but in some cases the approximation is not very accurate. An empirical application on Dutch Supply and Use Tables are in accordance with these findings.
In this paper two new heuristics have been developed that are easier to apply than GRP and better preserve the temporal dynamics of the preliminary series than Denton *PFD*. These heuristics are iterative procedures that can be applied to univariate as well as multivariate problems. One of the heuristics adds weighting factors to the objective function of Denton *PFD*, the other heuristic repeatedly applies first order Taylor linearization of the *GRP* objective function.
An important benefit of both heuristics is easiness: they are based on standard quadratic-linear problems. For large, practical problems, we can use very simple programs, avoiding the use of sophisticated, and black-box software and still obtain close to optimal results within a small number of iterations.
From empirical application it follows that both heuristics considerably better preserve the movements of the initial time-series than Denton PFD. The Taylor linearization heuristic outperforms the weighted Denton PFD approach. HGRP is a good and simple heuristic, but it is not a strong competitor of 'true' GRP, as instead TLGRP is.

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Appendix A. Augmented Denton PFD

The 'Augmented Modified Denton PFD' benchmarking procedure is grounded on an objective function seemingly different from, but equivalent to, the one of the classical 'Modified Denton PFD' benchmarking procedure.

A.1 The univariate case

Where the classical, univariate Denton PFD problem is given by

$$\min_{x_t} f(\mathbf{x}) = \sum_{t=2}^{n} \left(\frac{x_t}{p_t} - \frac{x_{t-1}}{p_{t-1}}\right)^2, \quad \text{subject to } \mathbf{Ax} = \mathbf{b} \tag{A.1}$$

the 'Augmented Modified Denton PFD' benchmarking procedure considers the following constrained minimization problem

$$\min_{x_t} f(\mathbf{x}) = \sum_{t=2}^{n} \left(\frac{x_t}{p_t} - \frac{x_{t-1}}{p_{t-1}}\right)^2 + (\mathbf{a}_{1.}^T\mathbf{x} - b_1)^2, \quad \text{subject to } \mathbf{Ax} = \mathbf{b}. \tag{A.2}$$

What has been done was adding the term $(\mathbf{a}_{1.}^T\mathbf{x} - b_1)^2$ to the objective function, i.e. the squared difference between the left hand side and the right hand side of the first temporal aggregation constraint. As it is immediately recognized, due to the imposed temporal constraints, this term does not give any contribution to the original function, i.e. its value will be zero in the optimal solution.

Nevertheless, as we shall see, its use results in a system whose solution requires the inversion of an (n x n) matrix, instead of (n+N x n+N) as in the expression (7) for the standard Modified Denton *PFD* benchmarking procedure.

In matrix form the benchmarking according to the Augmented Denton *PFD* procedure can be expressed as the following quadratic-linear constrained minimization problem:

$$\min_{\mathbf{x}} (\mathbf{Qx} - \mathbf{h})'(\mathbf{Qx} - \mathbf{h}), \text{subject to} \mathbf{Ax} = \mathbf{b}, \tag{A.3}$$

where $\mathbf{Q} = \begin{bmatrix} \Delta_\mathbf{n}\mathbf{P}^{-1} \\ \mathbf{a}_1^{\mathrm{T}} \end{bmatrix}$is a (n × n) matrix of rank n, and $\mathbf{h} = [\mathbf{0}'_{\boldsymbol{n-1}} \quad b_1]$is a (n × 1) vector.

Now, let's define $\mathbf{y} = \mathbf{Qx}$ and $\mathbf{B} = \mathbf{AQ}^{-1}$. The above constrained minimization problem can be written as

$$\min_{\mathbf{y}} (\mathbf{y} - \mathbf{h})'(\mathbf{y} - \mathbf{h}), \text{subject to } \mathbf{By} = \mathbf{b},$$

whose solution is

$$\hat{\mathbf{y}} = \mathbf{h} + \mathbf{B}'(\mathbf{BB}')^{-1}(\mathbf{b} - \mathbf{Bh})$$

from which we derive the benchmarked values

$$\hat{\mathbf{x}} = \mathbf{Q}^{-1}\hat{\mathbf{y}} \tag{A.4}$$

A.2 The multivariate case

The application of the augmented modified Denton *PFD* to the multivariate case is similar to the univariate case. The multivariate Denton *PFD* problem is given by

$$\min_{x_{jt}} f(\mathbf{x}) = \sum_{j=1}^{m}\sum_{t=2}^{n}\left(\frac{x_{jt}}{p_{jt}} - \frac{x_{jt-1}}{p_{jt-1}}\right)^2,$$
$$\text{subject to } \mathbf{Hx} = \mathbf{y}_{\mathrm{a}}.$$

Its augmented form is:

$$\min_{x_{jt}} f(\mathbf{x}) = \sum_{i=1}^{m}\sum_{t=2}^{n}\left(\frac{x_{jt}}{p_{jt}} - \frac{x_{jt-1}}{p_{jt-1}}\right)^2 + (\mathbf{H}^{*}\mathbf{x} - \mathbf{y}_{\boldsymbol{a}}^{*})^2,$$
$$\text{subject to } \mathbf{Hx} = \mathbf{y}_{a},$$

where $\mathbf{H}^{*}$and $\mathbf{y}_{a}^{*}$ are a subset of *m* rows from $\mathbf{H}$ and $\mathbf{y}_{a}$ that belong to the first temporal aggregation constraint of each time-series.

Let's define $\mathbf{p}_{\mathrm{j}}$ as the (*n* x 1) vector of preliminary values of time series *j* and $\mathbf{P}_j = \mathrm{diag}(\mathbf{p}_j)$. Further, the first temporal aggregation constraint of time-series *j* will be denoted $(\mathbf{h}_j^{*})'\mathbf{x} = (\mathbf{y}_a)_j^{*}$.

In analogy with the univariate case in Appendix A.1, the problem can be written as

$$\min_{\mathbf{x}} (\mathbf{Qx} - \mathbf{h})'(\mathbf{Qx} - \mathbf{h}), \text{subject to } \mathbf{Ax} = \mathbf{b},$$

where $\mathbf{Q} = \mathrm{diag}(\mathbf{Q}_j)$,$\mathrm{h} = [\mathbf{h}_1', \cdots, \mathbf{h}_m']'$, $\mathbf{A} = \mathbf{H}$ and $\mathbf{b} = \mathbf{y}_a$, with $\mathbf{Q}_j = \begin{bmatrix} \Delta_n \mathbf{P}_j^{-1} \\ (\mathbf{h}_j^{*})' \end{bmatrix}$and $\mathbf{h}_j = [\mathbf{0}_{n-1}' \quad (y_a)_j^{*}]$. Thus, the reconciled values can be obtained from (A.4).

However, note, that this is only possible if (A.4) can be computed. This is the case if $(\mathbf{BB}')^{-1}$exists. If some of the constraints are linearly dependent than the inverse of $\mathbf{BB}'$ cannot be computed. This problem can be solved by an appropriate factorization of the coefficient matrix $\mathbf{A}$.
It follows that the computation of the benchmarked requires the computation of the inverse of **Q**, a (*mn* x *mn*) matrix. Notice that the computation can be simplified because **Q** is a block-diagonal matrix. It consists of *m* square blocks, each of which of dimension *n*. Each block can be inverted independently of the other blocks. Basically, this means that *m* times an (*n* x *n*) matrix needs to be inverted.