THE ULTIMATE ENERGY INPUT-OUTPUT MODEL

Zeus Guevara[a,b] *, João F. D. Rodrigues[c], Tiago Domingos[b]

[a] Center for Sustainable Energy Systems, Universidade de Lisboa, C8 building, Campo Grande, 1749–016 Lisbon, Portugal.

[b] MARETEC, Department of Mechanical Engineering, Instituto Superior Técnico, Universidade de Lisboa, Avenida Rovisco Pais, 1, 1049-001 Lisbon, Portugal.

[c] Institute of Environmental Sciences, Leiden University, P.O.Box 9518, 2300 RA Leiden, The Netherlands

* Corresponding author: zeus.guevara@tecnico.ulisboa.pt

Prepared for the 23rd International Input Output Conference in Mexico City

Abstract

Conventional energy input-output models were developed about 40 years ago and have not been significantly improved since. These conventional models offer a limited description of energy flows in the economy. This paper introduces a novel energy input-output model, the primary-to-final energy input-output model (PF-EIO model), which is equivalent to the "standard" hybrid-unit model and can also replicate the form of the direct impact coefficient model. However it provides a better description of energy flows according to the processes of energy conversion and the levels of energy use in the economy. The model characterizes the vector of total energy output as a function of eight variables, including two energy efficiency indicators. Furthermore, the PF-EIO model enables additional improvements in the description of energy flows, which leads to the development of the primary-to-useful energy input-output model, which is the first ever EIO model to include useful energy flows. The proposed models are especially suitable to evaluate energy decoupling, energy use and energy efficiency trends in the economy. Finally, as the PF-EIO model is undeniably superior to conventional models, it should be considered the standard in energy input-output analysis.

Keywords: energy input-output analysis; embodied energy; energy efficiency

Notation

A bold lower case letter ($\boldsymbol{x}$) corresponds to a vector. Bold Greek and capital letter ($\boldsymbol{Z}$) describes a matrix or sub-matrices. Non-bold Latin and Greek letters (x_i, ψ and z_{ij}) represent matrix entries, vector elements, scalars and indexes.

A vector with a hat ($\hat{\boldsymbol{x}}$) represents a diagonal matrix, whose diagonal elements are the elements of vector $\boldsymbol{x}$. An apostrophe on a vector or matrix ($\boldsymbol{x}'$ and $\boldsymbol{Z}'$) denotes the vector or matrix transpose.

$\boldsymbol{i}$ is a vector of ones of a consistent length (or summation vector). $\boldsymbol{o}$ and $\boldsymbol{O}$ are a vector and matrix of zeros.

The superscript $*$ on a variable denotes a hybrid-unit version of that variable. The superscripts E and U correspond to elements of the energy sector and the adjunct energy sector, respectively.

1 INTRODUCTION

Any economic system "could not exist without large and incessant flows of energy" [1] as energy is the driving force of any activity and process in nature [2]. It is obvious then that when there is no energy, there will not be any economic activity; and when there is energy, it is possible to carry out economic activities. However, the relationship between energy use and the output of these activities is not simple [3, 4]. Furthermore, the analysis of energy use by an economy is also important because energy use has significant environmental impacts (>85% of global CO_2 emissions, IPCC [5]).

Because of the simplicity of the representation of the economic system [6, 7], the input-output method [8, 9] represent a propitious theoretical framework to model the relationships between energy use and economic activities. This method has been extensively used for environmental applications (including energy analysis) in recent decades [10-12].

Energy input-output analysis (EIO analysis) was developed around the late 1960's and the 1970's to account for the energy flows in the economy [8, 13, 14]. The EIO technique has relied on two conventional models (the hybrid-unit and the direct impact coefficient models), which offer a limited and undetailed representation of energy flows in the economy. In addition, these models have not been significantly improved in the last four decades.

The present paper introduces a novel EIO model that significantly improve the analysis of energy flows in the economy, the primary-to-final energy input-output model (PF-EIO model). The proposed model has a detailed representation of energy flows according to the processes of energy conversion and use in the economy. It is based on the pioneering work of Guevara et al. [15], who built a model of primary energy use in Portugal for decomposition analysis. In addition, the PF-EIO model is expanded to also include the consumption of useful energy, never before included in EIO analysis.

The paper is structured as follows: Section 2 presents a description and discussion of the conventional EIO analysis. The proposed EIO model and its expansion are introduced in Section 3. In Section 4, the advantages and issues of the proposed models are discussed. Finally, Section 5 present the general conclusions of the work.

2 Theoretical background: Energy input-output analysis

Energy input-output analysis (EIO) is designed to account for the energy flows in the economy [8, 13, 14]. It is normally used for the following applications: 1) direct and indirect energy requirements of the economy, i.e. Net energy analysis [16]; 2) the energy cost of goods and services for final demand [17, 18]; 3) the effect of alternative energy conversion technologies [8, 19]; and 4) changes in energy use through structural decomposition [20].

2.1 Conventional EIO models

There are two conventional energy input-output models: the hybrid-unit EIO model (HEIO model) and the direct impact coefficient EIO model (DIC-EIO model).

2.1.1 The hybrid-unit energy input-output model

The HEIO model was originally developed by Bullard and Herendeen [14] and Bullard and Herendeen [21][1] based on the conservation of embodied energy, which establishes that energy embodied in the output of an industry is equal to the energy embodied in input products plus any external energy input to this industry. This model is considered the standard for EIO analysis [8, 13, 23, 24].

The HEIO model starts from the basic input-output identity, i.e. $\boldsymbol{x} = \boldsymbol{Zi} + \boldsymbol{f}$. The n industries, represented in $\boldsymbol{Z}$, consist of r energy industries and $n - r$ non-energy industries (for convenience, energy industries are placed first in the index of industries).

It is possible to determine a similar identity for energy flows in physical units, as

$$\boldsymbol{g} = \boldsymbol{Ei} + \boldsymbol{h} \qquad [2.1]$$

where vector $\boldsymbol{g}$ is the total energy use (i.e. output of energy industries); matrix $\boldsymbol{E}$ represents the energy flows from energy industries to all producing industries (energy and non-energy); and vector $\boldsymbol{h}$ represent the energy deliveries to final demand.

The monetary-unit rows of energy industries in $\boldsymbol{Z}$, $\boldsymbol{f}$ and $\boldsymbol{x}$ are substituted by the corresponding physical-unit rows of $\boldsymbol{E}$, $\boldsymbol{h}$ and $\boldsymbol{g}$, respectively (Figure 2-1), in order to construct a system of hybrid-units:

$$\boldsymbol{x}^* = \boldsymbol{Z}^*\boldsymbol{i} + \boldsymbol{f}^*$$

where $\boldsymbol{x}^*$ is the hybrid-unit vector of total industrial output, $\boldsymbol{Z}^*$ is the hybrid-unit matrix of interindustry flows; and $\boldsymbol{f}^*$ is the hybrid-unit vector of final demand.

		Industries						Final demand	Total output
		Energy			Non-energy				
		1		r	$r+1$	$\cdots$	n		
Energy	1	e_{11}	$\cdots$	e_{1r}	$e_{1(r+1)}$	$\cdots$	e_{1n}	h_1	g_1
		$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
	r	e_{r1}	$\cdots$	e_{rr}	$e_{r(r+1)}$	$\cdots$	e_{rn}	h_r	g_r
Non-energy	$r+1$	$z_{(r+1)1}$	$\cdots$	$z_{(r+1)r}$	$z_{(r+1)}$	$\cdots$	$z_{(r+1)n}$	$f_{(r+1)}$	$x_{(r+1)}$
	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
	n	z_{n1}	$\cdots$	z_{nr}	$z_{n(r+1)}$	$\cdots$	z_{nn}	f_n	x_n
					$\boldsymbol{Z}^*$			$\boldsymbol{f}^*$	$\boldsymbol{x}^*$

Rules of substitution			
Rows	$\boldsymbol{Z}^*$	$\boldsymbol{f}^*$	$\boldsymbol{x}^*$
$1 \le i \le r$	$z_{ij}^* = e_{ij}$	$f_i^* = h_i$	$x_i^* = g_i$
$r < i \le n$	$z_{ij}^* = z_{ij}$	$f_i^* = f_i$	$x_i^* = x_i$

Figure 2-1 Graphical representation of the hybrid-unit identity. Note: In hybrid-unit systems the column sums of $\boldsymbol{Z}^*$ are meaningless.

[1] The work of Bullard and Herendeen [14] was contemporary to other attempts to build a consistent energy input-output model by for example Krenz [18], Wright [17] and Wright [22].

The model is then solved for $\boldsymbol{x}^*$ through the hybrid-unit technical coefficient matrix, $\boldsymbol{A}^* = \boldsymbol{Z}^*\widehat{\boldsymbol{x}^*}^{-1}$.

$$\boldsymbol{x}^* = (\boldsymbol{I} - \boldsymbol{A}^*)^{-1}\boldsymbol{f}^* = \boldsymbol{L}^*\boldsymbol{f}^* \qquad [2.2]$$

where $\boldsymbol{A}^*$ and $\boldsymbol{L}^*$ are the hybrid-unit versions of matrices $\boldsymbol{A}$ and $\boldsymbol{L}$ of he basic input-output model.

Eq. [2.2] is a hybrid-unit version of the basic input-output model. However, the objective of energy input-output analysis is to have a total energy requirements matrix $\boldsymbol{\alpha}$ that enables the calculation of the energy requirements to meet final demand, i.e. the matrix that solves the equation:

$$\boldsymbol{g} = \boldsymbol{\alpha}\boldsymbol{f}^*$$

As seen in Figure 2-1, $x_i^* = g_i$ for $1 \leq i \leq r$, so it is possible to establish a simple relationship, $\boldsymbol{g} = \boldsymbol{K}\boldsymbol{x}^*$, where $\boldsymbol{K}$ is a bridge matrix of size $r \times n$ with entries $K_{ij} = 1$ for $i = j$ (i.e. energy industries) and $K_{ij} = 0$ for $i \neq j$. $\boldsymbol{K}$ extracts the elements of $\boldsymbol{x}^*$ that correspond to the output of energy industries.

Consequently, the total energy requirements matrix ($\boldsymbol{\alpha}$) can be calculated as[2]

$$\boldsymbol{\alpha} = \boldsymbol{K}(\boldsymbol{I} - \boldsymbol{A}^*)^{-1} = \boldsymbol{K}\boldsymbol{L}^*$$

By partitioning $\boldsymbol{\alpha}$, it is possible to separate total energy output in two components: one caused by final demand of energy and the other by final demand of non-energy products, so

$$\boldsymbol{g} = [\boldsymbol{\alpha}_\theta \quad \boldsymbol{\alpha}_\tau]\begin{bmatrix}\boldsymbol{h}\\ \boldsymbol{f}_n\end{bmatrix}$$

or

$$\boldsymbol{g} = \boldsymbol{\alpha}_\theta\boldsymbol{h} + \boldsymbol{\alpha}_\tau\boldsymbol{f}_n \qquad [2.3]$$

To further describe the structure of the sub-matrices of the total energy requirements matrix ($\boldsymbol{\alpha}$), the structure of the hybrid-unit Leontief inverse matrix ($\boldsymbol{L}^*$) should be discussed first.

$\boldsymbol{L}^*$ is composed by four sub-matrices, see Bullard and Herendeen [14] and Casler and Wilbur [13]:

$$\boldsymbol{L}^* = \begin{bmatrix}\boldsymbol{L}_\theta^* & \boldsymbol{L}_\tau^*\\ \boldsymbol{L}_\pi^* & \boldsymbol{L}_\psi^*\end{bmatrix}$$

where

- $\boldsymbol{L}_\theta^*$ is the sub-matrix of energy transactions between energy industries per unit of final energy demand.
- $\boldsymbol{L}_\tau^*$ is the sub-matrix of direct energy use by non-energy industries per unit of final non-energy demand.
- $\boldsymbol{L}_\pi^*$ is the sub-matrix of non-energy transactions from non-energy to energy industries per unit of final energy demand.
- $\boldsymbol{L}_\psi^*$ is the sub-matrix of Interindustry transactions between non-energy industries per unit of final demand.

---

[2] This is a pragmatic and intuitive approach to matrix $\boldsymbol{\alpha}$. Bullard and Herendeen [14] obtained an equivalent expression by solving the embodied energy conservation equation $\alpha_{kj}x_j = \sum \alpha_{ki}x_i + g_{kj}$. This equation states that the energy of type k embodied in the output of industry j ($\alpha_{kj}x_j$) is equal to the energy embodied in the inputs of this industry ($\alpha_{ki}x_i$) plus any exogenous energy input (g_{kj}).

As the total energy requirements matrix of the hybrid-unit model ($\boldsymbol{\alpha}$) corresponds to the rows of energy industries of the $\boldsymbol{L}^*$. This indicates that $\boldsymbol{\alpha}$ is only composed by the sub-matrices $\boldsymbol{L}_\theta^*$ and $\boldsymbol{L}_\tau^*$, i.e.

$$[\boldsymbol{\alpha}_\theta \quad \boldsymbol{\alpha}_\tau] = [\boldsymbol{L}_\theta^* \quad \boldsymbol{L}_\tau^*]$$

Consequently, non-energy transactions in the economy in $\boldsymbol{L}_\pi^*$ and $\boldsymbol{L}_\psi^*$ are indirectly accounted for by $\boldsymbol{\alpha}_\theta$ and $\boldsymbol{\alpha}_\tau$.

In conclusion, $\boldsymbol{\alpha}$ accounts for every transaction in the economy. However, it is not able to separate the effect of energy (i.e. conversion efficiency or direct energy intensity) from non-energy transactions, i.e. both energy and non-energy production processes are aggregated together.

2.1.2 The direct impact coefficient energy input-output model

The DIC-EIO model was developed in the 1960's [8, 25]. It is simpler and requires less detailed data than the HEIO model. Because of this, it is still widely supported and used [24, 26-28] even though it is less consistent than the HEIO model, see Miller and Blair [8].

The model starts by calculating a direct energy intensity matrix ($\boldsymbol{T}$),

$$\boldsymbol{T} = \boldsymbol{E}\hat{\boldsymbol{x}}^{-1} \qquad [2.4]$$

whose elements t_{kj} correspond to the direct use of energy of type k ($1 \leq k \leq r$) to produce a monetary unit of output of industry j ($1 \leq j \leq n$).

Therefore, from Eq. [2.1] and the input-output relationship (i.e. $\boldsymbol{x} = \boldsymbol{Lf}$)

$$\boldsymbol{g} - \boldsymbol{h} = \boldsymbol{Tx} = \boldsymbol{TLf}$$

where matrix $\boldsymbol{TL}$ is the total interindustry energy coefficient matrix.

To account for the energy deliveries to final demand $\boldsymbol{h}$, Miller and Blair [8] (similar to Cruz [26] and Proops [28]) proposed a relationship with final demand as: $\boldsymbol{h} = \widetilde{\boldsymbol{Q}}\boldsymbol{f}$ where $\widetilde{\boldsymbol{Q}}$ is the matrix of inverse prices of energy of type k to final demand (p_{kj}), whose element $\tilde{q}_{kj} = 1/\ p_{kj}$ for $i = j$ (i.e. energy carriers) and $\tilde{q}_{kj} = 0$ for $i \neq j$.

Total energy use is therefore obtained as

$$\boldsymbol{g} = \left(\boldsymbol{TL} + \widetilde{\boldsymbol{Q}}\right)\boldsymbol{f} = \boldsymbol{\beta f}$$

where $\boldsymbol{\beta}$ is the matrix of total energy requirements (analogous to $\boldsymbol{\alpha}$) obtained through the direct impact coefficient approach.

The total energy requirements matrix ($\boldsymbol{\beta}$) combines two variables that are conceptually and technically different: direct and indirect energy embodied in final demand in $\boldsymbol{TL}$ and energy prices $\widetilde{\boldsymbol{Q}}$. Therefore, in the DIC-EIO model, it is better to analyze separately the production-related energy use (through $\boldsymbol{TL}$) and the energy deliveries to final demand (through $\widetilde{\boldsymbol{Q}}$ or alternative variables) as done, for example, by Cruz [26], Guevara et al. [15], Wachsmann et al. [29] and Chai et al. [30].

2.1.3 Other models

Kagawa and Inamura [31], extending the wok of Lin and Polenske [32], developed a model that decomposes non-energy and energy transactions in the economy by applying to the conventional HEIO model a hierarchical system with feedback loops from non-energy sectors. This model successfully accounts for non-energy and energy transactions, separately. However it cannot provide detail to energy transactions (e.g. levels of energy use, conversion efficiency or energy intensity).

Liang et al. [33] proposed a model with the aim of improving the economy-wide HEIO model. Their model consists of an input-output model with energy demand by non-energy industries and final consumers as final energy demand. This model is equivalent to the input-output model of the energy sector in Section 3.1.1 hence it cannot be considered as an energy input-output model for the whole economy.

2.2 Main conceptual issues of conventional EIO analysis

Issue 1: Constant returns to scale and fixed technical coefficients: Energy input-output analysis shares the analytical problems of standard input-output analysis [13]. The most significant are the assumption of constant return to scale and fixed technical coefficients, which equate the average to the marginal energy intensities [14].This is not true for most producing sectors because there is a share of direct energy use that is independent from production (e.g. lighting and powering security systems) [34].

Issue 2: Accounting for primary, final or useful energy: The HEIO and the DIC-EIO models do not clearly represent the primary, final, useful and service levels of energy use, see Appendix A and Schmidt (2008). These models only account for either the primary or the final level (the selection between them depends on the aim of the study).

Issue 3: Energy conservation principle: Miller and Blair [8] prove that the HEIO model complies with the energy conservation principle while this is not always true for the DIC-EIO model.

Issue 4: Energy conversion and energy efficiency: The total energy requirement matrices ($\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$) are not able to provide detailed information about energy conversion processes in the economy, separately (i.e. primary-to-final, final-to-useful and useful-to-service, see Appendix A). These matrices include in aggregated form all conversion efficiencies and the efficiency of energy use i.e. most indicators of energy performance in the economy cannot be isolated. Consequently, they oversimplify the mechanisms through which energy is used in the economy.

Issue 5: Representation of the energy sector in input-output data: The quality of analysis of energy flows through the input-output greatly depends on the detail of the representation of the industries in the energy sector. In this respect, the energy sector in most available input-output databases is highly aggregated. For example, the 1995-2010 make and use EUROSTAT tables for EU countries [35] include only two energy products: 1) Electricity, gas, steam and hot water; and 2) Coke and refined petroleum products. Furthermore for economies that import most of their primary energy sources, there is no clear representation of primary energy industries in input-output data.

3 Methodology

The present section introduces the primary-to-final energy input-output model (PF-EIO model) and the primary-to-useful energy input-output model (PU-EIO model), as improvements to conventional models. The proposed models are able to account in detail for the processes of energy conversion and direct / indirect energy use to meet final demand in the economy.

3.1 The primary-to-final energy input-output model

The section describes the development of the PF-EIO model. The model starts with an independent input-output model of the energy sector in physical units, which is then coupled to the input-output model of the rest of the economy in monetary units.

3.1.1 An input-output model of the energy sector

An isolated input-output model of the energy sector is built. In this model, the output of energy industries (e.g. cogeneration plants, oil refineries or geothermal power stations) is used as inputs for conversion processes (i.e. energy production processes) by energy industries and as energy products for direct energy demand (i.e. intermediate demand for energy by non-energy industries and final energy demand).

The model develops from an expression of interindustry flows for the energy sector in physical units, analogous to the basic input-output relationship, i.e.

$$\boldsymbol{x}^E = \boldsymbol{Z}^E \boldsymbol{i} + \boldsymbol{F}^E \boldsymbol{i} \qquad [3.1]$$

where $\boldsymbol{x}^E$ is the vector of total energy output, $\boldsymbol{Z}^E$ is the matrix of transactions between energy industries (inter-energy-industry transactions) and $\boldsymbol{F}^E$ is the matrix of direct energy demand. The system consists of n industries, of which r are energy industries and $n - r$ are non-energy industries, and a sector of final demand.

Eq. [3.1]describes the energy flows in the economy, hence it is equivalent to the equation of energy flows ($\boldsymbol{g} = \boldsymbol{E}\boldsymbol{i} + \boldsymbol{h}$, Eq. [2.1]) of the conventional energy input-output analysis. The correspondences between the elements of Eqs. [3.1] and [2.1] are the following: Vector $\boldsymbol{g}$ is equivalent to $\boldsymbol{x}^E$. Moreover, if the matrix of interindustry energy transactions ($\boldsymbol{E}$) is partitioned by energy and non-energy industries, i.e. $\boldsymbol{E} = [\boldsymbol{E_e} \quad \boldsymbol{E_n}]$ then $\boldsymbol{Z}^E$ is equivalent to $\boldsymbol{E_e}$ and $\boldsymbol{F}^E$ is composed by the sub-matrix $\boldsymbol{E_n}$ and the vector of energy deliveries to final demand ($\boldsymbol{h}$), i.e. $\boldsymbol{F}^E = [\boldsymbol{E_n} \quad \boldsymbol{h}]$. Figure 3-1 exemplifies these correspondences for an economy with three energy industries ($r = 3$).

	Energy industries			Direct energy demand: Intermediate demand (non-energy industries)				Direct energy demand: Final demand	Total output
Energy industries	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	$\cdots$	e_{1n}	h_1	g_1
	e_{21}	e_{22}	e_{23}	e_{24}	e_{25}	$\cdots$	e_{2n}	h_2	g_2
	e_{31}	e_{32}	e_{33}	e_{34}	e_{35}	$\cdots$	e_{3n}	h_3	g_3
	$\boldsymbol{Z}^E$			$\boldsymbol{F}^E$					$\boldsymbol{x}^E$

Figure 3-1 Input-output table of interindustry and final flows of energy goods.

Analogously, it is assumed that the amount of energy inputs from energy industry i to the energy industry j will increase proportionally with the total output of energy industry J[3], so:

[3] For this, it is assumed that the energy production function of energy industry j is of perfect complements (i.e. there is no substitution between energy inputs) and has constant energy returns to scale [8]

$$\boldsymbol{Z}^E = \boldsymbol{A}^E \widehat{\boldsymbol{g}}$$

where the term $\boldsymbol{A}^E$ is the matrix of technical energy coefficient of the energy sector, which describes the amount of energy inputs from industry i per unit of output of energy industry j.

Note: For standardization, $\boldsymbol{g}$ is used instead of $\boldsymbol{x}^E$.

Introducing the technical energy coefficient matrix into Eq. [3.1], we obtain

$$\boldsymbol{g} = \boldsymbol{A}^E \boldsymbol{g} + \boldsymbol{F}^E \boldsymbol{i}$$

The previous expression is then solved for the vector of total output $\boldsymbol{g}$

$$\boldsymbol{g} = (\boldsymbol{I} - \boldsymbol{A}^E)^{-1} \boldsymbol{F}^E \boldsymbol{i} = \boldsymbol{L}^E \boldsymbol{F}^E \boldsymbol{i} \qquad [3.2]$$

where the matrix $\boldsymbol{L}^E$ of size $r \times r$ is the total requirements matrix of the energy sector. This matrix accounts for the energy requirements of the production processes of the energy sector, therefore it describes the primary-to-final conversion processes (see Appendix A) in the economy. For this reason, it can be referred to as the total primary-to-final energy conversion requirements matrix.

3.1.2 The input-output model of the rest of the economy

To continue the development of the energy input-output model, the input-output model for the rest of the economy should be described. We start with the total output in monetary units for the whole economy, i.e.

$$\boldsymbol{x} = \boldsymbol{L}\boldsymbol{f} \qquad [3.3]$$

where is the vector of total output of the economy (including energy and non-energy industries), $\boldsymbol{L}$ is the inverse Leontief matrix and $\boldsymbol{f}$ is the vector of final demand.

The elements of Eq. [3.3] are partitioned in the following form

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_e \\ \boldsymbol{x}_n \end{bmatrix} \qquad \boldsymbol{L} = \begin{bmatrix} \boldsymbol{L}_\theta & \boldsymbol{L}_\tau \\ \boldsymbol{L}_\pi & \boldsymbol{L}_\psi \end{bmatrix} \qquad \boldsymbol{f}^* = \begin{bmatrix} \boldsymbol{f}_e \\ \boldsymbol{f}_n \end{bmatrix}$$

where

$\boldsymbol{x}_e$ and $\boldsymbol{x}_n$ are the sub-vectors of total output in monetary units of energy and non-energy industries, respectively

$\boldsymbol{L}_\theta$, $\boldsymbol{L}_\tau$, $\boldsymbol{L}_\pi$ and $\boldsymbol{L}_\psi$ are respectively the sub-matrices of direct and indirect

- energy requirements of energy industries
- energy requirements of non-energy industries
- non-energy requirements of energy industries
- non-energy requirements of non-energy industries

$\boldsymbol{f}_e$ and $\boldsymbol{f}_n$ are the sub-vectors of final demand in monetary units of energy and non-energy industries, respectively

As the energy transactions in the economy ($\boldsymbol{L}_\theta$ and $\boldsymbol{L}_\tau$,) are somehow dealt with by the input-output model of the energy sector (Eq. [3.2]), these are removed from the input-output model of the economy (Eq. [3.3]) to avoid double counting. Therefore, the input-output model of the rest of the economy is defined as

$$\boldsymbol{x}_n = \boldsymbol{L}_\pi \boldsymbol{f}_e + \boldsymbol{L}_\psi \boldsymbol{f}_n \qquad [3.4]$$

3.1.3 The complete PF-EIO model

The coupling of the input-output models of the energy sector and the rest of the economy is achieved by internalizing the rest of the economy into the energy sector's model.

First, the matrix of direct energy demand is decomposed as

$$\boldsymbol{F}^E = \boldsymbol{C}^E \widehat{\boldsymbol{r}^E}$$

where $\boldsymbol{C}^E$ is the matrix of composition (dimensionless) of direct energy demand and $\boldsymbol{r}^E$ is the vector in physical units of aggregated direct energy demand, which corresponds to the column sum of $\boldsymbol{F}^E$.

Partitioning the components of the previous expression, we obtain

$$\boldsymbol{F}^E = [\boldsymbol{C}_n^E \quad \boldsymbol{c}_h^E] \begin{bmatrix} \widehat{\boldsymbol{r}_n^E} & \boldsymbol{O} \\ \boldsymbol{o}' & r_h^E \end{bmatrix} \qquad [3.5]$$

where $\boldsymbol{C}_n^E$ and $\boldsymbol{r}_n^E$ are the matrix of composition and vector of aggregated direct energy demand by non-energy industries and $\boldsymbol{c}_h^E$ and r_h^E are the vector of composition and aggregated final energy demand (also known as residential energy demand). Notice that $\boldsymbol{h} = \boldsymbol{c}_h^E r_h^E$ and $\boldsymbol{E_n} = \boldsymbol{C}_n^E \boldsymbol{r}_n^E$, see above.

Inserting the previous expression into Eq. [3.2], the total output of energy industries is defined as

$$\boldsymbol{g} = \boldsymbol{L}^E \boldsymbol{C}_n^E \boldsymbol{r}_n^E + \boldsymbol{L}^E \boldsymbol{h} \qquad [3.6]$$

Note: For simplicity, $\boldsymbol{h}$ is used instead of $\boldsymbol{c}_h^E r_h^E$

The direct energy demand of non-energy industries ($\boldsymbol{r}_n^E$) is related to the production technology and final demand of these industries. Therefore, this component is related to the non-energy transactions of the rest of the economy (accounted for by Eq. [3.4]). To determine an expression for $\boldsymbol{r}_n^E$ in terms of the components of the input-output model of the rest of the economy, the direct energy intensity concept is used.

The direct energy intensity is the amount of direct energy use by a non-energy industry required to produce a unit of economic output of this industry (similar to Eq. [2.4]). It is mathematically represented as

$$\boldsymbol{T}^E = \widehat{\boldsymbol{r}_n^E} \widehat{\boldsymbol{x}_n}^{-1}$$

$\boldsymbol{T}^E$ establishes a link between the models of the energy sector (Eq. [3.6]) and the rest of the economy (Eq. [3.4]) and closes the primary-to-final energy input-output model.

$$\boldsymbol{g} = \boldsymbol{L}^E \boldsymbol{C}_n^E \boldsymbol{T}^E \left(\boldsymbol{L}_\pi \boldsymbol{f}_e + \boldsymbol{L}_\psi \boldsymbol{f}_n\right) + \boldsymbol{L}^E \boldsymbol{h}$$

Rearranging, the terms, we obtain

$$\boldsymbol{g} = \boldsymbol{L}^E \boldsymbol{C}_n^E \boldsymbol{T}^E \boldsymbol{L}_\psi \boldsymbol{f}_n + \boldsymbol{L}^E \boldsymbol{h} + \boldsymbol{L}^E \boldsymbol{C}_n^E \boldsymbol{T}^E \boldsymbol{L}_\pi \boldsymbol{f}_e$$

Furthermore, it is possible to find a relationship between final energy demand in physical units ($\boldsymbol{h}$) and in monetary units ($\boldsymbol{f}_e$) based on energy prices as

$$\boldsymbol{f}_e = \widehat{\boldsymbol{p}}\boldsymbol{h}$$

where $\boldsymbol{p}$ is a vector of average energy prices.

Therefore, the final form of the PF-EIO model is

$$\boldsymbol{g} = \underbrace{\boldsymbol{L}^E \boldsymbol{C}_n^E \boldsymbol{T}^E \boldsymbol{L}_\psi \boldsymbol{f}_n}_{Production-related} + \underbrace{(\boldsymbol{L}^E + \boldsymbol{L}^E \boldsymbol{C}_n^E \boldsymbol{T}^E \boldsymbol{L}_\pi \widehat{\boldsymbol{p}})\boldsymbol{h}}_{Residential} \quad [3.7]$$

where $\boldsymbol{L}^E$, $\boldsymbol{C}_n^E$ & $\boldsymbol{h}$ are in physical units, $\boldsymbol{L}_\psi$, $\boldsymbol{L}_\pi$& $\boldsymbol{f}_n$ are in monetary units and $\boldsymbol{T}^E$ & $\widehat{\boldsymbol{p}}$ are in mixed units.

The PF-EIO model accounts for the total energy use in the economy, so does the HEIO model. However the two models have radically different forms. Therefore the PF-EIO model should have a degree of correspondence with the HEIO model. To explore this, the following section presents a comparison of the HEIO and the PF-EIO models applied to a simple economy.

3.1.4 Numerical example

The sample economy consists of six producing sectors and one category of final demand. Table 3-1 show the interindustry transactions in monetary units and Table 3-2 presents the energy flows in physical units.

Table 3-1 Monetary interindustry transactions of a sample economy (Units MUSD)

	Oil	Gas	Electricity	Manufacturing	Services	Materials	Final demand	Total output
Oil	0	0	12.5	6	3.5	2.4	10.5	34.9
Gas	0	0	6	3.15	0.75	0.45	2	12.35
Electricity	1.5	0.7	0.5	3.75	6.3	1.5	9.9	24.15
Manufacturing	0.5	0.3	2	50	100	30	200	382.8
Services	0.8	1	2	130	50	20	150	413.8
Materials	0.7	0.2	1	80	5	0	5	131.9

Table 3-2 Energy flows (Units TJ)

	Oil	Gas	Electricity	Manufacturing	Services	Materials	Final demand	Total output
Oil	0	0	250	100	50	40	150	590
Gas	0	0	150	70	15	10	40	285
Electricity	10	5	5	25	35	10	55	145

Interindustry transactions and energy flows data are used to build the HEIO and the PF-EIO models.

The example shows that the sub-matrix $\boldsymbol{\alpha}_\theta$ of the hybrid-unit total energy requirements matrix ($\boldsymbol{\alpha}$) is approximately equal to the component $\boldsymbol{L}^E + \boldsymbol{L}^E \boldsymbol{C}_n^E \boldsymbol{T}^E \boldsymbol{L}_\pi \widehat{\boldsymbol{p}}$ of the PF-EIO model:

$$\boldsymbol{\alpha}_\theta = \begin{bmatrix} 1.0359 & 0.0381 & 1.9249 \\ 0.0211 & 1.0225 & 1.1509 \\ 0.0195 & 0.0205 & 1.1006 \end{bmatrix} \text{MJ/MJ}$$

$$\boldsymbol{L}^E + \boldsymbol{L}^E \boldsymbol{C}_n^E \boldsymbol{T}^E \boldsymbol{L}_\pi \widehat{\boldsymbol{p}} = \begin{bmatrix} 1.0357 & 0.0381 & 1.9222 \\ 0.0211 & 1.0224 & 1.1495 \\ 0.0194 & 0.0204 & 1.1000 \end{bmatrix} \text{MJ/MJ}$$

Note: The error between coefficients kl of these matrices is negligible, i.e. $\varepsilon_{kl} < |0.6\%|$.

Moreover the sub-matrix $\boldsymbol{\alpha}_\tau$ is equal to the matrix product $\boldsymbol{L}^E \boldsymbol{C}_n^E \boldsymbol{T}^E \boldsymbol{L}_\psi$:

$$\boldsymbol{\alpha}_\tau = \begin{bmatrix} 1.0389 & 0.7549 & 1.1679 \\ 0.5839 & 0.3871 & 0.5205 \\ 0.2370 & 0.2131 & 0.2542 \end{bmatrix} \text{MJ/MUSD}$$

$$\boldsymbol{L}^E \boldsymbol{C}_n^E \boldsymbol{T}^E \boldsymbol{L}_\psi = \begin{bmatrix} 1.0393 & 0.7556 & 1.1681 \\ 0.5841 & 0.3875 & 0.5205 \\ 0.2371 & 0.2133 & 0.2542 \end{bmatrix} \text{MJ/MUSD}$$

Note: The error between coefficients kl of these matrices is negligible, i.e. $\varepsilon_{kl} < |0.2\%|$

The example shows that the PF-EIO and the HEIO are approximately equivalent. In this respect, Appendix B presents the theoretical demonstration of the exact equivalence of the PF-EIO and the HEIO models through the development of a hybrid-unit partitioned EIO model. In addition, the demonstration in Appendix B explains that the minor numerical divergence between both models in the present example is caused by the use of the sub-matrices $\boldsymbol{L}_\psi$ and $\boldsymbol{L}_\pi$ of the inverse Leontief matrix in monetary units ($\boldsymbol{L}$) instead of the correspondent sub-matrices of the hybrid-unit inverse Leontief matrix ($\boldsymbol{L}^*$).

3.2 The primary-to-useful energy input-output model

The section describes the development of the PU-EIO model, which is based on the PF-EIO model with the inclusion of an input-output model of an adjunct energy sector. This adjunct energy sector is a theoretical sector that implements final-to-useful energy conversions in the economy [36]. It includes all end-use devices/technologies that convert energy carriers into useful work flows (e.g. electric motors, light bulbs or boilers). These end-use technologies are distributed in the infrastructure of non-energy industries and final consumers

3.2.1 An input-output model of the adjunct energy sector

An isolated input-output model of the adjunct energy sector is built. In this model, secondary energy flows (e.g. electricity, oil derivatives or natural gas) are used as inputs for final-to-useful conversion processes by end-use energy technologies to produce useful energy flows (e.g. mechanical drive, lighting or heat flow) for direct useful energy demand (i.e. intermediate demand for useful energy by non-energy industries and final useful energy demand). Moreover, there are neither transactions of useful energy flows to energy industries of the energy sector and between end-use energy technologies of the adjunct energy sector; nor transactions from non-energy industries to the adjunct energy sector.

The input-output model of the adjunct energy sector aims to construct a simple relationship as:

$$\boldsymbol{e} = \boldsymbol{L}^U \boldsymbol{F}^U \boldsymbol{i}$$

where $\boldsymbol{e}$ is the vector of total direct use of secondary energy flows by the economy, $\boldsymbol{L}^U$ is the matrix of total requirements of the adjunct energy sector and $\boldsymbol{F}^U$ is the matrix of direct useful energy demand of by non-energy industries and final consumers.

The matrix $\boldsymbol{L}^U$ accounts for the secondary energy requirements of the conversion processes of the adjunct energy sector, therefore it describes the final-to-useful conversion processes (see Appendix A) in the economy. For this reason, it can be referred to as the total final-to-useful energy conversion requirements matrix.

By definition the adjunct energy sector is present in every non-energy sector. As a result, the direct energy demand by non-energy industries and final consumers coincides with the direct use of secondary energy flows, i.e. the vector $\boldsymbol{e}$ is equivalent to $\boldsymbol{F}^E\boldsymbol{i}$ in Eq. [3.1]. The model consists therefore of connecting the direct energy demand with the direct useful energy demand by non-energy industries and final consumers.

$$\boldsymbol{F}^E = \boldsymbol{L}^U\boldsymbol{F}^{U,W} \quad [3.8]$$

Note: Because of the different characteristics of the adjunct energy sector compared to the conventional energy sector (primary-to-final energy conversions), the $\boldsymbol{L}^U$ is constructed with a different procedure as $\boldsymbol{L}^E$, depending on the type of available data. Appendix E presents a procedure to build the $\boldsymbol{L}^U$ matrix from useful energy flow data from the useful work accounting methodology developed by Ayres et al. [37], Serrenho et al. [38] and others.

3.2.2 The complete PU-EIO model

First, the matrix of direct energy demand and direct useful work demand are decomposed as (see Eq. [3.5])

$$\begin{aligned}\boldsymbol{F}^E &= \boldsymbol{C}_n^E\widehat{\boldsymbol{r}_n^E} + \boldsymbol{h}\\ \boldsymbol{F}^U &= \boldsymbol{C}_n^U\widehat{\boldsymbol{r}_n^U} + \boldsymbol{u}\end{aligned} \quad [3.9]$$

where $\boldsymbol{C}_n^U$ is the matrix of composition (dimensionless) of direct useful energy demand by non-energy industries, $\boldsymbol{r}_n^U$ is the vector in physical units of aggregated direct useful energy demand, which corresponds to the column sum of $\boldsymbol{F}^U$, and $\boldsymbol{u}$ is the vector of final useful energy demand.

Furthermore, to determine an expression for $\boldsymbol{r}_n^U$ in terms of the components of the input-output model of the rest of the economy, the direct useful energy intensity concept is used. This is the amount of direct useful energy use by a non-energy industry required to produce a unit of economic output of this industry. It is mathematically represented as

$$\boldsymbol{T}^U = \widehat{\boldsymbol{r}_n^U}\widehat{\boldsymbol{x}_n}^{-1} \quad [3.10]$$

Combining Eqs. [3.8], [3.9] and [3.10], we obtain

$$\begin{aligned}\boldsymbol{C}_n^E\boldsymbol{T}^E &= \boldsymbol{L}^U\boldsymbol{C}_n^U\boldsymbol{T}^U\\ \boldsymbol{h} &= \boldsymbol{L}^U\boldsymbol{u}\end{aligned}$$

Finally, integrating the previous expressions into the PF-EIO model, the final form of the PU-EIO model is

$$\boldsymbol{g} = \underbrace{\boldsymbol{L}^E\boldsymbol{L}^U\boldsymbol{C}_n^U\boldsymbol{T}^U\boldsymbol{L}_\psi\boldsymbol{f}_n}_{Production-related} + \underbrace{(\boldsymbol{L}^E + \boldsymbol{L}^E\boldsymbol{L}^U\boldsymbol{C}_n^U\boldsymbol{T}^U\boldsymbol{L}_\pi\widehat{\boldsymbol{p}})\boldsymbol{L}^U\boldsymbol{u}}_{Residential} \quad [3.11]$$

where $\boldsymbol{L}^E$, $\boldsymbol{L}^U$, $\boldsymbol{C}_n^U$ & $\boldsymbol{u}$ are in physical units, $\boldsymbol{L}_\psi$, $\boldsymbol{L}_\pi$& $\boldsymbol{f}_n$ are in monetary units and $\boldsymbol{T}^U$ & $\widehat{\boldsymbol{p}}$ are in mixed units.

Note: There is no HEIO equivalent model that accounts for useful energy flows. To construct such a model more research on transactions and non-energy flows of the adjunct energy sector is needed.

4 Discussion

As proven in the previous section, the PF-EIO and the HEIO models are equivalent, i.e. account for the same information. However the PF-EIO and the PU-EIO models organize the energy transactions and flows according to the energy conversion processes in the economy.

In this section, the advantages and issues of the PF-EIO and PU-EIO models are discussed. Nonetheless, as the components of the proposed models are radically different as those of the HEIO model, these components are described in detail first.

4.1 The components of PF-EIO and PU-EIO models

This section gives detail into the components of the proposed models

$\boldsymbol{L}^E$ Total requirements matrix of the energy sector or the Structure and efficiency of the primary-to-final energy conversion processes in the economy

The total primary-to-final energy conversion requirements matrix connects the direct energy demand with the total primary and secondary energy requirements to meet this demand (Eq. [3.2]). The element l_{ik}^E is the amount of energy flow form energy industry i which is required to produce a unit of secondary energy flow by energy industry k (e.g., the amount of natural gas required by thermoelectric power plants to produce a kWh of electricity).

The k^{th} column of $\boldsymbol{L}^E$ represents the energy input structure of total energy per unit of secondary energy flow to energy industry k. In addition, the summation of this k^{th} column, if not zero, is the total amount of energy required to produce a unit of secondary energy flow by energy industry k for the use of non-energy industries and final consumers. This is defined as the inverse of the overall conversion efficiency of energy industry k (ν_k), i.e.

$$\sum_i l_{ik}^E = \frac{1}{\nu_k}$$

Notice that ν_k is smaller than the conversion efficiency of the actual energy process carried out by energy industry k (e.g. oil refining) because it also accounts for the use of secondary energy by the energy sector (e.g. the amount electricity used for lighting in oil refinery).

$\boldsymbol{L}^U$ Total requirements matrix of the adjunct energy sector or the Structure and efficiency of the final-to-useful energy conversion processes in the economy

The total final-to-useful energy conversion requirements matrix connects the demand for useful energy with the secondary energy requirements to meet this demand. The element l_{kl}^U is the amount of secondary energy flow from energy industry k which is required to produce a unit of useful work flow by end-use energy technology l (e.g., the amount of natural gas required by industrial boiler to produce a TJ of high temperature heat).

The l^{th} column of $\boldsymbol{L}^U$ represents the energy input structure of total secondary energy per unit of useful energy flow to end-use energy technology l. In addition, the summation of this l^{th} column, if not zero, is the total amount of secondary energy required to produce a unit of useful energy flow by end-use energy technology l for the use of non-energy industries and final consumers. This is defined as the inverse of the overall final-to-useful conversion efficiency of end-use energy technology (ϵ_l).

$\boldsymbol{C}_n^E$ Demand composition of production-related direct energy demand

This component describes the composition of direct energy use by non-energy industries. The element $c_n^E|_{kj}$ is the fraction of the energy demand by non-energy industry j which is provided by energy industry k (e.g. the share of natural gas in the energy use of the chemical industry).

$\boldsymbol{C}_n^U$ Demand composition of production-related useful energy demand

This component describes the composition of direct useful energy use by non-energy industries. The element $c_n^U|_{lj}$ is the fraction of the useful energy demand by non-energy industry j which is provided by end-use energy technology l (e.g. the share of mechanical drive from an electric motor in the useful energy use by the chemical industry).

$\boldsymbol{T}^E$ Direct energy intensity

The direct energy intensity factor represents the total amount of direct energy use per unit of total output by non-energy industries. This component accounts for several variables of the economic system hence it is highly aggregated. Guevara [39] explains the direct energy intensity as a function of the following variables: 1) the inverse final-to-useful aggregate efficiency (ϵ^{-1}) ; 2) the inverse useful-to service aggregate efficiency (μ^{-1}); 3) the aggregate price of non-energy products; and 4) the demand for energy services.

$\boldsymbol{T}^U$ Useful energy intensity

The useful work intensity factor represents the total amount of useful work use per unit of total output by non-energy industries. This component is less aggregated than $\boldsymbol{T}^E$ though it is composed by the following variables: 1) the inverse useful-to service aggregate efficiency (μ^{-1}); 2) the aggregate price of non-energy products; and 3) the demand for energy services.

Note: if the industrial classification of direct energy demand (or direct useful energy demand) is equivalent to the classification of non-energy industries of the rest of the economy, $\boldsymbol{T}^E$ and $\boldsymbol{T}^U$are diagonal matrices.

$\boldsymbol{L}_\psi$ Structure of the rest of the economy

This components represents the interindustry transaction of the rest of the economy. The element $l_\psi\big|_{ij}$ represents the amount of purchases from non-energy industry i that non-energy industry j uses to provide a unit of product to final demand.

$\boldsymbol{L}_\pi$ Structure of non-energy inputs to the energy sector

This components accounts for the non-energy input structure of the energy sector (i.e. non-energy transactions to energy industries). The element $l_\pi|_{ik}$ represents the amount of purchases from non-energy industry i that energy industry k uses to produce a monetary unit of energy product to final demand.

$\widehat{\boldsymbol{p}}$ Average energy prices

There are two options to determine the values of this component: 1) the actual average prices faced by final consumers (i.e. $\widehat{\boldsymbol{p}} = \widehat{\boldsymbol{h}}\widehat{\boldsymbol{f}_e^{-1}}$), or the economy-wide average energy prices (i.e. $\widehat{\boldsymbol{p}} = \widehat{\boldsymbol{g}}\widehat{\boldsymbol{x}_e^{-1}}$). In Appendix C, it is shown that the economy-wide average energy prices give the smallest error between the PF-EIO model and HEIO models.

$\boldsymbol{f}_n$, $\boldsymbol{h}$ and $\boldsymbol{u}$ Scale components of final demand

$\boldsymbol{f}_n$ is the total expenditure of final consumers in non-energy products. $\boldsymbol{h}$ and $\boldsymbol{u}$ correspond to energy and useful energy deliveries to final demand in physical units. These components are related to the level of production in the economic system, hence to the size of the economy.

4.2 Advantages and issues of the PF-EIO and PU-EIO models

The PF-EIO and PU-EIO models have the following advantages compared to conventional models:

- Energy (in physical units) and non-energy transactions (in monetary units) in the economy are accounted for separately. Moreover, energy transactions are organized according to the processes of energy use and conversion.
- The energy sector can be represented in more detail, i.e. energy industries and energy carriers can have lower degree of aggregation. The level of detail in the energy sector can be further improved by building the input-output model of the energy sector under the product-by-industry approach (Miller and Blair [8], Ch. 5). In addition, non-energy-use and non-marketable energy carriers can be included (see Appendix D and Guevara et al. [15]).
- The models comply with the energy conservation principle (not always true for the DIC-EIO model), i.e. primary-to-final and final-to-useful energy conversion losses are accounted for $\boldsymbol{L}^E$ and $\boldsymbol{L}^U$, respectively.
- The primary, final and useful levels of energy use are represented, which improves the understanding of energy use mechanisms in the economy. Moreover, the primary, secondary and useful energy embodied in final demand can be estimated.
- The models are able to isolate up-to three indicators of energy efficiency and to account for them separately: $\boldsymbol{L}^E$, $\boldsymbol{L}^U$ and $\boldsymbol{T}^E/\boldsymbol{T}^U$ give insight into primary-to-final conversion efficiencies, final-to-useful conversion efficiencies and the efficiency of production-related energy use, respectively.
- The models are especially suitable to evaluate energy use, energy decoupling and energy efficiency trends in the economy in combination with structural decomposition analysis [15, 20].
- The models in Eqs. [3.7] and [3.11], allows the use of available monetary input-output data ($\boldsymbol{L}_\psi$ and $\boldsymbol{L}_\pi$) without the need of building a hybrid-unit input-output system.

However, the models cannot address the following issues of conventional EIO models:

- As in basic input-output analysis (see Leontief [7], Miller and Blair [8], Suh [11]): 1) constant returns to scale and fixed technical coefficients are considered to build the energy and non-energy requirements matrices ($\boldsymbol{L}^E$, $\boldsymbol{L}_\psi$ and $\boldsymbol{L}_\pi$); 2) There is not substitution between inputs to the production process, i.e. all inputs are perfect complements. 3) Resources are not constrained, i.e. supply is infinite and perfectly elastic, and are not underused, i.e. efficient use of resources.
- The service level of energy use is not represented, i.e. it is aggregated in the direct intensity matrices ($\boldsymbol{T}^E$ or $\boldsymbol{T}^U$) and final energy demand vectors ($\boldsymbol{h}$ or $\boldsymbol{u}$).
- The economic value of the services that energy industries provide is neglected due to the substitution of monetary by physical units.
- The PU-EIO model assumes an independent adjunct energy sector, i.e. the adjunct energy sector do not use inputs from non-energy industries of the rest of the economy. Even though there is not enough research to contradict this assumption, this issue can lead to a distorted representation of the adjunct energy sector.
- The PU-EIO model relies on the same assumptions made by the methodology to account for useful energy in the economy hence it involves the consequences of these assumptions.

4.3 Summary

The PF-EIO model has a detailed description of the primary and final levels of energy use and of the primary-to-final energy conversion in the economy. Moreover, the PU-EIO model extends the former model to include the useful level and the final-to-useful energy conversion stage of the economy. Table 4-1 summarizes and compares the characteristics of the ultimate and conventional EIO models.

Table 4-1 Characteristics of different energy input-output models

EIO model	Energy efficiency indicators	Levels of energy use[a]	Energy and economic transactions are accounted for by	Energy sector detail	Other features[b]			
					1	2	a	b
Conventional								
HEIO model	-	P or F	$\boldsymbol{\alpha}_\theta$ and $\boldsymbol{\alpha}_\tau$	Medium	Y	Y	X	XX
DIC-EIO model	$\boldsymbol{T}$	P or F	$\boldsymbol{T}$, $\boldsymbol{L}$ and $\widetilde{\boldsymbol{Q}}$	Low	-	-	-	-
Ultimate								
PF-EIO model	$\boldsymbol{L}^E$ and $\boldsymbol{T}^E$	P and F	$\boldsymbol{L}^E$, $\boldsymbol{C}_n^E$, $\boldsymbol{T}^E$, $\boldsymbol{L}_\psi$, $\boldsymbol{L}_\pi$ and $\widehat{\boldsymbol{p}}$	High	Y	Y	X	X
PU-EIO model	$\boldsymbol{L}^E$, $\boldsymbol{L}^U$ and $\boldsymbol{T}^U$	P, F and U	$\boldsymbol{L}^E$, $\boldsymbol{L}^U$, $\boldsymbol{C}_n^U$, $\boldsymbol{T}^U$ $\boldsymbol{L}_\psi$, $\boldsymbol{L}_\pi$ and $\widehat{\boldsymbol{p}}$	The highest	Y	Y	XX	X

[a] P – Primary, F – Final and U – Useful

[b] Advantages
1. Hybrid-units
2. Energy conservation

Disadvantages
a. Complex construction
b. Data intensive

The table shows that the PF-EIO and PU-EIO models can significantly improve the analysis of economy-wide studies of energy use, compared to conventional models, because they provide detailed information about the processes of energy conversion and use in the economy.

5 CONCLUSIONS

The present paper introduces the primary-to-final energy input-output model (PF-EIO model). This model is introduced as an improvement to the two conventional models in energy input-output analysis, i.e. the hybrid-unit (HEIO) and direct impact coefficient (DIC-EIO) models, which have been extensively used in the last 40 years without significant improvements.

The PF-EIO model is equivalent to the standard HEIO model and can also replicate the form of the DIC-EIO model. Nevertheless, it presents several advantages with respect to these conventional models. Remarkably, it provides a detailed description of energy flows and economic transactions according to the energy conversion processes, and the primary and final levels of energy use in the economy.

The model characterizes the vector of total energy output as a function of eight variables: Efficiency and structure of primary-to-final energy conversion (i.e. energy sector); Direct energy demand composition; Direct energy intensity; Economic structure; Non-energy input structure of the energy sector, Average energy prices to final consumers; Final energy demand; and Final demand for non-energy products. Moreover, it separately accounts for them, which makes the PF-EIO model especially suitable to evaluate energy decoupling, energy use and energy efficiency trends in the economy.

Furthermore, the PF-EIO enables the inclusion of other levels of energy use and energy transition stages in the economy. This led to the development of the primary-to-useful energy input-output model (PU-EIO model), which is the first ever energy input-output model to include useful energy flows and the final-to-useful energy conversion stage.

The development of the PF-EIO model can be also extended to improve other hybrid-unit input-output applications, for example the accounting of material flows, GHG emissions, exergy flows, and energy-related CO_2 emissions. Finally, as the PF-EIO model is undeniably superior to conventional models, it should be considered the standard in energy input-output analysis.

Acknowledgement: We recognize the financial support of the MIT Portugal Program and FCT through PhD scholarship SFRH/BD/51297/2010 to Zeus Guevara. We would also like to thank the support of AdI through the project Energy Wars (QREN7929).

REFERENCES

[1] Smil V. Energy at the Crossroads: Global Perspectives and Uncertainties. Cambridge, MA: MIT Press, 2003.

[2] Stern DI. The role of energy in economic growth. Annals of the New York Academy of Sciences, 2011; 1219(1): 26-51.

[3] Smil V. Energy in the twentieth century: Resources, conversions, costs, uses, and consequences. Annual Review of Energy and the Environment, 2000; 25(1): 21-51.

[4] Ockwell DG. Energy and economic growth: Grounding our understanding in physical reality. Energy Policy, 2008; 36(12): 4600-4604.

[5] IPCC. Climate Change 2007: Mitigation of Climate Change. Cambridge: Cambridge University Press, 2007.

[6] Christ CF. A review of input-output analysis. In: National Bureau of Economic Research, editor. Input-Output Analysis: An Appraisal. Princeton: Princeton University Press, 1955. pp. 137-182.

[7] Leontief W. Some basic problems of empirical input-output analysis. In: National Bureau of Economic Research, editor. Input-Output Analysis: An Appraisal. Princeton: Princeton University Press, 1955. pp. 137-182.

[8] Miller RE, Blair PD. Input-Output Analysis: Foundations and Extensions. 2nd ed. Cambridge: Cambridge University Press, 2009.

[9] Leontief WW. Quantitative input and output relations in the economic systems of the United States. Review of Economics and Statistics, 1936; 18(3): 105-125.

[10] Dietzenbacher E, Lenzen M, Los B, Guan D, Lahr ML, Sancho F, et al. Input-output analysis: The next 25 years. Economic Systems Research, 2013; 25(4): 369-389.

[11] Suh S. Handbook of Input-Output Analysis Economics in Industrial Ecology. Dordrecht: Springer, 2009.

[12] Duchin F, Steenge AE. Input-output analysis, technology and the environment. In: Van den Bergh JCJJ, editor. Handbook of environmental and resource economics. Cheltenham: Edward Elgar Publishing, 1999.

[13] Casler S, Wilbur S. Energy input-output analysis: A simple guide. Resources and Energy, 1984; 6(2): 187-201.

[14] Bullard CW, Herendeen RA. The energy cost of goods and services. Energy Policy, 1975; 3(4): 268-278.

[15] Guevara Z, Rodrigues JFD, Sousa T. A Structural Decomposition Analysis of Primary Energy Use in Portugal. 22nd International Input–Output Conference. Lisbon: IIOA, 2014.

[16] Bullard CW, Penner PS, Pilati DA. Net energy analysis: Handbook for combining process and input-output analysis. Resources and Energy, 1978; 1(3): 267-313.

[17] Wright DJ. Good and services: An input-output analysis. Energy Policy, 1974; 2(4): 307-315.

[18] Krenz JH. Energy per dollar value of consumer goods and services. IEEE Transactions on Systems, Man and Cybernetics, 1974; SMC-4(4): 386-388.

[19] Park S-H. An input-output framework for analysing energy consumption. Energy Economics, 1982; 4(2): 105-110.

[20] Su B, Ang BW. Structural decomposition analysis applied to energy and emissions: Some methodological developments. Energy Economics, 2012; 34(1): 177-188.

[21] Bullard CW, Herendeen RA. Energy impact of consumption decisions. Proceedings of the IEEE, 1975; 63(3): 484-493.

[22] Wright DJ. The natural resource requirements of commodities. Applied Economics, 1975; 7(1): 31-39.

[23] Griffin JM. Energy input-output modelling: Problems and prospects. Washington, DC: General Electric Co., 1976.

[24] Hawdon D, Pearson P. Input-output simulations of energy, environment, economy interactions in the UK. Energy Economics, 1995; 17(1): 73-86.
[25] Rose A, Casler S. Input–output structural decomposition analysis: A critical appraisal. Economic Systems Research, 1996; 8(1): 33-62.
[26] Cruz LM. Energy-environment-economy interactions: An input-output approach applied to the Portuguese case In: ISEE, editor. 7th Biennial Conference of the International Society for Ecological Economics. Tunisia: ISEE, 2002.
[27] Ardent F, Beccali M, Cellura M. Chapter 22: Application of the IO methodology to the energy and environmental analysis of a regional context. In: Suh S, editor. Handbook of Input-Output Analysis Economics in Industrial Ecology. Dordrecht: Springer, 2009. pp. 435-458.
[28] Proops JLR. Input-output analysis and energy intensities: A comparison of some methodologies. Applied Mathematical Modelling, 1977; 1(4): 181-186.
[29] Wachsmann U, Wood R, Lenzen M, Schaeffer R. Structural decomposition of energy use in Brazil from 1970 to 1996. Applied Energy, 2009; 86(4): 578-587.
[30] Chai J, Guo J-E, Wang S-Y, Lai KK. Why does energy intensity fluctuate in China? Energy Policy, 2009; 37(12): 5717-5731.
[31] Kagawa S, Inamura H. A structural decomposition of energy consumption based ona a hybrid rectangular input-output framework: Japan's case. Economic Systems Research, 2001; 13(4): 339-363.
[32] Lin X, Polenske KR. Input–output anatomy of China's energy use changes in the 1980's. Economic Systems Research, 1995; 7(1): 67-84.
[33] Liang S, Wang C, Zhang T. An improved input–output model for energy analysis: A case study of Suzhou. Ecological Economics, 2010; 69(9): 1805-1813.
[34] Eichhammer W. Industrial energy efficiency. In: Cleveland CJ, editor. Encyclopedia of Energy Volumen 3. New York, NY: Elsevier, 2004. pp. 383-393.
[35] EUROSTAT. ESA 95: Supply, use and input-output tables [Online database]. Luxembourg: European Comission, 2014. Available: http://epp.eurostat.ec.europa.eu/portal/page/portal/esa95_supply_use_input_tables/data/workbooks [Accessed Feb 2014]
[36] Santos J, Domingos T, Sousa T, Serrenho A. National accounts with a broader energy sector. Lisbon: University of Lisbon, 2013.
[37] Ayres RU, Ayres LW, Warr B. Exergy, power and work in the US economy, 1900-1998. Energy, 2003; 28(3): 219-273.
[38] Serrenho A, Warr B, Sousa T, Ayres RU, Domingos T. Useful work accounting in Portugal, 1856 - 2009. Conference Useful work accounting in Portugal, 1856 - 2009, Portland, OR.
[39] Guevara Z. Three-level energy decoupling: Energy decoupling at the primary, final and useful levels of energy use [Ph.D. Dissertation]. Lisboa: University of Lisbon, 2014.
[40] Haberl H. The global socioeconomic energetic metabolism as a sustainability problem. Energy, 2006; 31(1): 87-99.
[41] Nakićenović N, Grübler A. Energy conversion, conservation, and efficiency. Energy, 1993; 18(5): 421-435.
[42] Haas R, Nakicenovic N, Ajanovic A, Faber T, Kranzl L, Müller A, et al. Towards sustainability of energy systems: A primer on how to apply the concept of energy services to identify necessary trends and policies. Energy Policy, 2008; 36(11): 4012-4021.
[43] Pachauri S, Spreng D. Energy use and energy access in relation to poverty. Economic and Political weekly, 2004; 39(3): 271-278.
[44] Wirl F. Impact of regulation on demand side conservation programs. Journal of Regulatory Economics, 1995; 7(1): 43-62.
[45] UNDP. World Energy Assessment: Energy and the Challenge of Sustainability. New York, NY: United Nations Development Programme, 2000.
[46] UNDP. World Energy Assessment: Overview: 2004 Update. New York, NY: United Nations Development Programme, 2004.

[47] Cullen JM, Allwood JM. The efficient use of energy: Tracing the global flow of energy from fuel to service. Energy Policy, 2010; 38(1): 75-81.
[48] Haberl H. The energetic metabolism of societies: Part I: Accounting concepts. Journal of Industrial Ecology, 2001; 5(1): 11-33.
[49] Serrenho A. Useful work as and energy end-use accounting method: Historical and economic transitions and european patterns [Ph.D. Dissertation]. Lisboa: University of Lisbon, 2013.
[50] IEA. Energy balances of OECD countries: Documentation for beyond 2020 files [Dataset]. Paris: International Energy Agency, 2011.
[51] Bhattacharyya SC. Energy Economics: Concepts, Issues, Markets and Governance. Dundee: Springer, 2011.
[52] IEA. What are the methods of calculation of primary energy equivalent? [Web page]. Paris: International Energy Agency, 2014. Available: http://www.iea.org/statistics/ [Accessed 7 May 2014]
[53] UN. Concepts and methods in energy statistics with special reference to energy accounts and balances: A technical report. New York, NY: United Nations, 1982.
[54] Lightfoot HD. Understand the three different scales for measuring primary energy and avoid errors. Energy, 2007; 32(8): 1478-1483.
[55] Cegonho N, Delgado Domingos J, Domingos T. Um novo método de cálculo de energia primária e a sua aplicação a Portugal. In: MEFTE, editor. IV Conferência Nacional em Mecânica dos Fluidos, Termodinâmica e Energia. Lisboa2012.
[56] Cullen JM, Allwood JM, Borgstein EH. Reducing energy demand: What are the practical limits? Environmental Science & Technology, 2011; 45(4): 1711-1718.
[57] Orecchini F, Naso V. Energy Systems in the Era of Energy Vectors. London: Springer, 2012.
[58] Percebois J. Is the concept of energy intensity meaningful? Energy Economics, 1979; 1(3): 148-155.
[59] Nakićenović N, Gilli PV, Kurz R. Regional and global exergy and energy efficiencies. Energy, 1996; 21(3): 223-237.
[60] Bandi V. Decentralized energy services delivery in developing countries, Case study; India [M.Sc. Dissertation]. Helsinki: Aalto University, 2010.
[61] EU. Directive 2006/32/EC of the European Parliament and of the Council of 5 April 2006 energy end-use efficiency and energy services and repealing Council Directive 93/76/EEC. Official Journal of the European Union, 2006; 114: 64-85.
[62] Bierens HJ. The inverse of a partitioned matrix [Ebook]. Harrisburg, PA: Penn State University, 2014.
[63] Faliva M, Zoia MG. On a partitioned inversion formula having useful applications in econometrics [Ebook]. Geneva: Université de Genève, 2000.
[64] Tian Y, Takane Y. The inverse of any two-by-two nonsingular partitioned matrix and three matrix inverse completion problems. Computers & Mathematics with Applications, 2009; 57(8): 1294-1304.
[65] Henderson HV, Searle SR. On deriving the inverse of a sum of matrices. Siam Review, 1981; 23(1): 53-60.
[66] Miller KS. On the inverse of the sum of matrices. Mathematics Magazine, 1981; 54(2): 67-72.
[67] Tylavsky DJ, Sohie GRL. Generalization of the matrix inversion lemma. Proceedings of the IEEE, 1986; 74(7): 1050-1052.
[68] Guo J, Lawson AM, Planting MA. From make-use to symmetric input-output tables: An assessment of alternative technology assumptions 14th International Conference on Input-Output Techniques. Montreal: IIOA, 2009.
[69] UN. Handbook of Input-Output Table Compilation and Analysis. New York, NY: United Nations, 1999.
[70] EUROSTAT. Manual of supply, use and input-output tables. Luxembourg: European Comission, 2008.
[71] Grubler A. Energy transitions research: Insights and cautionary tales. Energy Policy, 2012; 50(0): 8-16.

[72] Valero A. Exergy accounting: Capabilities and drawbacks. Energy, 2006; 31(1): 164-180.
[73] Ayres RU, Warr B. The Economic Growth Engine. How Energy and Growth Drive Material Prospertity. Cheltenham: Edward Elgar, 2009.
[74] Lovins AB. Energy efficiency: Taxonomic overview. In: Cleveland CJ, editor. Encyclopedia of Energy Volumen 2. New York, NY: Elsevier, 2004. pp. 383-401.
[75] Serrenho AC, Sousa T, Warr B, Ayres RU, Domingos T. Decomposition of useful work intensity: The EU (European Union)-15 countries from 1960 to 2009. Energy (In Press), 2014.
[76] Palma M. Improvements on useful work accounting [M.Sc. Dissertation]. Lisbon: University of Lisbon, 2014.

Appendix A: The levels of energy use

The levels of energy use describe the flow of energy along all stages of energy conversion and use in the economy. These levels are defined based on the metabolism approach, which describes the economy as a physical input-output system drawing energy from the environment, performing internal physical processes (i.e. energy conversion or transfer) and dissipating low-grade waste heat to the environment [40].

Energy flows are traditionally classified into three levels of energy use: primary, final and useful with two stages of energy conversion: primary-to-final and final-to-useful. However, Nakićenović and Grübler [41] argue that this classification truncates the analysis at the last stage of energy conversion and hence does not include actual delivered energy services, see also Haas et al. [42], Pachauri and Spreng [43], and Wirl [44]. Therefore, a service level of energy use and a useful-to-service transition stage should be included (Figure A-1).

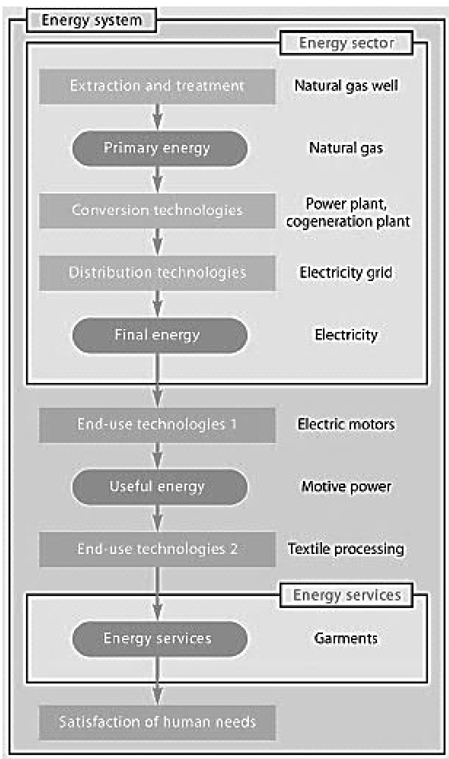

Figure A-1 Diagram of energy flows with major energy- and efficiency-related terms. Note: From UNDP [45] and UNDP [46]

UNDP [45] and Cullen and Allwood [47] place the service level in flow diagram of an energy system as the output of a dissipative energy transfer performed by a passive system (i.e. end-use device which dissipates most of the energy it receives).

In the following sections, the levels of energy use and the transition stages between them are explained.

A.1 Primary level of energy use

Primary energy corresponds to energy carriers as they are recovered or gathered from the natural environment [41, 48], i.e. natural resources such as mined coal, collected biomass or crude oil. As there are many forms of energy carriers in the natural environment, primary energy is given in terms of the energy content of these energy carriers, which leads to consistency issues [49]. In the case of fossil fuels, the energy content is usually determined by the enthalpy or heating thermodynamic potential [49, 50]. In the case of renewable and non-conventional energy carriers, the energy content (or primary energy equivalent) is estimated according to two main accounting methods, i.e. the partial substitution method and the physical content method [51-55].

A.2 Final level of energy use and the primary-to-final conversion stage

Final or secondary energy is the flow of energy carriers that is available for direct use by consumers. A consumer is a unit of the economic system (i.e. industries and households) that requires energy services for production or consumption, e.g. industries or households [48].

The final level of energy use accounts for the energy content in output products of the energy sector, e.g. oil derivatives, electricity, biodiesel or geothermal heat. It also accounts for energy carriers in secondary form produced by decentralized generation systems, which are not part of the energy sector, e.g. residential solar thermal boilers or stand-alone wind turbines. Additionally, the International Energy Agency does not include the direct use of secondary energy carriers by the energy sector as final energy flows [50].

The primary-to-final conversion is the first energy transformation stage in the economy, where primary energy sources are upgraded into more useful forms of energy through conversion processes [56]. In addition to conversion process, such as oil refining and coal-fuelled electric generation, this conversion stage usually includes other operation processes of the energy sector, e.g. extraction, storage and distribution [45, 49, 57].

A.3 Useful level of energy use and the final-to-useful conversion

Useful energy (mainly heat, motion, and light) is the last form of energy flows that is directly used to provide energy services [56]. It is obtained from the conversion of secondary energy carriers by end-use conversion devices (or end-use energy technology), for example, motor engines, boilers, ovens and lamps [41]. Because useful energy is situated immediately before the level of satisfied energy needs and is independent from the evolution of energy conversion technology, it is should be included in economy-wide energy accounting [48, 49, 58].

The final-to-useful conversion is the second and last energy conversion stage in the economy. This stage is usually carried in the exact location where energy services are required (in the household or in a factory). The final-to-useful conversion stage consists of a large share of one-step conversions, e.g. electricity into motion by an electric motor or natural gas into steam by a boiler, and of two-step conversions, e.g. gasoline into motion by a car engine into air conditioning low temperature heat. Moreover, this conversion stage is difficult to estimate since every sector, every energy carrier, end-use energy technology and every energy service must be considered [59].

A.4 Service level of energy use and the useful-to-service dissipative transfer

Haberl [48] defines energy services as immaterial services, whose provision involves the use of energy[4]. For example, heating of a room, moving commodities from one point to another in a defined time period, or transforming material inputs into a piece of furniture. The problem with this definition is that it is not possible to distinguish energy services from other goods and services in the economy [43] (everything needs energy to be produced). Consequently, energy services sometimes are classified into two categories [42, 43]: 1) direct, such as lighting, ironing, drilling, melting sands to form glass, etc.; and 2) indirect, i.e. the energy embodied in food, shoes, building, vehicles, etc. Nevertheless, as consumers do not consume non-energy products and services due to their embedded energy, the indirect services are evaluated in terms of the direct energy services used to produce them. Therefore, only direct energy services are considered part of the service level of energy use.

The Useful-to-service dissipative transfer between the useful and the service levels of energy use consists of a series of dissipative processes [48]. These processes are performed by passive systems[5] that "holds or traps useful energy for a time to provide a level of final service" [56].

This level of energy use and its preceding transition stage from the useful level is fundamental for the analysis of energy performance of an economy because it appears to be the "weakest link" of the energy chain from primary to services [41] so the largest improvements are expected to be at the useful-to-service transition stage [59]. Nevertheless, because of the issues in the definition of each energy service and the boundaries of its passive system, the service level is excluded in most studies [41].

---

[4] In this case, energy services do not correspond to the services that energy companies provide as in Bandi [60] and EU [61]

[5] A passive system is a last technical component of the energy chain, whose purpose is not converting the flow of useful energy into another energy form. In providing energy services, useful energy is eventually dissipated to the natural environment as low-grade heat [47]. Therefore, a passive system requires a continuous supply of useful energy to maintain a constant level of service.

Appendix B: Hybrid-unit partitioned EIO model

This appendix presents the theoretical demonstration of the exact equivalence of the PF-EIO and the HEIO models through the development of the hybrid-unit partitioned EIO model.

The basic input-output model relationship in hybrid-units is expressed by

$$\boldsymbol{x}^* = \boldsymbol{A}^*\boldsymbol{x}^* + \boldsymbol{f}^*$$

or, alternatively,

$$(\boldsymbol{I} - \boldsymbol{A}^*)\boldsymbol{x}^* = \boldsymbol{f}^* \qquad \text{[B.1]}$$

where $\boldsymbol{x}^*$is the vector of total hybrid-unit output, $\boldsymbol{A}^*$ is the matrix of hybrid-unit technical coefficients and $\boldsymbol{f}^*$ is the vector o hybrid-unit final demand.

The elements of Eq. [B.1] are partitioned in the following form

$$\boldsymbol{x}^* = \begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{x}_n \end{bmatrix} \qquad \boldsymbol{A}^* = \begin{bmatrix} \boldsymbol{A}^*_\theta & \boldsymbol{A}^*_\tau \\ \boldsymbol{A}^*_\pi & \boldsymbol{A}^*_\psi \end{bmatrix} \qquad \boldsymbol{f}^* = \begin{bmatrix} \boldsymbol{h} \\ \boldsymbol{f}_n \end{bmatrix}$$

where

- $\boldsymbol{g}$ and $\boldsymbol{x}^*_n$ are the sub-vectors of total energy output of energy and non-energy industries, respectively
- $\boldsymbol{A}^*_\theta$, $\boldsymbol{A}^*_\tau$, $\boldsymbol{A}^*_\pi$ and $\boldsymbol{A}^*_\psi$ are respectively the sub-matrices of technical coefficients of 1) transactions between energy industries; 2) transactions from energy industries to non-energy industries; 3) transactions from non-energy industries to energy industries; and 4) transactions between non-energy industries
- $\boldsymbol{h}$ and $\boldsymbol{f}_n$ are the sub-vectors of final demand of energy and non-energy industries, respectively

Rewriting Eq. [B.1] gives

$$\left(\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} - \begin{bmatrix} \boldsymbol{A}^*_\theta & \boldsymbol{A}^*_\tau \\ \boldsymbol{A}^*_\pi & \boldsymbol{A}^*_\psi \end{bmatrix}\right)\begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{h} \\ \boldsymbol{f}_n \end{bmatrix}$$

or

$$\begin{aligned} (\boldsymbol{I} - \boldsymbol{A}^*_\theta)\boldsymbol{g} - \boldsymbol{A}^*_\tau\boldsymbol{x}_n &= \boldsymbol{h} \\ -\boldsymbol{A}^*_\pi\boldsymbol{g} + (\boldsymbol{I} - \boldsymbol{A}^*_\psi)\boldsymbol{x}_n &= \boldsymbol{f}_n \end{aligned} \qquad \text{[B.2]}$$

Solving the system of two variables for $\boldsymbol{g}$, we obtain

$$\boldsymbol{g} = \left(\boldsymbol{I} - \boldsymbol{A}^*_\theta - \boldsymbol{A}^*_\tau(\boldsymbol{I} - \boldsymbol{A}^*_\psi)^{-1}\boldsymbol{A}^*_\pi\right)^{-1}\left(\boldsymbol{h} + \boldsymbol{A}^*_\tau(\boldsymbol{I} - \boldsymbol{A}^*_\psi)^{-1}\boldsymbol{f}_n\right)$$

Then, separating total energy use for fulfilling final energy demand and for production-related energy demand, the equation of total energy use is obtained as

$$\boldsymbol{g} = \begin{cases} \left(\boldsymbol{I} - \boldsymbol{A}_\theta^* - \boldsymbol{A}_\tau^*\left(\boldsymbol{I} - \boldsymbol{A}_\psi^*\right)^{-1}\boldsymbol{A}_\pi^*\right)^{-1}\boldsymbol{h} \\ + \\ \left(\boldsymbol{I} - \boldsymbol{A}_\theta^* - \boldsymbol{A}_\tau^*\left(\boldsymbol{I} - \boldsymbol{A}_\psi^*\right)^{-1}\boldsymbol{A}_\pi^*\right)^{-1}\boldsymbol{A}_\tau^*\left(\boldsymbol{I} - \boldsymbol{A}_\psi^*\right)^{-1}\boldsymbol{f}_n \end{cases} \quad \text{[B.3]}$$

To developed the model further, two themes must be explained

The inverse Leontief matrix of a partitioned hybrid-input-output model

The solution of the hybrid-unit system of Eq. [B.1] is

$$\boldsymbol{x}^* = \boldsymbol{L}^*\boldsymbol{f}^*$$

where $\boldsymbol{L}^*$ is the hybrid-unit Leontief inverse matrix .

This matrix is partitioned as follows

$$\boldsymbol{L}^* = \begin{bmatrix} \boldsymbol{L}_\theta^* & \boldsymbol{L}_\tau^* \\ \boldsymbol{L}_\pi^* & \boldsymbol{L}_\psi^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} - \boldsymbol{A}_\theta^* & -\boldsymbol{A}_\tau^* \\ -\boldsymbol{A}_\pi^* & \boldsymbol{I} - \boldsymbol{A}_\psi^* \end{bmatrix}^{-1}$$

From Bierens [62] , Faliva and Zoia [63] and Tian and Takane [64], we obtain

$$\boldsymbol{L}_\psi^* = \left(\boldsymbol{I} - \boldsymbol{A}_\psi^* - \boldsymbol{A}_\pi^*(\boldsymbol{I} - \boldsymbol{A}_\theta^*)^{-1}\boldsymbol{A}_\tau^*\right)^{-1} \quad \text{[B.4]}$$

$$\begin{aligned} \boldsymbol{L}_\pi^* &= \left(\boldsymbol{I} - \boldsymbol{A}_\psi^*\right)^{-1}\boldsymbol{A}_\pi^*\left(\boldsymbol{I} - \boldsymbol{A}_\theta^* - \boldsymbol{A}_\tau^*\left(\boldsymbol{I} - \boldsymbol{A}_\psi^*\right)^{-1}\boldsymbol{A}_\pi^*\right)^{-1} \\ &= \boldsymbol{L}_\psi^*\boldsymbol{A}_\pi^*(\boldsymbol{I} - \boldsymbol{A}_\theta^*)^{-1} \end{aligned} \quad \text{[B.5]}$$

and

$$\begin{aligned} \boldsymbol{L}_\tau^* &= \left(\boldsymbol{I} - \boldsymbol{A}_\theta^* - \boldsymbol{A}_\tau^*\left(\boldsymbol{I} - \boldsymbol{A}_\psi^*\right)^{-1}\boldsymbol{A}_\pi^*\right)^{-1}\boldsymbol{A}_\tau^*\left(\boldsymbol{I} - \boldsymbol{A}_\psi^*\right)^{-1} \\ &= (\boldsymbol{I} - \boldsymbol{A}_\theta^*)^{-1}\boldsymbol{A}_\tau^*\boldsymbol{L}_\psi^* \end{aligned} \quad \text{[B.6]}$$

The inverse of a matrix sum

According to Henderson and Searle [65], Miller [66] and Tylavsky and Sohie [67][6], the term $\left(\boldsymbol{I} - \boldsymbol{A}_\theta^* - \boldsymbol{A}_\tau^*\left(\boldsymbol{I} - \boldsymbol{A}_\psi^*\right)^{-1}\boldsymbol{A}_\pi^*\right)^{-1}$ can be decomposed as

$$\left(\boldsymbol{I} - \boldsymbol{A}_\theta^* - \boldsymbol{A}_\tau^*\left(\boldsymbol{I} - \boldsymbol{A}_\psi^*\right)^{-1}\boldsymbol{A}_\pi^*\right)^{-1} = (\boldsymbol{I} - \boldsymbol{A}_\theta^*)^{-1} + \boldsymbol{W}$$

where

$$\boldsymbol{W} = \left[\boldsymbol{I} - (\boldsymbol{I} - \boldsymbol{A}_\theta^*)^{-1}\boldsymbol{A}_\tau^*(\boldsymbol{I} - \boldsymbol{A}_\psi^*)^{-1}\boldsymbol{A}_\pi^*\right]^{-1}(\boldsymbol{I} - \boldsymbol{A}_\theta^*)^{-1}\boldsymbol{A}_\tau^*(\boldsymbol{I} - \boldsymbol{A}_\psi^*)^{-1}\boldsymbol{A}_\pi^*(\boldsymbol{I} - \boldsymbol{A}_\theta^*)^{-1}$$

[6] The general formula of the inverse of a matrix sum is $(\boldsymbol{A} + \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} - (\boldsymbol{I} + \boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}$

Moreover, introducing the expressions of $\boldsymbol{L}_{\pi}^{*}$ and $\boldsymbol{L}_{\psi}^{*}$ in Eqs. [B.4] and [B.5], an alternative expression for $\boldsymbol{W}$ is obtained:

$$\boldsymbol{W} = (\boldsymbol{I} - \boldsymbol{A}_{\theta}^{*})^{-1}\boldsymbol{A}_{\tau}^{*}\boldsymbol{L}_{\pi}^{*}$$

Therefore,

$$\left(\boldsymbol{I} - \boldsymbol{A}_{\theta}^{*} - \boldsymbol{A}_{\tau}^{*}\left(\boldsymbol{I} - \boldsymbol{A}_{\psi}^{*}\right)^{-1}\boldsymbol{A}_{\pi}^{*}\right)^{-1} = (\boldsymbol{I} - \boldsymbol{A}_{\theta}^{*})^{-1} + (\boldsymbol{I} - \boldsymbol{A}_{\theta}^{*})^{-1}\boldsymbol{A}_{\tau}^{*}\boldsymbol{L}_{\pi}^{*} \qquad \text{[B.7]}$$

Inserting Eqs. [B.4], [B.6] and [B.7] to Eq. [B.3] and separating total energy use for fulfilling final energy demand and for production-related energy demand, the expression of the vector of total energy use has the following form:

$$\boldsymbol{g} = [(\boldsymbol{I} - \boldsymbol{A}_{\theta}^{*})^{-1} + (\boldsymbol{I} - \boldsymbol{A}_{\theta}^{*})^{-1}\boldsymbol{A}_{\tau}^{*}\boldsymbol{L}_{\pi}^{*}]\boldsymbol{h} + (\boldsymbol{I} - \boldsymbol{A}_{\theta}^{*})^{-1}\boldsymbol{A}_{\tau}^{*}\boldsymbol{L}_{\psi}^{*}\boldsymbol{f}_{n}$$

From the input-output model of the energy sector in section 3.1.1, it could be inferred that $(\boldsymbol{I} - \boldsymbol{A}_{\theta}^{*})^{-1} = \boldsymbol{L}^{E}$ and $\boldsymbol{A}_{\tau}^{*} = \boldsymbol{C}_{n}^{E}\boldsymbol{T}^{E}$ of the PF-EIO model, therefore the final form of the PF-HEIO model is

$$\boldsymbol{g} = \underbrace{\boldsymbol{L}^{E}\boldsymbol{C}_{n}^{E}\boldsymbol{T}^{E}\boldsymbol{L}_{\psi}^{*}\boldsymbol{f}_{n}}_{Production-related} + \underbrace{(\boldsymbol{L}^{E} + \boldsymbol{L}^{E}\boldsymbol{C}_{n}^{E}\boldsymbol{T}^{E}\boldsymbol{L}_{\pi}^{*})\boldsymbol{h}}_{Residential} \qquad \text{[B.8]}$$

The previous expression shares most of the elements with the PF-EIO model, except for the hybrid-unit sub-matrices of the hybrid-unit inverse Leontief matrix (i.e $\boldsymbol{L}_{\psi}^{*}$ and $\boldsymbol{L}_{\pi}^{*}$). Therefore, as the example in Section 3.1.4 shows, the elements $\boldsymbol{L}_{\psi}$ and $\boldsymbol{L}_{\pi}\widehat{\boldsymbol{p}}$ (from the input-output model in monetary units) are approximately equal to $\boldsymbol{L}_{\psi}^{*}$ and $\boldsymbol{L}_{\pi}^{*}$, respectively.

Furthermore, the special case where the energy sector is independent from the rest of the economy is discussed.

If the energy sector is independent (i.e. the energy sector does not use any inputs from non-energy industries), then $\boldsymbol{Z}_{\pi}^{*} = \boldsymbol{0}$, $\boldsymbol{A}_{\pi}^{*} = \boldsymbol{0}$ and $\boldsymbol{L}_{\pi}^{*} = \boldsymbol{0}$ hence

$$\boldsymbol{g} = \boldsymbol{L}^{E}\boldsymbol{f}_{e}^{*} + \boldsymbol{L}^{E}\boldsymbol{C}_{n}^{E}\boldsymbol{T}^{E}\boldsymbol{L}_{\psi}^{*}\boldsymbol{f}_{n}^{*}$$

Under conditions of energy sector independence, $\boldsymbol{L}_{\psi}^{*} = \boldsymbol{L}_{\psi}$, i.e. the sub-matrices with sub-script ψ of the inverse Leontieff hybrid-unit and monetary matrices are equal (from Eq. [B.4]). So, the HEIO model and the PF-EIO models are exactly equal.

Appendix C: The average price of energy

The present appendix aims to determine the values of vector $\widehat{\boldsymbol{p}}$ that lead to the smallest error between the sub-matrix $\boldsymbol{\alpha}_\theta$ of the HEIO and the matrix expression $\boldsymbol{L}^E - \boldsymbol{L}^E\boldsymbol{C}_n^E\boldsymbol{T}^E\boldsymbol{L}_\pi\widehat{\boldsymbol{p}}$ of the PF-EIO model.

There are two options to determine the values of vector $\widehat{\boldsymbol{p}}$: 1) the actual average prices faced by final consumers (i.e. $\widehat{\boldsymbol{p}} = \widehat{\boldsymbol{h}}\widehat{\boldsymbol{f}_e^{-1}}$), or the economy-wide average energy prices (i.e. $\widehat{\boldsymbol{p}} = \widehat{\boldsymbol{g}}\widehat{\boldsymbol{x}_e^{-1}}$).

The sample economy in Section 3.1.4 will be used (Table 3-1 and Table 3-2) to compare both options. Moreover, the non-energy inputs to the energy sector, described by the sub-matrix $\boldsymbol{Z}_\pi$, are varied to change the level of independence of the energy sector. The level of independence of the energy sector (ESI) correspond to the share of non-energy inputs in the total input structure of the energy sector, i.e.

$$ESI = \frac{\sum \boldsymbol{Z}_\pi}{\sum \boldsymbol{Z}_\pi + \sum \boldsymbol{Z}_\theta} \times 100\%$$

where $\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{Z}_\theta & \boldsymbol{Z}_\tau \\ \boldsymbol{Z}_\pi & \boldsymbol{Z}_\psi \end{bmatrix}$ is the interindustry transaction matrix in monetary units (Table 3-1).

Interindustry transactions and energy flows data are used to build the HEIO and the PF-EIO models. Table C-1 presents the errors between entries of the sub-matrix $\boldsymbol{\alpha}_\theta$ of the HEIO and the matrix expression $\boldsymbol{L}^E - \boldsymbol{L}^E\boldsymbol{C}_n^E\boldsymbol{T}^E\boldsymbol{L}_\pi\widehat{\boldsymbol{p}}$ of the PF-EIO model depending on the value options for vector $\widehat{\boldsymbol{p}}$.

Table C-1 Effect of the value of $\widehat{\boldsymbol{p}}$ in the PF-EIO model

ESI	Error between entries $(\boldsymbol{\alpha}_\theta)_{kj} - (\boldsymbol{L}^E + \boldsymbol{L}^E\boldsymbol{C}^{E,S}\boldsymbol{T}^E\boldsymbol{\pi}^M\widehat{\boldsymbol{p}})_{kj}$	
	Avg. prices to final consumers $\widehat{\boldsymbol{p}} = \widehat{\boldsymbol{h}}\widehat{\boldsymbol{f}_e^{-1}}$	Economy-wide avg. prices $\widehat{\boldsymbol{p}} = \widehat{\boldsymbol{g}}\widehat{\boldsymbol{x}_e^{-1}}$
100%	0%	0%
80%	<|1.2%|	<|0.4%|
70%	<|1.6%|	<|0.5%|
60%	<|4%|	<|0.7%|
50%	<|6%|	<|0.8%|
40%	<|8%|	<|1.2%|
30%	<|10%|	<|1.5%|

The table shows that the use of economy-wide average prices ($\widehat{\boldsymbol{p}} = \widehat{\boldsymbol{g}}\widehat{\boldsymbol{x}_e^{-1}}$) in the PF-EIO model gives the smallest error with respect to the HEIO model at any level of energy sector independence.

Appendix D: A product-by-industry input-output model of the energy sector

A product-by-industry approach is used to build an isolated input-output model of the energy sector. For this approach, energy technologies (e.g. cogeneration, oil refinery or geothermal power generation) use energy carriers (e.g. diesel or coal) in conversion processes to produce one or more energy carriers for consumption of energy technologies and direct energy demand.

Data on energy flows in physical units from energy balances (e.g. IEA [50]) are arranged into a make-use table framework (Figure D-1).

The energy use matrix ($\boldsymbol{U}^E$) represents the amount of a specific energy carrier that is used as an input by an energy technology and also the amount that is delivered to direct energy demand (i.e. energy use by non-energy sectors). The energy make matrix ($\boldsymbol{V}^E$) shows the total supply of all energy commodities that are produced by a particular energy technology. $\boldsymbol{Y}^E$ is the matrix of direct energy demand by carrier and by direct demand category. $\boldsymbol{d}^E$ corresponds to the domestic production of primary energy carriers while $\boldsymbol{m}^E$ accounts for imports of primary and final energy carriers. The vector $\boldsymbol{w}^E$ represents the energy conversion losses in the production processes of energy technologies (all elements of this vector have a negative value). Finally, $\boldsymbol{q}^E$ and $\boldsymbol{x}^E$ are the vectors of total energy output by carrier and by technology, respectively.

		Energy carriers				Energy technologies				Direct Demand			Total output	
		1	2	$\cdots$	n_E	1	2	$\cdots$	n_T	1	$\cdots$	n_F		
Energy carriers	1					u_{11}^E	u_{12}^E	$\cdots$	$u_{1n_T}^E$	y_{11}^E	$\cdots$	$y_{1n_F}^E$	q_1^E	
	2					u_{21}^E	u_{22}^E	$\cdots$	$u_{2n_T}^E$	y_{21}^E	$\cdots$	$y_{2n_F}^E$	q_2^E	
	$\vdots$					$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\boldsymbol{q}^E$
	n_E					$u_{n_E1}^E$	$u_{n_E2}^E$	$\cdots$	$u_{n_En_T}^E$	y_{n_E1}	$\cdots$	$y_{n_En_F}^E$	$q_{n_E}^E$	
							$\boldsymbol{U}^E$				$\boldsymbol{Y}^E$			
Energy technologies	1	v_{11}^E	v_{12}^E	$\cdots$	$v_{1n_E}^E$								x_1^E	
	2	v_{21}^E	v_{22}^E	$\cdots$	$v_{2n_E}^E$								x_2^E	
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$								$\vdots$	$\boldsymbol{x}^E$
	n_T	$v_{n_T1}^E$	$v_{n_T2}^E$	$\cdots$	$v_{n_Tn_E}^E$								$x_{n_T}^E$	
				$\boldsymbol{V}^E$										
Domestic primary production		d_1^E	d_2^E	$\cdots$	$d_{n_E}^E$									
				$\boldsymbol{d}^E$										
Primary / final imports		m_1^E	m_2^E	$\cdots$	$m_{n_E}^E$									
				$\boldsymbol{m}^E$										
Conversion losses						w_1^E	w_2^E	$\cdots$	$w_{n_T}^E$					
								$\boldsymbol{w}^E$						
Total inputs		q_1^E	q_2^E	$\cdots$	$q_{n_E}^E$	x_1^E	x_2^E	$\cdots$	$x_{n_T}^E$					
				$(\boldsymbol{q}^E)'$				$(\boldsymbol{x}^E)'$						

Figure D-1 Make-use framework for product-by-industry data of the energy sector

The system consists of n_T energy technologies (e.g. oil refineries or wind power generation), n_E energy carriers (e.g. crude oil or electricity) and n_F non-energy sectors of direct energy demand (e.g. household or iron and steel industry). Furthermore, the total of n_E energy carriers are classified into three categories: (1) n_P primary energy carriers, endogenous and imported raw energy sources for conversion into final energy carriers (e.g. wind or crude oil); (2) n_C final energy carriers, energy products for direct use of economic sectors (e.g. electricity or fueloil); and (3) n_N non-energy-use energy carriers, outputs of the energy sector with non-energy uses (e.g. lubricant or paraffin).

The relationship between the total energy use by carrier and the direct energy demand is determined under the Industry Technology Assumption [8, 68-70] as

$$\boldsymbol{q}^E = \left(\boldsymbol{I} - \boldsymbol{U}^E(\widehat{\boldsymbol{x}}^E)^{-1}\boldsymbol{V}^E\widehat{\boldsymbol{q}^E}^{-1}\right)^{-1}\boldsymbol{Y}^E\boldsymbol{i}$$

where $\boldsymbol{L}^E = \left(\boldsymbol{I} - \boldsymbol{U}^E(\widehat{\boldsymbol{x}}^E)^{-1}\boldsymbol{V}^E\widehat{\boldsymbol{q}^E}^{-1}\right)^{-1}$ is the total product-by-product requirements matrix of the energy sector and $\boldsymbol{Y}^E\boldsymbol{i}$ is the direct demand for energy carriers.

Appendix E: An input-output model of the adjunct energy sector

This Appendix presents the construction of an input-output model of the adjunct energy sector based on the useful energy data obtained through the useful work accounting methodology[7].

In this model, the adjunct energy sector uses n_E final energy carriers as inputs to deliver n_U flows of useful work to non-energy sectors. Conversion technologies of the adjunct energy sector are equal in number to the flows of useful work (n_U), hence one conversion technology produces one and only one useful work flow. Moreover, there are neither transactions of useful work flows between conversion technologies nor transactions from non-energy sectors to the adjunct energy sector.

Useful work flows are defined by useful work category (e.g. lighting or hot temperature heat) and the type of final energy carrier (e.g. gasoline or electricity) that is used in the conversion process. Examples of useful work flows are: 1) lighting obtained from a kerosene lamp; 2) mechanical work obtained from an electric motor; or 3) low temperature heat obtained from a gas boiler.

Useful work flows ($\boldsymbol{F}^U$) and second-law final-to-useful efficiencies ($\boldsymbol{\epsilon}^{U,W}$) data are obtained from useful work accounting [37, 75, 76] in the form shown in Figure E-1.

The matrix $\boldsymbol{F}^{U,W}$ of size $n_U \times n_F$ is the direct useful work demand by type of useful work flow and by demand category. There is a matrix of second law final-to-useful efficiency ($\boldsymbol{\epsilon}^{U,W}$) with the same characteristics.

The input-output model of the adjunct energy sector aims to construct a simple relationship as:

$$\boldsymbol{F}^E = \boldsymbol{L}^U \boldsymbol{F}^{U,W}$$

where matrix $\boldsymbol{F}^E$ of of size $n_E \times n_F$ is the direct use of final energy carriers by the adjunct energy sector, matrix $\boldsymbol{L}^U$ of size $n_E \times n_U$ is the total final-to-useful energy conversion requirements matrix, and matrix $\boldsymbol{F}^{U,W}$ of size $n_U \times n_F$ is the direct demand of useful work flows by non-energy industries and final consumers.

---

[7] Useful work accounting (UWA) is an exergy accounting methodology [71, 72] at the useful level of energy use. This methodology was developed departing from the work of Ayres et al. [37] who introduced the concepts of useful work and second law final-to-useful efficiency in long-term energy transition studies.

Useful work or useful exergy (U^W) measures the effective amount of exergy (E^X) from final energy carriers that is delivered to a final function [73] , e.g. the mechanical work from electricity used by an elevator motor or the exergy of the heat provided by a gas heater. It is therefore closely related to energy services in the economy.

$$U^W = \epsilon E^X$$

On the other hand, the second law final-to-useful efficiency (ϵ) represents the fraction of an exergy input that is converted into useful work. The value of ϵ is a characteristic of each end-use energy technology and is subject to thermodynamic limits [74].

UW flows	UW category	Final energy carrier	Direct useful work demand 1	2	⋯	$\boldsymbol{n_F}$
1	1	1	f_{11}^U	f_{21}^U	⋯	$f_{1n_F}^U$
⋮		⋮	⋮	⋮	⋱	⋮
$\boldsymbol{n_E}$		n_E	$f_{n_E1}^U$	$f_{n_E2}^U$	⋯	$f_{n_En_F}^U$
$\boldsymbol{n_E+1}$	2	1	⋮	⋮	⋯	⋮
⋮		⋮	⋮	⋮	⋱	⋮
$\boldsymbol{2\cdot n_E}$		n_E	⋮	⋮	⋯	⋮
⋮	⋮	⋮	⋮	⋮	⋱	⋮
⋮	n_W	1	⋮	⋮	⋯	⋮
⋮		⋮	⋮	⋮	⋱	⋮
$\boldsymbol{n_U=n_W\cdot n_E}$		n_E	$f_{n_U1}^U$	$f_{n_U2}^U$	⋯	$f_{n_Un_F}^U$
					$\boldsymbol{F}^{U,W}$	

Figure E-1 Presentation of useful work accounting output data

According to the definition of useful work, direct exergy demand can be calculated from useful work flows by:

$$Final\ exergy = \frac{1}{\epsilon_{ij}} \cdot F_{ij}^{U,W}$$

The relationships requires a Hadamard or element-wise product (⊗) of matrix $\boldsymbol{F}^{U,W}$ by the inverse of matrix $\boldsymbol{\epsilon}^{U,W}$. The resulting matrix $\boldsymbol{F}^{X,W}$ of size $n_U \times n_F$ is the direct exergy demand by non-energy sectors per type of useful work flow.

$$\boldsymbol{F}^{X,W} = (\boldsymbol{\epsilon}^{U,W})^{-1} \otimes \boldsymbol{F}^{U,W}$$

The matrices $(\boldsymbol{\epsilon}^{U,W})^{-1}$ and $\boldsymbol{F}^{U,W}$ are rearranged into equivalent matrices $(\boldsymbol{\epsilon}^{U})^{-1}$ size $n_U \times (n_U \cdot n_F)$ and $\boldsymbol{F}^{U}$ of size $(n_U \cdot n_F) \times n_F$, respectively, so as to substitute the Hadamard product by a normal matrix multiplication.

$$\boldsymbol{Y}^{X,W} = (\boldsymbol{\epsilon}^{U})^{-1}\, \boldsymbol{F}^{U}$$

The final exergy demand per type of useful work flow ($\boldsymbol{Y}^{X,W}$) is transformed into final exergy demand per type of final energy carrier ($\boldsymbol{Y}^{X,E}$) through a bridge matrix $\boldsymbol{G}^{EC}$, whose architecture is shown in Figure E-2.

$$\boldsymbol{Y}^{X,E} = \boldsymbol{G}^{EC}\boldsymbol{Y}^{X,W} \quad \text{[E-1]}$$

UW flows	1	2	$\cdots$	$\boldsymbol{n_E}$	$\boldsymbol{n_E+1}$	$\boldsymbol{n_E+2}$	$\cdots$	$\boldsymbol{2\cdot n_E}$	$\cdots$		$\cdots$		$\boldsymbol{n_U}$
UW category		1				2					n_U		
Final energy carrier	1	2	$\cdots$	n_E	1	2	$\cdots$	n_E	$\cdots$	1	2	$\cdots$	n_E
1	1	0	$\cdots$	0	1	0	$\cdots$	0		1	0	$\cdots$	0
2	0	1	$\cdots$	0	0	1	$\cdots$	0		0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\boldsymbol{n_E}$	0	0	$\cdots$	1	0	0	$\cdots$	1		0	0	$\cdots$	1
							$\boldsymbol{G}^{EC}$						

Figure E-2 Architecture of the bridge matrix $\boldsymbol{G^{EC}}$

Furthermore, $\boldsymbol{F}^{X,E}$ is transformed from exergy into energy values by pre-multiplying by the inverse of the exergy factor vector $\boldsymbol{\phi}^E$ whose element ϕ_k^E is the exergy factor of the final energy carrier k.

$$\boldsymbol{F}^E = \left(\widehat{\boldsymbol{\phi}}^E\right)^{-\boldsymbol{1}} \boldsymbol{F}^{X,E} \qquad \text{[E-2]}$$

Finally, Eqs. [E-1] and [E-2] are combined to define the total final energy requirements matrix of the adjunct energy sector ($\boldsymbol{L}^U$)

$$\boldsymbol{L}^U = \left(\widehat{\boldsymbol{\phi}}^E\right)^{-\boldsymbol{1}} \boldsymbol{G}^{EC} (\boldsymbol{\epsilon}^U)^{-1}$$