A quantity output-driven model with heterogeneous intermediate and final outputs: towards a generalised input-output framework

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Abstract

The traditional quantity output-driven (Leontief) model is based on the assumption that outputs are homogeneous. This assumption is considered fundamental for the model to operate properly.

Although the actual heterogeneity of goods can partially be overcome by disaggregating input-output tables, such assumption constitutes a limitation of the modelling exercise. Also, in order to comply to this assumption, secondary production must be reallocated to other sectors instead of counting such production within the same sector, for example as a different (heterogeneous) final good. Thus, reallocation methods have been used to build symmetrical input-output tables according to the homogeneous goods assumption.

This paper aims to explore whether the homogeneous goods assumption can be dropped and, if so, to explore how would a quantity output-driven IO model work.

In this paper, the assumptions required by the traditional quantity output-driven (Leontief) model are reviewed together with the previous methods to account for secondary production. It is found that some methods applicable to physical input-output tables are already able to deal with simultaneously produced heterogeneous final outputs (e.g. disposals to nature).

In the analytical section of this paper, the usage of the homogeneous goods assumption is deconstructed. First, by illustrating how to deal with PIOTs and MIOTs with heterogeneous *intermediate* production. Second, by illustrating how to deal with PIOTs and MIOTs with heterogeneous *final* production. Building on the learnings from these sections, a generalised quantity output-driven model is suggested. It is demonstrated that the traditional quantity output-driven (Leontief) model

is a particular case of the generic quantity output-driven developed in this paper.

The generic quantity output-driven model makes it possible to build and analyse MIOTs and PIOTs without requiring to reallocate secondary production to the corresponding sector, i.e. secondary products can be considered within the intersectoral matrix and/or as final outputs. This enhances the analytical possibilities of IOA and opens the door to rethink how secondary production should be treated. Finally, this model is particularly interesting for Industrial Ecology, since enables researchers to trace the physical activity of the economy as is, i.e. each sector producing simultaneously different types of disposals to nature (e.g. emissions).

Contents

1 Introduction

Since the beginning of input-output analysis, the homogeneous goods assumption is considered a fundamental assumption in order to build the quantity output-driven (Leontief) model (Leontief, 1941; Miller and Blair, 2009; Suh, 2004).

However, being able to use a quantity output-driven model without using the homogeneous goods assumption would have the analytical advantage to be able to deal with heterogeneous outputs within the IO model itself, i.e., the model could potentially integrate secondary production within the intersectoral matrix and, also, as different final outputs, enhancing the analytical potential of IOA. Additionally, such model would be able to represent the production process *as is*, since secondary production is intrinsic within the productive structure, both in terms of producing different goods simultaneously and of producing several emissions and wastes as by-products (Suh, 2004; Xu and Zhang, 2009; Altimiras-Martin, 2014).

This paper aims to explore whether the homogeneous goods assumption can be dropped and, if so, to explore how would a quantity output-driven IO model work.

In the introductory section of this paper, first, the underlying assumptions behind the traditional quantity output-driven model are reviewed in section 1.1. Then, the analytical differences between monetary input-output tables (MIOTs) and physical input-output tables (PIOTs) are reviewed in section 1.2 to clarify the concept of secondary production (section 1.2.1), to review the differences between PIOTs and MIOTs since quantity output-driven models should be applicable to both types of tables (section 1.2.2), and to review of the use of IO methods adapted to PIOTs with disposals to nature since such tables entail heterogeneous final outputs (section 1.2.3). Finally, the previous methods used to reallocate secondary production in IOA are reviewed (section 1.3).

In the analytical section of this paper, the usage of the homogeneous goods assumption is deconstructed. First, by illustrating how to deal with PIOTs and MIOTs with heterogeneous *intermediate* production (section 2.1.1). Second, by illustrating how to deal with PIOTs and MIOTs with heterogeneous *final* production (section 2.1.2). Building on the learnings from these sections, a generalised quantity output-driven model is suggested in section 2.2. Finally, in section 2.3, it is illustrated how to partially use the homogeneous goods assumption on the generic model to simplify its operation. In section 2.3.3, it is shown that the traditional quantity output-driven model is a particular case of the generic quantity output-driven developed in section 2.2

Finally, the results are discussed in section 3.

1.1 On the underlying assumptions of the quantity output-driven (Leontief) model

IOA relies on monetary input-output tables (MIOTs) as accounting framework[1]. MIOTs capture the monetary flows between sectors, not the price nor quantity exchanged (the flow equals the quantity times the price). To emphasise this distinction, monetary flows will have a superscripted f, and quantity flows will have a superscripted q.

Typically, a MIOT entails the intersectoral flows bought (or sold) between sectors z_{ij}^f constituting the intersectoral matrix ($\mathbf{Z}^f$), the primary inputs provided to the economic system[2] (v_j^f) constituting the value added vector ($\mathbf{v}^{f\prime}$) and final outputs sold to the exogenous final demand[3] (f_i^f) constituting the final demand vector ($\mathbf{v}^f$).

So, each monetary flow has a price ($p_{z_{ij}}$, p_{v_j} or p_{f_i}) associated to it and the following relations between monetary and quantity flows can be established:

$$z_{ij}^f = p_{z_{ij}} \cdot z_{ij}^q \tag{1}$$

$$v_j^f = p_{v_j} \cdot v_j^q \tag{2}$$

$$f_i^f = p_{f_i} \cdot f_i^q \tag{3}$$

Additionally, the double-entry bookkeeping principle guarantees that sectoral inputs equal sectoral outputs. Since the flows that enter a sector, also exit it, the double entry relationship leads to the concept of the total outputs[4] x_i^f, as follows

$$\mathbf{Z}^f \cdot \mathbf{i} + \mathbf{f}^f = \left(\mathbf{i} \cdot \mathbf{Z}^f + \mathbf{v}^{f\prime}\right)' = \mathbf{x}^f \tag{4}$$

Focussing on the outputs of the economy, equation 4 can be rewritten as:

$$\sum_{j=1}^{n} z_{ij}^f + f_i^f = x_i^f \tag{5}$$

or

$$\mathbf{Z}^f \cdot \mathbf{i} + \mathbf{f}^f = \mathbf{x}^f \tag{6}$$

A MIOT for an n sectors economy is presented in table 1.

[1] Hybrid and Physical IOTs can also be used as accounting frameworks. The PIOTs is reviewed in detail in section 1.2. Hybrid tables and models are not reviewed since they are a combination of monetary and physical IOTs and models, which are already reviewed.

[2] Imports can be accounted as primary inputs or negative final outputs. In this paper, for simplicity, imports are not considered; only value added is considered as input.

[3] Final demand can be decomposed between gross capital formation, household consumption and exports. In this paper, they are represented aggregately for simplicity.

[4] Also known as total inputs or sectoral throughput.

	Sector 1	...	Sector n	Final demand	Total outputs
Sector 1					
⋮		$\mathbf{Z}^f$		$\mathbf{f}^f$	$\mathbf{x}^f$
Sector n					
Value added		$\mathbf{v}^{f\prime}$			
Total inputs		$\mathbf{x}^{f\prime}$			

Table 1: Monetary Input-Output Table with n sectors.

The quantity output-driven model devised by Leontief (1941) relies on three key assumptions:

1. The homogeneous goods assumption
2. The unitary price assumption
3. The proportionality assumption (also known as linearity or constant returns to scale assumption).

This paper aims to explore whether these assumptions are indissociable, i.e. whether we can only use some of these assumptions. And fundamentally, which assumptions are strictly required to perform Input-Output Analysis. These questions will be explored in depth in section 2. Below, these three assumptions are reviewed as they have been traditionally used in IOA.

1.1.1 The homogeneous goods assumption

The homogeneous goods assumption implies that each sector produces a single type of output, regardless whether intermediate or final. Algebraically, the flow values from each row (i.e. z_{ij}, f_i and x_i) refer to a single type of good. This assumption has two key implications:

1. Each good can be identified unequivocally to a specific sector. This implication is extremely useful analytically because it means that all values from each row refer to the same product. If this was not the case, e.g., each value corresponded to a different product, the researcher should remember which values corresponded to which product, complicating the analysis of input-output tables.
2. A single price exist for all products produced by each sector.

Thanks to the second implication, all prices from each row are equal. Algebraically,

$$p_{z_{i1}} = \ldots = p_{z_{in}} = p_{f_i} = p_{x_i} = p_i \qquad \forall i = [1, n] \tag{7}$$

Then, using equations 1 and 3, equation 5 representing the relationship between *monetary* flows becomes the following equation representing the *quantity* flows

$$\sum_{j=1}^{n} p_i \cdot z_{ij}^q + p_i \cdot f_i^q = p_i \cdot x_i^q \tag{8}$$

One is tempted to simplify p_i from each side of the equation but this would not solve the issue that the actual values from the MIOT correspond to monetary flows and not to the quantity flows, i.e. the quantity values remain unknown. In other words, the homogeneous goods assumption makes it easier to identify the correspondence between monetary values and underlying type of good, but it does not solve the issue that the actual values of the MIOT are in monetary terms and neither prices nor quantities are known. This issue is actually solved by using the unitary price assumption.

1.1.2 The unitary price assumption

The unitary price assumption makes all prices equal one, implying a change of accounting units and making the monetary flows equal the quantity flows. Consequently, equations 1, 2, and 3 become

$$z_{ij}^f = 1 \cdot z_{ij}^q \tag{9}$$
$$v_j^f = 1 \cdot v_j^q \tag{10}$$
$$f_i^f = 1 \cdot f_i^q \tag{11}$$

This is a counter-intuitive assumption but it allows researchers to consider the monetary flows as quantity flows. According to Leontief (1986, pp. 22–23)[5],

> All figures [in the value transactions table]... can also be interpreted as representing physical quantities of the goods or services to which they refer. This only requires that the physical unit in which the entries... are measured be redefined as being equal to that amount of output of that particular sector that can be purchased for $1 at [base year] prices... In practice the structural matrices are usually computed from input-output tables described in value terms... In any case, the input coefficients [A] — for analytical purposes... must be interpreted as ratios of two quantities measured in *physical units* [emphasis added].

For example, if a each kilo of good A is sold at $p = 5[\frac{\$}{1kg}]$, and two kilos were sold, i.e. $q = 2[kg]$, the monetary flow is $5[\frac{\$}{1kg}] \cdot 2[kg] = 10\$$. Using the

[5] As quoted in Miller and Blair (2009)

unitary price assumption, only a change of units happens. The same example becomes: $p = 1[\frac{\$}{\frac{1}{5}kg}]$ (note the change in the units), and now the quantity becomes $q = 10[\frac{1}{5}kg]$, so the monetary flow remains 10. The quantity value has changed from two kilos $q = 2[kg]$ to ten fifths of a kilo $q = 10[\frac{1}{5}kg]$, but it does not matter since units are a convention, i.e. we can measure the same quantity in different units. The key consequence is that the monetary flows can be considered quantity flows without requiring to know the prices nor the quantities.

1.1.3 The proportionality assumption

The proportionality assumption means that *intermediate* production is proportional to *total* production by means of the technical coefficients matrix **A**. Although the unitary price assumption has been applied and the monetary flows can now be considered quantity flows, the same f superscript is maintained because the values used are the ones from the actual MIOT, which correspond to the monetary values. So,

$$\mathbf{Z}^f = \mathbf{A} \cdot < \mathbf{x}^f > \Leftrightarrow \mathbf{A} = \mathbf{Z}^f \cdot < \mathbf{x}^f >^{-1} \tag{12}$$

1.1.4 The quantity output-driven (Leontief) model

Finally, the Leontief model is derived by substituting equation 12 in the output side of equation 4:

$$\mathbf{A} \cdot \mathbf{x}^f + \mathbf{f}^f = \mathbf{x}^f \tag{13}$$

Rearranging,

$$\mathbf{x}^f = (\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{f}^f \tag{14}$$

where

$$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} \tag{15}$$

is known as the Leontief inverse matrix.

Thus, the proportionality assumption is key because it transforms the output side of equation 4 into a set of n linear equations with n unknowns, which is solvable.

The other implication of the proportionality assumption is that final demand is also proportional to total outputs (c.f. equation 14) and, thus, it is also proportional to intermediate production as well.

The proportionality assumption is also applied to the primary inputs. It is assumed that primary inputs (i.e. the value added) are required proportionally to the total amount of sectoral inputs, i.e. there is an input coefficient ($\mathbf{c}^v$) relating the value added values and total inputs, as follows

$$\mathbf{c}^{v\prime} = \mathbf{v}' \cdot < \mathbf{x}^f >^{-1} \tag{16}$$

So, the Leontief model relies on equation 14, which enables researchers to run the model by providing a new final demand vector. The equation provides a new total outputs vector, which is used to recalculate the value added vector (using equation 16) and intermediate production (using equation 13).

Since the Leontief model uses the unitary price assumption, turning monetary flows into quantity flows, and the proportionality assumption on the output relationship, the model is traditionally called the quantity output-driven model.

It is worth noting that the Leontief model works with *homogeneous* outputs and *heterogeneous* inputs. The homogeneous goods assumption implies that outputs are homogeneous but each row produces a different product, so the model is able to cope with the relationship between different products.

Another implication of the homogeneous (output) goods assumption is that the model cannot deal with secondary production. Hence, several methods have been developed to reallocate secondary production, either before building the symmetrical MIOT or even to be able to deal with secondary production within symmetrical MIOT. Such methods are reviewed in section 1.3.

1.2 On MIOTs and PIOTs

Monetary Input-Output Tables (MIOTs) and Physical Input-Output Tables (PIOTs) have the same accounting relationship (total inputs equal total outputs) and have a similar structure: they both have primary inputs, intermediate products, and final products. PIOTs do not require prices so it might be thought that their treatment should be simpler. However, PIOTs have heterogeneous outputs because sectors produce their products and several disposals to nature (i.e. waste or emissions)[6], so the conventional Leontief model cannot be used in this case (Suh, 2004; Xu and Zhang, 2009; Altimiras-Martin, 2014).

[6] In this paper, only PIOTs with disposals to nature, i.e. with heterogeneous final outputs, are considered. The properties and operation of PIOTs without disposals to nature are analysed in Weisz and Duchin (2006).

1.2.1 On secondary production and intrinsic vs. price heterogeneity

A key point to clarify is whether all heterogeneous goods can be considered secondary production, either within a MIOT or a PIOT. To examine this question, two questions might be posed:

1. Is all secondary production heterogeneous?
   Yes, because if it is secondary, it is different from the primary production and, thus, heterogeneous. It must be noted that several different products can be produced simultaneously by a sector. And these products can simultaneously be actual goods, emissions and/or waste.
2. Can all heterogeneous goods be considered secondary production?
   Yes, because a sector producing heterogeneous goods means that it produces a product (which can be considered the primary product) and other heterogeneous goods, which constitute the secondary production.

Thus, heterogeneous goods and secondary production can be used interchangeably, both within the MIOT and PIOT frameworks.

A key difference between PIOTs and MIOTs is that MIOTs have an extra layer of complexity due to the fact that they are based on monetary flows rather than physical flows. As seen in section 1.1.1, the unitary price assumption is required in order to consider the monetary flows in MIOTs as physical flows. In PIOTs, this assumption is not required.

This leads to a sub-categorisation of heterogeneous goods:

- products can be intrinsically heterogeneous meaning that products are materially (i.e. physically) different. This can happen in PIOTs (e.g. a sector produces several types of goods, wastes and emissions) and MIOTs (e.g. a sector produces several types of goods).
- products can be heterogeneous due to their (different) price despite being materially equal — i.e. homogeneous from a physical perspective. For instance, the same fuel (i.e. an homogeneous good) can have different prices depending to which sector it is sold, so it is heterogeneous. This type of heterogeneity only applies to MIOTs.

1.2.2 Analytical differences between MIOTs and PIOTs

A PIOT can be intuitively constructed adopting the same structure as a MIOT by using the principle of mass conservation ("the materials that come in, go out") and using physical units instead of monetary ones (hence, the superscript q for quantity will be used instead of the superscript f used in the MIOT case, e.g. table 1).

So, PIOTs also have three quadrants: the first one consists of the total amount of primary resources required by the economy $\mathbf{r}^{q\prime}$, the second quadrant relates the material exchanges between the sectors of the economy $\mathbf{Z}^q$, and the third represents the final outputs of the system: the final goods for consumption $\mathbf{f}^q$ and the corresponding sectoral emissions disposed to nature $\mathbf{w}^q$ due to *total* sectoral production (*both* intermediate and final).

The disposals to nature can represent different material flows, depending on the materials traced: e.g. waste, pollution and even non-pollutant emissions. In any case, PIOTs have at least two different final outputs final goods and the corresponding disposals to nature, i.e. they produce *heterogeneous final goods*. To highlight this difference relative to MIOTs, the total output of an IOT with heterogeneous final goods is indicated as $\underline{\mathbf{x}}^q$. A PIOT following this notation is presented in table 2.

In this section it is assumed that the intermediate and final production of goods within a PIOT are homogeneous and only the disposals to nature are heterogeneous (compared to the goods produced by each sector). This assumption will be removed in section 2.3.

	Sector 1	...	Sector n	Final demand	Waste	Total outputs
Sector 1						
⋮		$\mathbf{Z}^q$		$\mathbf{f}^q$	$\mathbf{w}^q$	$\underline{\mathbf{x}}^q$
Sector n						
Resources		$\mathbf{r}^{q\prime}$				
Total inputs		$\underline{\mathbf{x}}^{q\prime}$				

Table 2: Structure of an PIOT with two heterogeneous final outputs (final goods $\mathbf{f}$ and emissions $\mathbf{w}$). All components are in physical units.

1.2.3 Treatment of disposals to nature in PIOTs using output-driven models

The fact that this table contains *two heterogeneous final outputs* posed a great challenge to be able to calculate the primary resources and disposals to nature due to a given final demand (even if intermediate and final goods are assumed homogeneous).

Below, the superscript q is removed for simplicity since PIOTs do not contain monetary flows.

This is because the output accounting relationship now includes an heterogeneous final output — disposal to nature $\mathbf{w}$ —, so equation 6 should be rewritten as

$$\underline{\mathbf{x}} = \mathbf{Z} \cdot \mathbf{i} + \mathbf{f} + \mathbf{w} \tag{17}$$

The technical coefficients matrix must be redefined accordingly as

$$\underline{\mathbf{A}} = \mathbf{Z} \cdot \hat{\underline{\mathbf{x}}}^{-1} \tag{18}$$

which, inserted in equation 17, leads to

$$\underline{\mathbf{x}} = (\mathbf{I} - \underline{\mathbf{A}})^{-1} \cdot (\mathbf{f} + \mathbf{w}) \tag{19}$$

This last equation poses a great issue: how to calculate total outputs when only **f** is known? (one knows the initial pair **f** and **w** but not the emissions associated with any other final demand; that is why Hubacek and Giljum (2003) and Giljum and Hubacek (2004) were forced to estimate the emissions exogenously). In short, the issue with these early methods is that they treat emissions as exogenous while they are endogenous, since they are generated according to the total amount of intermediate and final goods produced — i.e., emissions are the by-products of production so they can be considered secondary production.

Suh (2004) suggested three methods to operate a PIOT, of which only the second one gathered correct results (Altimiras-Martin, 2014). Suh's method consists on a change of units of the PIOT that subtracts the disposals to nature from the final outputs. This change of units turns the PIOT with heterogeneous final outputs (**f** and **w**) into a PIOT with a single output (**f**) and a negative primary input (**w**). Thanks to this transformation, the original PIOT with heterogeneous final outputs becomes a PIOT with homogeneous final outputs and the Leontief could be applied as usual. So, this transformation enables researchers to apply the traditional Leontief model to a table where the model could not be initially applied.

However, from a modelling perspective, Suh's method is a workaround that could be avoided by applying the proportionality assumption correctly; Xu and Zhang (2009) were the first ones to do that, i.e. to consider disposals to nature proportional to the total outputs. This idea makes sense both from the productive perspective and from the theoretical IOA perspective. From a productive perspective, a sector uses a certain amount of primary resource (and intermediate goods) to produce goods. However, the material that are not included in the goods are disposed to nature. If goods are produced proportionally to the resources consumed, so must be the disposals to nature (i.e. emissions and wastes). From a theoretical perspective, the proportionality assumption is not well applied in equation 19 because **w** should also be proportional to $\underline{\mathbf{x}}$.

As suggested in Xu and Zhang (2009), **w** can be related to $\underline{\mathbf{x}}$ as follows:

$$\mathbf{w} = \mathbf{E} \cdot \underline{\mathbf{x}} \tag{20}$$

In other words, the emissions are related to the sectoral throughput and, thus, they are endogenously determined since $\mathbf{w}$ is a function of $\underline{\mathbf{x}}$, solving the analytical issue mentioned above.

$\mathbf{E}$ can be found by diagonalising both sides of the equation

$$\mathbf{E} = \hat{\mathbf{w}} \cdot \hat{\underline{\mathbf{x}}}^{-1} \tag{21}$$

The assumption that intermediate production is proportional to total outputs is maintained. However, since total output units include emissions, the technical coefficients matrix is

$$\underline{\mathbf{A}} = \mathbf{Z} \cdot \hat{\underline{\mathbf{x}}}^{-1} \tag{22}$$

By combining equations 20 and 22 into 17, Xu and Zhang derive a new Leontief inverse matrix which includes the emissions or, in other words, endogenises the emission generation in the production structure:

$$\underline{\mathbf{x}} = \underline{\mathbf{A}} \cdot \underline{\mathbf{x}} + \mathbf{f} + \mathbf{E} \cdot \underline{\mathbf{x}} \tag{23}$$

$$\underline{\mathbf{x}} = (\mathbf{I} - \underline{\mathbf{A}} - \mathbf{E})^{-1} \cdot \mathbf{f} \tag{24}$$

$$\underline{\mathbf{x}} = \underline{\mathbf{L}} \cdot \mathbf{f} \tag{25}$$

where

$$\underline{\mathbf{L}} = (\mathbf{I} - \underline{\mathbf{A}} - \mathbf{E})^{-1} \tag{26}$$

Xu and Zhang were therefore able to relate the final production $\mathbf{f}$ to $\underline{\mathbf{x}}$ by means of a modified Leontief inverse $\underline{\mathbf{L}}$ by applying rigorously the proportionality assumption.

In fact, the proportionality assumption can be extended to any amount of disposals to nature, i.e. for any number of heterogeneous final outputs. According to Altimiras-Martin (2014), for the case of a PIOTs with m *heterogeneous final products*, final demand $\mathbf{f}$ is chosen to drive the model and the other $m-1$ disposals to nature — the different emissions of the PIOT represented as $\mathbf{w}_k$ with $k = [1, ..., m-1]$ — are considered proportional to total outputs and endogenised in the production structure.

Since in table 3 there are m heterogeneous final outputs, equation 23 is modified to accommodate the $m-1$ emissions (one of the final heterogeneous final outputs has to drive the model, hence the $m-1$) :

$$\underline{\mathbf{x}} = \mathbf{Z} \cdot \mathbf{i} + \mathbf{f} + \mathbf{w}_1 + \mathbf{w}_2 + \ldots + \mathbf{w}_{m-1} \tag{27}$$

The technical coefficients matrix is calculated as in the single emission case (using equation 22). The proportionality assumption is applied to all

	Sector 1	...	Sector n	Final outputs				Total outputs
Sector 1								
$\vdots$		$\mathbf{Z}$		$\mathbf{f}$	$\mathbf{w}_1$	...	$\mathbf{w}_{m-1}$	$\underline{\mathbf{x}}$
Sector n								
Resources		$\mathbf{r}'$						
Total inputs		$\underline{\mathbf{x}}'$						

Table 3: Structure of a PIOT with m heterogeneous final outputs.

heterogeneous final outputs except to the one driving the model ($\mathbf{f}$ in this case), so equation 20 is generalised to

$$\mathbf{w}_k = \mathbf{E}_k \cdot \underline{\mathbf{x}} \qquad \text{for } k = 1, ..., m-1 \tag{28}$$

Using equation 28 in equation 27:

$$\underline{\mathbf{x}} = \mathbf{A} \cdot \underline{\mathbf{x}} + \mathbf{f} + \mathbf{E}_1 \cdot \underline{\mathbf{x}} + \mathbf{E}_2 \cdot \underline{\mathbf{x}} + \ldots + \mathbf{E}_{m-1} \cdot \underline{\mathbf{x}} \tag{29}$$

$$\underline{\mathbf{x}} = (\mathbf{I} - \mathbf{A} - \mathbf{E}_1 - \mathbf{E}_2 - \ldots - \mathbf{E}_{m-1})^{-1} \cdot \mathbf{f} = \underline{\mathbf{L}} \cdot \mathbf{f} \tag{30}$$

The Leontief inverse resembles the one in equation 26 but now includes $m-1$ heterogeneous final outputs (emissions in this particular case):

$$\underline{\mathbf{L}} = (\mathbf{I} - \mathbf{A} - \mathbf{E}_1 - \mathbf{E}_2 - \ldots - \mathbf{E}_{m-1})^{-1} \tag{31}$$

The main difference between the methods developed by Suh (2004) and Xu and Zhang (2009) is that Suh's relies on a change of unit altering the structure of the PIOT from a heterogeneous outputs one to a homogeneous final outputs PIOT while Xu and Zhang's relies on applying the proportionality assumption to the disposals to nature. So, while Suh's method relies on the traditional Leontief model, Xu and Zhang's constitutes is in fact a new IO model able to deal with heterogeneous final outputs (Altimiras-Martin, 2014).

The consequence of these methodological differences is that both methods reveal different structures (i.e. different technical coefficients and Leontief inverse matrices), gathering different results when performing structural analyses such as backward and forward linkage analysis. The PIOT transformed by the second method in Suh (2004) reveals the structure of the economy as if no emissions were produced, so only the model developed by Xu and Zhang (2009) should be used if the *complete* physical structure of the economy (i.e. tracing goods and emissions) is to be analysed (Altimiras-Martin, 2014).

So, the model developed by Xu and Zhang (2009) and extended in Altimiras-Martin (2014) demonstrates that it is possible to operate IOTs with heterogeneous final outputs, at least in the case where the flows represent quantities. The next questions are whether it is possible to apply this model to MIOTs (i.e. with IOTs with monetary flows) and whether it makes sense analytically. The answers to these are sought in section 2.

It is important to note that Xu and Zhang (2009) and Altimiras-Martin (2014) implicitly assumed that intermediate production is homogeneous and it is homogeneous to one of the final outputs (the final demand $\mathbf{f}$ in the case above). In section 2, it will also be explored whether this assumption is mandatory to use IO models or whether it is possible to use IO models with heterogeneous intermediate production.

1.3 Methods to account for secondary goods production

The methods aiming to reallocate secondary production so as to build MIOTs that comply to the homogeneous goods assumption are reviewed in section 1.3.1. Then, the only method reallocating secondary production within the intersectoral matrix is reviewed in section 1.3.2.

1.3.1 Methods in SUTs

Since the homogeneous goods assumption has been considered as fundamental in order to use the traditional quantity output-driven (Leontief) model, most methods have focussed to reallocate secondary production so as to build symmetrical monetary input-output tables that have sectors producing outputs as homogeneous as possible.

Initially, secondary products (and the inputs required for their production) were reallocated to the sector producing such goods as a primary output by constructing a reallocation table, which was added to the transaction matrix, inducing double counting (Miller and Blair, 2009).

The development of the commodity-by-industry framework, also known as the Supply and Use Tables (SUT) framework, made it possible to reallocate secondary production more efficiently. Despite this more sophisticated framework, reallocating secondary production is still challenging. Production processes are varied and secondary production is sometimes induced due for different reasons. In some cases, secondary production is intrinsic to the production process: e.g., a company might produce two different types of fuels simultaneously because its primary input can only be decomposed in this two types of fuel, at a given proportion. In other cases, secondary production is not tight to the inputs used by the sector. This two cases require different treatment within the SUT framework, the former through a commodity-based technology model and the latter an industry-based technology model (Miller and Blair, 2009, chap. 5). But the economic system entails both at the same time, so mixed models can also be used.

At least seven models to integrate the information from the Supply and Use Tables and build symmetrical MIOTs exist (Jansen and Raa, 1990). Each of these methods have its particularities (Raa et al., 1984; Jansen and Raa, 1990; Raa and Rueda-Cantuche, 2003), so different statistical offices have adopted different methods.

1.3.2 Methods in SIOTs

Stone (1961) developed a method to deal with secondary production within the intersectoral matrix, as presented in Nakamura and Kondo (2009, sec. 3.2.2). It is argued that the identity matrix within the traditional Leontief inverse matrix (recall equation 15) represents the production of the sector itself. Since it is an identity matrix, each sector only produces a single output.

However, Stone (1961) suggested that the identity matrix could be substituted by a matrix indicating the secondary production of each sector (the value outside the diagonal would indicate secondary production).

Nakamura and Kondo (2009, sec. 3.2.2) show how to use this approach within the Leontief model. By using this method, "an increase in the final demand for the primary product of a sector with a by-product would increase the supply of the by-product, and would reduce its supply from the sector that produces it as the primary product" (Nakamura and Kondo, 2009, pp. 92). So, this method sets a precedent in dropping the homogeneous goods assumption.

Note that this method deals with secondary production within the intersectoral matrix, but cannot account for secondary production sold as final outputs.

2 Towards a generalised output-driven model without homogeneous goods assumption

2.1 Deconstructing the homogeneous goods assumption

The homogeneous goods assumption regarding intermediate production and final production will be challenged in sections 2.1.1 and 2.1.2, respectively. In both cases, PIOTs will be examined first since they are less complex because their flows are quantities and, then, the homogeneous goods assumption will be reviewed in MIOTs.

2.1.1 Dealing with heterogeneous intermediate goods

In PIOTs Let's assume a PIOT *without* disposals to nature. All flows are in physical units, and it represents an economy extracting primary resources $\mathbf{r}$, exchanging goods $\mathbf{Z}$ and providing final goods for its consumers $\mathbf{f}$, where sectoral total inputs equal total outputs $\mathbf{x}$. This PIOTs is presented in table 4.

	Sector 1 ... Sector n	Final demand	Total outputs
Sector 1 ⋮ Sector n	$\mathbf{Z}$	$\mathbf{f}$	$\mathbf{x}$
Primary resources	$\mathbf{r}'$		
Total inputs	$\mathbf{x}'$		

Table 4: Physical Input-Output Table with n sectors and one final output.

The output accounting relationship would be the same as in equation 6, except for the type of flows, which are already in physical units:

$$\mathbf{Z} \cdot \mathbf{i} + \mathbf{f} = \mathbf{x} \tag{32}$$

This relationship is maintained even if the intermediate products are heterogeneous. In physical terms, the sum of its physical units (e.g. kilograms) holds despite the product heterogeneity. Also, the mass balance principle holds (since matter cannot disappear), so total inputs equals total outputs.

Under this circumstances, which assumptions are required to be able to build an IO model for this particular PIOT?

Let's assume the homogeneous goods assumption is not used, i.e. intermediate products are heterogeneous and the final good produced is also different from intermediate production. To emphasise this difference, the intersectoral matrix will be noted $\mathbf{Z}^{het}$.

Note that, since flows are already in quantities, the unitary price assumption is not required.

Finally, the proportionality assumption is required to establish a relationship between the intermediate flows and final demand. In this case, this assumption requires the same as in the traditional Leontief model: to create a technical coefficients matrix $\mathbf{A}^{het}$, as follows:

$$\mathbf{Z}^{het} = \mathbf{A}^{het} \cdot < \mathbf{x} > \Leftrightarrow \mathbf{A} = \mathbf{Z}^{het} \cdot < \mathbf{x} >^{-1} \tag{33}$$

As in the traditional Leontief model, the quantity IO model for heterogeneous intermediate goods is built by inserting equation 33 in 32 to find the traditional Leontief inverse matrix, as in the traditional Leontief model, except that here the homogeneous goods assumption is not used. Thus,

$$\mathbf{x} = (\mathbf{I} - \mathbf{A}^{het}) \cdot \mathbf{f} \tag{34}$$

where

$$\mathbf{L}^{het} = (\mathbf{I} - \mathbf{A}^{het})^{-1} \tag{35}$$

would be the equivalent to the Leontief inverse matrix.

What is the analytical meaning of this model? This model is operated as a traditional Leontief (quantity, output-driven) model, with the only difference that each value of the IOT represents the quantity of a different product. This difference affects the interpretations of the Leontief inverse matrix $\mathbf{L}^{het}$ only.

In particular, the technical coefficient matrix can be understood as the direct requirements to produce each final good (this would be equivalent to multiply the technical coefficients matrix by a unitary vector representing the production of a particular final good). Thus, each column of the technical coefficients matrix represents the direct requirements of the sector in order to produce its final good. Since the homogeneous good assumption only affects sectoral outputs, the technical coefficients matrix represents the heterogeneous inputs required by each sector in order to produce its outputs, both in the traditional Leontief model and in this case, where heterogeneous intermediate outputs are considered.

The case of the Leontief inverse matrix is slightly different. The Leontief matrix is also known as the total requirements matrix, i.e. it reveals the total amount of outputs required economy-wide to produce each final output. Therefore, each column of the Leontief matrix represents the total production of each sector in order to produce the final good of that column sector. When the homogeneous good assumption is used, the Leontief inverse matrix provides the information on the total amount of goods produced and since only one good is produced by each sector, it is known how much of this product is produced. However, if the homogeneous good assumption is *not* used, the Leontief inverse matrix *still* provides the information on the total amount of goods produced but since each sector produces several heterogeneous intermediate outputs, the composition of the total requirements is unknown, i.e. it is unknown how much of each intermediate good is required in total. However, this is not a limiting issue since the model can be used to recalculate the flows associated to any given final demand and find the unknown mix composition. It only affects the interpretation of the production multipliers as an aggregate because the cannot be associated to a single intermediate product.

These differences are illustrated in the following numerical example.

Numerical example Let's imagine an economy with two sectors, the agricultural and a manufacturing sector. The agricultural sector sells almond oil to the manufacturing sector and apples to final consumers and keeps part of the almonds and apples for itself, to have seeds for next year. The manufacturing sector sells shovels to the agricultural sector, tables to final consumers, and screwdrivers to itself (to be able to assemble the shovels and tables). These flows are presented in table 5 in kilogram units.

In this economy, the sectors are as interdependent as in a normal economy, e.g. the manufacturing sector relies on the agricultural sector to provide oil to grease and produce its tools and, similarly, the agricultural sector relies on the manufacturing to provide the tools to dig the earth.

	Agric.	Manuf.	Final demand	Total outputs
Agriculture	5	8	20	33
Manufacturing	3	6	25	34
Resources	25	20		
Total inputs	33	34		

Table 5: Two sector PIOT with heterogeneous intermediate goods and a single final output. All values in physical units: kilograms.

Using equation 33 and 35, $\mathbf{A}^{het} = \left(\begin{smallmatrix} 0.152 & 0.235 \\ 0.091 & 0.175 \end{smallmatrix}\right)$ and $\mathbf{L}^{het} = \left(\begin{smallmatrix} 1.216 & 0.347 \\ 0.134 & 1.253 \end{smallmatrix}\right)$.

Regarding the interpretation of $\mathbf{A}^{het}$, the direct requirements to produce one unit (kilo) of apples (the agricultural final output) are 0.152 kilos of seeds and 0.091 kilos of shovels, which makes sense both analytically and the description of the economy functioning.

Regarding the interpretation of $\mathbf{L}^{het}$, the total (direct and indirect) requirements to produce one unit (kilo) of apples are 1.216 kilos of products from the agricultural sector and 0.134 kilos from the manufacturing sector.

The composition of the total requirements is partially known because of the direct requirements. In other words, from the 1.216 kilos produced by the agricultural sector in order to produce 1 kilo of apples, 1 kilo correspond to the apples and 0.152 correspond to the seeds; the composition of the remaining 0.064 kilos is an unknown mix between almond oil and seeds (the two intermediate goods produced by the agricultural sector). Similarly, from the 0.134 kilos required from the manufacturing sector, 0.091 correspond to the shovels directly required, but the remaining 0.043 is an unknown mix between shovels and screwdrivers (the two intermediate goods produced by the manufacturing sector).

A new state corresponding to a new given final demand can be calculated following the same operational steps as in the traditional Leontief model. The new state can also be interpreted as a differential state, since the proportionality assumption is used. The state of table 5 corresponding to a new final demand $\mathbf{f} = \binom{1}{0}$ is presented in table 6.

	Agric.	Manuf.	Final demand	Total outputs
Agriculture	0.184	0.033	1	1.216
Manufacturing	0.111	0.024	0	0.134
Resources	0.921	0.079		
Total inputs	1.216	0.134		

Table 6: Primary resources, intermediate and total production required to produce one agricultural final unit in the economy represented in table 5. All values in physical units: kilograms.

Following the discussion about the interpretation of $\mathbf{L}^{het}$, it is important to note that the limitation on knowing the exact mix of intermediate production can be removed by calculating the new state of the PIOT.

For example, $\mathbf{L}^{het}$ shows that 1.216 kilos are to be produced by the agricultural sector in order to produce 1 kilo of apples. Thank to $\mathbf{A}^{het}$, it was known that at least 0.152 kilos of seeds were required but it was unknown which mix of the remaining 0.064 was required (1.216-1-0.152). In table 6, it can be seen that this remainder is composed by 0.032 kilos (0.184-0.152) of seeds and 0.032 of almond oil.

In MIOTs In this section, it is sought to build a quantity output-driven model for a MIOT considering heterogeneous intermediate production.

The structure of the MIOT is the same as in table 1 and the relationship between monetary and quantity flows is also the same as in equations 1–3. Using these relationships in equation 5, the generic relationship between output quantities is:

$$\sum_{j=1}^{n} p_{z_{ij}} \cdot z_{ij}^{q} + p_{f_i} \cdot f_i^{q} = x_i^{f} \tag{36}$$

In equation 36, total outputs need to remain as monetary flows since total outputs are composed of heterogeneous goods with different prices but this does not affect the development of the model.

Note that since the homogeneous goods assumption is avoided, there is no price homogeneity as in the traditional Leontief model (recall equations 7 and 8).

The key question is whether the unitary price assumption requires the homogeneous goods assumption to be applied previously. Recalling from section 1.1 that the unitary price assumption only implies a unit change in the quantity flow, the answer is *no* and the unitary price assumption can be applied on heterogeneous goods. It only means that each intermediate (and final) good will be measured in its own quantity units, so equations 9–11 are still valid. So, by using the unitary price assumption on heterogeneous intermediate goods, the MIOT monetary flows can be considered as quantity flows.

Finally, the proportionality assumption can be applied as in the traditional Leontief model, i.e. assuming intermediate production proportional to total outputs by means of the technical coefficients matrix and using it to derive the Leontief inverse matrix, as in equations 12 and 14. The key difference is that each intermediate (and final) value corresponds to a different type of output.

The implications for the interpretation of the technical coefficients matrix $\mathbf{A}$ and the Leontief inverse matrix $\mathbf{L}$ are the same as for the previous case, i.e. for a PIOT with heterogeneous intermediate goods.

The interpretation of $\mathbf{A}$ is the same as in the case of homogeneous goods since $\mathbf{A}$ reveals the input requirements of each sector, which are heterogeneous even when applying the homogeneous goods assumption.

The interpretation of $\mathbf{L}$ has the same caveat as in the PIOT case. $\mathbf{L}$ reveals the total (i.e. direct and indirect) requirements of the economic system to produce its final outputs. In particular, each of its columns can be interpreted as the total production (intermediate and final) generated by each sector in order to produce a unit of the column sector's final good. Since intermediate products are heterogeneous, it cannot be unknown which products are exactly produced by examining $\mathbf{L}$ alone. However, as seen in the previous numerical example, this is easily overcome by recalculating the IOT for the new final demand. Then, it can be ascertained what is the exact composition of the total outputs.

To illustrate how to use and interpret the quantity output-driven model for heterogeneous intermediate outputs, a numerical example is provided below.

Numerical example The numerical example is based on a two sector economy producing the same products as in the previous example. The MIOT is represented in table 7.

The equations driving the model are the same as in the traditional Leontief model and the model for PIOTs with heterogeneous intermediate goods (as

	Agric.	Manuf.	Final demand	Total outputs
Agriculture	5	8	20	33
Manufacturing	3	6	25	34
Resources	25	20		
Total inputs	33	34		

Table 7: Two sector MIOT with heterogeneous intermediate goods and a single final output. All values in monetary units ($).

seen in the previous section). Thus, the technical coefficient and Leontief matrices are the same as for table 5: $\mathbf{A} = \left(\begin{smallmatrix} 0.152 & 0.235 \\ 0.091 & 0.175 \end{smallmatrix}\right)$ and $\mathbf{L} = \left(\begin{smallmatrix} 1.216 & 0.347 \\ 0.134 & 1.253 \end{smallmatrix}\right)$.

The interpretation of **A** is the same as in the PIOT with heterogeneous intermediate goods but with the difference that quantity units are now $. So, to produce 1 $ of apples, 0.152 $ of seeds and 0.091 $ of shovels are required.

Again, the interpretation of **L** is the same as in the PIOT with heterogeneous intermediate goods but with the difference that quantity units are now $. So, to produce 1 $ of apples, the agricultural sector produces 1.216 $ of intermediate and final goods and the manufacturing sector produces 0.134 $ of intermediate goods. The exact composition of the total outputs is unknown by looking at **L** but can be found by calculating the primary and intermediate flows corresponding the production of 1 $ of apples. This calculation would lead to the same results presented in table 6 since the equations to run the model for a MIOT with heterogeneous intermediate goods are the same than the equation to run the model for a PIOT with heterogeneous intermediate goods from the previous section.

Conclusion In this section, it was assumed that intermediate production was heterogeneous and, thus, the homogeneous goods assumption was not used. Considering heterogeneous intermediate goods also implies that they are heterogeneous from the the final output (except in the case where the final output is homogeneous to one of the intermediate goods). However, no simultaneous heterogeneous final outputs were considered and, thus, the PIOT had no disposals to nature (simultaneous heterogeneous final outputs will considered in the next section).

It has been shown that the homogeneous goods assumption can be avoided in the MIOT and PIOT case. Thus, the proportionality assumption is the only assumption required to build a IO model and the unitary price assumption is only required for MIOTs.

PIOTs and MIOT with heterogeneous intermediate production use the same equations as the quantity output-driven (Leontief) model. However, the

Leontief inverse matrix does not provide as much info as in the homogeneous goods case since it does not indicate the exact composition of the total sectoral production. This caveat can be overcome by calculating the intermediate flows associated to the new final demand (as illustrated in the numerical examples).

Thus, from an analytical perspective, the homogeneous goods assumption is not mandatory for intermediate production. However, it simplifies the analysis of MIOTs and PIOTs because it implies a one-to-one correspondence between sectors and goods produced. When IOTs with heterogeneous intermediate production are used, the researcher must remember which product each value refers to, which complicates the analysis and can lead to confusions.

On the other hand, being able to remove the homogeneous goods assumption might open the door to new applications and accounting of secondary production, since the models for heterogeneous intermediate production can model the production of different goods simultaneously within symmetrical MIOTs and PIOTs.

2.1.2 Dealing with heterogeneous final goods

The concept of heterogeneous final outputs refers to the idea that several heterogeneous final outputs coexist, not to the fact that a single final output is heterogeneous compared to the intermediate production. The latter situation corresponds to the case of heterogeneous intermediate goods since different intermediate products also differ from the final product (as seen in the two examples of the previous section).

In the following sections, the idea of heterogeneous final outputs is explored first within the PIOT framework and then within a MIOT. In both cases, it is assumed that intermediate production is homogeneous to simplify the analysis.

In PIOTs A PIOT with homogeneous intermediate products and heterogeneous final products has several final outputs by definition. If a simple case where only two final outputs are considered, for example, the final goods produced by each sector and the emissions generated simultaneously, the PIOT has the same structure as the case examined in section 1.2.3. Thus, the model for such type of tables was first developed by Xu and Zhang (2009).

In Xu and Zhang (2009), the PIOT is treated as a MIOT in physical units with the added disposals to nature (i.e. emission and wastes released to nature). This implicitly implies that each sector generates intermediate and final homogeneous goods except for the disposals to nature, since the homogeneous goods assumption is implied when applying IO models to

MIOTs. The homogeneous intermediate goods assumption makes it easier to understand this model since it is clear that the disposals to nature are the only heterogeneous flow which impedes applying the conventional Leontief model to the PIOT.

Recalling section 1.2.3, a PIOT with homogeneous intermediate goods and heterogeneous final outputs (with one of the final outputs homogeneous to intermediate goods) was presented in table 2 and the analytical difficulty posed by equation 19 was solved in equation 20 by applying rigorously the proportionality assumption to the heterogeneous final outputs (i.e. to the disposals to nature).

This model was further extended in Altimiras-Martin (2014) for the case of m simultaneous heterogeneous final outputs. A PIOT with such structure was presented in table 3. The equations constituting the IO model able to deal with m heterogeneous final outputs were presented in the same section (equations 27 to 31).

So, dealing with heterogeneous final outputs within PIOTs requires a different formulation from the traditional Leontief model, whereby the heterogeneous final outputs are endogenised within the Leontief inverse matrix.

This difference alters the interpretation of the new Leontief inverse matrix (equation 31). Each column of the new Leontief inverse represents the total (intermediate and final) production generated by each sector to produce a unit of the final output driving the model produced corresponding to the column's sector. So, it now includes all final heterogeneous final output produced, even if they are disposals to nature. The interpretation of the technical coefficients matrix remains the same: each column represents the (direct) intermediate requirements to produce a unit of the final output driving the model produced corresponding to the column's sector. Thus, the $m-1$ flows of the heterogeneous final outputs do not affect the interpretation of the technical coefficient matrix.

Numerical example Table 8 represents a three sector PIOT with six heterogeneous final outputs: final goods $\mathbf{f}$, two waste types and three emission types: solid waste $\mathbf{w}_1$, waste for incineration $\mathbf{w}_2$, emissions to air $\mathbf{w}_3$, emissions to water $\mathbf{w}_4$, and emissions to soil $\mathbf{w}_5$

Next, the amount of each type of emission generated to produce one unit of final goods by the services sector is calculated. First, the technical coefficients matrix is calculated using equation 22: $\underline{\mathbf{A}} = \begin{pmatrix} 0.176 & 0.082 & 0.127 \\ 0.076 & 0.366 & 0.314 \\ 0.038 & 0.013 & 0.042 \end{pmatrix}$

Using equation 28, the heterogeneous final output coefficient matrices for each emission are:

$\mathbf{E}_1 = \begin{pmatrix} 0.115 & 0 & 0 \\ 0 & 0.100 & 0 \\ 0 & 0 & 0.191 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} 0.063 & 0 & 0 \\ 0 & 0.019 & 0 \\ 0 & 0 & 0.042 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 0.057 & 0 & 0 \\ 0 & 0.080 & 0 \\ 0 & 0 & 0.064 \end{pmatrix}$, $\mathbf{E}_4 =$

	Agri.	Man.	Ser.	$\mathbf{f}$	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$	$\mathbf{w}_4$	$\mathbf{w}_5$	$\underline{\mathbf{x}}$
Agriculture	153	190	30	20	100	55	50	125	147	870
Manufacturing	66	845	74	658	230	45	185	145	62	2310
Services	33	29	10	67	45	10	15	25	2	236
Resources $\mathbf{r}'$	618	1246	122							
Total inputs $\underline{\mathbf{x}}'$	870	2310	236							

Table 8: PIOT with six heterogeneous final outputs (in million tons): final goods, two waste types and three emission types: solid waste $\mathbf{w}_1$, waste for incineration $\mathbf{w}_2$, emissions to air $\mathbf{w}_3$, emissions to water $\mathbf{w}_4$, and emissions to soil $\mathbf{w}_5$.

$\begin{pmatrix} 0.144 & 0 & 0 \\ 0 & 0.063 & 0 \\ 0 & 0 & 0.106 \end{pmatrix}$ and $\mathbf{E}_5 = \begin{pmatrix} 0.169 & 0 & 0 \\ 0 & 0.027 & 0 \\ 0 & 0 & 0.008 \end{pmatrix}$.

Using equation 31, $\underline{\mathbf{L}} = (\mathbf{I} - \underline{\mathbf{A}} - \mathbf{E}_1 - \mathbf{E}_2 - \mathbf{E}_3 - \mathbf{E}_4 - \mathbf{E}_5)^{-1} = \begin{pmatrix} 4.124 & 1.039 & 1.555 \\ 1.190 & 3.256 & 2.145 \\ 0.314 & 0.147 & 1.987 \end{pmatrix}$.

So, the total throughput of each sector of the economy $\underline{\mathbf{x}}$ for the services sector to produce one unit of final goods is

$$\underline{\mathbf{x}} = \begin{pmatrix} 4.124 & 1.039 & 1.555 \\ 1.190 & 3.256 & 2.145 \\ 0.314 & 0.147 & 1.987 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.55 \\ 2.14 \\ 1.99 \end{pmatrix}$$

Finally, using equation 28, the corresponding emissions are: $\mathbf{w}_1 = \begin{pmatrix} 0.18 \\ 0.21 \\ 0.38 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 0.10 \\ 0.04 \\ 0.08 \end{pmatrix}$, $\mathbf{w}_3 = \begin{pmatrix} 0.09 \\ 0.17 \\ 0.13 \end{pmatrix}$, $\mathbf{w}_4 = \begin{pmatrix} 0.22 \\ 0.13 \\ 0.21 \end{pmatrix}$ and $\mathbf{w}_5 = \begin{pmatrix} 0.26 \\ 0.06 \\ 0.02 \end{pmatrix}$. For each unit of final goods produced by the services sector, the agricultural sector produces 0.18 units of solid waste ($\mathbf{w}_1$), 0.10 units of incineration waste ($\mathbf{w}_2$), 0.09 units of air emissions ($\mathbf{w}_3$), 0.22 units of water emissions ($\mathbf{w}_4$) and 0.26 units of soil emissions ($\mathbf{w}_5$), and so on for the emissions of the manufacturing and services sectors.

It is interesting to note that the Leontief inverse matrix $\underline{\mathbf{L}}$ has some values well above 1 in the diagonal. This is due to the production of the heterogeneous final outputs. For example, the agricultural sector produces only 20 million tons of final goods while it generates a total of 477 million tons of disposals to nature. In other words, the excess of 1 is not only due to the direct and indirect intermediate production to produce the final good but also to the amount of heterogeneous final outputs (emissions in this case) generated simultaneously with the final good.

In MIOTs While models for PIOTs with heterogeneous final outputs had been previously developed (as seen in the previous section), this is not the case for MIOTs.

This section aims to build a IO model applicable to a MIOT with m heterogeneous final outputs. Such MIOT would entail an added value vector as primary input ($\mathbf{v}'$), an intersectoral matrix ($\mathbf{Z}'$) and m heterogeneous final outputs, i.e. m final demand vectors $\mathbf{f}_1$ to $\mathbf{f}_m$. The total output (and inputs) are represented by an underlined x to highlight the difference from the conventional IOT structure (i.e. with a single final output). A MIOT with this structure is presented in table 9.

	Sector 1	...	Sector n	Final outputs	Total outputs
Sector 1					
⋮		$\mathbf{Z}$		$\mathbf{f}_1$ ... $\mathbf{f}_m$	$\underline{\mathbf{x}}$
Sector n					
Added value		$\mathbf{v}'$			
Total inputs		$\underline{\mathbf{x}}'$			

Table 9: Structure of a MIOT with m heterogeneous final outputs.

The output accounting relationship corresponding to table 9 is

$$\mathbf{Z} \cdot \mathbf{i} + \mathbf{f}_1 + \ldots + \mathbf{f}_m = \underline{\mathbf{x}} \tag{37}$$

Since heterogeneous final outputs are considered, the homogeneous goods assumption cannot be applied as done in the traditional Leontief model. However, for simplicity, it is assumed that intermediate production is homogeneous.

The issue is then: can the unitary price assumption be applied for heterogeneous goods? The answer, as in the heterogeneous intermediate goods MIOT case (c.f. section 2.1.1), is *yes*. The fact that goods are heterogeneous is irrelevant in the sense that the unit of each flow can be independently redefined and unitary prices can be considered for all monetary flows.

The difference with the previous notation is that the price equations for the final goods need to accommodate for the m heterogeneous vectors. So, equation 3 becomes:

$$f_{ik}^f = p_{f_{ik}} \cdot f_{ik}^q \quad \text{for } i = [1, n] \text{ and } k = [1, m] \tag{38}$$

which becomes, when applying the unitary prices assumption,

$$f_{ik}^f = 1 \cdot f_{ik}^q \quad \text{for } i = [1, n] \text{ and } k = [1, m] \tag{39}$$

Then, the proportionality assumption must be applied to build the IO model. However, it must be decided beforehand which of the heterogeneous final outputs should drive the model. In the previous case of a PIOT with disposals to nature, the "intuitive" driver was the final output corresponding to the final good production because the researcher wants to know what are the emissions associated to a given final production. However, this distinction is analytically irrelevant because *all* heterogeneous final outputs are linearly related to total production (Altimiras-Martin, 2014).

Here, final output $\mathbf{f}_1$ is chosen as the final output driving the model for notational convenience. So, $\mathbf{f}_2$ to $\mathbf{f}_m$ will need to be made proportional to total outputs by creating $m-1$ heterogeneous output coefficient matrices $\mathbf{\Phi}_k$, as follows

$$\mathbf{f}_k = \mathbf{\Phi}_k \cdot \underline{\mathbf{x}} \qquad \text{for } k = 2, ..., m \tag{40}$$

To calculate the heterogeneous output coefficient matrices $\mathbf{\Phi}_k$, each side of equation 40 is diagonalised and post-multiplied by $\underline{\mathbf{x}}^{-1}$, resulting in

$$\mathbf{\Phi}_k = \hat{\mathbf{f}}_k \cdot \hat{\underline{\mathbf{x}}}^{-1} \qquad \text{for } k = 2, ..., m \tag{41}$$

The proportionality assumption affects the intersectoral matrix in the usual manner and the technical coefficients matrix is defined as:

$$\underline{\mathbf{A}} = \mathbf{Z} \cdot \hat{\underline{\mathbf{x}}}^{-1} \tag{42}$$

Then, using equations 40 and 42 in 37,

$$\underline{\mathbf{x}} = \underline{\mathbf{A}} \cdot \underline{\mathbf{x}} + \mathbf{f}_1 + \mathbf{\Phi}_2 \cdot \underline{\mathbf{x}} + \ldots + \mathbf{\Phi}_m \cdot \underline{\mathbf{x}} \tag{43}$$

Rearranging,

$$\underline{\mathbf{x}} = (\underline{\mathbf{I}} - \underline{\mathbf{A}} - \mathbf{\Phi}_2 - \ldots - \mathbf{\Phi}_m)^{-1} \cdot \mathbf{f}_1 \tag{44}$$

$$\underline{\mathbf{x}} = \underline{\mathbf{L}} \cdot \mathbf{f}_1 \tag{45}$$

where

$$\underline{\mathbf{L}} = (\underline{\mathbf{I}} - \underline{\mathbf{A}} - \mathbf{\Phi}_2 - \ldots - \mathbf{\Phi}_m)^{-1} \tag{46}$$

So, equation 44 constitutes the IO model for heterogeneous final output applicable to MIOTs.

The interpretation of $\underline{\mathbf{A}}$ is equivalent to the previous cases since the heterogeneous final outputs do not affect its definition.

Each column of $\underline{\mathbf{L}}$ is usually interpreted as representing the total production (intermediate and final) of each sector in order to produce the final good produced by that column's sector. However, in the current case, the definition must be refined to accommodate to the different heterogeneous final outputs. In particular, the new Leontief inverse matrix represents the total production (intermediate and final) of each sector in order to produce the final good produced by that column's sector *that is actually driving the model.*

MIOTs with heterogeneous final outputs pose a new challenge because, depending on which final demand is chosen to drive the model, a different Leontief inverse matrix will be found, inducing different results when performing a structural analysis such as backward linkage analysis. The paradox of having a single MIOT with several production structures (m different structure to be precise in the current case) makes sense precisely because each final output is different (heterogeneous) and requires a different production structure to be produced.

Numerical example A numerical example of a MIOT with three sectors (agriculture, manufacturing and services) producing homogeneous intermediate goods. Intermediate production is as follows: the agriculture produces soy-beans, which are sold to the agriculture (as seeds for next crop), to the manufacturing (to produce oil to lubricate its machines) and to the services sector (to be sold in restaurants); The manufacturing sector sells tables to the three sectors to help in the productive process; and the service sector sells soy porridge to the agricultural, manufacturing and services sectors. The table has six heterogeneous products: agriculture produces apples ($\mathbf{f}_1$) and tomatoes ($\mathbf{f}_2$), the manufacturing produces screwdrivers ($\mathbf{f}_3$) and chairs ($\mathbf{f}_4$) and the services sector produces sweets ($\mathbf{f}_5$) and soy milkshake ($\mathbf{f}_6$). The MIOT corresponding to this economy is presented in table 10.

	Agri.	Man.	Ser.	$\mathbf{f}_1$	$\mathbf{f}_2$	$\mathbf{f}_3$	$\mathbf{f}_4$	$\mathbf{f}_5$	$\mathbf{f}_6$	$\underline{\mathbf{x}}$
Agriculture	153	190	30	220	277	0	0	0	0	870
Manufacturing	66	845	74	0	0	658	667	0	0	2310
Services	33	29	10	0	0	0	0	67	97	236
Added Value $\mathbf{v}'$	618	1246	122							
Total inputs $\underline{\mathbf{x}}'$	870	2310	236							

Table 10: MIOT with six heterogeneous final outputs (in \$): apples ($\mathbf{f}_1$), tomatoes ($\mathbf{f}_2$), screwdrivers ($\mathbf{f}_3$), chairs ($\mathbf{f}_4$), sweets ($\mathbf{f}_5$) and soy milkshake ($\mathbf{f}_6$).

Below, the heterogeneous goods produced while producing one unit of apples ($\mathbf{f}_1$) is calculated.

First, the technical coefficients matrix is calculated using equation 42: $\underline{\mathbf{A}} = \begin{pmatrix} 0.176 & 0.082 & 0.127 \\ 0.076 & 0.366 & 0.314 \\ 0.038 & 0.013 & 0.042 \end{pmatrix}$

Using equation 41, the heterogeneous final output coefficient matrices are:

$\mathbf{\Phi}_2 = \begin{pmatrix} 0.318 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{\Phi}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.285 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{\Phi}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.289 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{\Phi}_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.284 \end{pmatrix}$ and $\mathbf{\Phi}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.411 \end{pmatrix}$.

Using equation 46, $\underline{\mathbf{L}} = (\mathbf{I} - \underline{\mathbf{A}} - \mathbf{\Phi}_2 - \mathbf{\Phi}_3 - \mathbf{\Phi}_4 - \mathbf{\Phi}_5 - \mathbf{\Phi}_6)^{-1} = \begin{pmatrix} 3.955 & 7.656 & 11.051 \\ 10.500 & 42.247 & 55.505 \\ 1.073 & 3.124 & 8.054 \end{pmatrix}$.

So, the total output of each sector of the economy $\underline{\mathbf{x}}$ for the agricultural sector to produce one unit of apples is

$$\underline{\mathbf{x}} = \begin{pmatrix} 3.955 & 7.656 & 11.051 \\ 10.500 & 42.247 & 55.505 \\ 1.073 & 3.124 & 8.054 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.955 \\ 10.500 \\ 1.073 \end{pmatrix}$$

Finally, using equation 40, the heterogeneous final outputs produced at the same time than one unit of apples are: $\mathbf{f}_2 = \begin{pmatrix} 1.259 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{f}_3 = \begin{pmatrix} 0 \\ 2.991 \\ 0 \end{pmatrix}$, $\mathbf{f}_4 = \begin{pmatrix} 0 \\ 3.032 \\ 0 \end{pmatrix}$, $\mathbf{f}_5 = \begin{pmatrix} 0 \\ 0 \\ 0.305 \end{pmatrix}$ and $\mathbf{f}_6 = \begin{pmatrix} 0 \\ 0 \\ 0.441 \end{pmatrix}$. For each unit of apples produced by the agricultural sector, the economy also produces 1.259 units of tomatoes ($\mathbf{f}_2$), 2.991 units of screwdrivers ($\mathbf{f}_3$), 3.032 units of chairs ($\mathbf{f}_4$), 0.305 units of sweets ($\mathbf{f}_5$) and 0.441 units of soy milkshake ($\mathbf{f}_6$).

It is key to note that the same results could not be reproduced using the traditional quantity output-driven model, not only because it cannot be applied to a MIOT with heterogeneous final outputs but also because it implies that only one final output coefficient exists, i.e. all final outputs are produced with the same structure and at the same rate. Precisely, the model adapted to MIOTs with heterogeneous final outputs makes it possible to calculate the production of several final outputs with their own production structure.

Conclusion In this section, it has been shown that the treatment of heterogeneous final goods do require a new formulation of the quantity output-driven model, both in the case of PIOTs and MIOTs, by endogenising the production of heterogeneous final outputs within the Leontief inverse matrix (see equations 31 and 46, respectively).

The equations constituting the model for MIOTs with heterogeneous final outputs are equivalent to the ones constituting the model for a PIOT with heterogeneous final outputs despite the MIOT requiring an extra assumption: the unitary price assumption.

The interpretation of $\underline{\mathbf{A}}$ and $\underline{\mathbf{L}}$ are also equivalent in both cases. The interpretation of $\underline{\mathbf{A}}$ is in fact equivalent to the interpretation of as in the traditional Leontief model but the interpretation of $\underline{\mathbf{L}}$ needs to accommodate for the inclusion of the heterogeneous final outputs within the Leontief inverse matrix.

In the MIOT and PIOT cases, the model for heterogeneous final outputs makes sense intuitively and analytically, opening the door for new analysis within industrial ecology (e.g. tracing the generation of several emissions) and economics (e.g. tracing secondary production as final outputs).

2.2 A generalised quantity output-driven model (with heterogeneous intermediate production and heterogeneous final outputs)

Section 2.1.1 analysed whether it was possible to build a model with heterogeneous *intermediate* products, first for a PIOT and then for a MIOT. Then, section 2.1.2 analysed whether it was possible to build a model with heterogeneous *final* products, first for a PIOT and then for a MIOT. The analyses were made separately in case one option was not possible. Since both options are possible, this section aims to combine both approaches to suggest a generalised framework underlying the quantity output-driven model. In other words, it is sought to formulate a generic quantity output-driven model with minimal assumption requirements using the learnings from sections 2.1.1 and 2.1.2.

In particular, the idea is to build a quantity output-driven model without knowing whether the underlying accounting framework is a PIOT or a MIOT that can deal with the production of heterogeneous *intermediate and final* outputs (i.e. the most general case when not using the homogeneous goods assumption).

To start with, a generic n sector IOT with n heterogeneous intermediate outputs and m heterogeneous final outputs is presented in table 11. The units of the table can indistinctly be monetary or physical.

	Sector 1 ... Sector n	Final outputs	Total outputs
Sector 1 ⋮ Sector n	$\mathbf{Z}^{het}$	$\mathbf{f}_1 \ \dots \ \mathbf{f}_m$	$\underline{\mathbf{x}}$
Primary inputs	$\mathbf{p}'$		
Total inputs	$\underline{\mathbf{x}}'$		

Table 11: n sector IOT with n heterogeneous intermediate outputs and m heterogeneous final outputs (either in monetary or physical units).

The IOT table can represent either monetary units (turning it into a MIOT) or physical units (turning it into a PIOT). In any case, it is assumed that it has been built either using the double-entry bookkeeping principle (the underlying accounting principle of MIOTs) or the mass balance principle (the underlying accounting principle of PIOTs). Thus, in both cases, total outputs equal total inputs.

Also, in both cases, the output accounting relationship corresponding to table 11 is:

$$\underline{\mathbf{x}} = \mathbf{Z}^{het} \cdot \mathbf{i} + \mathbf{f}_1 + \ldots + \mathbf{f}_m \tag{47}$$

By definition, table 11 entails heterogeneous final outputs; however, the homogeneous goods assumption could still be applied to the intermediate production. It is not applied to keep the model as general as possible.

Before applying the proportionality assumption, the attentive reader would like to know whether the table is a MIOT or a PIOT in order to examine whether it is required and possible to apply the unitary prices assumption. However, the previous learning from sections 2.1.1 and 2.1.2 indicates that the unitary price assumption is applicable to the case of heterogeneous *intermediate and final* goods and it is a "transparent" assumption. In other words, it is implicitly (and automatically) applied when considering monetary flows as quantity flows without requiring any modification from the flow values or IOT structure[7]. Thus, the output accounting relationship (equation 47) is maintained regardless whether the IOT under examination is a PIOT or a MIOT and the unitary price assumption does not need to be explicitly applied.

Before applying the proportionality assumption, it must be decided which final output will drive the model. In theory, the researcher can select any of the heterogeneous final outputs since all final outputs will be proportional to total outputs when applying the proportionality assumption. Here, final good $\mathbf{f}_1$ is selected to simplify the notation.

Finally, the proportionality assumption is applied to transform the set of n equations with $n^2 + n \cdot m$ unknowns (corresponding to intermediate and final production unknowns) into a solvable set of n (linear) equations with n unknowns.

To do that, three steps are required.

First, the heterogeneous intermediate production is considered proportional to the total outputs, leading to the definition of the technical coefficients

[7] The only difference is that the values of a PIOT will reflect the actual quantity of the flow while the values of a MIOT imply a different quantity produced of that particular product, as explained in section 1.1.2.

matrix $\underline{\mathbf{A}}$:

$$\mathbf{Z}^{het} = \underline{\mathbf{A}}^{het} \cdot \hat{\underline{\mathbf{x}}} \Leftrightarrow \underline{\mathbf{A}}^{het} = \mathbf{Z}^{het} \cdot \hat{\underline{\mathbf{x}}}^{-1} \tag{48}$$

This shrinks the unknowns from intersectoral flows from n^2 to n unknowns from the $\underline{\mathbf{x}}$ vector.

Second, the heterogeneous final outputs are also considered proportional to total outputs, shrinking the unknowns from final output flows from $n \cdot (m-1)$ to n unknowns from the $\underline{\mathbf{x}}$ vector.

To do that, the $m-1$ are linearly related to the total outputs by means of $m-1$ heterogeneous output coefficient matrices $\mathbf{\Phi}_k$, as follows

$$\mathbf{f}_k = \mathbf{\Phi}_k \cdot \underline{\mathbf{x}} \qquad \text{for } k = 2, ..., m \tag{49}$$

To calculate the heterogeneous output coefficient matrices $\mathbf{\Phi}_k$, each side of equation 49 is diagonalised and post-multiplied by $\underline{\mathbf{x}}^{-1}$, resulting in

$$\mathbf{\Phi}_k = \hat{\mathbf{f}}_k \cdot \hat{\underline{\mathbf{x}}}^{-1} \qquad \text{for } k = 2, ..., m \tag{50}$$

Then, using equations 49 and 48 in 47,

$$\underline{\mathbf{x}} = \underline{\mathbf{A}}^{het} \cdot \underline{\mathbf{x}} + \mathbf{f}_1 + \mathbf{\Phi}_2 \cdot \underline{\mathbf{x}} + \ldots + \mathbf{\Phi}_m \cdot \underline{\mathbf{x}} \tag{51}$$

Rearranging,

$$\underline{\mathbf{x}} = (\underline{\mathbf{I}} - \underline{\mathbf{A}}^{het} - \mathbf{\Phi}_2 - \ldots - \mathbf{\Phi}_m)^{-1} \cdot \mathbf{f}_1 \tag{52}$$

where

$$\underline{\mathbf{L}}^{generic} = (\underline{\mathbf{I}} - \underline{\mathbf{A}}^{het} - \mathbf{\Phi}_2 - \ldots - \mathbf{\Phi}_m)^{-1} \tag{53}$$

Third, the primary inputs are also linearly related to total outputs (i.e. total inputs) by means of input coefficients ($\mathbf{c}^p$) defined as

$$\mathbf{c}^{p\prime} = \mathbf{p}' \cdot \hat{\underline{\mathbf{x}}}^{-1} \tag{54}$$

So, equation 58 constitutes a quantity output-driven model tracing heterogeneous intermediate and final outputs without requiring to know whether the underlying flows are in quantity or monetary units. This equation can be used to drive the model into a new state induced by a new final demand $\mathbf{f}_1^*$, and then equations 48, 54 and 49 can be used to calculate, respectively, the intermediate, primary input and heterogeneous final output flows corresponding to $\mathbf{f}_1^*$. The issue of whether such model is practical is left for the discussion section 3.

It must be noted that thanks to the proportionality assumption, the model can also be understood either in absolute terms, where the new state associated to a new final demand is considered the actual new state of the system, and relative (or differential) terms, where where the new state associated to a new final demand is considered the variation compared to the initial state.

2.3 Constraining the generic model with the homogeneous goods assumption

The generic model for heterogeneous intermediate and final products can be partially or totally constrained by the homogeneous goods assumption, leading to simplifications of the model.

2.3.1 Model with homogeneous intermediate production and heterogeneous final outputs

Applying the homogeneous goods assumption to intermediate production does not change the formulation of the model because, even if the rows of $\mathbf{Z}^{het}$ are considered homogeneous, the output accounting relationship remains the same as in equation 47. So, the corresponding Leontief inverse matrix $\underline{\mathbf{L}}^{het\ final}$ follows the same formulation as in equation 59. The only difference is that there is a one-to-one correspondence between sectors and intermediate products, both in the cases where the IOT is a PIOT and a MIOT.

In fact, it can even be assumed that one of the heterogeneous final outputs is homogeneous to the intermediate production. This approach was the one implicitly used in Xu and Zhang (2009) and Altimiras-Martin (2014) to analyse a PIOT.

2.3.2 Model with heterogeneous intermediate production and homogeneous final outputs

If the heterogeneous final outputs of the generalised model are constrained by the homogeneous goods assumption, the heterogeneous final outputs can be aggregated as a single final demand and the resulting table can be analysed with the traditional quantity output-driven (Leontief) model. So,

$$\mathbf{f}_{agg} = \sum_{k=1}^{m} \mathbf{f}_k \tag{55}$$

and equation 47 becomes

$$\underline{\mathbf{x}} = \mathbf{Z}^{het} \cdot \mathbf{i} + \mathbf{f}_{agg} \tag{56}$$

Assuming the unitary price assumption if required (in the case that the IOT is a MIOT) and applying the proportionality assumption, the intermediate production is related to total outputs as in equation 48.

Substituting equation 48 in 56:

$$\underline{\mathbf{x}} = \underline{\mathbf{A}}^{het} \cdot \underline{\mathbf{x}} + \mathbf{f}_{agg} \tag{57}$$

Rearranging,

$$\underline{\mathbf{x}} = (\underline{\mathbf{I}} - \underline{\mathbf{A}}^{het})^{-1} \cdot \mathbf{f}_{agg} \tag{58}$$

where

$$\underline{\mathbf{L}}^{het\ interm} = (\underline{\mathbf{I}} - \underline{\mathbf{A}}^{het})^{-1} \tag{59}$$

2.3.3 Model with homogeneous intermediate production and final outputs (traditional quantity output-driven (Leontief) model)

Finally, the generic model is constrained by assuming that intermediate and final production are homogeneous using the homogeneous goods assumption as in the traditional Leontief model.

So, $\mathbf{Z}^{het}$ becomes $\mathbf{Z}$ and the heterogeneous final goods can be aggregated as in equation 55. Thus, the generic equation 47 becomes

$$\underline{\mathbf{x}} = \mathbf{Z} \cdot \mathbf{i} + \mathbf{f}_{agg} \tag{60}$$

which is the output accounting relationship from which the Leontief model is built by applying the unitary price assumption and the proportionality assumption (as seen in section 1.1).

However, equation 60 was derived from the generic model, which can be applied to a IOT in monetary or physical units. So, this relationship can in fact be applied to a PIOT or a MIOT indistinctly. Using it on a MIOT implies the use of the unitary price assumption, but this assumption is not required when applied to a PIOT.

3 Conclusion and discussion

The homogeneous goods assumption implies a one-to-one correspondence between sectors and the goods produced within the considered system (Miller and Blair, 2009), creating an extremely practical analytical framework. However, this comes at the expense of having to re-allocate secondary production using different methods when building the symmetrical monetary input-output table (MIOT) (Miller and Blair, 2009)and making it difficult to deal with the actual production structure where different by-products, emissions and wastes are generated by the very same production process.

In this paper, it is demonstrated that the homogeneous good assumption, usually thought to be a fundamental assumption to build a quantity output-driven IO model (Miller and Blair, 2009; Suh, 2004), is in fact dispensable. Waiving this assumption makes it possible to consider symmetrical MIOTs and PIOTs with heterogeneous intermediate and final goods, expanding the analytical possibilities of IOA. Waiving the homogeneous goods assumption opens the door to devise new analyses, applications and accounting methods to build symmetrical IOTs, either by tracing secondary production as intermediate and/or final production.

The traditional assumptions used in the quantity output-driven model (Leontief, 1941) were first examined in section 1.1. Then, in section 1.2.3, it was shown that models to deal with heterogeneous final outputs within the PIOT framework had already been devised (Xu and Zhang, 2009; Altimiras-Martin, 2014) (although, traditionally, secondary production is reallocated using the Supply and Use Tables (c.f. section 1.3)).

In section 2.1.1, it is demonstrated that IO quantity output-driven models can cope with heterogeneous *intermediate* production when applied PIOTs and MIOTs, and, in section 2.1.2, it is demonstrated that IO quantity output-driven models can cope with heterogeneous *final* production when applied PIOTs and MIOTs.

Using the learnings from these two sections, a generalised quantity output-driven model is suggested in section 2.2. It is shown that such model only requires the proportionality assumption in order to be built. It is also shown that the unitary price assumption is only required in the case that flows are in monetary units, and this assumption is in fact "transparently" applied, i.e. it does not require a reformulation of the generic model. It is the first time a generic quantity output-driven model able to deal with secondary production, both as intermediate and final production, is suggested.

The generic model is then constrained by applying the homogeneous goods assumption on intermediate production (section 2.3.1), final production (section 2.3.2) and intermediate and final production (section 2.3.3). It is found that, in the last case, it leads to the equivalent formulation of the traditional quantity output-driven model.

The formulation of a generic quantity output-driven model has key implications for how secondary production is treated. As seen in section 1.3, most reallocation methods use the Supply and Use Tables framework to build symmetrical MIOTs with sectors that are as homogeneous as possible. By waiving the homogeneous goods assumption, it is possible to rethink how symmetrical MIOTs are built and analysed because the quantity output-driven model can now accommodate for secondary production either within the intersectoral matrix and/or as final outputs. Regarding the study of the physical structure of the economy (e.g. within the Industrial Ecology field), the application of a generic model is straight forward, since goods production is indissociable from the generation of a multitude of by-products and disposals to nature.

The generic form of the model as devised in section 2.2 might be impractical for economic analysis or deemed too complex as analytical tool since each value corresponds to a different type of good. On the other hand, the generic model opens the possibility to analyse secondary production in a different manner than previously allowed and, if deemed too complex, it can be partially constrained with the homogeneous goods assumption, either to work with homogeneous intermediate products (as illustrated in section 2.3.1) or homogeneous final outputs (as illustrate in section 2.3.2).

A limitation of the generic model is that it implies a commodity-based technology. In other words, when the production of a given heterogeneous final output increases (e.g., the final output driving the model), the production of all heterogeneous final outputs also increases. For example, imagine the furniture industry produces several heterogeneous final outputs (chairs, tables, closets,...). If using the generic model, the system will be driven by only one of the heterogeneous final outputs (e.g. amount of chairs produced), but the model will consider systematically the production of the other final outputs in the same proportion they have been built throughout the accounting year. This is not necessarily a caveat since it reflects the current structure of the economic system, i.e. some final outputs are required in certain proportions. So, the advantage of the generic model can also be considered a disadvantage, since it is not possible to analyse the economic activity induced by a single final output when directly related to other heterogeneous final outputs.

It is key to note that this disadvantage does not exist when the IOT represents exclusively processes whose technology is commodity-based, e.g. the generation of by-products and disposals to nature.

To sum up, despite the limitation described above, the generic quantity output-driven model developed in this paper (section 2.2) constitutes a new quantity output-driven IO model whereby secondary production as intermediate and/or final production can be explicitly analysed in PIOTs and MIOTs, enhancing the analytical potential of IOA and providing a new framework to construct symmetrical PIOTs and MIOTs entailing secondary production.

Bibliography

Altimiras-Martin, A., 2014. Analysing the structure of the economy using physical input-output tables. Economic Systems Research 26, 463–485.

Giljum, S., Hubacek, K., 2004. Alternative Approaches of Physical Input-Output Analysis to Estimate Primary Material Inputs of Production and Consumption Activities. Economic Systems Research 16, 301–310.

Hubacek, K., Giljum, S., 2003. Applying physical input-output analysis to estimate land appropriation (ecological footprints) of international trade activities. Ecological Economics 44, 137–151.

Jansen, P.K., Raa, T.T., 1990. The Choice of Model in the Construction of Input-Output Coefficients Matrices. International Economic Review 31, 213–227.

Leontief, W., 1941. The structure of American economy, 1919-1929.

Leontief, W.W., 1986. Input-output economics. Oxford University Press US. 2nd edition.

Miller, R.E., Blair, P.D., 2009. Input-output analysis: foundations and extensions. Cambridge University Press, Cambridge UK. 2nd ed. edition.

Nakamura, S., Kondo, Y., 2009. Waste Input-Output Analysis. Eco-efficiency in industry and science, Springer Netherlands, Dordrecht.

Raa, T.t., Chakraborty, D., Small, J.A., 1984. An Alternative Treatment of Secondary Products in Input-Output Analysis. The Review of Economics and Statistics 66, 88–97.

Raa, T.t., Rueda-Cantuche, J.M., 2003. The Construction of Input–Output Coefficients Matrices in an Axiomatic Context: Some Further Considerations. Economic Systems Research 15, 439–455.

Stone, R., 1961. Input-output and national accounts. Organisation for European Economic Co-operation, Paris.

Suh, S., 2004. A note on the calculus for physical input-output analysis and its application to land appropriation of international trade activities. Ecological Economics 48, 9–17.

Weisz, H., Duchin, F., 2006. Physical and monetary input-output analysis: What makes the difference? Ecological Economics 57, 534–541.

Xu, Y., Zhang, T., 2009. A new approach to modeling waste in physical input-output analysis. Ecological Economics 68, 2475–2478.