Measuring I/O-Production in DIGITAL ECONOMY

Jean-François Emmenegger, jean-francois.emmenegger@unifr.ch, Fribourg, corresponding author,
Hassan Ahmed Nour Eldin, eldin@uni-wuppertal.de, Zurich, Adnan Abu Nabout (DFG)

IIOA Conference in Malaysia, 28.08 - 2.09.2022

Abstract: *Productivity is an all-power measure of a national economy and stays beside the main aggregates of the system of national accounts, as the national income Y (a proxy of GDP), the total output X and the circulating capital K, which is the total value of the means of production. – In this paper a notion of the measurement of the productivity of an economy, also called the productiveness of the economy, is proposed. Two algebraic properties have to exist. The first property (a) has to guarantee that the theorem of Perron-Frobenius [3] can be applied to the appropriate matrices of a production system, as the commodity flow matrix, appearing in Input-Output Tables*[1]*. The second property (b) has to guarantee the productivity of the economy, i.e., the vector of final demand (Table 1) is semi-positive.*

Keywords: Input-Output Table (IOT), productiveness of an economy, commodity flow matrix, Perron-Frobenius theorem, Frobenius number

Consider a schematic Input-Output Table (Table 1)[2], where the transactions z_{ij} in monetary terms are recorded at basic prices in the *supply-table* of the central part, described by the $n \times n$ *commodity flow matrix* $\mathbf{Z} = (z_{ij})$. On the right side is the use-table of final consumption, which is in reality a matrix, but here it is aggregated to a single $n \times 1$ vector $\mathbf{f} = (f_i)$ of final demand. Then, there is on the lower side the value-added part, which is here represented by the $1 \times n$ row vector $\mathbf{v} = (v_j)$.

Products (CPA)	buying sectors						final	total
selling sectors	S_1	S_2	...	S_j	...	S_n	demand	output
S_1	z_{11}	z_{12}	...	z_{1j}	...	z_{1n}	f_1	x_1
S_2	z_{21}	z_{22}	...	z_{2j}	...	z_{2n}	f_2	x_2
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$
S_i	z_{i1}	z_{i2}	...	z_{ij}	...	z_{in}	f_i	x_i
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$
S_n	z_{n1}	z_{n2}	...	z_{nj}	...	z_{nn}	f_n	x_n
value added	v_1	v_2	...	v_j	...	v_n	$V = F$	
total outlays	x_1	x_2	...	x_j	...	x_n		X

Table 1: A symmetric IOT of n producing sectors and a sector of final demand

On the one hand, we work with national Input-Output Tables, the German IOT-2013, the Austrian IOT-2015 and the Swiss IOT-2014, which differ one from the other, not only with respect of the number n of sectors but also of their structure. On the other hand, we work with uniform Input-Output Tables, issued from the WIOD project, proposed by the Groningen Growth and Development Centre, Faculty of Economics and Business, University of Groningen, Netherlands. Erik Dietzenbacher et al. (2013) [7] have developed

[1]The Perron-Frobenius theorem says that for a non-negative irreducible $n \times n$ matrix, there exists always a maximal, real and positive eigenvalue, the Frobenius number, to which is associated a positive eigenvector.

[2]The notations follow the usual standards. The products or product groups are classified in the columns according to the CPA system, the economic activities are grouped in homogenous branches in the rows according to the NACE classification but are then adapted in the symmetric Input-Output Tables also to the CPA system. Germany, Austria and Switzerland operate in this way, see Nathani [6].

the World Input-Output Database (WIOD) of World InputOutput Tables (WIOTs). The WIOT constitute the core of the World InputOutput Database (WIOD), they all have the same structure and are composed of the same $n = 36$ product groups in the rows, and the same $n = 36$ homogenous economic activities (branches) in the columns, classified according to the CPA system.

We may expect that all the production sectors i participate with their goods to the *means of production.* But this is not the case. In all the national and WIOD IOTs mentioned above, there is a vertical sector of homogenous branches and a horizontal sector of products, situated generally in the last row and the last columns, corresponding to "Activities of households as employees of domestic personal", with no entries, where all the entries are zero. Moreover in the Austrian IOT2015 there is the branch $n = 65$ and the product $n = 65$, called "Services provided by extraterritorial organisations and bodies", showing exclusively zero entries. Even the corresponding total output is zero, $x_{65} = 0$. For this reason the vectors of total output of the considered input-output matrices $\mathbf{Z}$ are positive, $\mathbf{x} > \mathbf{o}$.

We have to calculate the Frobenius numbers of semi-positive matrices. For this reason we can eliminate the sector with zero entries in the row and in the column and total output equal to zero, because they contribute nothing to the present economy. This reduction doesn't change the Frobenius number to be calculated, according to the existing theorems.

In the *commodity flow matrix* $\mathbf{Z} = (z_{ij})$ all the components are greater or equal to zero, $z_{ij} \geq 0$, $i, j \in \{1, ..., n\}$. For this reason matrix $\mathbf{Z} = (z_{ij}) \geq \mathbf{0}$ is clearly semi-positive and are generally reducible. A matrix which is not reducible is irreducible. We define now the reducibility of a matrix.

A *non-negative* matrix $\mathbf{A}$ is *reducible* if it can be brought with permutations, operated by permutation matrix $\mathbf{P}$ of the rows and columns to "canonical form", see Gantmacher ([4], p. 411), presented here:

$$\widetilde{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \widetilde{\mathbf{A}}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{22} & \mathbf{0} & \mathbf{0} \\ ... & ... & ... & ... \\ \widetilde{\mathbf{A}}_{k1} & \widetilde{\mathbf{A}}_{k2} & ... & \widetilde{\mathbf{A}}_{kk} \end{bmatrix} \tag{1}$$

We come back to the fact that in any case all the total outputs are positive, $x_i > 0$, and therefore there is a positive vector of total outputs, $\mathbf{x} > \mathbf{o}$. For this reason the reducible semi-positive coefficients matrix $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$ can be computed. An extension of the Perron-Frobenius theorem applies, Lemma A.10.3, see Emmenegger et al. ([2], p. 476), stating that the maximal eigenvalue of a semi-positive reducible matrix $\widetilde{\mathbf{A}}$ is positive, $\lambda_A > 0$[3].

Now, we formulate a specific Sraffa [9] price model, see Emmenegger & al. ([2], p. 110–118), where all the workers obtain exclusively subsistence wages, consisting of commodities like a bundle of agricultural products as wages. All the surplus goes as profits to the entrepreneurs. There is the semi-positive matrix $\mathbf{S} \geq \mathbf{0}$. There is a semi-positive vector of surplus $\mathbf{d} \geq \mathbf{o}$, and the vector of total output $\mathbf{q} = \mathbf{S}\mathbf{e} + \mathbf{d} > \mathbf{o}$. We set up the specific Sraffa price model, leading to an eigenvalue equation, using the coefficients matrix $\mathbf{C} = \mathbf{S}\hat{\mathbf{q}}^{-1} \geq \mathbf{0}$, which is reducible or irreducible as matrices $\mathbf{Z}$, $\mathbf{A}$,

$$\mathbf{S}'\mathbf{p}(1 + R) = \hat{\mathbf{q}}\mathbf{p} \quad \Leftrightarrow \quad \mathbf{C}'\mathbf{p}(1 + R) = \mathbf{p} \quad \Leftrightarrow \quad \mathbf{C}'\mathbf{p} = \lambda_C \mathbf{p}. \tag{2}$$

[3]The eigenvalue $\lambda_A > 0$ is equal to the maximal eigenvalue of one of the submatrices $\widetilde{\mathbf{A}}_{ll}, l = 1, .., k$ of the "canonical form" (1).

Angain Lemma A.10.3 (Emmenegger & al., [2], p. 476) applies. The Frobenius number of $\mathbf{C}'$ is positive, $\lambda_C = 1/(1+R) > 0$. The special request $\mathbf{x} > \mathbf{o}$ and $\mathbf{q} > \mathbf{o}$ for IOTs with the identity $\mathbf{x} = \hat{\mathbf{q}}\mathbf{p} > \mathbf{o}$ leads to positive price vectors $\mathbf{p} > \mathbf{o}$. We calculate the price p_i of every bundle of commodities i. The value of the *means of production* is termed as *circulating capital* $K = \mathbf{e}'(\mathbf{S}'\mathbf{p})$, the value of the total output is $X = \mathbf{e}'(\hat{\mathbf{q}}\mathbf{p})$.

Every *CPA product group* is characterized by four specific *attributes*: quantity q_i, price p_i, value $x_i = p_i q_i$, and object e_i, $i = 1, ..., n$. These attributes can be combined to the following vectors: the vector of total outputs of quantities $\mathbf{q} = [q_1, ..., q_n]' > \mathbf{o}$, the price vector $\mathbf{p} = [p_1, .., p_n]' > \mathbf{o}$, the vector of values $\mathbf{x} = \hat{\mathbf{q}}\mathbf{p} = [x_1, .., x_n]' > \mathbf{o}$ and the vector of objects $\mathbf{e} = [1, .., 1]'$, the object giving the unit of production for one period.

The symmetric I-O-Tables (Table 1) are set up in a way that the total outputs are the total outlays. There is an accounting identity and therefore an equilibrium equation between the *use* of the n CPA product groups, leading to the vector $\mathbf{x}_I = \mathbf{Ze}$ together with the final demand (consumption) $\mathbf{f} \geq \mathbf{o}$, and the total *supply*, leading to the vector $\mathbf{y}_I = \mathbf{Z}'\mathbf{e}$ together with the value added $\mathbf{v}$. There is the *accounting identity*:

$$\mathbf{x} = \mathbf{Ze} + \mathbf{f} = \mathbf{x}_I + \mathbf{f} = \mathbf{y}_I + \mathbf{v} = \mathbf{Z}'\mathbf{e} + \mathbf{v} = \mathbf{y}. \tag{3}$$

In algebraic developments the *commodity flow matrix* $\mathbf{Z}$ is generally presented in λ_F-normalized form, where $\lambda_F > 0$ is the Frobenius number of matrix $\mathbf{Z}$. For this reason, the initial matrix $\mathbf{Z}$ is divided by λ_F. On gets the λ_F-normalized matrix $\mathbf{Z}_{\lambda_F} = \mathbf{Z}/\lambda_F$, which for reasons of mathematical elegance is again noted as $\mathbf{Z}$. Thus there is $\mathbf{Z}_{\lambda_F} = \mathbf{Z}/\lambda_F \Rightarrow \mathbf{Z}$. In the sequence $\mathbf{Z}$ notes the λ_F-normalized *flow commodity matrix* $\mathbf{Z}_{\lambda_F}$.

The Input-Output Table represents the periodic commodity flow in *monetary terms*. For our purposes it is necessary to attain through the four above presented attributes of the bundle of commodities (CPA product group) a presentation in *physical terms*. For every value z_{ij} in *monetary terms* there is a bundle of commodities (in German: Produktmix [6], p. 11) s_{ij} in *physical terms* with the equation $z_{ij} = p_i s_{ij}$ for the CPA product group $i = 1, ..., n$ and the production sectors[4] $j = 1, ..., n$. The price vector $\mathbf{p}$ is then diagonalized to the diagonal matrix $\hat{\mathbf{p}}$. The equation $z_{ij} = p_i s_{ij}$ is written in matrix form $\mathbf{Z} = \hat{\mathbf{p}}\mathbf{S}$. Now matrix $\mathbf{S}$ is also λ_F-normalized. The *coefficients matrices* $\mathbf{C} = \mathbf{S}\hat{\mathbf{q}}^{-1}$ and $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$ are defined. Matrices $\mathbf{A}$ and $\mathbf{C}$ are similar and have for this reason *identical eigenvalues*, $\lambda_A = \lambda_C$ (**Corollary 1**).

Even if the national Statistical Offices do not deliver flow commodity matrices $\mathbf{S}$ in physical terms in IOTs, there are exclusively flow commodity matrices $\mathbf{Z}$ in monetary terms IOTs, we combine the specific Sraffa price model (2) with the accounting system (3) through equation $\mathbf{Z} = \hat{\mathbf{p}}\mathbf{S}$. Then there is between the vector of *final demand* $\mathbf{f} \geq \mathbf{o}$ and the vector of surplus $\mathbf{d} \geq \mathbf{o}$ the equation $\mathbf{f} = \hat{\mathbf{p}}\mathbf{d} \geq \mathbf{o}$. This means that vector $\mathbf{f}$ is dependent of the price vector $\mathbf{p} > \mathbf{o}$ and the vector of surplus $\mathbf{d} \geq \mathbf{o}$ of the Sraffa price model,

Consider the specific Sraffa price model (2) $\mathbf{S}'\mathbf{p}(1+R) = \hat{\mathbf{q}}\mathbf{p} \Leftrightarrow \mathbf{C}'\mathbf{p}(1+R) = \mathbf{p}$, where all the profits goes to the entrepreneurs, and where the matrix $\mathbf{C}$ is irreducible. The price vector $\mathbf{p}$ is a Frobenius eigenvector, associated to the Frobenius number $\frac{1}{1+R}$.

We call R the *productivity of the economy* or also the productiveness[5]. The following Lemma shows that the productivity R of an economy is directly computed in terms of a boundary value problem, see Emmenegger et al. (2020), pp. 399–408.

[4]If the bundles of commodities are subdivided down to the level of single products like wheat or coal, the *physical terms* become *physical measure units* like kilogram, liter, etc.

[5]To our knowledge the term *productivity of the economy* for this number R has been proposed by Helmut Knolle [5] in the year 2010 in a similar context of another specific Sraffa price model with irreducible matrix, a so called standard system with standard commodity.

Lemma 1 *: Consider the irreducible flow commodity matrix* $\mathbf{Z} = \hat{\mathbf{p}}\mathbf{S}$ *with the Frobenius eigenvalue* λ_F *of* $\mathbf{Z}$*. The productivity* R *is the scalar product of the normalized right eigenvector* $\mathbf{s}_1$*,* $\mathbf{Z}\mathbf{s}_1 = \lambda_F \mathbf{s}_1$*,* $\mathbf{s}_1'\mathbf{e} = 1$*, and the* λ_F-*normalized vector* $(\mathbf{v}/\lambda_F)$ *of total value added. The Frobenius number* $\frac{1}{1+R}$ *is also the quotient of the scalar products* $\mathbf{s}_1'\mathbf{y}_I$ *and* $\mathbf{s}_1'\mathbf{y}$ *with* $\mathbf{y}$*,* $\mathbf{y}_I = \mathbf{y} - \mathbf{v}$*,*

$$R = \mathbf{s}_1'(\frac{\mathbf{v}}{\lambda_F}) = \frac{\mathbf{s}_1'\mathbf{v}}{\mathbf{s}_1'\mathbf{y}_I}, \qquad \frac{\mathbf{s}_1'\mathbf{y}_I}{\mathbf{s}_1'\mathbf{y}} = \frac{1}{1+R}. \tag{4}$$

Proof: We start from the Sraffa price model (2): $\mathbf{q} = \mathbf{S}\mathbf{e}+\mathbf{d}$, $\mathbf{S}'\mathbf{p}(1+R) = \hat{\mathbf{q}}\mathbf{p} = \mathbf{y}$ and setting $\lambda_C := 1/(1+R) \Rightarrow \mathbf{S}'\mathbf{p} = \lambda_C \hat{\mathbf{q}}\mathbf{p}$. By definition: $\mathbf{Z}' = \mathbf{S}'\hat{\mathbf{p}}$ and $\mathbf{y}_I = \mathbf{Z}'\mathbf{e} = \mathbf{S}'\hat{\mathbf{p}}\mathbf{e} = \mathbf{S}'\mathbf{p}$. Then, one gets with (2) $\mathbf{y} = \hat{\mathbf{q}}\mathbf{p}$ and $\mathbf{s}_1'\mathbf{e} = 1$: $\lambda_F = \lambda_F \mathbf{s}_1'\mathbf{e} = (\mathbf{s}_1'\mathbf{Z}')\mathbf{e} = \mathbf{s}_1'(\mathbf{Z}'\mathbf{e}) = \mathbf{s}_1'\mathbf{y}_I = \lambda_F$. For this reason, one obtains: $\lambda_F = \mathbf{s}_1'\mathbf{y}_I = \mathbf{s}_1'\mathbf{S}'\mathbf{p} = \mathbf{s}_1'\,(\lambda_C\hat{\mathbf{q}}\mathbf{p}) = \lambda_C\,(\mathbf{s}_1'\mathbf{y}) = \lambda_C\,\mathbf{s}_1'(\mathbf{y}_I+\mathbf{v}) \Rightarrow$ $\lambda_C = \frac{\mathbf{s}_1'\mathbf{y}_I}{\mathbf{s}_1'\mathbf{y}} = \frac{\mathbf{s}_1'\mathbf{y}_I}{\mathbf{s}_1'\mathbf{y}_I + \mathbf{s}_1'\mathbf{v}} = \frac{\lambda_F}{\lambda_F + \mathbf{s}_1'\mathbf{v}} = \frac{1}{1+\mathbf{s}_1'(\frac{\mathbf{v}}{\lambda_F})} = \frac{1}{1+R}$, this gives the productivity $R = \mathbf{s}_1'(\mathbf{v}/\lambda_F) = \mathbf{s}_1'\mathbf{v}/\mathbf{s}_1'\mathbf{y}_I$. ▲

Corollary 1 *: The productivity* R *of an economy computed in Lemma 1 is obtained from the Frobenius number* λ_A *of the input-output coefficients matrix* $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$*,* $R = \frac{1}{\lambda_A} - 1$*.*

Proof: Consider the equations: $\mathbf{Z} = \hat{\mathbf{p}}\mathbf{S}$, $\mathbf{C} = \mathbf{S}\hat{\mathbf{q}}^{-1}$, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, $\hat{\mathbf{x}} = \hat{\mathbf{p}}\hat{\mathbf{q}}$. Compute $\mathbf{C} = \mathbf{S}\hat{\mathbf{q}}^{-1} = (\hat{\mathbf{p}}^{-1}\mathbf{Z})\hat{\mathbf{q}}^{-1} = (\hat{\mathbf{p}}^{-1}\hat{\mathbf{x}})\mathbf{A}\hat{\mathbf{q}}^{-1} = (\hat{\mathbf{p}}^{-1}\hat{\mathbf{p}})\hat{\mathbf{q}}\mathbf{A}\hat{\mathbf{q}}^{-1} = \hat{\mathbf{q}}\mathbf{A}\hat{\mathbf{q}}^{-1} \Rightarrow \mathbf{C} = \hat{\mathbf{q}}\mathbf{A}\hat{\mathbf{q}}^{-1}$. Thus, the irreducible provided matrices $\mathbf{A}$ and $\mathbf{C}$ are similar and have therefore identical Frobenius numbers, $1 > \lambda_A = \lambda_C > 0$. Thus, $R = \frac{1}{\lambda_A} - 1$. ▲

We illustrate this presentation by two computational examples.

Example 1 : *Consider an economy consisting of* $n = 2$ *sectors, from which one knows the following matrices and vectors,*

$$\mathbf{S} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \tag{5}$$

Compute the vector $\mathbf{q} = \mathbf{S}\mathbf{e} + \mathbf{d} > \mathbf{o}$*, the coefficients matrix* $\mathbf{C} = \mathbf{S}\hat{\mathbf{q}}^{-1} > \mathbf{0}$*, the Frobenius number* $\lambda_C > 0$ *of the matrix* $\mathbf{C}$*, the productivity* $R = (\frac{1}{\lambda_C}) - 1 > 0$ *and the price eigenvector* $\mathbf{p} = [1, p_2]'$ *of matrix* $\mathbf{C}$*.*

Then, compute the commodity flow matrix $\mathbf{Z} = \hat{\mathbf{p}}\mathbf{S}$ *and its Frobenius number* λ_F*, as well as the* λ_F-*normalized matrices* $\mathbf{Z}$ *and* $\mathbf{S}$*. Corresponding to these* λ_F-*normalized matrices one computes the vectors* $\mathbf{x}$*,* $\mathbf{y}$*,* $\mathbf{x}_I$*,* $\mathbf{y}_I$ *and* $\mathbf{v}$*. One then computes the normalized right eigenvector* $\mathbf{s}_1$ *of* $\mathbf{Z}$*,* $\mathbf{Z}\mathbf{s}_1 = \mathbf{s}_1$*,* $\mathbf{s}_1'\mathbf{e} = 1$*. Then, one verifies the quotient* $\frac{1}{1+R} = (\mathbf{s}_1'\mathbf{y}_I)/(\mathbf{s}_1'\mathbf{y})$ *and the equation* $R = \mathbf{s}_1'\mathbf{v}$*.*

Solution of Example 1: One gets,

$$\begin{aligned} \mathbf{q} = \mathbf{S}\mathbf{e} + \mathbf{d} &= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix} \\ \mathbf{C} = \mathbf{S}\hat{\mathbf{q}}^{-1} &= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{4}{5} \\ \frac{1}{9} & \frac{2}{5} \end{bmatrix} \\ \mathbf{Z} = \hat{\mathbf{p}}\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} &= \begin{bmatrix} 3 & 4 \\ 3 & 6 \end{bmatrix}, \quad \mathbf{f} = \hat{\mathbf{p}}\mathbf{d} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}. \end{aligned} \tag{6}$$

Here, the Frobenius number of matrix $\mathbf{C}$ has been calculated, $\lambda_C = 2/3$, the productivity is $R = (1/\lambda_C) - 1 = 1/2$ and the price vector $\mathbf{C}$ is $\mathbf{p} = [1,3]'$. The Frobenius number of matrix $\mathbf{Z}$ is $\lambda_F = 8.275$. Now one computes the λ_F-normalized matrices $\mathbf{Z}$ and $\mathbf{S}$,

$$\mathbf{Z} \;\Rightarrow\; \mathbf{Z} = \frac{1}{\lambda_F}\begin{bmatrix} 3 & 4 \\ 3 & 6 \end{bmatrix}, \quad \mathbf{S} \;\Rightarrow\; \mathbf{S} = \frac{1}{\lambda_F}\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}. \tag{7}$$

In order to keep elegant mathematics, as explained above, one does not change the notation of the λ_F-normalized matrices and vectors, keeping the initial ones. One computes,

$$\mathbf{x}_I = \frac{1}{8.275}\begin{bmatrix} 3 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.846 \\ 1.088 \end{bmatrix}, \mathbf{y}_I = \frac{1}{8.275}\begin{bmatrix} 3 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.725 \\ 1.208 \end{bmatrix}. \tag{8}$$

The λ_F-normalization does not alter the matrix $\mathbf{C}$. One computes the normalized right eigenvectors of matrix $\mathbf{Z}$ and gets: $\mathbf{s}_1 = [0.431, 0.569]'$, $\mathbf{s}_1'\mathbf{e} = 1$. Then, one computes,

$$\mathbf{x} = \frac{1}{\lambda_F}(\mathbf{Ze} + \mathbf{f}) = \frac{1}{\lambda_F}(\begin{bmatrix} 3 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}) = \begin{bmatrix} 1.088 \\ 1.813 \end{bmatrix} = \mathbf{y}, \, \mathbf{v} = \mathbf{y} - \mathbf{y}_I = \begin{bmatrix} 0.363 \\ 0.604 \end{bmatrix}. \tag{9}$$

We continue, computing:

$$\frac{1}{1+R} = \frac{\mathbf{s}_1'\mathbf{y_I}}{\mathbf{s}_1'\mathbf{y}} = \frac{\begin{bmatrix} 0.431 & 0.569 \end{bmatrix}\begin{bmatrix} 0.725 \\ 1.208 \end{bmatrix}}{\begin{bmatrix} 0.431 & 0.569 \end{bmatrix}\begin{bmatrix} 1.088 \\ 1.813 \end{bmatrix}} = \frac{2}{3}, \; R = \mathbf{s}_1'\mathbf{v} = \mathbf{s}_1'\begin{bmatrix} 0.363 \\ 0.604 \end{bmatrix} = \frac{1}{2}. \;\blacktriangle \tag{10}$$

Example 2 : *Compute the coefficients matrix* $\mathbf{A} = \mathbf{Z}\mathbf{x}^{-1}$ *in monetary terms from Example 1 and confirm the equality of the Frobenius numbers,* $\lambda_A = \lambda_C = 2/3$.

Solution of Example 2: We continue, computing:

$$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 3 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{15} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{4}{15} \\ \frac{1}{3} & \frac{6}{15} \end{bmatrix}. \tag{11}$$

Then, one gets the characteristic polynomial $P_2(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - (11/15)\lambda + (2/45)$ and computes the Frobenius numbers of matrices $\mathbf{A}$ and $\mathbf{C}$, $\lambda_A = \lambda_C = 2/3$, using the property that $\mathbf{A}$ and $\mathbf{C}$ are similar. $\blacktriangle$

The national Statistical Offices do not produce commodity flow matrices $\mathbf{S}$ in physical terms, but the national IOTs contain commodity flow matrices $\mathbf{Z}$ in monetary terms. Thus **Lemma 1** is not applicable. But, inspired by **Corollary 1**, we compute the productivity R of national economies from coefficients matrices $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, calculated from the flow commodity matrices $\mathbf{Z}$ in monetary terms, contained in the national $n \times n$ Input-Output Tables (IOT).

We have chosen the countries: Germany, Austria and Switzerland[6] of different sizes. The three IOTs are set up from an $n \times n$ *commodity flow matrix* $\mathbf{Z}$, an $n \times m$ *matrix of final use* and a $k \times n$ *matrix of value added*[7], as presented in Table 2.

[6] Statistisches Bundesamt, Deutschland, Input-Output Tabelle 2013 (Revision 2014, Stand: August 2017, Stand: 12.06.2018); Austrian IOT 2015 (with National Accounts, Version: May 2018), Swiss IOT 2014, Bundesamt für Statistik, Schweiz, (Stand: 2018).

[7] Zero rows and columns could be extracted from the semi-positive matrices $\mathbf{Z} \geq \mathbf{0}$ of the German IOT 2013 and the Swiss IOT 2014 in order to get irreducible matrices, (*) $(\mathbf{I}+\mathbf{A})^{n-1} > \mathbf{0}$, according to the above discussed method, and the Perron-Frobenius theorem applied. For the Austrian IOT 2015 the criterion (*) did not work after extractions, the Frobenius number $\lambda_A = 0.5868$ with positive eigenvectors have been calculated.

Subject	Germany	Austria	Switzerland
year	2013	2015	2014
commodity flow matrix	72×72	65×65	49×49
No: (Laufende Nr.)			
matrix of final uses	72×12	65×15	49×24
matrix of value added	10×72	16×65	4×49

Table 2: Characteristics of national IOTs

One identifies the characteristic quantities of the elements of *national accounting*, obtainable form the IOTs. There is the *final consumption of households* C, the *total investments* I, the *government expenditures* G, the *exports* E and the *imports* M, Table 3. To these five characteristics correspond: the vector of final consumption of household $\mathbf{c} = (c_i)$, the vector of total investments $\mathbf{i} = (i_i)$, the vector of government expenditures $\mathbf{g} = (g_i)$, the vector of exports $\mathbf{e}_x = (e_{xi})$, $i = 1, ..., n$, and the row vector of imports $\mathbf{m}' = (m_j)$, $j = 1, ..., n$.

One constitutes the vector of final consumptions (use) $\mathbf{f} = \mathbf{c}+\mathbf{i}+\mathbf{g}+(\mathbf{e}_x-\mathbf{m}')$, summing it up, one gets,

$$\mathbf{e}'\mathbf{f} = \mathbf{e}'(\mathbf{c}+\mathbf{i}+\mathbf{g}+(\mathbf{e}_x-\mathbf{m}')) = F = C + I + G + (E - M). \tag{12}$$

Then, one computes the vector of total output $\mathbf{x} = \mathbf{Ze} + \mathbf{f} > \mathbf{o}$ and the *coefficients matrix* $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$. We apply Theorem A.12.1, see Emmenegger et al. [2] (or equivalently, Theorem 1.5, p, 39, in Ashmanov [1]). Further, the coefficients matrix $\mathbf{A}$ has its Frobenius number in the open interval $]0, 1[$, $0 < \lambda_A < 1$. This also means, that the Leontief Inverse $(\mathbf{I} - \mathbf{A})^{-1}$ exists. Consequently, there is a productive Leontief model $\mathbf{f} \geq \mathbf{0}$ (1). One can compute the positive vector $\mathbf{x} > \mathbf{o}$ of total output according to equation (3). The vector $\mathbf{x}$ is also computable from the vector of final demand (use) $\mathbf{f} \geq \mathbf{o}$, $\mathbf{x} = (\mathbf{A} - \mathbf{I})^{-1}\mathbf{f} > \mathbf{o}$.

The Frobenius numbers λ_A of the three coefficient matrices $\mathbf{A}$ (D, A, CH) are computed, from which are deduced the *productivities of the economy*, respectively the *productiveness* of the national economies. One indicates in Table 4 the *circulating capital* $K = \mathbf{e}'\mathbf{Ze}$, the *total final use (final demand)* of an economy $F = \mathbf{e}'\mathbf{c} = \mathbf{e}'(\mathbf{c}+\mathbf{i}+\mathbf{g}+(\mathbf{e}_x-\mathbf{m}'))$, the *total output* $X = K + F$ and the GDP equal to $Y = F - M$.

Countries	final consumption (use) of commodities consump. of households C	 total investments I	 governm. expend. G	 export E	import M	productivity R
Germany (EURO)	1'472'436	551'462	593'728	1'257'691	1'049'077	0.6771
Austria (EURO)	188'848	81'955	68'033	167'908	162'473	0.7043
Switzerland (CHF)	345'035	158'682	77'777	351'212	282'987	1.1518
1 EURO=1.15 CHF						

Table 3: Aggregates of national accounting and the productivity

The structure and the size of the three official national IOTs (D, A, CH) are different from each other (Table 2) as well as the content of their sectors, the presently obtained coefficients of productivity R, (Table 3), measuring the *productivity of the economy*, are not directly comparable, although the main aggregates G, I, C, E, M, Y computed form these national IOTs are exactly the official values published by the respective countries Germany, Austria, Switzerland for the indicated years. One observes that the measure of productivity

countries	total circulation capital K	total final use F	Gross domestic product $Y \sim GDP$	total output X
Germany (EURO)	2'831'297	3'875'317	2'826'240	6'706'614
Austria (EURO)	328'767	506'745	344'272	835'512
Switzerland (CHF)	662'275	932'706	649'718	1'594'980
1 EURO =1.15 CHF				

Table 4: Other aggregates of the accounting systems (G, A, CH)

R is sensitive to the structure of the IOTs. – Indeed, it is known that model reduction influences the Frobenius number, see Emmenegger et al. [2], pp. 425-427, Figure 10.12, Figure 10.3, Figure 10.14.

For this reason, we have looked for IOTs of a common structure. We rely therefore on the Input-Output Tables, issued from the WIOD project, proposed by the Groningen Growth and Development Centre, Faculty of Economics and Business, University of Groningen, Netherlands, which has developed the World Input-Output Database (WIOD) see Dietzenbacher & al. [7]. In his article Dietzenbacher *"describes the construction of the World Input-Output Tables (WIOTs) that constitute the core of the World InputOutput Database. WIOTs are available for the period 1995-2009 and give the values of transactions among 35 industries in 40 countries plus the 'Rest of the World' and from these industries to households, governments and users of capital goods in the same set of countries. The article describes how information from the National Accounts, Supply and Use Tables and International Trade Statistics have been harmonized, reconciled and used for estimation procedures to arrive at a consistent time series of WIOTs"*. These World InputOutput Tables (WIOTs) proposed by the University of Groningen have the advantage that their structure is unified across all the analysed countries. The WIOTs have uniformly $n = 35$ product groups, based on the standard classification of the ISCI, and industry-by-industry production sectors, see the publication *"International Standard Industrial Classification of All Economic Activities"*, [10]. Thus, the economic structure becomes comparable from one country to the other across these WIOTs.

– The format of the OECD [8] harmonised national Input-Output Tables, reveals 36×36 *input-output matrices* $\mathbf{Z}$ for the three considered IOTs of the countries: DE-OECD-IOT-2015, A-OECD-IOT-2015, CH-OECD-IOT-2014. The *productivity of the economy* of the three *input-output matrices* $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$ of the countries Germany, Austria, Switzerland have been recalculated, obtaining the coefficients of productivity $R_{DE} = 0.9859$, $R_A = 0.9040$, $R_{CH} = 0.7546$.

The perspective of getting monetary and physical data in matrices as $\mathbf{Z}$ and $\mathbf{S}$ opens the way to formulate the economic systems of production in terms of boundary value problems as it is the case in physics and technical sciences.

References

[1] Ashmanov, S. A., *Introduction in Mathematical Economics*, (in Russian language) - Moscow: Nauka, 296 pages, (1984).

[2] Emmenegger, J.-F., Chable D., Nour Eldin H. A., Knolle, H., *Sraffa and Leontief Revisited: Mathematical Models of a Circular Economy*, de Gruyter-Oldenbourg, München-Berlin, (Januar 2020).

[3] Frobenius, G., *Über Matrizen aus nicht negativen Elementen*, Berliner Bericht, Seiten 456-477, (1912).

[4] Gantmacher, F. R., *"Matrizentheorie"*, Berlin, Heidelberg, New York,Tokyo, Springer-Verlag, (1986).

[5] Knolle, Helmut, *Vollbeschäftigung bei Nullwachstum: Für eine soziale und ökologische Wirtschaft*, Im Jahrbuch Denknetz, Verlag: Edition 8, Postfach 3522, 8021 Zürich, (2010).

[6] Nathani, C., Schmid, Ch., van Nieuwkoop, R., *Schätzung einer Input-Output-Tabelle der Schweiz 2008*, Schlussbericht an das Bundesamt für Statistik, Schweizerisches Bundesamt für Statistik, Rüschlikon/Bern, (2011).

[7] Dietzenbacher Erik, Los Bart, Stehrer Robert, Timmer Marcel & de Vries Gaaitzen *The Construction of World Input-Output Tables in the WIOD Project*, Economic Systems Research, 25:1, 71-98, DOI: 10.1080/09535314.2012.761180. (2013).

[8] Wikipedia, OECD, $https://www.oecd.org/sti/ind/input-outputtables.htm.$ (2021,08.27)

[9] Sraffa, P., *Production of Commodities by means of Commodities*, Cambridge: Cambridge University Press, (1960).

[10] United Nations, *International Standard Industrial Classification of All Economic Activities (ISIC)*, Statistical Papers, No. 4, Rev. 4, (2008).