

From SUT to SIOT and Backward: Exploring the Reverse Transformations under Product Technology and Industry Technology Assumptions

One of the main aims of constructing input-output balance models is to assess an impact of exogenous changes in net final demand (at constant prices, certainly) on simultaneous behavior of an economy. Nowadays, two approaches to constructing input-output coefficients are widely used in practice, namely, one based on so-called product technology assumption and another based on so-called industry technology assumption. These approaches provide direct transforming supply and use tables (SUT) to symmetric input-output tables (SIOT).

Focus of attention in the study is concentrated on analyzing the reverse transformations that link exogenous changes of final demand in SIOT with corresponding changes of the production and intermediate consumption matrices in initial SUT. Material balance equation, classical Leontief equation and product (or commodity) technology model form the system of equations with production and intermediate consumption matrices as unknowns. It is shown that this system has the solution that guarantees the exogenous changes in final demand to be at constant prices.

In turn, material balance equation, classical Leontief equation and industry technology model constitute another system of equations (with the same unknowns) that can be also resolved with respect to production matrix and intermediate consumption matrix. However, exogenous varying the final demand in obtained solution leads to quantity changes in the intermediate consumption matrix and to price changes in the production matrix. This type of economy's response to exogenous changes in final demand seems to be implausible artifact that is out of economic sense. Thus, there are some certain doubts about plausibility of underlying background for an industry technology assumption and a fixed product sales structure assumption that are widely used for transforming SUT to SIOT.

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JEL Classification: C67; D57

1. An introduction

Consider a common form of demand-driven input-output model written as follows:

$$\mathbf{x} = \mathbf{L}\mathbf{y} \quad (1)$$

where \mathbf{x} is N -dimensional column vector of product outputs, N is the number of products being produced in the economy, \mathbf{y} is N -dimensional column vector of final demand, and \mathbf{L} is nondegenerate square matrix of order N (in particular, the Leontief inverse). Clearly, formula (1) expresses the functional dependence of product outputs on final demand components, in which \mathbf{y} plays the role of independent vector variable. Thus, it is implicitly presumed in (1) that vector \mathbf{y} can be varied arbitrarily while all components of the final demand undergo quantity (not price!) changes.

However, vector \mathbf{y} is a part of material balance equation

$$\mathbf{x} = \mathbf{z} + \mathbf{y} \quad (2)$$

where \mathbf{z} is column vector of intermediate product inputs with dimensions $N \times 1$. Since $\mathbf{y} = \mathbf{x} - \mathbf{z}$, as it follows from (2), the changes in final demand exert an influence on the differences between product outputs and product inputs. Therefore, exogenous variations of final demand vector \mathbf{y} lead to corresponding changes in \mathbf{x} and \mathbf{z} .

Key idea of Leontief input-output analysis is to eliminate variable \mathbf{z} from equation (2) by substitution of the linear linking relation

$$\mathbf{z} = \mathbf{A}\mathbf{x} \quad (3)$$

where square matrix \mathbf{A} of order N is known in special literature as (Leontief) technical coefficients matrix. The technical coefficients are usually calculated on a base of the given supply and use table for certain time period (say, period 0) that contains supply (production) matrix \mathbf{X}_0 and use (intermediate consumption) matrix \mathbf{Z}_0 of the same dimension $N \times M$ where M is the number of industries in the economy. Letting $\mathbf{A} = \mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0)$ be a square matrix of order N , and then solving system (2), (3) with respect to the product outputs vector \mathbf{x} leads to Leontief demand-driven input-output model (1) with transformation matrix

$$\mathbf{L} = [\mathbf{E}_N - \mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0)]^{-1}$$

where \mathbf{E}_N is an identity matrix of order N .

Thus, constructing demand-driven input-output model for analytic purposes comes down to a choice of appropriate pattern for technical coefficients matrix. There are many ways to define the coefficients pattern known in special literature. "It is standard to derive input-output constructs from alternative assumptions" (Kop Jansen and ten Raa, 1990, p. 214). Nevertheless, two approaches to constructing input-output coefficients are most widely used in practice, namely, one based on so-called product technology assumption and another based on so-called industry technology assumption (see Eurostat, 2008; United Nations, 2018).

Notice that $\mathbf{x} = \mathbf{X}\mathbf{e}_M$ and $\mathbf{z} = \mathbf{Z}\mathbf{e}_M$ where \mathbf{X} and \mathbf{Z} are matrix variables of the same dimension as production matrix \mathbf{X}_0 and intermediate consumption matrix \mathbf{Z}_0 , respectively, and \mathbf{e}_M is $M \times 1$ summation column vector with all entries equal to one. Main scope of the paper is to assess and interpret the consequences of exogenous final demand varying in input-output model (1) for production matrix \mathbf{X} and intermediate consumption matrix \mathbf{Z} at choosing input-output coefficients $\mathbf{A} = \mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0)$ under product technology assumption and, in turn, under industry technology assumption. The Leontief demand-driven input-output model

$$\mathbf{X}\mathbf{e}_M = [\mathbf{E}_N - \mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0)]^{-1} \mathbf{y} \quad (4)$$

and material balance equation

$$\mathbf{X}\mathbf{e}_M = \mathbf{Z}\mathbf{e}_M + \mathbf{y} \quad (5)$$

serve as a toolbox for analyzing concomitant changes in product outputs and intermediate inputs following the changes in final demand.

2. Product technology

The product technology pattern (or the commodity technology model – see, e.g., Kop Jansen and ten Raa, 1990) can be presenting in our denotations as follows:

$$\mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0) = \mathbf{Z}_0 \mathbf{X}_0^{-1}. \quad (6)$$

Obviously, the pattern is valid if number of products N coincides with number of industries M , i.e., $N = M = K$, and square production matrix \mathbf{X}_0 of order K is invertible. The latter does not seem to be too restrictive because the actual production matrices use to be strictly diagonally dominant in practice.

Substituting the pattern (6) into Leontief demand-driven input-output model (4) yields

$$\mathbf{X}\mathbf{e}_K = (\mathbf{E}_K - \mathbf{Z}_0 \mathbf{X}_0^{-1})^{-1} \mathbf{y} = \mathbf{X}_0 (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}.$$

Since this equation should be fulfilled at any vector of final demand, it is possible to express the production matrix \mathbf{X} in left-hand side as

$$\mathbf{X} = \mathbf{X}_0 \langle (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y} \rangle = \mathbf{X}_0 \hat{\mathbf{q}} \quad (7)$$

where angled bracketing around vector's designation (or putting a “hat” over vector's symbol) denotes a diagonal matrix with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697).

Substitution the production matrix (7) into material balance equation (5) and rearrangement the terms gives

$$\begin{aligned} \mathbf{Z}\mathbf{e}_K &= \mathbf{X}\mathbf{e}_K - \mathbf{y} = \mathbf{X}_0 (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y} - \mathbf{y} = [\mathbf{X}_0 (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} - \mathbf{E}_K] \mathbf{y} = \\ &= [\mathbf{X}_0 - (\mathbf{X}_0 - \mathbf{Z}_0)] (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y} = \mathbf{Z}_0 (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}. \end{aligned}$$

This equation should be fulfilled at any final demand vector \mathbf{y} , hence, intermediate consumption matrix \mathbf{Z} in left-hand side can be determined as

$$\mathbf{Z} = \mathbf{Z}_0 \langle (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y} \rangle = \mathbf{Z}_0 \hat{\mathbf{q}}. \quad (8)$$

It is easy to show that $\mathbf{q} = \mathbf{e}_K$ at $\mathbf{y} = \mathbf{y}_0 = \mathbf{X}_0 \mathbf{e}_K - \mathbf{Z}_0 \mathbf{e}_K$. Indeed, it follows directly from $\mathbf{y}_0 = (\mathbf{X}_0 - \mathbf{Z}_0) \mathbf{e}_K$ so that $\mathbf{q} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}_0 = \mathbf{e}_K$, or $\hat{\mathbf{q}} = \mathbf{E}_K$.

Thus, at choosing input-output coefficients according to *product technology* pattern an arbitrary variation of final demand vector leads to the changes in production and intermediate consumption matrices described by simple *postmultiplication* formulas $\mathbf{X} = \mathbf{X}_0 \hat{\mathbf{q}}_{PT}$, $\mathbf{Z} = \mathbf{Z}_0 \hat{\mathbf{q}}_{PT}$

where $\mathbf{q}_{\text{PT}} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}$, and subindex ‘‘PT’’ means an association of the vector with product technology pattern. Note that all K components of vector \mathbf{q}_{PT} are dimensionless.

3. Industry technology

Kop Jansen and ten Raa (1990) studied the industry technology model that in our denotations becomes

$$\mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0) = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1}. \quad (9)$$

It is easy to see that this pattern for technical coefficients matrix is valid at any combinations of number of products N and number of industries M .

Substituting industry technology pattern (9) into the Leontief demand-driven input-output model (4) and rearranging the terms yields

$$\begin{aligned} \mathbf{X} \mathbf{e}_M &= \left(\mathbf{E}_N - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \right)^{-1} \mathbf{y} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \right)^{-1} \mathbf{y} = \\ &= \left\langle \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \right)^{-1} \mathbf{y} \right\rangle \mathbf{X}_0 \mathbf{e}_M \end{aligned}$$

where the obvious quasi-commutativity property $\hat{\mathbf{a}} \mathbf{b} = \hat{\mathbf{b}} \mathbf{a}$ for a pair of N -dimensional column vectors \mathbf{a} and \mathbf{b} is used. The latter equation should be fulfilled at any vector of final demand, as earlier. Therefore,

$$\mathbf{X} = \left\langle \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \right)^{-1} \mathbf{y} \right\rangle \mathbf{X}_0 = \hat{\mathbf{p}} \mathbf{X}_0. \quad (10)$$

Substitution the production matrix (10) into material balance equation (5) and rearrangement the terms gives

$$\begin{aligned} \mathbf{Z} \mathbf{e}_M &= \mathbf{X} \mathbf{e}_M - \mathbf{y} = \left[\langle \mathbf{X}_0 \mathbf{e}_M \rangle \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \right)^{-1} - \mathbf{E}_N \right] \mathbf{y} = \\ &= \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \langle \mathbf{X}_0 \mathbf{e}_M \rangle + \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \right) \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \right)^{-1} \mathbf{y} = \\ &= \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \right)^{-1} \mathbf{y}. \end{aligned}$$

Again, as earlier, this equation should be fulfilled at any final demand vector \mathbf{y} , hence, intermediate consumption matrix \mathbf{Z} in left-hand side can be derived as

$$\mathbf{Z} = \mathbf{Z}_0 \left\langle \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \right)^{-1} \mathbf{y} \right\rangle = \mathbf{Z}_0 \hat{\mathbf{q}} = \mathbf{Z}_0 \left\langle \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \mathbf{p} \right\rangle. \quad (11)$$

On checking the initial condition, setting $\mathbf{y} = \mathbf{y}_0 = \mathbf{X}_0 \mathbf{e}_M - \mathbf{Z}_0 \mathbf{e}_M$ in (10) gives the following equation with respect to unknown vector \mathbf{p} :

$$\langle \mathbf{X}_0 \mathbf{e}_M \rangle \mathbf{p} - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \mathbf{p} = \mathbf{X}_0 \mathbf{e}_M - \mathbf{Z}_0 \mathbf{e}_M.$$

Since $\langle \mathbf{X}_0 \mathbf{e}_M \rangle \mathbf{e}_N = \mathbf{X}_0 \mathbf{e}_M$ and $\mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \mathbf{e}_N = \mathbf{Z}_0 \langle \mathbf{X}'_0 \mathbf{e}_N \rangle^{-1} \mathbf{X}'_0 \mathbf{e}_N = \mathbf{Z}_0 \mathbf{e}_M$, the solution to this equation is $\mathbf{p} = \mathbf{e}_N$, from which and (11) it directly follows that

$$\mathbf{q} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \mathbf{p} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \mathbf{e}_N = \mathbf{e}_M.$$

Thus, at choosing input-output coefficients according to *industry technology* pattern (9) an arbitrary variation of final demand vector induces the changes in production and intermediate consumption matrices respectively described by *premultiplication* formula $\mathbf{X} = \hat{\mathbf{p}}_{IT} \mathbf{X}_0$ (in contrast to product technology case) and *postmultiplication* formula $\mathbf{Z} = \mathbf{Z}_0 \hat{\mathbf{q}}_{IT}$ (as in product technology case) where

$$\mathbf{p}_{IT} = \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \right)^{-1} \mathbf{y}, \quad \mathbf{q}_{IT} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \mathbf{p}_{IT}, \quad (12)$$

and subindex “IT” means an association of the vector with industry technology pattern. Note that all components of vectors \mathbf{p}_{IT} and \mathbf{q}_{IT} are dimensionless in accordance with (10) and (11), respectively.

4. Interpretation of the results

There are some distinctions between the formal results obtained above in Section 2 (product technology case) and Section 3 (industry technology case). The substantial distinction lies in the expressions derived for production matrix, namely, $\mathbf{X} = \mathbf{X}_0 \hat{\mathbf{q}}_{PT}$ in product technology case whereas $\mathbf{X} = \hat{\mathbf{p}}_{IT} \mathbf{X}_0$ in industry technology case. Recall that $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ are diagonal matrices of appropriate orders.

To clarify a role of vector \mathbf{p} in arbitrary varying of final demand together with production and intermediate consumption matrices, consider Leontief price model

$$\mathbf{p}' = \mathbf{p}' \mathbf{A} + \mathbf{v}' \langle \mathbf{e}'_K \mathbf{X}_0 \rangle^{-1}$$

where \mathbf{p} is price index vector with dimensions $K \times 1$, $\mathbf{A} = \mathbf{Z}_0 \langle \mathbf{e}'_K \mathbf{X}_0 \rangle^{-1}$ is technical coefficients matrix, and \mathbf{v} is value added vector with dimensions $K \times 1$ (see, e.g., Miller and Blair, 2009). Substituting technical coefficients matrix \mathbf{A} into the price model yields the financial balance equation $\mathbf{p}' \langle \mathbf{e}'_K \mathbf{X}_0 \rangle = \mathbf{p}' \mathbf{Z}_0 + \mathbf{v}'$ from which $\mathbf{X} = \hat{\mathbf{p}} \langle \mathbf{e}'_K \mathbf{X}_0 \rangle$ and $\mathbf{Z} = \hat{\mathbf{p}} \mathbf{Z}_0$ (note that in Leontief price model the production matrix is assumed to be diagonal). Hence, *price changes* in production matrix and intermediate consumption matrix are described by the initial matrices \mathbf{X}_0 and \mathbf{Z}_0 *premultiplying* by diagonal matrix of price indices.

Futhermore, for establishing a role of vector \mathbf{q} in arbitrary varying of final demand together with production and intermediate consumption matrices, consider Ghosh quantity model

$$\mathbf{q} = \mathbf{B}\mathbf{q} + \langle \mathbf{X}_0 \mathbf{e}_K \rangle^{-1} \mathbf{y}$$

where \mathbf{q} is quantity index vector with dimensions $K \times 1$, and $\mathbf{B} = \langle \mathbf{X}_0 \mathbf{e}_K \rangle^{-1} \mathbf{Z}_0$ is allocation coefficients matrix (see Miller and Blair, 2009; Motorin, 2017). Substituting allocation coefficients matrix \mathbf{B} into the quantity model yields the material balance equation $\langle \mathbf{X}_0 \mathbf{e}_K \rangle \mathbf{q} = \mathbf{Z}_0 \mathbf{q} + \mathbf{y}$ from which $\mathbf{X} = \langle \mathbf{X}_0 \mathbf{e}_K \rangle \hat{\mathbf{q}}$ and $\mathbf{Z} = \mathbf{Z}_0 \hat{\mathbf{q}}$ (in Ghosh quantity model the production matrix is also assumed to be diagonal). Therefore, *quantity changes* in production matrix and intermediate consumption matrix are described by the initial matrices \mathbf{X}_0 and \mathbf{Z}_0 *postmultiplying* by diagonal matrix of quantity indices.

Thus, arbitrary variations of final demand vector in input-output model (1) within a *product technology* pattern (PT-model) are translated into the *quantity changes* in production and intermediate consumption matrices. Each component of vector $\mathbf{q}_{\text{PT}} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}$ represents an index of output growth in corresponding industry induced by a change of final demand as well as an index for intermediate inputs growth in the same industry caused by the industry output growth. In other words, the PT-model (1) with exogenous final demand operates *at constant prices* because $\hat{\mathbf{p}}_{\text{PT}} = \mathbf{E}_K$.

In turn, arbitrary variations of final demand vector in input-output model (1) within a *industry technology* pattern (IT-model) are transferred to the *price changes* in production matrix and at the same time to the *quantity changes* in intermediate consumption matrix. Note that each entry of vector \mathbf{p}_{IT} represents a price index for output of corresponding product that does not vary along the row of all producing-and-consuming industries. As it follows from the first formula (12), this price-induced output change is a part of response to exogenous change of final demand in IT-model (1). The another part of the response is related to the quantity changes in intermediate consumption matrix, whence follows that each element of vector in the second formula (12), $\mathbf{q}_{\text{IT}} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \mathbf{p}_{\text{IT}}$, should be considered as an index for *intermediate inputs growth* in corresponding industry caused by the industry component of *price-induced output change* mentioned above (?!). This mixed situation of combining price and quantity changes in a presence of the direct linkage (12) between vectors \mathbf{p}_{IT} and \mathbf{q}_{IT} seems to be an implausible artifact that is out of economic sense.

5. Concluding remarks

Practical applications of demand-driven input-output model (1) are actually based on a principal opportunity to arbitrarily vary the final demand vector in right-hand side of (1) provided that all its components undergo quantity changes. This seems to be a necessary requirement for

constructing the proper transformation matrix \mathbf{L} applicable to solving various problems in main branches of modern input-output theory such as multiplier analysis, impact analysis, structural decomposition analysis, value-added chain analysis, etc.

Formally, the demand-driven input-output model (1) within a product technology pattern (6) is fully (mathematically and economically) consistent with the above requirement. As it is shown in Section 2 and 3, the model do operates at constant prices. Nevertheless, well-known (but not indisputable!) problem of negative cell entries in the product technology (see, e.g., United Nations, 2018, Annex B to Chapter 12) does not allow yet to recommend a product technology approach as universal way of transforming the supply and use tables into symmetric input-output tables.

To continue, the demand-driven input-output model (1) within an industry technology pattern (9) do certainly violate the above requirement in general with the exception of a case when the production matrix is diagonal (because for diagonal output matrix, obviously, a price index and a growth index are indiscernible). Unfortunately, model operating generates an information and logic “price'n'quantity” gap between initial supply and use table and resulting input-output table. It casts some doubt on plausibility of an industry technology assumption and a fixed product sales structure assumption (see Eurostat, 2008) often used in converting supply and use tables to symmetric input-output tables.

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