Dynamics of an input output reaction di¤usion model.

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Abstract. A input output reaction di¤usion model is considered. A matrix depending of the input output matrix and the growth and profit rates and the Turing numbers and di¤usion coe¢ cients for the determination of the stability of the steady state is deduced.

The pioneer work on the dynamic of input output models comes from the book of Harrod of 1936. Some of these models already have been considered in (Semmler, W. and Flaschel, P. and Franke, R. 1997) at sectorial level have been considered in (Semmler, W. and Flaschel, P. 1989). Here, the main novel in this type of models is the introduction of pertubation terms in each economic branch in form of di¤usions dependent of the consumption and the wages; these di¤usions can be modeled with the Laplacian operator in several variables. In general the dynamic of type of reaction di¤usion are within the following scheme of partial di¤erentials equations:

$$\frac{\partial u_i(x_1,...,x_n,t)}{\partial t} = D_i \sum_{j=1}^{n} \frac{\partial^2 u_i(x_1,...,x_n,t)}{\partial x_j^2} + f_i(x_1,...,x_n,t).$$

Here, $i = 1, 2, ..., m$. The expression $\sum_{j=1}^{n} \frac{\partial^2 u_i(x_1,...,x_n,t)}{\partial x_j^2}$ is the Laplacian of u_i and it is denoted by 4 u_i.The vector that only includes to the functions f_j is the reaction vector and the one that includes the Laplacian is the di¤usion vector. In most of the dynamic economic models is frequent to find the reaction system single. The dynamics of these models is richer if di¤usions are considered. The di¤usions that set out in this type of models are those that are included in the consumption vector and in the vector of wages for the habitual input output models. It is a form to endogenisation these variables and of not considering them exogenous or constant. The techniques for the investigation of this type of dynamical systems can be found in for example in books like (Folland, G. B., 1992), (Strauss, W. A. 1992). The main objective in the economic systems that are investigated consists of finding the steady state and making the investigation on the stability of this type of systems with some matrix associated to these systems.

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There are two diffusions considered in the model. The first diffusion suppose that the changes of the production time react according to the accelerated production changes (second derivative) with respect to the consumtion and with respect to the wages ($D_X \triangle X(C,W,t)$). The second diffusion assume that the changes of the prices at the time are perturbed with respect to the accelerated changes of the prices with respect to the consumption and with respect to the wages ($D_P \triangle P(C,W,t)$). When both diffusions are suppose to be zero the system has been investigated by Semmler, W. et al 1997.

The model can be constructed with the usual laws of the excess of demand and the excess of rent. The model is the following system of reaction diffusion:

$$\begin{aligned}
\frac{\partial X(C,W,t)}{\partial t} &= D_X \triangle X(C,W,t) + d_{11}\left(\begin{array}{c} A(\tau)X(C,W,t) + gA(\tau)X(C,W,t) \\ +C - X(C,W,t) \end{array}\right) \\
&\quad - d_{12}\left(A(\tau)' P(C,W,t) + rA(\tau)' P(C,W,t) + W - P(C,W,t)\right), \\
\frac{\partial P(C,W,t)}{\partial t} &= D_P \triangle P(C,W,t) + d_{21}\left(\begin{array}{c} A(\tau)X(C,W,t) + gA(\tau)X(C,W,t) \\ +C - X(C,W,t) \end{array}\right) \\
&\quad + d_{22}\left(A(\tau)' P(C,W,t) + rA(\tau)' P(C,W,t) + W - P(C,W,t)\right), \\
\tau &= 1950, 1960, 1970, 1975, 1978, 1980, 1985, 1990, 1993;
\end{aligned}$$

with the boundary conditions :

$$\begin{aligned}
X(C,W,t) &= 0, \\
P(C,W,t) &= 0,
\end{aligned}$$

when $C_j = 0$ or 1 or $W_j = 0$ or 1and the other variables stay in the interval $(0,1)$ for every $j = 1,2,...,\frac{d}{2}$. To the supposition that is an competitive economy of total labour and basic consumer goods, the production and the prices fall to zero when some consumption either wage falls to 0 or when some sector monopolizes to the consumption or the wage to 1, for any time t.

Briefly define is had $X(C,W,t) \equiv X$, $P(C,W,t) \equiv P$.

C, W are vectors of consumption and wages of $\frac{d}{2} \times 1$:

$$\begin{aligned}
C &= \left(C_1, ..., C_{\frac{d}{2}}\right)', \\
W &= \left(W_1, ..., W_{\frac{d}{2}}\right)'.
\end{aligned}$$

Of this form X and P are vectors of production and prices of $\frac{d}{2} \times 1$:

$$
\begin{aligned}
X &= \left(X_1, ..., X_{\frac{d}{2}} \right)', \\
P &= \left(P_1, ..., P_{\frac{d}{2}} \right)'.
\end{aligned}
$$

The symbols $\Delta X(C,W,t)$, $\Delta P(C,W,t)$, mean vector Laplacians and this are defined as:

$$
\begin{aligned}
\Delta X(C,W,t) &= \left(\Delta X_1, ..., \Delta X_{\frac{d}{2}} \right)', \\
\Delta P(C,W,t) &= \left(\Delta P_1, ..., \Delta P_{\frac{d}{2}} \right)',
\end{aligned}
$$

and of course

$$
\begin{aligned}
\Delta X_i &= \sum_{j=1}^{\frac{d}{2}} \frac{\partial^2 X_i}{\partial C_j^2} + \frac{\partial^2 X_i}{\partial W_j^2}, \\
\Delta P_i &= \sum_{j=1}^{\frac{d}{2}} \frac{\partial^2 P_i}{\partial C_j^2} + \frac{\partial^2 P_i}{\partial W_j^2}.
\end{aligned}
$$

D_X, D_P are the diagonals matrices of diffusion coefficients

$$
\begin{aligned}
D_X &= diag \left(D_{X1}, ..., D_{X\frac{d}{2}} \right), \\
D_P &= diag \left(D_{P1}, ..., D_{P\frac{d}{2}} \right),
\end{aligned}
$$

$A(\tau)$ it is an input output matrix of technical coefficients of $\frac{d}{2} \times \frac{d}{2}$.
d_{ij} they are the first diagonals of adjustment of positive components or zero:

$$
d_{ij} = diag \left(d_{ij1}, ..., d_{ij\frac{d}{2}} \right).
$$

When $d_{11} = d_{22} = 0$; $d_{12} \neq 0$, $d_{21} \neq 0$ the neoclassic case is had.
When $d_{12} = d_{21} = 0$; $d_{11} \neq 0$, $d_{22} \neq 0$, the Keynesian case is had.
When all the matrices are different from zero has the case postKeynesian.
If in addition the matrices are defined:

$$
\begin{aligned}
\mathsf{A} &= \begin{pmatrix} d_{11} & -d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} (1+g)\,A - I & 0 \\ 0 & (1+r)\,A^{0} - I \end{pmatrix}, \\
\mathsf{B} &= \begin{pmatrix} d_{11} & -d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} C \\ W \end{pmatrix}, \\
\Delta\zeta &= \begin{pmatrix} \Delta X \\ \Delta P \end{pmatrix}, \\
D &= diag\left(d_{1X}, ..., d_{\frac{d}{2}X}, d_{1P}, ..., d_{\frac{d}{2}P} \right).
\end{aligned}
$$

The model can be written in compact form like a system of nonhomogenous reaction diffusion:

$$
\frac{\partial \zeta}{\partial t} = D\Delta\zeta + \mathsf{A}\zeta + \mathsf{B}.
$$

The non homogeneity of the this equation consist of the B term, because it depends on the independiantes variables C and W.

It is possible to be observed that unlike the usual dynamic input output models of reaction, the vector of prices and the vector of production depend on the consumption and the wages in endogenous form.

In order to solve previous the nonhomogenous system, one looks first a solution of a combined part of the reaction with the diffusion; that is, one solve the equation of the steady state

$$
D\Delta\zeta^{*} + \mathsf{A}\zeta^{*} + \mathsf{B} = 0;
$$

with the same conditions of border that before.

After solving this equation, a change of variables $\zeta = \zeta^{*} + v$ will transform the first system in the simplest system $\frac{\partial v}{\partial t} = D\Delta v + \mathsf{A}v$, since ζ^{*} does not depend explicitly on t.

When $\mathsf{A} = (\alpha_{ij})$, then the components of the previous equation are:

$$
\begin{aligned}
\Delta X_i^{*} &= -d_{iX}^{-1}\left[(d_{11i}C_i - d_{12i}W_i) + \sum_{j=1}^{\frac{d}{2}} \alpha_{ij}X_j^{*} + \sum_{j=\frac{d}{2}+1}^{d} \alpha_{ij}P_j^{*} \right], \\
\Delta P_i^{*} &= -d_{iP}^{-1}\left[(d_{21i}C_i + d_{22i}W_i) + \sum_{j=1}^{\frac{d}{2}} \alpha_{\frac{d}{2}+i,j}X_j^{*} + \sum_{j=\frac{d}{2}+1}^{d} \alpha_{\frac{d}{2}+i,j}P_j^{*} \right]. \\
i &= 1, 2, ..., \frac{d}{2}.
\end{aligned}
$$

Given the boundary conditions, these equations can be solved with a multiple Fourier sine series:

$$X_i^* = \sum_{\substack{n_1 n_2 \ldots n_{\frac{d}{2}} m_1 m_2 \ldots m_{\frac{d}{2}} \\ \text{integers}}} a^i_{n_1 n_2 \ldots n_{\frac{d}{2}} m_1 m_2 \ldots m_{\frac{d}{2}}} \prod_{j=1}^{\frac{d}{2}} \sin(n_j \pi C_j) \sin(m_j \pi W_j),$$

$$P_i^* = \sum_{\substack{n_1 n_2 \ldots n_{\frac{d}{2}} m_1 m_2 \ldots m_{\frac{d}{2}} \\ \text{integers}}} b^i_{n_1 n_2 \ldots n_{\frac{d}{2}} m_1 m_2 \ldots m_{\frac{d}{2}}} \prod_{j=1}^{\frac{d}{2}} \sin(n_j \pi C_j) \sin(m_j \pi W_j).$$

The symbol is brief $\underset{\text{integers}}{n_1 n_2 \ldots n_{\frac{d}{2}} m_1 m_2 \ldots m_{\frac{d}{2}}}$ to the symbol n, m.

The Fourier coefficients a^i_{nm} are determined through the numbers

$$A = \left(A^1_{nm}, \ldots, A^{\frac{d}{2}}_{nm} \right)', \qquad B = \left(B^1_{nm}, \ldots, B^{\frac{d}{2}}_{nm} \right)',$$

where

$$A^i_{n,m} = \frac{2^d d_{iX}^{-1} \sum_{j=0}^{d-1} (-1)^{d-j} \left(d_{11i} (-1)^{\sigma^j_{n_i}} - d_{12i} (-1)^{\sigma^j_{m_i}} \right)}{\pi^{n_1 + n_2 + \ldots + n_{\frac{d}{2}} + m_1 + m_2 + \ldots + m_{\frac{d}{2}}} \prod_{l=1}^{\frac{d}{2}} n_l m_l},$$

$$B^i_{n,m} = \frac{2^d d_{iP}^{-1} \sum_{j=0}^{d-1} (-1)^{d-j} \left(d_{21i} (-1)^{\sigma^j_{n_i}} + d_{22i} (-1)^{\sigma^j_{m_i}} \right)}{\pi^{n_1 + n_2 + \ldots + n_{\frac{d}{2}} + m_1 + m_2 + \ldots + m_{\frac{d}{2}}} \prod_{l=1}^{\frac{d}{2}} n_l m_l}.$$

and

$$\sigma^j_{n_i} = \sum_{j=0}^{d-i} n_j + m_j$$

n_i it is in each term of the sum

$$\sigma^j_{m_i} = \sum_{j=0}^{d-i} n_j + m_j$$

m_i it is in each term of the sum

Now

$$a = \left(a^1_{nm}, ..., a^{\frac{d}{2}}_{nm}\right)',$$

$$b = \left(b^1_{nm}, ..., b^{\frac{d}{2}}_{nm}\right)'.$$

From this, the vectors can be calculated solving the system of linear equations

$$\begin{pmatrix} a \\ b \end{pmatrix} = \left(\pi^2 \sum_{l=1}^{\frac{d}{2}} \left(n_l^2 + m_l^2\right) I - \mathcal{A}\right)^{-1} \begin{pmatrix} A \\ B \end{pmatrix}.$$

With these particular solutions the system that determines the dynamics of the model is the reaction system diffusion

$$\frac{\partial v}{\partial t} = D \Delta v + \mathcal{A} v.$$

Using again solutions in form of multiple sine series of Fourier, the dynamics of this system is determined by the eigenvalues of the matrix

$$\mathcal{A} - \pi^2 \sum_{l=1}^{\frac{d}{2}} \left(n_l^2 + m_l^2\right) D.$$

n_l, m_l correspond with the Turing numbers of the input output reaction diffusion dynamics.

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