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# Approximate Surrogate Production Functions: Do They Exist, For Large Systems? (Families of strongly curved and of nearly linear wage curves[1])

Summary:

*The Cambridge debate of the 60's showed conclusively that a meaningful aggregation of capital, so as to obtain a surrogate production function à la Samuelson, is not possible in general, with critical implications also for other variants of neoclassical theory. The framework for the demonstration is that of linear activity analysis: a finite number of methods of production in each industry, constant returns to scale, perfect competition, homogeneous labour and heterogeneous capital goods so that relative prices are determined, given distribution.*

*There is an individual wage curve in function of the rate of profit for each technique (one method employed in each industry). If these individual wage curves were straight lines, their envelope would define a wage curve resulting from all techniques, from which a surrogate production function could be derived, but all wage curves are straight only, if there is only one industry. And if wage curves are not straight, phenomena such as reswitching show that essential neoclassical hypotheses such as the inverse relationship between the level of the rate of profit and the intensity of capital of the technique chosen need not hold. A recent empirical investigation by Han and Schefold [Cambridge Journal of Economics 2006, 30.5, 737-765] has found one empirical example for reswitching and several for reverse capital deepening.*

*A rigorous derivation of surrogate production functions thus is ruled out also on empirical grounds, but the paradoxes seem not to be as frequent as the critics once thought, so that the question arises whether approximate surrogate production functions could be derived, with individual wage curves which would be sufficiently linear to construct approximate surrogate production functions, indicating a relationship between the intensity of capital and output per head which would be sufficiently precise to work with. The answer to this question seems to be mathematically surprisingly difficult. The question is open and, in the paper, arguments will be discussed pro and contra, in relation especially to "large" input-output systems.*

1 Another version of this paper has been offered for publication in the Festschrift for Ian Steedman.

## 1. *The surrogate production function*

Production functions came back in advanced economic research with the advent of endogenous growth theory. The new start was made without any significant attempt to contradict the older debate about capital theory which started with Robinson (1953-54) and culminated in a series of papers rejecting and criticising Samuelson's surrogate production function (Samuelson 1962). The debate had shown that a theoretically rigorous aggregation of capital and hence a logically stringent construction of the production function were impossible (Garegnani 1970, Harcourt 1972, Pasinetti 1966), with critical implications for marginal productivity theory and even for intertemporal general equilibrium theory (Garegnani 2003, Schefold 1997, 2005).

The Cambridge critique had been extended to empirical methods of estimating production functions by Anwar Shaikh (see Shaikh 1987). But these critiques did not prevent the extensive use of production functions both in the theory and in empirical work.

The gap between the theoretical and empirical applications of the production functions on the one hand and the theoretical and empirical critiques on the other has never been bridged. With a few exceptions, marginal productivity theorists reject the critique without seriously trying to demonstrate its shortcomings, while the community of their opponents cannot explain how it is possible to erect a theoretical edifice as vast as the new growth theory on illogical foundations.

One side regards the critique as irrelevant, the other cannot explain the apparent success of the prevailing theory. To confront the positions, a middle ground must be found for a better comparison of the relative merits of both. A mere empirical test could hardly be regarded as satisfactory. For we cannot verify; we can only fail to falsify a theoretical proposition, if we follow Popper's methodology in this context. We first need a theory of a less-than-fully-rigorous construction of the surrogate production function for the confrontation, since the theoretically perfect justification of the aggregation underlying the production function cannot exist (to this extent the critique is irrefutable). Appropriate criteria to judge the validity of such an approach have to be developed. It might turn out that the construction would not be absurd, but not sufficiently correct to serve its purpose. Or it might turn out to be hopeless. Or it might be adequate. The question is open.

The name of the surrogate production function already suggested that its originator Samuelson (1962) had something less than perfect in mind. We return to the old debate in order to find out to what extent the criteria for a rigorous construction may be relaxed without falling into arbitrariness and in such a way that aggregation might be justified (wider issues of the critique for general equilibrium theory shall here be ignored). The usual assumptions made for the construction of the surrogate production function are straightforward and shall not be questioned: one deals with a closed economy, with a linear technology and constant returns to scale and single product industries in which one commodity is produced by means of other commodities, used as circulating capital, and by means of labour of uniform quality. There is no reason to generalise at this stage, since the introduction of heterogeneous labour, of fixed capital and joint production and of variable returns to scale do not render the existence of the surrogate production function more likely. The assumption of perfect competition should be retained, since monopoly control or other forms of imperfect competition would render the task of demonstrating the working of the principle of marginal productivity more difficult. Even a set-theoretical description of technological alternatives does not eliminate the possibility of paradoxes of capital theory, as long as strict convexity is not postulated, and strict convexity is an extremely problematic assumption (see Schefold 1976).

Hence we assume a finite number of methods of production, available for the production in each industry in the form of a book of blueprints. Competition will then ensure that, at any given rate of profit, a certain combination of methods will be chosen, one in each industry, such that positive normal prices and a positive wage rate result, expressed in terms of a numéraire. The wage rate can then be drawn in function of the rate of profit for this combination of methods between a rate of profit equal to zero and a maximum rate of profit, and the 'individual' wage curve for this technique will be monotonically falling (see Han and Schefold 2006 for a more detailed description). If the choice of technique is repeated at each rate of profit, starting from zero, different individual wage curves will appear on the envelope of all possible wage curves, and the envelope will also be monotonically falling. Technical change is 'piecemeal' in that only one individual wage curve will be optimal in entire intervals, except at a finite number of switch points where generically only two wage curves intersect and where a change of technique generically takes place only in one industry, so that the two wage curves to the left and to the right of the switch point will have all other methods in all other industries in common. The intensity of capital and output per head change discontinuously at the switch points (they can be represented geometrically for a given individual wage curve $w(r)$, if the numéraire consists of the vector of output per head in the stationary state): output per head equals $w(0)$ and capital per head $k = (w(0) - w(r))/r$.

If many individual wage curves appear successively on the envelope, this envelope may be replaced by a smooth approximation, and each point on this modified envelope can be thought to represent one individual technique, represented by an individual wage curve. The surrogate production function then is defined by taking the tangent to this modified envelope (supposed to be convex to the origin): the slope of the tangent is equal to capital per head and the intersection of the tangent with the abscissa is equal to output per head, as in Diagram 1. If and only if the individual wage curves are linear, the construction is rigorous in that output per head and capital per head of techniques individually employed will be equal to those which we have just defined, and the paradoxes of capital theory (to be discussed presently) will then be absent.

However, the critique of the surrogate production functions starts from the observation that individual wage curves will in general not be linear and the envelope will not be necessarily convex to the origin; envelope $\hat{w}(r)$ in diagram 1 provides an example. Output per head at $\tilde{r}$ is given by $\tilde{w}(0)$, where $\tilde{w}(r)$ is the individual wage curve tangent to $\hat{w}(r)$ at $\tilde{r}$.

The phenomenon which has attracted most attention is that of reswitching and reverse capital deepening: there may be switch points on the original envelope such that the intensity of capital does not fall with the rate of profit (reverse capital deepening), and the individual wage curve may have appeared on the envelope already at a lower rate of profit (reswitching). It is also possible that capital per head rises with the rate of profit in the industry where the switch of methods of production takes place (reversed substitution of labour) and, surprisingly, reverse capital deepening (the perverse change of aggregate capital per head) and reverse substitution of labour (a perverse change of capital per head at the industry level) need not go together in systems with more than two industries. Returns of processes seem to be frequent: a process which is used in one industry in one interval of the rate of profit is used again in another interval, but not in between. This is a generalisation of reverse capital deepening. It can be shown to imply large changes of relative prices and capital values and it demonstrates that processes cannot be classed as being inherently more or less capital-intensive, prior to their use in specific systems and at specific levels of distribution. Finally, there is likely to be a divergence between output per head and capital per head in the individual industry and the corresponding values which follow from the definition of the surrogate production function; this divergence is called declination and it is illustrated in diagram 1: output per head would be $\hat{y}$ and $k = tg\alpha$, if the individual wage curve $\hat{w}(r)$ was linear, but since this is not the case, there is the declination $\tilde{w}(0) - \hat{y}$. Output per head equals $\hat{y}$ according to the definition of the surrogate production function, but real output per head is $\tilde{w}(0)$.

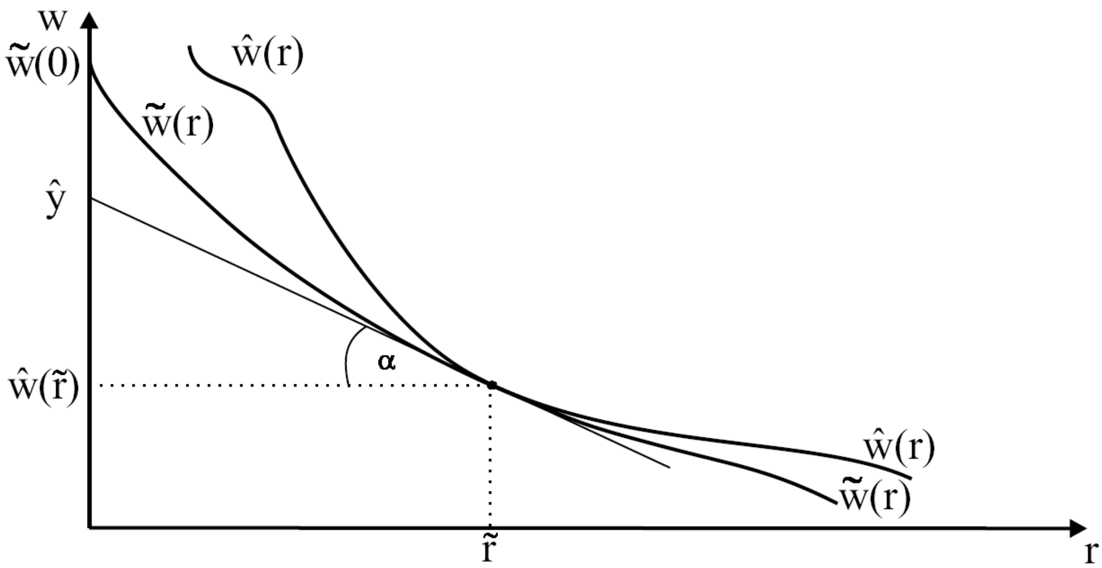


Diagram 1: Declination $\tilde{w}(0) - \hat{y}$. The surrogate production function yields output per head $\hat{y} = \hat{w}(\tilde{r}) + rtg\alpha,\ tg\alpha = -\hat{w}'(\tilde{r})$. Actual output per head equals $\tilde{w}(0)$ in the stationary state.

I confess that I once used to think (Schefold 1989 [1971], p. 298) that reverse capital deepening might be about just as likely (frequent) as 'normal' switches and that one would encounter 'many' individual wage curves succeeding each other on the envelope in a piecemeal fashion (it was conceded that reswitching might be unlikely in Schefold 1997, p. 480). I thought that envelopes would alternate in curvature, being partly convexed to the origin, partly not, and that the surrogate production function was not only theoretically not rigorous, but that the paradoxes would also have to appear in reality. Only, 'reality' was an illusive concept, for where does one find the book of blueprints for an economy? Only one technique, the one in actual use, seems to be measurable, and even this only at some level of aggregation in the form of an input-output table for a number of sectors which is small compared to the multitude of commodities.

A different picture emerges in Han and Schefold (2006), where it is assumed that techniques used in the past, as represented in corresponding input-output tables, could be used again, and that similarly the technique used in another country could be used at home. Comparing only two input-output tables in this manner results in a multitude of wage curves, since two methods (the foreign method or that of the past) are available as alternatives to the actual method employed in each industry so that $2^n$ alternative systems result, if both input-output tables are composed of $n$ sectors.

Han and Schefold (2006) analysed envelopes derived from nearly 500 pairs of input-output tables for economies (32 tables with 36 sectors). It was not possible to compute all the $2^{36}$=65536 wages curves for each of 496 pairs, but the envelopes were obtained by means of linear programming. Contrary to our expectations, reverse capital deepening and reverse substitution of labour are obtained only in a little less than 4 % of all switch points on the envelopes. Technical change is confirmed as piecemeal, but, also surprisingly, only about 10 wage curves appear on average on each envelope.

Similar empirical investigations would be welcome to confirm or question these results. Meanwhile, theoretical reflections on this peculiar outcome may be useful. The critics of neoclassical theory can point out that, for the first time, an empirical case of reswitching and many of reverse capital deepening have been found. But the frequency is not sufficient to destroy neoclassical hopes that the production function might survive as an approximation, similar perhaps not to the more rigorous laws of physics but to the empirical generalisations, supported by some theoretical considerations, which one finds in biology. The discussion then moves on a plane lower than that of the critique of pure theory for which Ian Steedman has given so many insightful examples.

We know the characteristics individual wage curves would have to have for a rigorous construction: they would have to be linear. The envelope would then become convex to the origin, declination would vanish and the intensity of capital would fall with any increase of the rate of profit.

The open question thus is whether the surrogate production function can be defined under assumptions which are sufficiently general to take the relevant aspects of real modern economies into account and sufficiently specific to rule out the paradoxes of the capital theory in a form which would render meaningless the theoretical analysis or its application. This construct - if it exists - could be called an 'approximate surrogate production function'.

The original surrogate production function had linear wage curves, and strictly linear wage curves imply that prices are equal to labour values (unless the numéraire is very special). Prices and values can differ substantially, as Ian Steedman and Judith Tomkins (1998) have pointed out. It would not only be ironic to fall back on a primitive form of the labour theory of value (Marx had prices of production as transformed labour values), but there is also a specific inconsistency implied by the assumption of prices equal to values: it can be shown that two techniques with linear wage curves, due to uniform organic compositions of capital, can not coexist at a switch point. For if their linear wage curves cross, a combination of the methods of the techniques will exist, with a wage curve dominating this point of intersection (Salvadori and Steedman 1988). The reason is that technical change on the envelope must be piecemeal. If we have a wage curve of a technique with uniform composition of capital on the envelope, more than one method must change in order to get to another technique which is also characterised by a uniform composition of capital.

A linear wage curve also results if the basket of goods defining the numéraire happens to be equal to Sraffa's standard commodity. As far as I can see, it has always been thought that this property could not be used to construct a surrogate production function, since the numéraire has to be the same for all techniques, hence at most one technique can be linearised in this manner. As an illustration, one can have a surrogate production function composed of at most two strictly linear wage curves. One starts with any technique with a uniform composition of capital - this yields one linear wage curve -, and one changes one method of production in one industry. This will create a second technique in which the labour theory of value does not hold. The corresponding wage curve can be linearised, however, by taking the standard commodity of the second technique as the numéraire in common for both; the wage curve of the first technique will then still be linear, if expressed in this standard. But a third strictly linear wage curve could not be added to the construction without generating yet other, non-linear wage curves which would in part be on the envelope.

These two constellations, which lead to linear wage curves, both concern the eigenvectors of the input matrix. If the labour theory of value holds and relative prises are constant, they must be equal to the relative prices formally obtained at a rate of profit equal to $-1$. They will then be equal to relative direct labour inputs. Hence, the labour vector must be the Frobenius eigenvector of the input matrix, if the labour theory of value holds. The standard commodity, on the other hand, is known to be the dual positive eigenvector. Schefold (1989 [1971]) also considered the other eigenvalues of the input matrix. A transformation, which will be used again here, showed that relative prices as functions of the rate of profit took a very simple form, related to the properties of Sraffa's standard system, if the eigenvalues other than the Frobenius eigenvalue are zero. Thirty years later, Christian Bidard showed in a seminal paper together with Tom Schatteman (Bidard and Schatteman 2001) that the eigenvalues other than the dominant eigenvalue will tend to zero for larger and larger random matrices. Both observations taken together suggest that 'random' large systems will exhibit wage curves of comparatively small and even curvature.

We thus have three properties on which the construction of approximate surrogate production functions might perhaps be based, because they lead to more linear wage curves and they thus reduce both the risk of the paradoxes and declination: they would be based on systems with prices not differing much from labour values, with numéraire vectors not differing much from the standard commodity and with matrices having small eigenvalues (except for the dominant one). However, there are two additional properties. One can observe that the magnitudes on which the paradoxes of capital depend are continuous functions of elements of the input matrix, of the labour vector and of the numéraire (though not of the rate of profit), so that each single small change of methods of production in different industries can only exert a small effect on the aggregates, and if the system is large and the changes are many, rare paradoxical changes will, as it were, disappear in the noise of frequent transitions (the numerical results in Han and Schefold 2006 had this character[2]). The fifth argument concerns declination only and is discussed in Schefold 2006): One can show that declination will diminish, if a positive rate of growth, $g$, is introduced, and declination disappears in the golden rule case $r = g$.

We concentrate on the first three arguments in this paper which concern the forms of the individual wage curves, hence they concern both the paradoxes and declination. Preliminary investigations have led me to the conviction that no single of these three properties can serve to justify the construction of an approximate surrogate production function. Whether combinations of them (or of all five effects) can do that is again our open question in a more developed form.

In a preliminary attempt to solve it, I propose to discuss 'families' of wage curves defined by some common properties of the techniques involved. The families will be called 'closed', if combinations of two techniques and their wage curves lead to a combined optimal technique and wage curve which still belongs to the same family.

Two such 'families' will be discussed in the remainder of this paper. One will be used to show that wage curves with extreme curvature are possible. The other, on the contrary, shall demonstrate (with less rigour, however) how near-linearity may be obtained.

## 2. *Circular production. A family of techniques.*

The techniques can be represented by Sraffa systems (Sraffa 1960) of the usual form:

$$(1 + r)\mathbf{Ap} + w\mathbf{l} = \mathbf{p},$$

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2 See table 2 in Han and Schefold (2006), where reverse capital deepening is of the order of magnitude of one percent.

where $\mathbf{A} = (a_{ij})$ is the input matrix, $\mathbf{l} = (l_i)$ is the (positive) labour vector (column), $\mathbf{p} = (p_i)$ is the vector of prices; $i, j = 1,...,n$; $w$ is the wage rate, $r$ is the rate of profit and $\mathbf{d} = (d_1,...,d_n)$ is the numéraire vector. The systems are assumed to be semi-positive, basic (indecomposable) and productive. Productivity can be ensured by assuming that there is a surplus with $\mathbf{eA} \leq \mathbf{e}$ ($\mathbf{e}$ is the summation vector). The prices expressed in this numéraire and the wage rate will then be positive for $0 \leq r \leq R$.

Circular systems are defined by the property that there is only one commodity input in each industry. The first industry thus produces the input for the second industry, the second industry the input for the third industry, and so on; the last industry produces the input to the first industry. The family is closed for a given number of sectors. Formally:

$$\mathbf{A} = \begin{pmatrix} 0, & 0, & \ldots, & a_1 \\ a_2, & 0, & \ldots, & 0 \\ & & \ldots, & \\ 0, & 0, & ..a_n, & 0 \end{pmatrix}.$$

We now put $\rho = 1 + r$. To calculate prices, we need the inverse of the following matrix

$$(\mathbf{I} - \rho\mathbf{A}) = \begin{pmatrix} 1, & 0, & \ldots, & -\rho a_1 \\ -\rho a_2, & 1, & \ldots, & 0 \\ & & \ldots & \\ 0, & \ldots, & -\rho a_n, & 1 \end{pmatrix}.$$

The corresponding determinant is $\det(\mathbf{I} - \rho\mathbf{A}) = 1 - \rho^n a_1 a_2 \cdot ... \cdot a_n$, where the sign follows from two considerations: we have $(-1)^n$ by multiplying the $-\rho a_i$. On the other hand, we have the factor $(-1)^{n-1}$, because the column indices in the product of the non-zero off-diagonal elements of the matrix represent a permutation of the row indices, obtained after $n-1$ steps: $a_{1n} = a_1,\ a_{21} = a_2,\ ...,\ a_{n,n-1} = a_{n-1}$, so that the combined factor equals $(-1)^n(-1)^{n-1} = -1$.

Next we calculate the adjoint of $\mathbf{I} - \rho\mathbf{A}$. We obtain

$$(\mathbf{I} - \rho\mathbf{A})_{Ad} = \begin{pmatrix} 1, & \rho^{n-1} a_1 a_3 \cdot \ldots \cdot a_n, & \ldots, & \rho a_1 \\ \rho a_2, & 1, & \ldots, & \rho^2 a_1 a_2 \\ \rho^2 a_2 a_3, & \rho a_3, & \ldots, & \rho^3 a_1 a_2 a_3 \\ & & \ldots, & \\ \rho^{n-1} a_2 \cdot \ldots \cdot a_n, & \rho^{n-2} a_3 \cdot \ldots \cdot a_n, & \ldots, & 1 \end{pmatrix}.$$

This may be verified by calculating backwards and confirming that

$$(\mathbf{I}-\rho\mathbf{A})(\mathbf{I}-\rho\mathbf{A})_{Ad} = \det(\mathbf{I}-\rho\mathbf{A})\mathbf{I}.$$

The inverse of the wage follows from $1/w = \mathbf{d}(\mathbf{I}-\rho\mathbf{A})^{-1}\mathbf{l}$.

Hence we have the following explicit formula for the wage rate in function of the rate of profit

$$w = \frac{1-\rho^n a_1 \cdot \ldots \cdot a_n}{b_0 + b_1\rho + \ldots + b_{n-1}\rho^{n-1}},$$

where

$$\begin{aligned}
b_0 &= d_1 l_1 + \ldots + d_n l_n \\
b_1 &= d_1 a_1 l_n + \ldots + d_n a_n l_{n-1} \\
&\ldots \\
b_{n-1} &= d_1 a l_2 / a_2 + \ldots + d_n a l_1 / a_1
\end{aligned}$$

and where $a = a_1 \cdot \ldots \cdot a_n$.

We proceed to construct wage curves of extreme properties by giving special values to the parameters of this circular system. It is convenient to restrict our attention to the case $a = 1$. Instead of observing prices and wage rates for $0 \le r \le R$, we then have $R = 0$ and observe prices for $-1 \le r \le 0$, hence with $0 \le \rho \le 1$; it is obvious how this formally simplified analysis can be extended to take account of $R > 0$.

1. It is instructive to see how a linear wage curve can be engendered in the simplest (but of course not the only) case. With $a_1 = \ldots = a_n = 1$, $d_1 = \ldots = d_n = 1$, $l_1 = \ldots = l_n = 1/n$ we have $b_0 = \ldots = b_{n-1} = 1$. We thus get a geometric series in the denominator of the wage curve which adds up to $(1-\rho^n)/(1-\rho)$, so that

$$w = 1 - \rho.$$

2. We then construct a wage curve which is nearly horizontal, by putting $a = 1$, $d_1 = l_1 = 1$ and $d_i = l_i = \varepsilon$ otherwise. This means that $b_0 \to 1$, and $b_i \to 0$ otherwise. We thus approximate the wage curve

$$w = 1 - \rho^n,$$

and we suppose that $\varepsilon \to 0$ sufficiently fast as $n \to \infty$. This means formally that we approximate $w(0) = 1$, $w(1) = 0$ and $w(\rho) \to 1$ for all $\rho < 1$. Since wage curves must be monotonically falling, this one is extreme in being nearly horizontal up to the maximum rate of profit. The wage curve is concave to the origin and exhibits an extreme Wicksell effect.

3. A seemingly small variation of the assumptions made in the first case leads on the contrary to a wage curve which tends to zero even for very small rates of profit. We assume $d_1 = ... = d_n = 1$, $l_1 = ... = l_n = 1/n$, and we let $a_1 \to \infty$, and $a_2 = 1/a_1 \to 0$ so that $a = 1$ as above. We then again have $b_0 = 1$, but $b_1$ (and possibly other coefficients in the denominator) tend to infinity, so that $w(0) = 1$, $w(1) = 1$, and $w(\rho) \to 0$ for all $\rho > 0$.

What we have demonstrated may also be expressed by saying that, for each point in the interior of $[0,1] \times [0,1]$, there is a wage curve which begins in $w(0) = 1$, which ends in $w(1) = 0$ and which passes through that point. Since wage curves of single product systems must be monotonically falling, this result provides the most extreme conceivable evidence of how wages curves can deviate from linearity. It is clear that these wage curves can give rise to very large declinations. Actual output per head equals one, but apparent output per head can get arbitrarily close to zero (case 3) or to infinity (case 2), while the linear wage curve simply is the diagonal of $[0,1] \times [0,1]$.

However, even here caution is necessary. One might think that one could combine such wage curves to construct extreme cases of reswitching, e.g. as follows: One takes a curve of type (2) above, $w_a = 1 - \rho^n$ (where $n = 20$, $d_1 = l_1 = 1$, $d_i = l_i = 0$; $i = 2, \ldots, n$, $a = 1$), and another, a variant of type (3), with

$$w_b = \frac{1 - \left((2/3)\rho\right)^{20}}{2/3 + \rho},$$

where $n = 20$, $d_1 = 1$, $l_1 = 2/3$, $l_n = 1$, $d_i = 0$, $l_i = 0$ otherwise, $a_1 = \ldots = a_{n-1} = 1$, $a_n = (2/3)^{20}$. These wage curves $w_a$ and $w_b$ are represented in Diagram 2 and look like a case of reswitching, but the impression is misleading: in the transition from $w_a$ to $w_b$, methods are changed in sectors $1, 2$ and $n$ so that $8$ wage curves are involved. The intersections of the wage curves $w_a$ ad $w_b$ cannot both be on the envelope. For, as is pointed out in Schefold (1997, p. 486), reswitching, as the result of the change of method in one industry, can take place only, if the use of at least one circulating capital good input increases and that of at least one other falls.

Since this proposition is important for the understanding of the theory, we here give an explicit proof which was omitted in the earlier presentation.

*Proposition:* If method $(\mathbf{a}_1, l_1)$ in the Sraffa system $(\mathbf{A}, \mathbf{l})$ is replaced by method $(\mathbf{a}_0, l_0)$, the two corresponding wage curves can intersect on the envelope at least twice, only if neither $\mathbf{a}_0 > \mathbf{a}_1$ nor $\mathbf{a}_0 < \mathbf{a}_1$.

*Proof:* Let $\hat{\mathbf{p}} = \mathbf{p}/w$ be the price of the original system in terms of the wage rate. Switch points are rates of profit for which

$$(\mathbf{a}_0 - \mathbf{a}_1)(1 + r)\hat{\mathbf{p}}(r) = l_1 - l_0.$$

If $\mathbf{a}_0 > \mathbf{a}_1$ or $\mathbf{a}_0 < \mathbf{a}_1$, the left hand side is a strictly monotonic function in $[0, R)$, hence there can be at most one switch point.

Processes in circular production systems have only one circulating capital good. Reswitching thus is excluded. This simple observation may surprise readers who remember Sraffa's (1960) example of 'wine' and 'oak chest', where reswitching is exemplified by means of dated labour inputs only, hence by means of a structure of production which seems even simpler than that of circular systems. In fact, reswitching was discovered by Irving Fisher prior to the First World War in the context of an Austrian model serving his critique of Böhm-Bawerk (Schefold 1999), and Sraffa's 'wine' and 'oak chest' example - which involves no basic commodity - also essentially is Austrian and must be interpreted as an implicit reference to that debate.

The comparison of wage curves derived from profiles of dated inputs of labour permits relatively easy constructions of cases of reswitching but they can be misleading for the same reason as apply in case of Diagram 2: to change an entire time-profile may mean to change several processes of production at once. If the processes can be changed independently of each other, further technical combinations arise, with wage curves which may dominate on the envelope, for technical change is piecemeal.

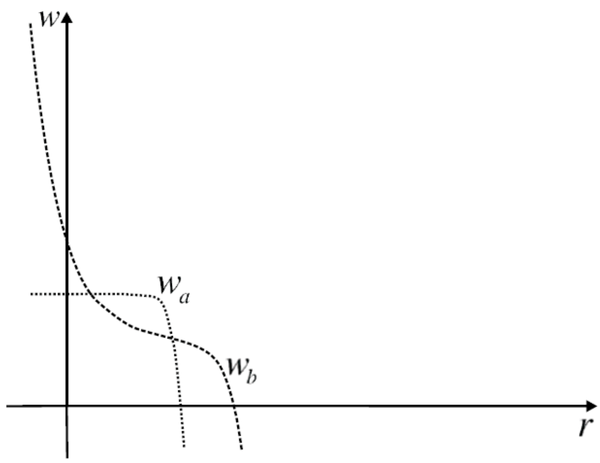


Diagram 2: Apparent case of reswitching with extreme wage curves explained in the text[3].

The example of Diagram 2 nevertheless is tempting, for if one tries to construct the usual examples of reswitching in two-sector models, one encounters wage curves which are close together, and if one derives wage curves from input-output tables, they usually turn out nearly linear (publications of wage curves derived from input-output systems are listed in Han and Schefold 2006). If we move from Austrian models to circular systems with similar, but more restricted time profiles, declination can be dramatic, but reswitching vanishes. The intriguing question is what happens as we move to more realistic systems.

3 The discontinuities of capital per head as a function of the rate of profit result from the fact that the technology is convex, but not strictly convex (Schefold 1976).

The family of techniques with circular production is of theoretical interest, but it is clearly not realistic. Circular production gets more and more hypothetical as $n$ is increased. In fact, as $n$ tends to infinity, the cyclical structure gets lost and the analysis would have to be conducted in a Hilbert space.

Nonetheless, the reader of Sraffa's book may get the impression that the results obtained from this example could easily be extended to systems with many inputs to each process, since the prices in Sraffa's example are calculated by means of dated inputs of labour, and a 'reduction to dated quantities of labour' can also be obtained for all basic systems, using Sraffa's formula for prices in terms of the standard commodity:

$$\mathbf{p} = \left(1 - \frac{r}{R}\right) \sum_{t=0}^{\infty} (1+r)^t \mathbf{A}^t \mathbf{l},$$

where the terms $\mathbf{A}^t\mathbf{l}$; $t = 1,2,...$; represent indirect labour, expended $t$ periods ago and embodied in the present product. The point of the exercise follows from Sraffa's analysis of the polynomial expressions

$$f_t(r) = \left(1 - \frac{r}{R}\right)(1+r)^t,$$

where $f_t(0) = 1$, and $f_t(R) = 0$. These polynomials measure the weight, due to interest, of indirect labour expended $t$ periods ago. Labour simply adds up, if the rate of interest is zero, but, because of the diminishing wage, the weight is lower at higher rates of profit, except in that a very sharp maximum arises at rates of interest close to the maximum because of the influence of $(1+r)^t$. The maximum increases dramatically and the curves become steeper and are pointed the more sharply, the closer one is to the maximum rate of profit and the higher $t$. One might thus think that labour inputs of a long time ago could exert a strong influence on the present at high rates of profit because of this effect of geometric growth of interest costs. One could thus be induced to think that wage curves, dramatically different from linearity, could be constructed by choosing appropriate time profiles for past labour inputs, in an exercise similar to the one which we have just executed for circular systems. But the impression is misleading to the extent that the effect is compensated by a geometric decline of the labour inputs; in fact $\mathbf{A}^t$ tends to zero with certain regularities. They prevent a simple reproduction of the extreme wage curves easily obtained in the Austrian case, especially, as soon as one has to deal with basic systems of a structure which is more complicated than that of circular production.

### *3. Systems with small non-dominant eigenvalues*

We now turn to a family of techniques of less extreme curvature so that realistic additional conditions to ensure quasi linearity of the wage curves may *perhaps* be found.

Some formal conditions to ensure this property will be discussed below. We now assume that the non-dominant eigenvalues of the input-output systems are small. As Bidard and Schatteman (2001) show, in the article already quoted, the non-dominant eigenvalues of so-called random matrices (with a random distribution of positive coefficients) have the property that the non-dominant eigenvalues all tend to zero as the number of sectors increases. However, the speed of convergence to zero of these non-dominant eigenvalues is not large enough to justify the approximation which we shall use below. We need not only the assumption that the individual non-dominant eigenvalues tend to zero, but also that the sum of their absolute values converges to zero, and this is a much stronger requirement. We shall here have to be content with a provisional definition of the family by postulating that the non-dominant eigenvalues are sufficiently small to be neglected in the calculation which follows, and we shall not analyse conditions under which this family might be closed.

For preparation, we introduce an example for which all eigenvalues except the dominant root are zero. Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \end{pmatrix}.$$

The Frobenius eigenvector here is $\mathbf{e} = (1,1,1)$; the Frobenius eigenvalue, $\mu_1$, obtained from $(\mu\mathbf{I} - \mathbf{A})\mathbf{e}^{\mathrm{T}} = \mathbf{0}$, equals $3/4$, and the corresponding maximum rate of profit is $R_1 = 1/3$, where $\mu_i = 1/(1+R_i)$. The two other eigenvalues, $\mu_2$ and $\mu_3$, are equal to zero, hence $R_2$ and $R_3$ are infinite.

We use the normalised eigenvectors for a transformation of the prices of this system with labour vector $\mathbf{l}$ (normalised so that $\mathbf{el} = 1$) and the vector of numéraire goods $\mathbf{d}$. We assume $\mathbf{q}_i\mathbf{l} \neq 0$ and postulate $\mathbf{q}_i\mathbf{l} = R_i/(1+R_i) = 1 - \mu_i$, $i = 1,2,3$. This yields

$$\mathbf{q}_1 = \mathbf{e}/4$$
$$\mathbf{q}_2 = (1,-1,0)/(l_1 - l_2)$$
$$\mathbf{q}_3 = (1,0,-1)/(l_1 - l_3).$$

The normalisations introduced here can be generalised and used to provide a simplified expression for all wage curves of basic simple product systems; a specific simplification results if the non-dominant eigenvalues are zero, as in this example. The point is that, even if the non-dominant eigenvalues are not zero, they may be sufficiently close to zero to be ignored. The economic interpretation is clear: if the non-dominant eigenvalues are close to zero, all processes in the economy are in essence nearly proportional to a single process. This single-process economy must then have properties similar to those of a one-good economy. Hence the property helps, if one is seeking economies with quasi-linear wage curves, and it appears, following Bidard and Schatteman, that large systems tend to have this property, if the coefficients are random.

Hence we start afresh, with $\mathbf{A} \geq \mathbf{0}$ basic, where $R_1,...,R_n$ are different 'large' maximum rates of profit (except for the 'true' maximum rate of profit $R_1$ which corresponds to the Frobenius eigenvalue). We have $(1+R_i)\mathbf{q}_i\mathbf{A} = \mathbf{q}_i$, $\mathbf{l} \geq \mathbf{0}$, $\mathbf{d} \geq \mathbf{0}$. With any of the associated eigenvectors we get (proof by inversion of the matrix)

$$\mathbf{q}_i\left(\mathbf{I} - (1+r)\mathbf{A}\right)^{-1} = \frac{1+R_1}{R_i - r}\mathbf{q}_i.$$

This is a generalisation of Sraffa's standard system where $\mathbf{q}_1 = \mathbf{q}(\mathbf{I} - \mathbf{A})$, $R_1 = R$ is the maximum rate of profit, with normalisation $\mathbf{ql} = 1$, $\mathbf{el} = 1$; this, taken as the numéraire, yields Sraffa's familiar linear wage curve:

$$1 = \mathbf{q}(\mathbf{I} - \mathbf{A})\mathbf{p} = r\mathbf{qAp} + \overline{w}\mathbf{ql} = (r/R)\mathbf{q}(\mathbf{I} - \mathbf{A})\mathbf{p} + \overline{w}\mathbf{ql} = (r/R) + \overline{w}\,.$$

One thus has the wage curve in terms of the standard commodity

$$\overline{w} = 1 - \frac{r}{R}.$$

We generalise Sraffa's normalisation by putting $\mathbf{q}_i\mathbf{l} = \dfrac{R_i}{1+R_i}$ (assuming $\mathbf{q}_i\mathbf{l} \neq 0$, which means that $\mathbf{l}$ is not an eigenvector of $\mathbf{A}$ and the labour theory of value does not hold). We choose a numéraire $\mathbf{d} > \mathbf{0}$, with $\mathbf{d} = \lambda_1\mathbf{q}_1 + ... + \lambda_n\mathbf{q}_n$.

We thus obtain a simplified formula for the inverse of the wage rate

$$\frac{1}{w} = \mathbf{d}\left(\mathbf{I} - (1+r)\mathbf{A}\right)^{-1}\mathbf{l} = \sum \lambda_i\mathbf{q}_i\left(\mathbf{I} - (1+r)\mathbf{A}\right)^{-1}\mathbf{l} = \sum \lambda_i \frac{1+R_i}{R_i - r}\mathbf{q}_i\mathbf{l} = \sum \lambda_i \frac{R_i}{R_i - r} = \sum \frac{\lambda_i}{1 - \dfrac{r}{R_i}}.$$

The numéraire $\mathbf{d}$ here can be chosen so that $w(0) = 1$ which is equivalent to $\sum \lambda_i = 1$. We shall show below that $\lambda_1 > 0$ and that the vector

$$\hat{\mathbf{q}} = \sum_{i=2}^{n} \lambda_i\mathbf{q}_i$$

is real. Obviously, the standard commodity represents the special case where $\lambda_1 = 1$, $\lambda_2 = ... = \lambda_n = 0$ so that $\mathbf{d} = \mathbf{q}_1 = \mathbf{q}(\mathbf{I} - \mathbf{A})$; then we have again

$$\overline{w} = 1 - \frac{r}{R}.$$

But the general formula for the wage is

$$w = \frac{1}{\dfrac{\lambda_1}{1-\dfrac{r}{R}} + \sum_{i=2}^{n} \dfrac{\lambda_i}{1-\dfrac{r}{R_i}}}.$$

Since $\lambda_1 > 0$, and since $w$ is real if $r$ is real (so that $\lambda_1/(1-r/R)$ is real), the second term in the denominator must also be real, as a sum of possibly complex terms. The wage curves $w$ and $\overline{w}$ intersect at the maximum wage rate and at the maximum rate of profit, for $w(0) = \overline{w}(0) = 1$ and $w(R) = \overline{w}(R) = 0$; both curves fall monotonically. However, we have $w(r) \equiv \overline{w}(r)$ only for $\lambda_2 = ... = \lambda_n = 0$.

We are now interested in a family of wage curves for which the absolute values of $R_2,...,R_n$ are large enough so that $r/R_i$ can be ignored for $0 \le r \le R$. Such a family of matrices exists as a family of approximations to the matrix we had as an example (where all eigenvalues except the Frobenius eigenvalue vanish). One might think that it could suffice to invoke a random property of the matrices and to postulate that they are large enough in order to ignore the influence of the non-dominant eigenvalues, but is not enough that $r/R_i$ diminish individually ($i = 2,...,n$) since $n$ and hence the number of the terms $r/R_i$ increases.

For the family of matrices for which this approximation is permissible, one obtains an approximate wage curve $\tilde{w}(r)$, putting $z = \lambda_2 + ... + \lambda_n$:

$$\tilde{w}(r) = \frac{1}{\dfrac{\lambda_1}{1-\dfrac{r}{R}} + \lambda_2 + ... + \lambda_n} = \frac{R-r}{R\lambda_1 + (R-r)z} = \frac{R-r}{R-zr},$$

where $\lambda_1 + z = 1$. $z$ must be real in the limit. It can be positive; we then must have $z < 1$, since $\lambda_1 + z = 1$. Or $z$ can be negative, with $\lambda_1 > 1$. Two cases result, represented by two hyperbolas which are drawn in Diagrams 3a and 3b.

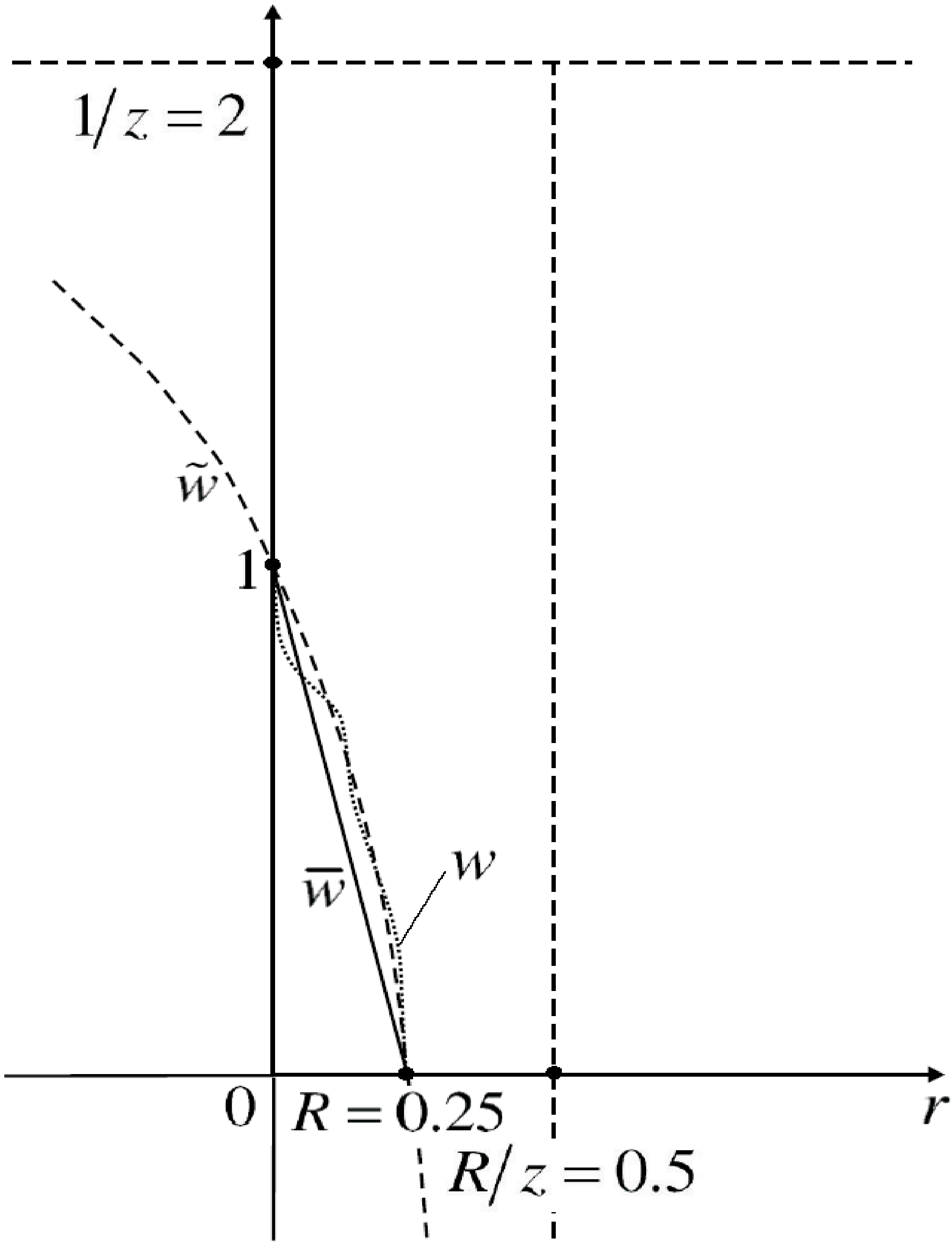


Diagram 3a: Wage curve $\tilde{w}$ with $z=0.5>0$, as a simplification of $w$ and possibly approximating $\overline{w}$. $R=0.25$ ($R$ is an expression of the output-capital ratio in a steady state with maximum rate of growth) and $z=0.5$[4].

4 If the capital output ratio is 4 and accordingly $R=1/4$, the wage curve becomes as steep as it is drawn here. An increase of the rate of profit by one percentage point requires a diminution of the real wage by 4 % of the maximum wage. It may come about through a rise of money prices, given a constant money wage.

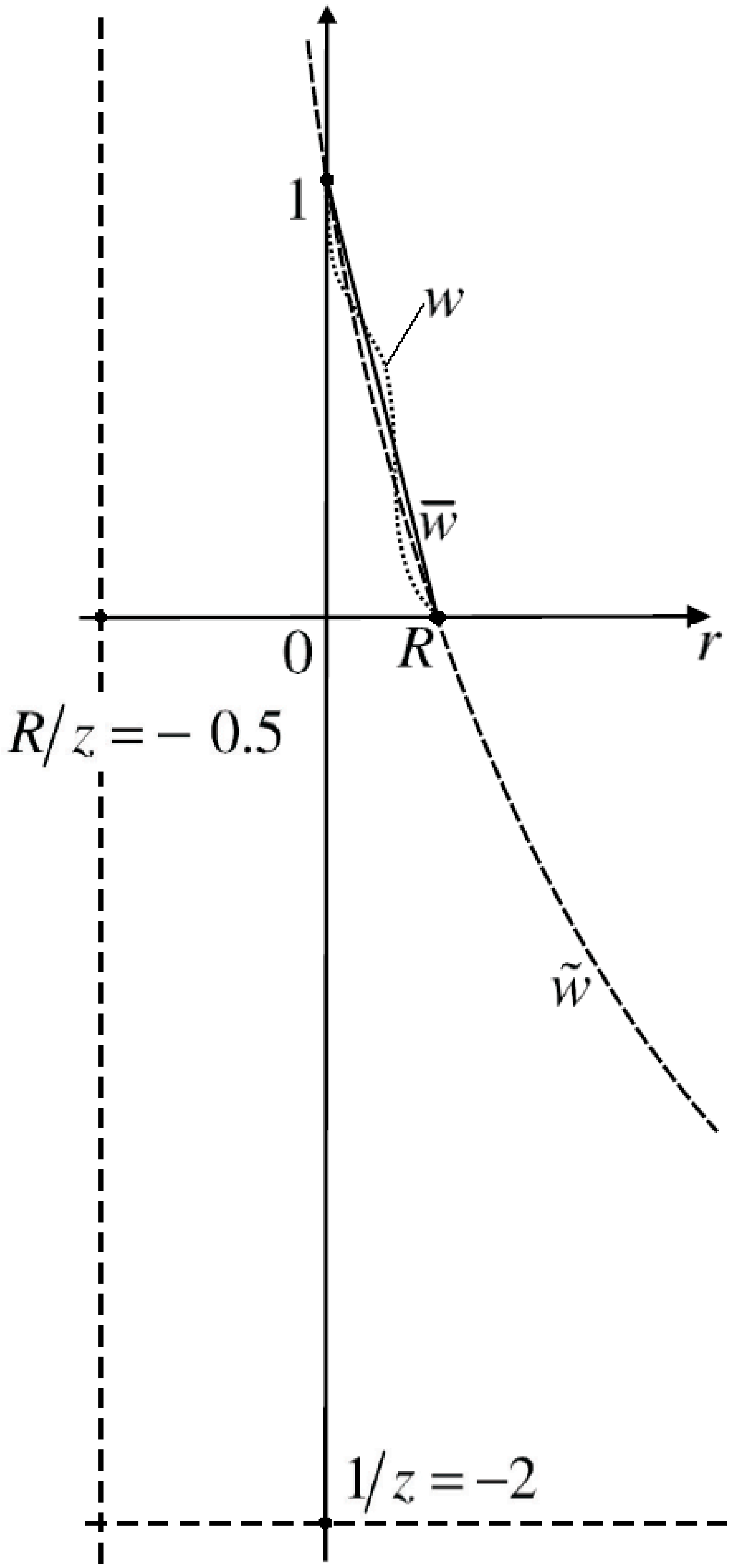


Diagram 3b: Wage curve $\tilde{w}$ with $z < 0$, as a simplification of $w$ and possibly approximating $\overline{w}$. Same $R$ as in Diagram 3a, but $z = -0.5$.

It is easily seen that $\tilde{w}$ will approximate $\overline{w}$ the better, the closer $z$ is to zero, for the asymptotes of the two hyperbolas (dotted lines) will then move to infinity and the wage curve $\tilde{w}$ will become linear. The case favourable for the construction of the surrogate production function and for neoclassical theory is obtained with $z < 0$, for the hyperbola will then be convex to the origin, and it will be relatively straight, if $|z|$ is small. It is clear that a positive $z$ implies $0 < z < 1$, since the wage curve cannot diverge to infinity for $0 \le r \le R$.

We thus identify two properties of the systems which lead together to almost linear wage curves: If the non-dominant eigenvalues of the matrix are small enough, a simple hyperbolic form of the wage curve results; it is, as it were, very smooth. Then it is important that $z$ be close to zero so that the hyperbola is 'stretched'. This happens, if $\lambda_1$ is close to 1, which means that the numéraire is close to the Frobenius eigenvector of the system. We turn to these relationships, but we first prove assertions made about $\lambda_1$ and $\hat{\mathbf{q}}$ above.

We recall a well-known property of the theory of non-negative matrices. If all $R_i$ are different and if $(1+R_1)\mathbf{A}\overline{\mathbf{p}} = \overline{\mathbf{p}} > \mathbf{0}$, we have $\mathbf{q}_i\overline{\mathbf{p}} = 0$, $i = 2,...,n$. For (we repeat the argument) we should otherwise have

$$\mathbf{q}_i\overline{\mathbf{p}}/(1+R_i) = \mathbf{q}_i\mathbf{A}\overline{\mathbf{p}} = \mathbf{q}_i\overline{\mathbf{p}}/(1+R_1)$$

and that would imply $R_i = R_1$, contradicting the assumption.

Now consider the representation of the numéraire in terms of the eigenvectors

$$\mathbf{d} = \sum_{i=1}^{n} \lambda_i \mathbf{q}_i = \lambda_1 \mathbf{q}_1 + \hat{\mathbf{q}},\ \hat{\mathbf{q}} = \sum_{i=2}^{n} \lambda_i \mathbf{q}_i.$$

We know that $\hat{\mathbf{q}}\overline{\mathbf{p}} = 0$, since $\mathbf{q}_i\overline{\mathbf{p}} = 0$; $i = 2,...,n$. Hence

$$\mathbf{dp} = (\lambda_1\mathbf{q}_1 + \hat{\mathbf{q}})\overline{\mathbf{p}} = \lambda_1\mathbf{q}_1\overline{\mathbf{p}},$$

hence $\lambda_1$ must be positive, since $\mathbf{q}_1$, $\overline{\mathbf{p}}$, $\mathbf{d}$ are positive vectors.

We now can conclude that $\hat{\mathbf{q}}$ must be real, since $\mathbf{d} = \lambda_1\mathbf{q}_1 + \hat{\mathbf{q}}$. And since $\hat{\mathbf{q}}\overline{\mathbf{p}} = 0$, we must either have $\hat{\mathbf{q}} = \mathbf{0}$ and $\lambda_1 = 1$ or $\hat{\mathbf{q}}$ has both positive and negative components. If $R_2,...,R_n$ tend to infinity, $z$ must tend to a real number, as is clear from our formula for $\tilde{w}(r)$. And if $\lambda_1$ is close to 1, $z$ must be close to zero. This confirms that we approximate the linear relationship, if $\mathbf{d}$ is close to $\mathbf{q}_1$ and $\hat{\mathbf{q}}$ is small in $\mathbf{d} = \lambda_1\mathbf{q}_1 + \hat{\mathbf{q}}$.

The point can be re-enforced by reverting to the example where all non-dominant eigenvalues are zero. We here assume $a_{ij} = 1/n$ for al $i, j$, hence $R = 0$, $\mu_1 = 1$, $\mu_2 = \ldots = \mu_n = 0$. The interest in this example derives not so much from the fact that the results apply to all similar matrices, i.e. to all matrices with the same spectrum, but, more generally, from the observation that the eigenvalues of non-negative matrices are continuous functions of the coefficients of the matrices so that the results apply as approximations to a neighbourhood of $\mathbf{A}$. We also assume $l_1 + \ldots + l_n = 1$. With $\rho = 1 + r$, we have for $0 \le \rho < 1$ and normalised eigenvectors:

$$\mathbf{q}_1(\mathbf{I} - \rho\mathbf{A})^{-1}\mathbf{l} = 1/(1-\rho), \text{ with } \mathbf{q}_1\mathbf{l} = 1,$$
$$\mathbf{q}_i(\mathbf{I} - \rho\mathbf{A})^{-1}\mathbf{l} = \mathbf{q}_i\mathbf{l} = 1;\ i = 2, \ldots, n;$$

assuming $\mathbf{q}_1\mathbf{l} \neq 0$, so that $\mathbf{d} = \lambda_1\mathbf{q}_1 + \ldots + \lambda_n\mathbf{q}_n$ implies

$$1/w = \lambda_1/(1-\rho) + \lambda_2 + \ldots + \lambda_n.$$

Clearly, $\mathbf{q}_1$ is proportional to $\mathbf{e}$; the chosen $\mathbf{q}_i$ are proportional to the difference of the unit vectors $\mathbf{e}_1 - \mathbf{e}_i$; $i = 2, \ldots, n$. Because of the normalisation $\mathbf{q}_i\mathbf{l} = 1$, the components $q_{ij}$ of $\mathbf{q}_i$ fulfil $q_{i1} = 1/(l_1 - l_i)$, $q_{ii} = 1/(l_i - l_1)$, $q_{ij} = 0$ otherwise. Hence $\mathbf{d} = \lambda_1\mathbf{q}_1 + \ldots + \lambda_n\mathbf{q}_n$ implies $d_i - \lambda_1 = \lambda_i/(l_i - l_1)$, therefore

$$\lambda_i = (d_i - \lambda_1)(l_i - l_1);\ i = 2, \ldots, n.$$

The interpretation is clear: the $\lambda_i$ are the smaller in absolute value, the closer are the $l_i$ to $l_1$ (the nearer we are to the labour theory of value with $l_i = 1/n$; $i = 1, \ldots, n$) and the closer are the $d_i$ to $\lambda_1$ (the nearer we are to the normalisation by means of the standard commodity, with $\mathbf{d}$ thus being proportional to $\mathbf{e}$). And if both $|d_i - \lambda_1|$ and $|\lambda_i - \lambda_1|$ are small, $\lambda_i$ will be small of the second order; $i = 2, \ldots, n$. With $\lambda_i$ sufficiently close to zero, the wage curve becomes $w = 1 - \rho$. This provides the mathematical proof that the properties of prices being close to values and of the numéraire being close to the standard re-enforce each other in the generation of quasi-linear wage curves.

## 4. *A link between the two families and an open conclusion*

The neoclassical economists who still use the production function are ignorant about the problems of capital theory or agnostic as to how they might be overcome or they hope that the change of relative prices with distribution are sufficiently moderate to permit the use of the production function as an approximation; hence they rely on the old argument that prices are close to values and/or the propositions of the numéraire are close to balanced proportions. Perhaps they also invoke a continuity argument.

New, by contrast is the proposition that small non-dominant eigenvalues also help. Why should we expect non-dominant eigenvalues to be small in a large class of systems? A complete mathematical answer to this question would presuppose a satisfactory solution to the inverse eigenvalue problem, applied to the whole spectrum of eigenvalues of a semipositive matrix. This problem seems not to have been solved yet (Minc 1988, p. 183). I offer some heuristic considerations.

It is easy to see that it suffices to analyse stochastic matrices, i.e. to assume $\mathbf{eA} = \mathbf{e}$, as was occasionally done above (Gantmacher 1966, p. 74) so that $dom\mathbf{A} = 1$. The other eigenvalues must then be interior points of the unit circle or they are complex numbers $z$ on the unit circle with $z = e^{2\pi ip/q}$; $p$, $q$ natural numbers (the case of imprimitive matrices, Gantmacher 1966, p. 70). This suggests that the unit circle would gradually be filled by the eigenvalues of non-negative matrices picked out at random, but Bidard and Schattemann show that the subdominant eigenvalues of random matrices tend to concentrate at the centre of the circle.

An intuitive observation, pointing in the direction of their curious result, is the following: let $\mathbf{A}$ be an indecomposable and primitive stochastic semipositive matrix with eigenvalues $\alpha_1,...,\alpha_n$; $\alpha_1 = dom\mathbf{A} = 1$. $\mathbf{A}^{n-1}$ will the also be stochastic, primitive and strictly positive with eigenvalues $\lambda_k = (\alpha_k)^{n-1}$. We still have $\lambda_1 = 1$, but the other $\lambda_k$ will be close to zero for large $n$.

Now consider the stochastic matrix $\mathbf{S} = \mu\mathbf{C} + (1-\mu)\mathbf{U}$; $0 \le \mu \le 1$; where $\mathbf{C}$ is the circular matrix of the first family discussed in section 2, with $c_{1n} = 1$, $c_{i,i-1} = 1$ for $i = 2,...,n$ and $c_{ij} = 0$ otherwise, and where $\mathbf{U} = \frac{1}{n}\mathbf{E}$, i.e. $u_{ij} = \frac{1}{n}$ for all $i,j$. Let $\gamma_k$ be an eigenvalue of $\mathbf{C}$; they are all on the unit circle, as we shall see. Let $\lambda_k$ be the eigenvalues of $\mathbf{U}$ and $\sigma_k$ the eigenvalues of $\mathbf{S}$, with $dom\mathbf{C} = \gamma_1 = 1$, $dom\mathbf{U} = \lambda_1 = 1$, $dom\mathbf{S} = \sigma_1 = 1$. Clearly, the $\sigma_i$ can be regarded as functions of $\mu$, given $\mathbf{C}$ and $\mathbf{U}$.

$\mathbf{S}$ is a weighted average of the representative matrices of the two families which we have introduced so that we may ask how the properties of the two families shift with the weight $\mu$. If one believes that the non-dominant eigenvalues move rapidly to zero as the dimension of the matrices $n$ increases, one expects $\sigma_k(\mu)$ to fall rapidly, as $\mu$ falls from one $(\mathbf{S} = \mathbf{C})$ to zero $(\mathbf{S} = \mathbf{U})$. But we can prove:

1. $\sigma_1(\mu) \equiv 1$,
2. $\sigma_k(\mu) = \mu\gamma_k$; $k = 2,...,n$.

The non-dominant eigenvalues therefore only fall in proportion to $\mu$, not faster, and their fall is not influenced by $n$. The absolute values $|\sigma_k|$ thus fall linearly from $1$ to $0$; $k = 2,...,n$.

The proof is not based on a direct calculation of the characteristic polynomial of $\mathbf{S}$ (which is cumbersome), but on a theorem about matrix rings. The matrix ring over the field of real numbers of matrix $\mathbf{A}$ consists of all polynomials $f(\mathbf{A})$, where $f(x)$ is any polynomial with real coefficients of the real variable $x$ and where $f(\mathbf{A})$ is the polynomial matrix resulting from the substitution of $x$ by $\mathbf{A}$. The theorem then asserts that the characteristic roots $\varphi_k$ of $f(\mathbf{A})$ are given by $f(\alpha_k)$, with $\alpha_k$ being the characteristic roots of $\mathbf{A}$ (Gröbner 1966, p. 157-8). $\mathbf{C}$ is a permutation matrix; the product $\mathbf{CA} = \overline{\mathbf{A}}$ transforms any matrix $\mathbf{A}$ in such a way that row $\mathbf{a}_i$ becomes row $i+1$ and row $n$ becomes the first row of $\overline{\mathbf{A}}$. A power $\mathbf{C}^m$ may, with $\mathbf{C}^{m-1}\mathbf{C} = \mathbf{C}^m$, be interpreted as a cyclical repetition, $m-1$ times, of this permutation on $\mathbf{C}$ itself. Hence $\mathbf{C}^n = \mathbf{I}$ is the unit matrix, and it is easily seen that $\mathbf{I} + \mathbf{C} + \mathbf{C}^2 + ... + \mathbf{C}^{n-1} = \mathbf{E}$. $\mathbf{S}$ is therefore equal to the polynomial matrix

$$\mathbf{S} = \mu\mathbf{C} + (1-\mu)\mathbf{U} = \mu\mathbf{C} + (1-\mu)\frac{1}{n}(\mathbf{I} + \mathbf{C} + ... + \mathbf{C}^{n-1});$$

the theorem yields for $k = 1,...,n$

$$\vartheta_k = \mu\gamma_k + (1-\mu)\frac{1}{n}\left[1 + \gamma_k + ... + (\gamma_k)^{n-1}\right].$$

If $k = 1$, we know that $\gamma_1 = dom\mathbf{C} = 1$, so that

$$\vartheta_1 = \mu + (1-\mu)\frac{n}{n} = 1,$$

as was already clear from $\mathbf{eS} = \mathbf{e}$.

The characteristic polynomial of $\mathbf{C}$ is (see section 2) $|\gamma\mathbf{I} - \mathbf{C}| = \gamma^n - 1 = 0$, hence the eigenvalues are unit roots $\gamma_k = e^{2\pi i(k-1)/n}$ for $k = 2,...,n$. Then, by addition of the geometric series,

$$\vartheta_k = \mu\gamma_k + (1-\mu)\frac{1}{n}\left\{1 - \left[(\gamma_k)^n\right]\right\}/(1-\gamma_k) = \mu\gamma_k,$$

since $(\gamma_k)^n = 1$. This completes the proof.

We thus have found by means of a counterexample that the non-dominant eigenvalues do not generally move more quickly to zero for larger systems with primitive input matrices. The corresponding wage curves therefore do not necessarily approximate the simple hyperbolic form encountered in section 3. Hence we cannot conclude that wage curves are quasi-linear and that surrogate production functions exist; large real systems are not generally certain to have small non-dominant eigenvalues, and numéraires need not be close to the Frobenius eigenvectors of all the techniques in the books of blueprints. The logical critique of the surrogate production function still stands and the present investigation leaves open the possibility that the approximations which we have constructed are not sufficiently good to be comparable with the accuracy of other econometric work which is less controversial.

Sufficient conditions for the existence of an approximate surrogate production function would have to define a family of wage curves such that the combined effects of non-zero non-dominant eigenvalues, of deviations from the labour theory of value and of the distance of the Frobenius eigenvector of each system from the common numéraire would - in some sense to be made precise - be small enough to justify economic predictions and other empirical applications of the production function. The family would essentially have to be closed so that the conditions would still be fulfilled for combinations of systems. Our discussion, however, is not entirely negative either. It indicates that conditions which would be sufficient and realistic might be found.

Anwar Shaikh has shown that econometric techniques to estimate production functions which were employed in the 'sixties produce spurious expressions of production functions - a Cobb-Douglas production function could be implied, even if there was no choice of technique at all, provided the distribution of income was constant. Without going into the merits or demerits of more recent econometric techniques, it can be seen that theoretical considerations here lead to the discovery of new criteria to estimate the validity of empirical work based on approximate surrogate production functions. We do not know what the result will be, if we start to measure the declination of the surrogate production functions considered here. The problem whether approximate surrogate production functions exist is open.

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