**Calculating wage-profit frontiers and supporting prices**

**in Leontief-Sraffa models**

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**Abstract**

In this paper we show how the wage-profit frontier in Leontief-Sraffa type input-output models can be determined. The wage rate is shown to be a polynomial function of the rate of profit, linearity arising in the case of special price or output proportions.

**1. Introduction**

Little is known about the distribution of income between wages and profits in input-output models of the Leontief-Sraffa type. The reason is that two different processes are involved which interconnect in a highly complex way. The first process concerns the relation between the wage-profit frontier, i.e. the relation between the rate of profit (r) and the wage rate (w) over their entire range. The second process concerns the role of variations in the structure of prices accompanying shifts in r and w. Exact results have been obtained only in the extreme cases where w or r is zero, or in cases characterized by special proportions in the price or production relations. Sraffa (1960) derived the wage-profit frontier in the special case where the economy’s surplus (definitions will follow) is exactly proportional to required total output. In that case, the influence of price movements is neutralized, and we obtain the *linear* relation.

[[3]](#footnote-3)

1. r = (1 – w)R

where r∈ [0,R] and w ∈ [0,1], with R the maximum rate of profit. Up to now no general approach to the problem of determining the income distribution over the entire range of r and w has been presented. Also it is unclear if linearization is really necessary to neutralize the role of price fluctuations.

In this paper our aim is to show that the wage-profit relation can be exactly determined over the entire range of w and r. Linearization is not necessary and basically a detour. Our point of departure will be the Sraffian price equation

1. 

where A is the nonnegative matrix of input coefficients and l > 0 the vector of direct labour input coefficients. A is assumed to be indecomposable and of full rank, with Perron-Frobenius eigenvalue λ(A) < 1.

2. **The model**

We shall depart from a standard open IO model. That is, we have the real output system

1. x = Ax + f

where x is the total output vector, and f > 0 the economy’s surplus or (real) net income, i.e. all production above the minimum necessary for replacement. [[4]](#footnote-4) Total employed labour, L, is defined as

1. L = lx

With p the vector of equilibrium prices, the accompanying price equation is

1. p = pA + wl

Usually, the wage rate is defined in terms of a monetary unit per unit of labour. Final demand f is given exogenously, which gives us x via equation (3), and L via (4). Prices are obtained directly via (5) with exogenous w.

We should note that the model admits an alternative formulation, the so-called extended form. Combining (3) and (4), we obtain

1. x = (A + [f/L]l)x

where A and l represent the technology in use, and where the values for f and L have to be supplied exogenously. However, once these values are known, they can be implemented in (6). The model thus has been reformulated, which now gives x as the right-hand Perron-Frobenius (PF) eigenvector of the matrix on the r.h.s. of (6). Let us define

1. M = A + [f/L]l

From (6) we then have that the PF eigenvalue of M, denoted by λ(M), is equal to 1. We also have straightforwardly that the determinant of matrix (I - M) is 0: [[5]](#footnote-5)

1. │I - M│ = 0

We can reformulate the price model analogously if we adopt a special numeraire for the wage. As mentioned above, traditionally the wage is expressed in terms of dollars, pounds, et cetera, per unit of labour. Let us define now the wage (per unit of labour) in terms income per head. If we follow Sraffa, we may standardize at national income and total labour force both at unity. [[6]](#footnote-6) Thus, pf = 1 and L = 1. This gives pf/L = p(f/L) = w = 1. So, using (1),(2) and (3), we obtain the following price equation

1. p = p(A + [f/L]l)

That is, the price vector p has been written as the left-hand PF eigenvector of matrix M. Before continuing, we introduce one more symbol, i.e. ω (omega). This new symbol stands for the share of the wage (expressed in the newly introduced numeraire) that is allocated to labour. Here, where any other claimant of the wage is absent, we have ω = 1. However, if such a claimant would appear, if if this claimant would be successful in claiming part of the wage, we would have 0 < ω < 1. Correspondingly, we can rewrite (9) as

1. p = p(A + ω[f/L]l)

clearly with ω = 1.

**2. Deriving the wage-profit function**

Let us now introduce a rate of profit in the above price equation. We then have

1. p = p[(1 + r)A + ω(f/L)l]

So, with

1. Mr,w = [(1 + r)A + ω(f/L)l]

we now have

1. p(I - Mr,w) = 0

We thus see that r and ω must be such that λ(Mr,w), the PF eigenvalue of matrix Mr,w, is equal to unity. From (13), we now have straightforwardly the following characteristic equation

(14) φ(r,ω) = │I - [(1+r)A + ω(f/L)l│ = 0

Equation (14) gives all combinations of r and ω that, after substituting in (12), result in matrices satisfying (13). Below we shall derive the general form of φ(r,ω), employing the fact that M(r,ω) is the sum of the full rank matrix (1+r)A and the positive matrix of rank 1, B = ω(f/L)l. We have

*Theorem 1:*

Let A, f, L, r and ω be as defined above. Then

(15) 

where the α’s and β’s are appropriate coefficients.

*Proof of Theorem 1:*

With B = (f/L)l and writing ej, gj, aj, and bj (j = 1,…,n) for, respectively, the jth column of the unit matrix and the matrices G = [I - Mr,w], A, and B, we have gj = ej – [(1+r)aj + ωbj]. Writing dj = ej – (1+r)aj, we have gj = dj – ωbj. We now split each column gj in a part dj and a part - ωbj, and apply the addition rule for determinants successively. This gives

(16) 

Continuing, we also split the remaining columns g2,…,gn of φ(r,ω) in two parts, i.e. a part d2 and a part – ωb2, etc. We now can write φ(r,ω) as the sum of determinants the jth column of which is either dj or - ωbj. We have 2n of such determinants. Because matrix B has rank 1, we straightforwardly have that the value of determinants which contain two or more columns of B is zero. There are  of such determinants. This leaves  = n + 1 determinants which are non-zero. We obtain the functional form of φ(r,ω) noting that the determinant that contains no columns of B, i.e. │d1, d2,…,dn│, is a polynomial of degree n in r. The n determinants that contain n-1 columns of matrix D and only one column of matrix B are polynomials of degree n-1 in r and the first degree in ω. This gives (15). ■

From (15) we have that ω is a function of r which can be written as the quotient of two polynomials, the nominator having degree n, the denominator degree (n-1). Substituting an admissible point (r,ω) in (12) and solving for prices p gives us the corresponding Sraffian prices, standardization via pf/L = 1 absolute prices. Below we shall illustrate the procedure with Sraffa’s numerical example.

In the case where f is proportional to x, the so-called standard system, the distribution reduces to a linear relation between r and w, i.e. (1), see further Pasinetti (1977, ch. 5), Abraham-Frois and Berrebi (1979, ch. 2), Kurz and Salvadori (1995, 2007) or Steenge (1995) on this. Similarly, also in the case where l in (2) is proportional to p, we have (1), see Pasinetti (1977, ch. 5).

**3. Sraffa’s frontier and prices**

Below we shall present an example using Sraffa’s data on his 'actual system′ (see also Pasinetti, 1977, chs. 2 and 5 on this system). We have, rounding in three decimals,



and



Final demand equals



Straightforward calculation using (2) gives



and



Substituting the values of f, l and L in φ(r, ω) gives:

(14) 

a polynomial of 3rd degree in r and first degree in w. Putting φ(r, w) = 0 and solving for w, we have

(15) 

w(r) is non-linear. To illustrate, the Taylor approximation of w(r) by a polynomial of degree 2 gives

(16) 

Before presenting the graph of the economically relevant part of the curve, it is useful to look at the wider picture. Figure 1 presents the graph of φ(r, ω) = 0 for the intervals -20 < r < 20 and -20 < ω < 20. The equation plots as three separate non-linear curves in the intervals mentioned:

Figure 1 about here

Economically relevant is the part of the left-hand curve delineated by the points (0,1) and (0,R). Figure 2 illustrates this graphically for the interval -0.25 < r < 0.70 (in our example, R = 0.483).

Figure 2 about here

Prices are obtained by substituting an admissible point (r,ω) in (8) and solving for prices p; standardization via (4) gives absolute prices.

**4. Final remarks and conclusion**

In this paper we have shown that the wage-profit frontier is completely determined in any Sraffian system over the whole range of r, the rate of profit, and ω, the share of wages in the net income. Therefore, there is no need to adopt special numeraires such as the standard commodity to neutralize the role of prices and to obtain a linear frontier. In fact, striving for linearity basically is a detour in obtaining insight in the properties of the income distribution determining mechanism. In addition, the standard system itself is just a special case of the general case we have discussed. There is nothing that makes that particular system more special than any other one. (Compare also Duménil and Lévy (1993) on this point. They point out (p. 52) that the standard commodity has the “aesthetical property” of generating a linear trade off between r and w, but is not a standard of value in any sense). Finally, it is not difficult to show that if f in (2) approaches the proportions of the Perron-Frobenius eigenvector of A (i.e. if the input-output system approaches the proportions of the standard system), w(r) approaches linearity.

**References**

Abraham-Frois, G. and E. Berrebi, 1979, Theory of Value, Prices and Accumulation: a Mathematical Integration of Marx, von Neumann and Sraffa. (Cambridge University Press, Cambridge).

Duménil, G. and D. Lévy, 1993, The Economics of the Profit Rate. (Edward Elgar, Aldershot).

Kurz, H.D. and N. Salvadori, 1995, Theory of Production: A long-period Analysis. (Cambridge University Press, Cambridge, New York and Melbourne).

Kurz, H.D. and N. Salvadori, 2007, On the collaboration between Sraffa and Besicovitch: the ′Proof of Gradient′, in: G. Chiodi and L. Ditta, eds., Sraffa or An Alternative Economics, (Palgrave Macmillan, Houndmills, etc., England), 260-274.

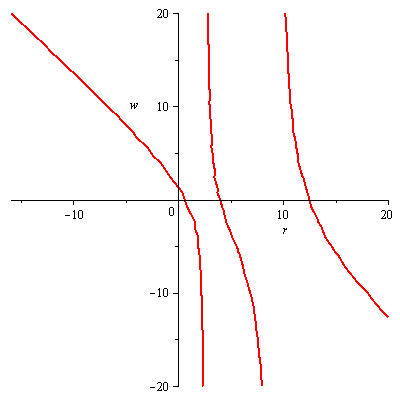
Pasinetti, L.L., 1977, Lectures on the Theory of Production. (Columbia University Press, New York).

Samuelson, P.A. and E.M. Etula, 2006, Testing to confirm that Leontief-Sraffa matrix equations for input/output must obey constancy of returns to scale, Economics Letters, vol. 90, 183-188.

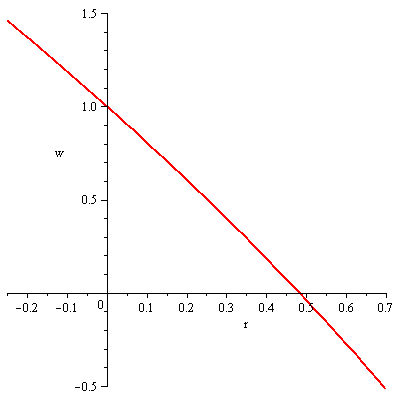
Sraffa, P., 1960, Production of Commodities by Means of Commodities. (Cambridge University Press, Cambridge).

Steenge, A.E., 1995, Sraffa and Goodwin: a unifying framework for standards of value in the income distribution problem, Journal of Economics, vol. 62, no 1, 55-75.

**Figure 1**: Relation between r and w, -20 < r < 20 and -20 < w < 20



**Figure 2**: Relation between r and w, 0.3 < r < 0.7 and -0.5 < w < 1.5



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3. For proofs, see Pasinetti (1977, ch. 5) or Abraham-Frois and Berrebi (1977, ch. 2). [↑](#footnote-ref-3)
4. Sraffa called this the actual system. For a recent discussion on the role of the constant returns to scale assumption in this system, see Samuelson and Etula (2006). [↑](#footnote-ref-4)
5. Here I is the unit matrix of appropriate dimension. [↑](#footnote-ref-5)
6. As we shall see later, there is no need to follow Sraffa in standardizing the values of pf and L both at unity. [↑](#footnote-ref-6)