A New Method for Triangulation of Input-Output Tables for Comparing Industrial Structures and Investigating Clusters of Industries

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Abstract. Understanding the industrial structure of a national or regional economy is one of the central issues in economics. The triangulation of an input-output table (IOT) can be used to understand the hierarchical structure of industrial sectors, and summarize and visualize IOTs in a way that facilitates specific analysis and interpretation. This paper proposes a new method to triangulate IOTs based on mixed integer programs for comparing the industrial structures of economies and investigating clusters of industries within an economy. The application of the proposed method to the Japanese IOTs demonstrates its usefulness and exemplifies how it provides insight into industrial structures.

Keywords: Triangularization, Industrial structure, Block-triangularity, Mixed integer program

JEL classification numbers: C61, C67, L16

1. Introduction

The entire industrial structure of a national or regional economy can be observed as its input-output table (IOT). The industry-by-industry part of an IOT is a square matrix which describes transactions among industrial sectors, and consists of thousands of numerical values. Because understanding the industrial structure of an economy is one of the central issues in economics, quantitative methods for summarizing and visualizing information archived in IOTs are indispensable. The triangulation of IOTs is one such method.

Primary, secondary and tertiary sectors in IOTs are traditionally arranged in that order. An IOT has many zero elements and it is a sparse matrix, because most sectors require input of products from a limited number of sectors. Therefore, an IOT may be triangulated, in the sense that most elements in the upper triangular part are zeros, by rearranging the sectors in a descending order of their degrees of fabrication. Stated differently, if a particular sequence of sectors converts an IOT into a lower triangular form, the sequence represents the order of fabrication. The history of the triangulation of input-output tables dates back to, at least, the research by the Planning Research Division of the US Air Force in 1950s, according to Haltia (1992, p. 223). Chenery and Watanabe (1958) compared hierarchical structure of sectors in US, Japan, Norway and Italy and showed that the hierarchies are quite similar among these countries. Simpson and Tsukui (1965) pointed out dependence and independence between clusters of sectors called block dependence and independence, based on data of US, Japan, Norway, Italy and Spain.

In most studies, the triangulation of an IOT is defined as an optimization problem to maximize the sum of elements in the lower triangular part by permuting sectors (e.g., Fukui (1986)). This is a combinatorial optimization problem and known as an NP-hard problem, which is difficult to solve (Charon and Hudry (2007)). The number of its feasible solutions is $n!$ if there are n sectors. The problem can be easily solved by enumerating all the $n!$ permutations of sectors and choosing an optimal solution from among them if n is very small; for example, $n! = 120$ if $n = 5$. However, such a brute force algorithm does not work even in the case of a moderate number of sectors; for example, $n! \approx 3.0 \times 10^{64}$ if $n = 50$. Sectors may be rearranged based on expert knowledge of technologies and relationship among sectors to triangulate an IOT to some degree. In contrast, objective methods are useful because there is hardly a clear one-way order of sectors and a hierarchical structure among sectors is complex.

Several algorithms specifically designed for the triangulation problem have been developed and proposed in the literature. Simpson and Tsukui (1965), Korte and Oberhofer (1970) and Fukui (1986) proposed heuristic algorithms in which substitutions of sectors, called ringshift permutations, are iterated. Optimal solutions are not necessarily found by performing these algorithms. In contrast, algorithms with which optimal solutions can be found for problems of moderate size have also been developed. Haltia (1992) and Östblom (1997) proposed an algorithm without ringshift permutation. The triangulation problem is equivalent to the linear ordering problem, and more efficient algorithms for the problem have been proposed in the literature of operations research (Grötschel, Jünger and Reinelt 1984a, Laguna, Marti and Campos 1999, Chiarini, Chaovalitwongse and Pardalos 2004).

Because the triangulation problem can be represented as an integer program as explained in Grötschel, Jünger and Reinelt (1984a, 1984b) and Chiarini, Chaovalitwongse and Pardalos (2004), it can be solved by a general-purpose algorithm for integer programs which is typically implemented in prevailing software. It can be said that most general-purpose algorithms are less efficient, i.e., more time-consuming, than special algorithms. On the other hand, general-purpose algorithms can be applied even when the original problem is extended or modified, for example, by adding constraints and changing its objective function, while special algorithms generally cannot. This paper proposes new extended methods for the triangulation of IOTs that are represented as mixed integer programs to which a general-purpose algorithm is applied. At the cost of computational efficiency, the new methods can properly take account of two relevant issues, which have been pointed out but not studied in a satisfactory manner so far.

One is related to comparisons of the industrial structures of multiple economies. The Spearman rank correlation coefficient has been used to quantify a similarity between the sequences of sectors that are obtained by the triangulation of IOTs (Chenery and Watanabe 1958, Grötschel, Jünger and Reinelt 1984b, Fukui 1986). As

Grötschel, Jünger and Reinelt (1984b) and Östblom (1997) pointed out, however, underlying structures might be similar even if the optimal sequences of sectors at hand look quite different. This is because an optimal solution to the triangulation problem is typically not unique. This paper proposes a new method to compare the structures of multiple economies, which does not suffer from the non-uniqueness of optimal solutions and is consistent to the Kendall rank correlation coefficient. The method is presented in detail in Section 3.1.

The other relevant issue is to consider clusters of sectors, which are called blocks in the literature of the triangulation. Simpson and Tsukui (1965) triangulated IOTs of developed countries by taking such blocks into consideration. They first divided sectors into four clusters or blocks following the physical characteristics of products, then triangulated the whole table by regarding each cluster as a sector, and finally triangulated sectors within each cluster. Block-triangulations, which mean triangulations with a consideration of blocks, were carried out in this manner in the literature. It has been recognized that a consideration of blocks contributes greatly to reducing computational burden, when blocks are mutually exclusive and defined in advance (e.g., Östblom (1993)). In contrast, scarce study has been performed on how sectors can be classified into clusters; Dietzenbacher (1996) is a rare exception. This paper proposes new methods for block-triangulation, for which expert knowledge of industrial structures or physical characteristics of products is not necessary. The method is described in detail in Section 3.2.

The construction of the remaining parts of this paper is as follows. In Section 2, the triangulation problem is formulated within the literature, and its representation as an integer program is explained. The new methods are proposed and applied to the Japanese Linked Input-Output Tables in Section 3. Section 4 concludes the paper.

2. Definition and representations of the triangulation problem

The triangulation problem is formulated and a representation of it as an integer program is explained in this section. The representation is equivalent to one utilized in the literature as we will see below. However, the process to derive it and its interpretation are useful for developing new methods proposed in this paper.

Let there be n industrial sectors and suppose that our target is to triangulate an $n \times n$ inter-industry transaction table or input coefficient matrix $\mathbf{A} = (A_{ij})$ in order to analyze the industrial structure of the economy that $\mathbf{A}$ describes and a hierarchy among sectors. We simply call $\mathbf{A}$ an IOT, hereafter. For ease of exposition, we define the set of natural numbers referring to these sectors as $N = \{1,2,\dots,n\}$. We then denote a permutation of n sectors by $\boldsymbol{\pi} = (\pi(1), \pi(2), \dots, \pi(n))$ and the set of all permutations of sectors by Π. Given a permutation $\boldsymbol{\pi} \in \Pi$, let $\mathbf{A}(\boldsymbol{\pi}) = (A_{ij}(\boldsymbol{\pi}))$ denote the IOT in which the sectors are permuted according to $\boldsymbol{\pi}$, that is,

$$A_{ij}(\boldsymbol{\pi}) = A_{\pi(i)\pi(j)} \quad (i,j \in N). \tag{1}$$

For any $n \times n$ matrix $\mathbf{M} = (M_{ij})$, we denote the sum of the elements in its lower triangular part by

$$\ell(\mathbf{M}) = \sum_{i=2}^{n} \sum_{j=1}^{i-1} M_{ij} = \sum_{i>j} M_{ij}. \tag{2}$$

Using these notations, the triangulation problem is formulated as a combinatorial optimization problem:

$$\left| \begin{array}{ll} \text{maximize} & \ell(\mathbf{A}(\boldsymbol{\pi})) \\ \text{subject to} & \boldsymbol{\pi} \in \Pi. \end{array} \right. \tag{3}$$

This problem can be easily solved by enumerating all the permutations and choosing an optimal solution from among them if n is very small. However, such a brute force algorithm in this manner fails even in the case of a moderate number of sectors.

The index called the degree of linearity has been used in the literature to represent how well an IOT is triangulated. Given an IOT, the degree of linearity of a permutation $\boldsymbol{\pi}$ is defined as

$$\lambda\big(\mathbf{A}(\boldsymbol{\pi})\big) = \frac{\sum_{i>j} A_{ij}(\boldsymbol{\pi})}{\sum_{i \neq j} A_{ij}(\boldsymbol{\pi})}. \tag{4}$$

The numerator is the same as the objective function of the triangulation problem (3). The denominator is the sum of all the off-diagonal elements. No diagonal elements are summed up in either the numerator or denominator. It can be rewritten as

$$\lambda\big(\mathbf{A}(\boldsymbol{\pi})\big) = \frac{\ell\big(\mathbf{A}(\boldsymbol{\pi})\big)}{\sum_{i \neq j} A_{ij}}. \tag{5}$$

This formula shows that a permutation which maximizes the objective function of the triangulation problem, $\ell\big(\mathbf{A}(\boldsymbol{\pi})\big)$, also maximizes the degree of linearity, $\lambda\big(\mathbf{A}(\boldsymbol{\pi})\big)$ because the denominator does not depend on $\boldsymbol{\pi}$.

A permutation $\boldsymbol{\pi}$ can be represented by the corresponding permutation matrix $\mathbf{P} = \big(P_{ij}\big)$ such that

$$P_{ij} = 1\{i = \pi(j)\} \quad (i, j \in N), \tag{6}$$

where $1\{\cdot\}$ is the indicator function such that $1\{Q\} = 1$ if the proposition Q is true and $1\{Q\} = 0$ otherwise. For example,

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{7}$$

when $n = 4$ and $\boldsymbol{\pi} = (4,3,1,2)$. An IOT $\mathbf{A}(\boldsymbol{\pi})$ in which the sectors are permuted according to $\boldsymbol{\pi}$ can be written as follows:

$$\mathbf{A}(\boldsymbol{\pi}) = \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}, \tag{8}$$

where $\mathbf{P}^{\mathrm{T}}$ is the transpose of $\mathbf{P}$. Defining a $\{0,1\}$-matrix $\mathbf{L} = (L_{ij})$ as

$$L_{ij} = 1\{i \geq j\} \quad (i, j \in N), \tag{9}$$

therefore, we can rewrite the objective function $\ell(\mathbf{A}(\boldsymbol{\pi}))$ of the triangulation problem (3) plus the sum of the diagonal elements of $\mathbf{A}$, $\sum_{i=1}^{n} A_{ii}$, as

$$\begin{aligned} \ell(\mathbf{A}(\boldsymbol{\pi})) + \sum_{i=1}^{n} A_{ii} &= \sum_{i=1}^{n}\sum_{j=1}^{i} A_{ij}(\boldsymbol{\pi}) = \sum_{i=1}^{n}\sum_{j=1}^{n} L_{ij}A_{ij}(\boldsymbol{\pi}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \big(\mathbf{L} \odot \mathbf{A}(\boldsymbol{\pi})\big)_{ij} = \mathbf{1}_n^{\mathrm{T}}\big(\mathbf{L} \odot \mathbf{A}(\boldsymbol{\pi})\big)\mathbf{1}_n \\ &= \mathbf{1}_n^{\mathrm{T}}(\mathbf{L} \odot \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P})\mathbf{1}_n, \end{aligned} \tag{10}$$

where the term $\sum_{i=1}^{n} A_{ii}$ is added to the left-hand side to make the following formulae compact, $(\cdot)_{ij}$ refers to the (i, j)-element of a matrix in the parentheses, $\mathbf{1}_n$ is an $n \times 1$ vector of unities, and $\odot$ denotes the Hadamard product of two matrices, e.g., $(\mathbf{C} \odot \mathbf{D})_{ij} = C_{ij}D_{ij}$ for two matrices of the same size, $\mathbf{C} = (C_{ij})$ and $\mathbf{D} = (D_{ij})$. In addition, the set of all the permutation matrices can be written as

$$\Pi_{\mathbf{P}} = \left\{\mathbf{P} \in \mathbb{R}^{n\times n} \middle| \mathbf{P}\mathbf{1}_n = \mathbf{1}_n,\ \mathbf{P}^{\mathrm{T}}\mathbf{1}_n = \mathbf{1}_n,\ P_{ij} \in \{0,1\}\ (i, j \in N)\right\}. \tag{11}$$

This set can be regarded as another representation of Π because there is a one-to-one correspondence between permutations and permutation matrices. Thus, the triangulation problem (3) is equivalent to the following, except that their objective values are different by a constant term, $\sum_{i=1}^{n} A_{ii}$:

$$\left|\begin{array}{ll} \text{maximize} & \mathbf{1}_n^{\mathrm{T}}(\mathbf{L} \odot \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P})\mathbf{1}_n \\ \text{subject to} & \mathbf{P}\mathbf{1}_n = \mathbf{1}_n, \\ & \mathbf{P}^{\mathrm{T}}\mathbf{1}_n = \mathbf{1}_n, \\ & P_{ij} \in \{0,1\}\ (i, j \in N). \end{array}\right. \tag{12}$$

This representation of the triangulation problem is very compact and concise. However, it is likely very difficult to solve because its objective function is not linear in variables, and thus an alternate, linear representation would be preferred.

Note that the objective function with an adjustment by a constant term (10) can be further rewritten as

$$\begin{aligned}\mathbf{1}_n^{\mathrm{T}}(\mathbf{L} \odot \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P})\mathbf{1}_n &= \mathrm{tr}(\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}\mathbf{L}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{A}\mathbf{P}\mathbf{L}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}) \\ &= \mathrm{tr}(\mathbf{A}(\mathbf{P}\mathbf{L}\mathbf{P}^{\mathrm{T}})^{\mathrm{T}}) = \mathbf{1}_n^{\mathrm{T}}(\mathbf{P}\mathbf{L}\mathbf{P}^{\mathrm{T}} \odot \mathbf{A})\mathbf{1}_n,\end{aligned} \tag{13}$$

where $\mathrm{tr}(\cdot)$ is the trace of a square matrix. The first and last equalities hold because

$$\mathrm{tr}(\mathbf{C}\mathbf{B}^{\mathrm{T}}) = \sum_{i=1}^{n}(\mathbf{C}\mathbf{B}^{\mathrm{T}})_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}B_{ij} = \mathbf{1}_n^{\mathrm{T}}(\mathbf{B} \odot \mathbf{C})\mathbf{1}_n,$$

and the second equality holds because $\mathrm{tr}(\mathbf{B}\mathbf{C}) = \mathrm{tr}(\mathbf{C}\mathbf{B})$ for any pair of $n \times n$ matrices $\mathbf{B} = (B_{ij})$ and $\mathbf{C} = (C_{ij})$. Note that each of the left- and right-hand sides of (13) is the sum of all the elements of a square matrix which is the Hadamard product of two matrices. As for the left-hand side, they are $\mathbf{L}$ and $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$. Because $\mathbf{L}$ is a $\{0,1\}$-matrix, the Hadamard product picks up elements, which should be summed up, from $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$. In contrast, those matrices are $\mathbf{P}\mathbf{L}\mathbf{P}^{\mathrm{T}}$ and $\mathbf{A}$ for the right-hand side. Because $\mathbf{P}\mathbf{L}\mathbf{P}^{\mathrm{T}}$ is also a $\{0,1\}$-matrix, the Hadamard product picks up elements from $\mathbf{A}$. In other words, the left- and right-hand sides of (13) represent the two different approaches to calculate the objective function $\ell(\mathbf{A}(\boldsymbol{\pi}))$ with an adjustment by a constant term. The expression on the left-hand side permutes sectors, and picks the elements to sum up from the lower triangular part of the IOT in which the sectors are permuted. In contrast, the other approach expressed on the right-hand side picks the elements to sum up from the original table, in which the sectors are not permuted.

The expression on the right-hand side of (13) motivates us to introduce another variable, $\mathbf{X}$, in order to give an alternate representation of the set of all permutations, Π, as follows.

$$\Pi_{\mathbf{X}} = \{\mathbf{X} \in \mathbb{R}^{n \times n} | \mathbf{X} = \mathbf{PLP}^{\mathrm{T}}, \quad \mathbf{P} \in \Pi_{\mathbf{P}}\}. \tag{14}$$

Using this variable, the objective function in (12) and (13) can be simplified and it is linear in variable $\mathbf{X}$:

$$\ell(\mathbf{A}(\boldsymbol{\pi})) + \sum_{i=1}^{n} A_{ij} = \mathbf{1}_n^{\mathrm{T}}(\mathbf{X} \odot \mathbf{A})\mathbf{1}_n = \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}A_{ij}. \tag{15}$$

The variable $\mathbf{X} = \mathbf{PLP}^{\mathrm{T}}$ is a $\{0,1\}$-matrix because $\mathbf{L}$ is a $\{0,1\}$-matrix defined in (9) and $\mathbf{P}$ ($\mathbf{P} \in \Pi_{\mathbf{P}}$) is a permutation matrix.

Note that there is a one-to-one correspondence between $\mathbf{X}$ and $\boldsymbol{\pi}$, because there is a one-to-one correspondence between $\mathbf{P}$ and $\boldsymbol{\pi}$, and $\mathbf{P}$ and $\mathbf{L}$ are nonsingular matrices. Note also that $\mathbf{X}$ can be obviously constructed by definition as $\mathbf{X} = \mathbf{PLP}^{\mathrm{T}}$, with the permutation matrix $\mathbf{P}$ corresponding to a given permutation $\boldsymbol{\pi}$. On the other hand, the permutation $\boldsymbol{\pi}$ that corresponds to a given matrix $\mathbf{X}$ can also be formed from the matrix, shown below. The (i,j)-element, X_{ij}, of $\mathbf{X}$ can be rewritten as

$$\begin{aligned} X_{ij} &= (\mathbf{PLP}^{\mathrm{T}})_{ij} = \sum_{h=1}^{n}\sum_{k=1}^{n} P_{ih}L_{hk}P_{jk} \\ &= L_{\pi^{-1}(i)\pi^{-1}(j)} = 1\{\pi^{-1}(i) \geq \pi^{-1}(j)\} \quad (i,j \in N), \end{aligned} \tag{16}$$

where the third equality holds because $P_{ih} = P_{jk} = 1$ if and only if $h = \pi^{-1}(i)$ and $k = \pi^{-1}(j)$. Therefore, $X_{ij} = 1$ if sector j is placed at the same position as sector i, i.e., $j = i$, or sector j precedes sector i in the permutation $\boldsymbol{\pi}$; and $X_{ij} = 0$ otherwise. Eq. (16) directly represents the relationship between $\mathbf{X}$ and $\boldsymbol{\pi}$. Summing up the right- and left-hand sides of (16) over $j \in N$, we have

$$\sum_{j=1}^{n} X_{ij} = \sum_{j=1}^{n} 1\{\pi^{-1}(i) \geq \pi^{-1}(j)\} = \pi^{-1}(i) \quad (i \in N). \tag{17}$$

The expression in the middle of (17) is the number of sectors which are placed at the same position as sector i or precedes sector i. The permutation $\boldsymbol{\pi}$ that corresponds to a given matrix $\mathbf{X}$ can thus be recovered from $\mathbf{X}$ as in (17); i.e., the rank of sector i, $\pi^{-1}(i)$, is equal to the sum of elements in the i-th row of $\mathbf{X}$.

Recall that the objective function of the triangulation problem can be represented by a formula linear in $\{0,1\}$-variables, X_{ij}, as in (15). On the other hand, the expression of $\Pi_{\mathbf{X}}$ in (14) has a nonlinear equation, $\mathbf{X} = \mathbf{PLP}^{\mathrm{T}}$, in its conditioning part. Thus, our next step is to derive another expression of $\Pi_{\mathbf{X}}$ conditioned with only linear inequalities, so that the triangulation problem may be represented as a (linear) integer program. The following system of linear inequalities is known for properly defining $\Pi_{\mathbf{X}}$ in the literature (Grötschel, Jünger and Reinelt (1984a, eqs. (10)–(12) on p. 1202), Chiarini, Chaovalitwongse and Pardalos (2004, eqs. (9)–(10) on p. 259)).

(F1) $X_{ii} = 1 \ (i \in N)$;

(F2) $X_{ij} + X_{ji} = 1 \ (i < j; i, j \in N)$; and

(F3) $0 \leq X_{ij} + X_{jk} - X_{ik} \leq 1 \ (i < j < k;\ i, j, k \in N)$.

This system can be derived by considering a binary relation on N; see Appendex A. Grötschel, Jünger and Reinelt (1984b) used a variant of this system to triangulate IOTs of European countries.

Therefore, the triangulation problem can be represented as the following (linear) integer program:

$$\left|\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}X_{ij} - \sum_{i=1}^{n} A_{ii} \\ \text{subject to} & X_{ii} = 1 \ (i \in N), \\ & X_{ij} + X_{ji} = 1 \ (i < j; i, j \in N), \\ & 0 \leq X_{ij} + X_{jk} - X_{ik} \leq 1 \ (i < j < k;\ i, j, k \in N), \\ & X_{ij} \in \{0,1\} \ (i, j \in N). \end{array}\right. \tag{18}$$

or equivalently,

$$\left|\begin{array}{ll}\text{maximize} & \displaystyle\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{\left(A_{ij}-A_{ji}\right)X_{ij}+A_{ji}\right\} \\ \text{subject to} & 0 \le X_{ij}+X_{jk}-X_{ik} \le 1 \;\; (i<j<k;\; i,j,k \in N), \\ & X_{ij} \in \{0,1\} \;\; (i<j;\; i,j \in N).\end{array}\right. \tag{19}$$

The last representation is an integer program, or more specifically, a $\{0,1\}$-program, which has $n(n-1)/2$ binary variables and $n(n-1)(n-2)/3$ inequality constraints.

Given an optimal solution to (19), the corresponding optimal solution to (18) can be easily obtained by letting $X_{ii}=1 \;\; (i \in N)$ and $X_{ji}=1-X_{ij} \;\; (i<j; i,j \in N)$. The corresponding optimal permutation $\boldsymbol{\pi}$ can then be retrieved as in (17), i.e.,

$$\pi^{-1}(i)=\sum_{j=1}^{n} X_{ij} \;\; (i \in N). \tag{20}$$

Turning to empirical applications, we used the input coefficient matrices obtained from the 1990-1995-2000 Japanese Linked Input-Output Tables (SB-MIAC (2005)). We discarded smaller elements by setting $A_{ij}=0$ if $A_{ij}<1/n \;\; (i,j \in N)$ as Simpson and Tsukui (1965) did. The number of sectors is $n=99$. We solved the integer program (19) for each of the three input coefficient matrices. Xpress Mosel language version 3.0.2 and Xpress Optimizer version 20.00.11 were used to solve the programs. The linear program relaxation provided integer solutions for the three IOTs, so that no global search phase based on the branch and bound method was performed.

The three input-output matrices were triangulated fairly well, as the degrees of linearity of the triangulated input coefficient matrices were 0.9437, 0.9442 and 0.9459 for the years 1990, 1995 and 2000, respectively. These large values, which are close to unity, mean that the inter-dependence among sectors can be summarized as a nearly one-directional hierarchy. Thus there is little multi-directional dependence, like feedback loops, with substantial inter-sector transactions.

To summarize how similar the optimal sequences of sectors are over time, we calculated the Spearman rank correlation coefficient, $\rho(s,t)$, and the Kendall rank correlation coefficient, $\tau(s,t)$, for $s,t = 1990, 1995, 2000$. The results are $\rho(1990{,}1995) = 0.859$, $\rho(1995{,}2000) = 0.875$ and $\rho(1990{,}2000) = 0.845$, and $\tau(1990{,}1995) = 0.701$, $\tau(1995{,}2000) = 0.730$ and $\tau(1990{,}2000) = 0.707$. These results may indicate that the hierarchies revealed by triangulation are similar and have changed only mildly during the period 1990–2000. Figure 1 shows the detailed results on how the hierarchies have changed over time in the form of a migration diagram, proposed by Grötschel, Jünger and Reinelt (1984b). There are more crosses in the upper half of Figure 1 than in the lower half. This implies that the rankings of sectors of high degrees of fabrication changed more drastically than those of low degrees of fabrication. For instance, the rankings of '#62 Precision instruments', '#51 Electronic computing equipment and accessory equipment' and '#52 Communication equipment' sectors range from 1 to 36, 1 to 40 and 8 to 50, respectively. In contrast, the difference between the maximum and minimum of rankings is less than 10 for most sectors in the lower half of Figure 1. The difference is, for example, $91 - 89 = 2$ for '#9 Crude petroleum and natural gas' sector. The top three sectors with regard to this difference are '#85 Public administration', '#30 Coal products' and '#80 Freight forwarding' sectors; the differences are 64, 53 and 50, respectively.

We will investigate whether or not these drastic changes are meaningful before giving interpretations to them, by applying the new method proposed in the next section.

3. Extensions of the triangulation problem and applications

3.1 Comparing two input-output tables

IOTs can be triangulated by solving the integer program explained in the previous section or by applying other existing algorithms for the triangulation problem. Inter-temporal or inter-regional comparisons of two (or more) IOTs have addressed interesting and important issues since the early studies, for instance, by Chenery and Watanabe (1958). In addition to comparing the whole sequences of sectors in two triangulated IOTs, the Spearman rank correlation coefficient has been used to quantify a similarity between the two sequences (Chenery and Watanabe (1958), Grötschel, Jünger and Reinelt (1984b), Fukui (1986), Östblom (1993)). However, it might be possible to improve the two sequences in the sense that the rank correlation becomes positively stronger without decreasing the degrees of linearity, as Grötschel, Jünger and Reinelt (1984b) and Östblom (1997) pointed out. This is because an optimal solution to a triangulation problem is not necessarily unique. This is actually the case for our application as explained later. If one only triangulates two IOTs and calculates the rank correlation coefficient without carefully seeing the optimal rankings or triangulated tables themselves, then the result would likely be misinterpreted. In this subsection, we propose an improved method for comparing hierarchies and industrial structures based on the triangulation of IOTs.

Suppose that we have IOTs for time periods 1 and 2, $\mathbf{A}^{(t)} = \left(A_{ij}^{(t)}\right)$ $(t = 1,2)$, where superscript '(t)' is introduced to indicate the time period (the method to be proposed can be easily applied for inter-regional comparisons although we focus on inter-temporal comparisons in this paper). Suppose also that we have solved the integer program (19) with $\mathbf{A} = \mathbf{A}^{(t)}$ for the two time periods $(t = 1,2)$, and then obtained an optimal solution $\bar{X}_{ij}^{(t)}$, the corresponding optimal permutation $\bar{\boldsymbol{\pi}}_{(t)}$, and the optimal value $\bar{M}_{(t)}$ (maximized objective value) of the program. The problem that we are proposing is to find two sequences of sectors which are mutually as close as possible, guaranteeing that the degrees of linearity are equal to the best attainable ones. The problem can be written as follows, with the concept of the "difference" allowed to be ambiguous.

$$\left|\begin{array}{ll}\text{minimize} & \text{difference between } \boldsymbol{\pi}_{(1)} \text{ and } \boldsymbol{\pi}_{(2)} \\ \text{subject to} & \ell\left(\mathbf{A}^{(t)}(\boldsymbol{\pi}_{(t)})\right) = \bar{M}_{(t)} \ \ (t = 1,2), \\ & \boldsymbol{\pi}_{(t)} \in \Pi \ \ (t = 1,2).\end{array}\right. \tag{21}$$

Let us define "the difference between $\boldsymbol{\pi}_{(1)}$ and $\boldsymbol{\pi}_{(2)}$" as the sum of absolute difference between the elements of $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$, i.e., $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|X_{ij}^{(2)} - X_{ij}^{(1)}\right|$. The advantage of this difference over others, e.g., the sum of squared differences, will be discussed later. We then introduce new variables, U_{ij}, V_{ij} $(i < j; i, j \in N)$, to represent the difference with a formula linear in variables, such that

$$U_{ij} - V_{ij} = X_{ij}^{(2)} - X_{ij}^{(1)}, \qquad U_{ij} \geq 0, \qquad V_{ij} \geq 0 \quad (i < j;\ i, \in N), \tag{22}$$

Note that $\left|X_{ij}^{(2)} - X_{ij}^{(1)}\right| = U_{ij} + V_{ij}$ if $U_{ij}V_{ij} = 0$ and (22) hold, given any pair of $X_{ij}^{(1)}$ and $X_{ij}^{(2)}$. This is a well-known technique in the field of operations research to deal with absolute values.

Employing this technique, we make the program (21) concrete and propose the following mixed integer program for a comparison of hierarchies among sectors in two IOTs.

$$\left|\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(U_{ij} + V_{ij}\right) \\ \text{subject to} & 0 \leq X_{ij}^{(t)} + X_{jk}^{(t)} - X_{ik}^{(t)} \leq 1 \ \ (i < j < k;\ i, j, k \in N; t = 1,2) \\ & U_{ij} - V_{ij} = X_{ij}^{(2)} - X_{ij}^{(1)} \ \ (i < j;\ i, j \in N) \\ & \displaystyle\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{\left(A_{ij}^{(t)} - A_{ji}^{(t)}\right)X_{ij}^{(t)} + A_{ji}^{(t)}\right\} = \bar{M}_{(t)} \ \ (t = 1,2) \\ & X_{ij}^{(t)} \in \{0,1\} \ \ (i < j; i, j \in N; t = 1,2) \\ & U_{ij} \geq 0, V_{ij} \geq 0 \ \ (i < j;\ i, \in N) \end{array}\right. \tag{23}$$

The first and fourth constraints form the same set of constraints in (19), and correspond to the constraints $\boldsymbol{\pi}_{(t)} \in \Pi$ $(t = 1,2)$ in (21). Then, $\mathbf{X}^{(1)} = \left(X_{ij}^{(1)}\right)$ and $\mathbf{X}^{(2)} = \left(X_{ij}^{(2)}\right)$ consistently represent two permutations of sectors. The second and fifth constraints and the objective function compose the technique to minimize the difference between $\boldsymbol{\pi}_{(1)}$ and $\boldsymbol{\pi}_{(2)}$, or $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|X_{ij}^{(2)} - X_{ij}^{(1)}\right|$. The third constraint corresponds to the constraints $\ell\left(\mathbf{A}^{(t)}\left(\boldsymbol{\pi}_{(t)}\right)\right) = \bar{M}_{(t)}$ $(t = 1,2)$ in (21), and guarantees that the degrees of linearity are equal to the best attainable ones.

The advantage of the program (23), as a specific form of the general program

(21), over other forms is at least twofold. One is that the objective function and constraints, except for integralities, are linear in variables, as is already explained. Because our strategy to solve the triangulation problem is not to develop specialized algorithms but to use a general algorithm, which is implemented in prevailing software, the linearity is almost a prerequisite. Therefore, we do not specify "the difference between $\boldsymbol{\pi}_{(1)}$ and $\boldsymbol{\pi}_{(2)}$" as a nonlinear formula, e.g., the sum of squared differences between the rankings, $\sum_{i=1}^{n}\left(\pi_{(1)}^{-1}(i)-\pi_{(2)}^{-1}(i)\right)^2$.

The other advantage is the consistency with the Kendall rank correlation coefficient. Note that, for any pair (i,j) such that $\neq j$ $(i,j \in N)$,

$$X_{ij}^{(2)}-X_{ij}^{(1)}=\begin{cases}1 & \text{if } \bar{\pi}_{(1)}{}^{-1}(i)<\bar{\pi}_{(1)}{}^{-1}(j),\ \bar{\pi}_{(2)}{}^{-1}(i)>\bar{\pi}_{(2)}{}^{-1}(j)\\ -1 & \text{if } \bar{\pi}_{(1)}{}^{-1}(i)>\bar{\pi}_{(1)}{}^{-1}(j),\ \bar{\pi}_{(2)}{}^{-1}(i)<\bar{\pi}_{(2)}{}^{-1}(j)\\ 0 & \text{otherwise}\end{cases} \tag{24}$$

because $X_{ij}^{(t)}=1\{\pi_{(t)}^{-1}(i)\geq\pi_{(t)}^{-1}(j)\}$ $(i,j\in N, t=1,2)$ by (16). Thus, the objective function of the proposed program (23), $Q=\sum_{i<j}\left|X_{ij}^{(2)}-X_{ij}^{(1)}\right|=\sum_{i<j}(U_{ij}+V_{ij})$, can be interpreted as the number of pairs (i,j) of sectors that disagree in the two rankings. Note also that the Kendall rank correlation coefficient between $\boldsymbol{\pi}_{(1)}$ and $\boldsymbol{\pi}_{(2)}$ is given by $\tau(\boldsymbol{\pi}_{(1)},\boldsymbol{\pi}_{(2)})=1-4Q/n(n-1)$ (see, for example, Kendall and Gibbons (1990), p. 5). It has thus been shown that the proposed program (23) maximizes the Kendall rank correlation coefficient between two sequences of sectors, guaranteeing that the degrees of linearity are equal to the best attainable ones.

A variant of the program (23) that is consistent with the Spearman rank correlation coefficient can also be formulated by specifying its objective function as the sum of squared differences between the rankings, $\sum_{i=1}^{n}\left(\pi_{(1)}^{-1}(i)-\pi_{(2)}^{-1}(i)\right)^2$. However, we will not employ this nonlinear formulation because a linear formulation is much preferred to nonlinear ones, as explain above.

We solved the proposed program (23) for the three pairs of years. The linear program relaxation provided integer solutions for the three pairs, similarly to the results in the previous section. The Kendall rank correlation coefficients for these pairs were

$\tau(1990{,}1995) = 0.914$, $\tau(1995{,}2000) = 0.957$ and $\tau(1990{,}2000) = 0.902$. These coefficients were much improved, or increased, by about 30%, compared to the case where the three IOTs were independently triangulated. These results seem to indicate that the hierarchies are quite similar and have only slightly changed during the period 1990–2000. Figures 2–4 show the detailed results of pair-wise optimization in the form of a migration diagram. It is found that the top 26 to 29 sectors agree in the two rankings for all the three pairs. This implies that the drastic changes of hierarchies which appear in the form of many crosses in the upper half of Figure 1 are not meaningful, and approximately the top 30 sectors have remained in their top positions over time. Such a stability of the rankings is not limited to those top-ranked sectors. The deference between the maximum and minimum of the rankings is less than 10, for 92 to 95 of the 99 sectors for all three pairs.

As shown in Figure 4, '#85 Public administration', '#20 Publishing, printing', '#30 Coal products', '#8 Coal mining', and '#54 Semiconductor devices and integrated circuits' sectors changed their positions in the rankings to a great extent from 1990 to 2000. Only the sector '#99 Activities not elsewhere classified' inputs the product of '#85 Public administration' sector, and the input coefficient, $A_{85,99}$, increased during 1990–2000. 'Public administration' sector was thus located just after 'Activities not elsewhere classified' sector in the ranking for 1995 and 2000 when these years were paired with 1990.

A significant amount of products of '#20 Publishing, printing' sector was purchased by 22 and 23 sectors, respectively in 1990 and 1995, while only by 17 sectors in 2000. The average of the relevant input coefficients, $A_{20,j}$ $(j \in N)$, decreased during 1990–2000, which seems due partly to the trend toward a paperless business. In addition, the amount of sector '#18 Pulp, paper, paperboard, building paper' used by 'Publishing, printing' sector represented by the input coefficient, $A_{18,20}$, increased during the period, between 1995 and 2000, in particular. 'Publishing, printing' sector was thus located just before 'Pulp, paper, paperboard, building paper' sector in the ranking for 2000 when it was paired with 1990 and 1995.

Sector '#30 Coal products' was located just after sector '#38 Pig iron and crude steel' in the ranking for 1990 because the latter sector is one of the largest consumers of coal products. The relevant input coefficient, $A_{30,38}$, decreased greatly, and its values were 0.069, 0.059 and 0.041 for 1990, 1995 and 2000, respectively. This change is due partly to the increase of recycling of waste plastics as substitutes for coal and coke, and results in a gradual upward change in the position of the 'Coal products' sector. In addition, sector '#8 Coal mining' was the second or third largest consumer of the service produced by sector '#76 Railway transport' during 1990–2000. This may be one of the reasons why 'Coal products' and 'Coal mining' sector changed their positions and were located just before 'Railway transport' sector in 2000.

'#54 Semiconductor devices and integrated circuits' sector was located just before '#32 Rubber products' sector in 1990, and then their positions were reversed in 1995 and 2000. The relevant input coefficient, $A_{32,54}$, decreased considerably, with values of 0.0132, 0.0071 and 0.0045 for 1990, 1995 and 2000, respectively. In addition, sector '#54 Semiconductor devices and integrated circuits' was located after '#63 Miscellaneous manufacturing products' and '#56 Heavy electrical equipment' sectors in 1995 and 2000. The relevant input coefficients, $A_{54,63}$ and $A_{54,56}$, increased significantly so that those values in 1995 and 2000 were 1.7 to 3.7 times as large as those in 1990. This change seems due partly to the trend of widespread use of embedded microcomputers.

To sum up, the hierarchies among sectors are fairly stable during the period 1990–2000. We can pick up substantial and meaningful changes in rankings of sectors by applying the method newly proposed for inter-temporal comparisons of triangulated IOTs.

3.2 Block triangularity

The computational burden for triangulating an IOT can be significantly reduced by grouping sectors into blocks according to their physical characteristics, and then triangulating each block separately, as suggested by Simpson and Tsukui (1965) and pointed out by Östblom (1993). In addition, the sequences of sectors obtained by triangulating an IOT may be interpreted more easily if the sectors are grouped into several blocks, because objective methods for triangulation with no grouping might sometimes provide results which are too sensitive to small differences among values in the IOT. Simpson and Tsukui (1965) analyzed the production structures of developed countries mainly by grouping industrial sectors into four blocks, Metal, Non-metal, Energy and Services, while they pointed out that the triangularity is improved if sectors are grouped into six blocks, where two blocks, Metal and Non-metal, are disaggregated into four categories, Metal final, Metal basic, Non-metal final and Non-metal basic. Östblom (1993) grouped sectors into five blocks, which are the same as the six blocks that Simpson and Tsukui (1965) suggested, except that Metal final and Metal basic are gathered into one category, in analyzing the Swedish economy.

We grouped the 99 industrial sectors into the following eight blocks: Mining, Construction (except for repair), Metal, Non-metal, Utilities (energy and water supply), Business services, Personal services and Others. The difference in grouping between this study and the previous studies is partly due to the difference in the number of available sectors and the characteristics of the Japanese economy and IOTs. For instance, some service sectors, such as restaurants and hotels, are totally consumed by final demand sectors, so that we define the two different blocks of service industries, while all the service sectors were grouped into one block in the previous studies. The demand for building construction and civil engineering is classified not as intermediate demand but as final demand in Japanese IOTs, because it is an investment, so that we define a block for construction sectors excluding repair, while Östblom (1993) classified the construction sector as a service industry.

We then propose two new methods to investigate block dependence and independence based on the grouping of sectors defined above. The first method is designed to check how closely sectors grouped in the same blocks can be gathered, guaranteeing that the degree of linearity does not become much worse than the best attainable one. Define a partition of the set of sectors, N, into $S+1$ mutually exclusive and exhaustive subsets, B_s $(s = 0,1,\dots,S)$, such that $B_0 \cup B_1 \cup \cdots \cup B_S = N$, and $B_0 \cap B_1 \cap \cdots \cap B_S = \emptyset$. For the grouping explained above, $S = 7$ and B_0 corresponds to Others.

Suppose that we have solved the triangulation problem (19), given an IOT, and then obtained an optimal solution $\bar{X}_{ij}$, the corresponding optimal permutation $\bar{\boldsymbol{\pi}}$, and the optimal value $\bar{M}$ of the program. Letting $|B|$ denote the number of elements of a set B, we propose the following mixed integer program to investigate how clearly a given grouping can be represented by a block-triangular structure.

$$
\left|
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{s=1}^{S} D_s/(|B_s|-1) \\
\text{subject to} & 0 \leq X_{ij} + X_{jk} - X_{ik} \leq 1 \ \ (i<j<k;\ i,j,k \in N) \\
& \displaystyle W_i = \sum_{j=1}^{i-1}\left(1 - X_{ji}\right) + \sum_{j=i+1}^{n} X_{ij} \ \ (j \in N) \\
& W_i - W_j \leq D_s \ \ (i,j \in B_s;\ \ s = 1,\dots,S) \\
& \displaystyle\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{\left(A_{ij} - A_{ji}\right)X_{ij} + A_{ji}\right\} \geq \alpha\bar{M} \\
& X_{ij} \in \{0,1\} \ \ (i<j; i,j \in N)
\end{array}
\right.
\tag{25}
$$

where α is a constant and $0 < \alpha \leq 1$.

The first and fifth constraints form the same set of constraints as in (19), the basic form of the triangulation problem. The fourth constraint plays a role to guarantee that the degree of linearity achieves, at least, the prescribed level, $\alpha\bar{M}$; we set $\alpha = 1.0$ or 0.95 in our empirical analysis below. Regarding the newly introduced variables, W_i's, in the second constraint of (25), it should first be noted that $\pi^{-1}(i) = W_i + 1 \ (i \in N)$ because $\pi^{-1}(i) = \sum_{j=1}^{n} X_{ij}$ by (20) by letting $X_{ii} = 1 \ (i \in N)$ and $X_{ji} = 1 - X_{ij} \ (i < j; i,j \in N)$. Thus, D_s in the third constraint of (25) gives an upper bound on the differences between rankings of sectors which are grouped in the s-th block, B_s. Because the inequality $D_s \geq |B_s| - 1$ holds, the optimal value of the program (25) cannot be smaller than the number of well-defined groups, *S*, which refers to the number of groups except for the group 'Others'. The optimal value attaining its lower bound *S* corresponds to the case that sectors can be arranged in a perfectly block-triangular form, where sectors grouped in the same block are gathered most compactly and any sectors grouped in a different block are not located between them. The optimal value divided by *S* may thus be used as a numerical measure of the degree of block-triangularity.

The second method that we propose is to explore blocks which are nearly mutually block-independent, without expert knowledge of industrial structures or physical characteristics of products. Therefore, the second method does not employ the $S+1$ blocks defined above at all. Let us explain our basic idea to develop the new method with a numerical example: $n = 5$ and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}.$$

Note that six of $n! = 120$ permutations, $(1,2,3,4,5)$, $(1,3,2,4,5)$, $(1,3,4,2,5)$, $(3,1,2,4,5)$, $(3,1,4,2,5)$ and $(3,4,1,2,5)$, are optimal solutions to the basic triangulation problem (19), and the corresponding optimal value is 15. In addition, two of the six optimal solutions, $(1,2,3,4,5)$ and $(3,4,1,2,5)$, display the block-triangularity of $\mathbf{A}$ as

$$\mathbf{A}\big((1,2,3,4,5)\big) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}, \quad \mathbf{A}\big((3,4,1,2,5)\big) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix},$$

while the other four optimal solutions do not, for example,

$$\mathbf{A}\big((1,3,4,2,5)\big) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}, \quad \mathbf{A}\big((3,1,4,2,5)\big) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}.$$

We here omit explicitly writing matrices for the other two optimal solutions, to save space.

We now introduce an $n \times n$ matrix $\mathbf{Y} = (Y_{ij})$ such that all the elements on the main diagonal and in the upper triangular part are zeros, i.e., $Y_{ij} = 0 \ (i \le j; i,j \in N)$, and elements in the lower triangular part are equal to the difference between the row and column indices, i.e., $Y_{ij} = i - j \ (i > j; i,j \in N)$. For our numerical example with $n = 5$,

$$\mathbf{Y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}.$$

Letting $\beta(\boldsymbol{\pi}) = \mathbf{1}_n^{\mathrm{T}}(\mathbf{Y} \odot \mathbf{A}(\boldsymbol{\pi}))\mathbf{1}_n$, it can be shown that $\beta(\boldsymbol{\pi}) = 33$ for the two permutations, $\boldsymbol{\pi} = (1,2,3,4,5)$ and $(3,4,1,2,5)$, which display the block-triangularity of $\mathbf{A}$, and $\beta(\boldsymbol{\pi}) > 33$ for the other four optimal solutions. Therefore, we formulate our method to explore blocks which achieve block-triangularity as

$$\left|\begin{array}{ll} \text{minimize} & \beta(\boldsymbol{\pi}) \\ \text{subject to} & \ell(\mathbf{A}(\boldsymbol{\pi})) \geq \alpha \bar{M}, \\ & \boldsymbol{\pi} \in \Pi, \end{array}\right. \tag{26}$$

where $\bar{M} = \max\{\ell(\mathbf{A}(\boldsymbol{\pi}))|\boldsymbol{\pi} \in \Pi\}$ and $\alpha \in (0,1]$ is a constant as in (25). Some remarks on the specification of the matrix $\mathbf{Y}$ are due before moving on to a representation of (26) as a mixed integer program. An indispensable characteristic of the matrix $\mathbf{Y}$ is that it is lower triangular, and $Y_{hi} < Y_{jk}$ if $h - i < j - k$ for any element in the lower triangular part. In other words, $Y_{ij} = \phi(i-j)$ with an arbitrary (strictly) increasing function ϕ may be a valid specification, e.g., $Y_{ij} = 10^{i-j}$. However, a linear function is preferred for specifying ϕ, to make the representation of (26) be linear in variables, as presented below. It should also be noted that the program (26) does not always succeed in finding an optimal solution which displays block-triangularity, while it did work well for the numerical example above. For instance, if the elements of $\mathbf{A}$, $A_{52} = A_{54} = 3$, are replaced with $A_{52} = A_{54} = 6$, the optimal solution to (26) is $\boldsymbol{\pi} = (1,3,4,2,5)$, but it is not a block-triangular sequence of sectors. Triangulating only the top four sectors can solve this difficulty because the program (26) suffers from the elements in the fifth row, which are not block-independent of the others. In a real IOT, blocks such as Mining, Utilities and Business services may prevent the program from finding a block-triangular sequence. It may thus be a reasonable solution to triangulate a part of the IOT which consists of only the sectors that do not belong to these blocks.

Introducing new variables Z_{ij}'s, we can formulate the following mixed integer program to investigate the block-triangularity of an IOT.

$$\left|\begin{array}{ll}\text{minimize} & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}Z_{ij} \\ \text{subject to} & 0 \le X_{ij} + X_{jk} - X_{ik} \le 1 \ \ (i < j < k;\ i,j,k \in N) \\ & W_i = \displaystyle\sum_{j=1}^{i-1}\left(1 - X_{ji}\right) + \sum_{j=i+1}^{n} X_{ij} \ \ (j \in N) \\ & Z_{ij} \ge W_i - W_j \ \ (i \ne j;\ i,j \in N) \\ & \displaystyle\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{\left(A_{ij} - A_{ji}\right)X_{ij} + A_{ji}\right\} \ge \alpha\bar{M} \\ & X_{ij} \in \{0,1\} \ \ (i < j; i,j \in N) \\ & Z_{ij} \ge 0 \ \ (i \ne j;\ i,j \in N)\end{array}\right. \tag{27}$$

The first, fourth and fifth constraints restrict X_{ij}'s to representing valid permutations of sectors with which the degrees of linearity are greater than a prescribed level, in common with the program (25). The second constraint is also the same as the one in the program (25), so that $W_i = \pi^{-1}(i) - 1 \ \ (i \in N)$. In order to understand how $\mathbf{Z} = \left(Z_{ij}\right)$ works in the program (27), the relationship between $\mathbf{L}$ and $\mathbf{X}$ should be recalled such that $\ell\left(\mathbf{A}(\boldsymbol{\pi})\right) + \sum_{i=1}^{n} A_{ii} = \mathbf{1}_n^{\mathrm{T}}\left(\mathbf{L} \odot \mathbf{A}(\boldsymbol{\pi})\right)\mathbf{1}_n = \mathbf{1}_n^{\mathrm{T}}(\mathbf{X} \odot \mathbf{A})\mathbf{1}_n$, as expressed in (10) and (15). That is, the objective function of the basic triangulation problem can be written with $\mathbf{X}$ instead of $\mathbf{L}$ and $\boldsymbol{\pi}$. The objective function, $\beta(\boldsymbol{\pi})$, of the program (26) can similarly be written with $\mathbf{Z}$ instead of $\mathbf{Y}$ and $\boldsymbol{\pi}$, given the third and sixth constraints in (27). Assuming that $A_{ij} \ge 0 \ \ (i,j \in N)$, the new variable Z_{hk} can be rewritten as follows, because $W_i - W_j = \pi^{-1}(i) - \pi^{-1}(j)$ $(i,j \in N)$:

$$Z_{ij} = \begin{cases} 0 & \left(\pi^{-1}(i) \le \pi^{-1}(j)\right) \\ \pi^{-1}(i) - \pi^{-1}(j) & \left(\pi^{-1}(i) > \pi^{-1}(j)\right) \end{cases}$$

or equivalently, the relationship between $\mathbf{Z}$ and $\mathbf{Y}$ can be written as $Z_{ij} = Y_{\pi^{-1}(i)\pi^{-1}(j)}$, which looks the same as the relationship between $\mathbf{X}$ and $\mathbf{L}$ in (16). Therefore, $\beta(\boldsymbol{\pi})$ can be written as $\beta(\boldsymbol{\pi}) = \mathbf{1}_n^{\mathrm{T}}\left(\mathbf{Y} \odot \mathbf{A}(\boldsymbol{\pi})\right)\mathbf{1}_n = \mathbf{1}_n^{\mathrm{T}}(\mathbf{Z} \odot \mathbf{A})\mathbf{1}_n$.

Turning to our empirical analyses, we merged the three blocks, Mining, Utilities and Business services, into one block B_L, and restricted them to positioning at the bottom of the IOT, by letting $X_{ij} = 0 \ \ (i < j;\ i \notin B_L, j \in B_L)$ and $X_{ij} = 1 \ \ (i < j;\ i \in$

$B_L, j \notin B_L$), to alleviate the difficulty pointed to above. Then, for the first method represented as the program (25), we set $S = 4$, and defined five blocks, Metal, Non-metal, Construction (except for repair), Personal services and Others, in addition to the block of low degree of fabrication, B_L, as shown in Table 1; see also Table B1 of Appendix B.

The input coefficient matrix of the year 2000 that is triangulated following the obtained optimal solution to (19) is shown in Figure B1 of Appendix B, and the ranking of sectors is presented in the column labeled 'Org' in Table 1. This is a base case in which no blocks are taken into account. The input coefficient matrix shown in Figure B1 has 1073 elements which are larger than or equal to $1/n$, of $n^2 = 9801$ elements. The degree of linearity is $\bar{\lambda} = 0.9459$, the largest element in the upper triangular part is 0.1344, and only seven elements are larger than 0.05 in the upper triangular part.

We solved the program (25) with the input coefficient matrix of the year 2000, with $\alpha = 1.0$ and 0.95. The optimality was unfortunately not verified within several hours for each value of α, while some integer solutions were at least found. The values of the objective function evaluated at the best integer solutions are 11.586 and 4.1128 for the cases with $\alpha = 1.0$ and 0.95, respectively, and the corresponding best bounds are 11.226 and 4.0, respectively. Because the objective value of the program (25) is bounded at 4.0 from below for any IOT, it almost achieves the lower bound for the case with $\alpha = 0.95$, while it does not for the case with $\alpha = 1.0$. The rankings of sectors corresponding to the best integer solutions are reported in the columns labeled 'G100' and 'G095' in Table 1. In the best integer solutions for the case with $\alpha = 0.95$, the four prescribed blocks, Personal services, Metal, Non-metal and Construction, were gathered so compactly that these blocks did not overlapp except for in the 36th to 38th sectors, where Metal and Non-metal blocks slightly overlapped. The input coefficient matrix that is triangulated following this solution is shown in Figure B2 of Appendex B. Considering that the optimality was not verified and looking at the individual elements, we found that these sectors can be interchanged in the sequence, which results in the one presented in the column labeled 'G.mod' in Table 1; see also Table B2 of Appendix B for the sequence of sectors. Therefore, sectors which belong to the same prescribed blocks can be gathered to constitute those blocks, without distorting the triangularity in the sense that the degree of linearity is kept larger than $\alpha\bar{\lambda} = 0.95 \times 0.9459$. In contrast, the objective value, 11.586, and the corresponding best bound, 11.226, are much larger than the lower bound, 4.0, for the case with $\alpha = 1.0$, so that the prescribed blocks substantially overlap.

We also solved the program (27) to explore what blocks can be formed without defining any blocks in advance. An optimal solution was found and the optimal value was 204.24 for the case with $\alpha = 1.0$. However, the difference between this optimal value and the corresponding value, 286.46, of the base case is not so large. This result implies that the block-triangular structure of the IOT may not be understood if we persist with the best attainable degree of linearity. In contrast, while the optimality was not verified within several hours for the case with $\alpha = 0.95$, the objective value was substantially improved, i.e., the value of the objective function evaluated at the best integer solution was 93.133 and the corresponding best bound was 25.221. Compared to the base case, these objective values imply that positive elements in the lower triangular part, except for the bottom block B_L, can be moved to positions close to the main diagonal, and may indicate that blocks are formed without needing to define any blocks in advance, at a little cost of the degree of linearity.

The input coefficient matrix that is triangulated following the solution to (27) for the case with $\alpha = 0.95$ is shown in Figure 5, and the sequence of sectors is presented in the column labeled 'E095' in Table 1; see also Table B3 of Appendix B. In Figure 5, several blocks are clearly displayed: some of them are mutually block-independent and others overlap, as summarized in the right-most column in Table 1. The fourth to twelfth sectors constitute a block of food, agriculture and fishery; we call it Block A. This block shows an advantage of the new method because the block includes hotels and restaurants (sectors #95 and #96) as well as several primary and tertiary sectors, which are classified in different blocks in the previous studies. In other words, such a block is difficult to find out without the new method proposed in this paper. Among the 17th to 38th sectors, the three blocks, 'B. Leather and textile product', 'C. Paper products', and 'D. Plastic, rubber, coal and petroleum products' were found. These three blocks are mildly dependent upon each other because of products made from different materials, such as clothes made of natural and chemical fiber, and office supplies made of paper and plastics. The overlap of the two blocks 'G. Machinery including ship and metal products, and building construction' and 'H. Wooden products' may come about for a similar reason, as both metal and wooden furniture are popular. It seems, in addition, that '#50 Household electronic and electric appliances' sector is located at the 24th position because its products are basically a complex combination of different materials such as iron and steel, copper, aluminum, and plastics. It is thus shown that the newly proposed method can find out blocks which possibly overlap and are difficult to extract by the existing methods, with little expert knowledge.

4. Concluding remarks

This paper has proposed a new method for the triangulation of input-output tables (IOTs), which is usually defined as a combinatorial optimization problem to find a permutation of sectors which maximizes the sum of elements in the lower triangular part of a given IOT from among the set of all permutations. We have employed a representation of the triangulation problem as an integer program along with existing algorithms which can be applied to general integer programs. At the cost of computational efficiency to some degree, we can easily modify and extend the triangulation problem by adding constraints and changing the objective function. This allows us to take account of relevant issues which are difficult to tackle with existing algorithms that are designed specific to just the triangulation problem. We have proposed modifications to consider both inter-temporal comparison and block-triangulation. The method for inter-temporal comparison provides sequences of sectors which are mutually as close as possible, and which is consistent to the Kendall rank correlation coefficient (a numerical measure of similarity of the sequences). The two methods for block-triangulation can be used, respectively, to evaluate the degree of block-triangularity of blocks which are defined based on experts' knowledge in advance, and to explore what blocks can be formed without experts' knowledge. The application of the proposed methods to the Japanese input-output tables has demonstrated their utility and illustrated how they provide insight into the hierarchical structure of sectors.

An important future direction for research is to acquire experience by applying the methods proposed in this paper to real world data, learning from practical problems which we might encounter, and investigating the characteristics of the methods. Modifications and further extensions of the methods based on experiences acquired are desirable. Applications of the methods to datasets of inter-sector material flows and energy flows as well as the standard monetary input-output tables will provide a useful tool to visualize how sectors are inter-related from various perspectives.

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Table 1. Rankings of sectors based on block-triangulation

Sector		Org	G100	G095	G.mod	Pre-defined blocks	E100	E095	Block found out
97	Other personal services	41	38	8	8	4 Personal services	8	1	
94	Amusement and recreational services	30	25	7	7	4 Personal services	2	2	
13	Tobacco	1	15	44	44	1 Non-metal	4	3	
96	Hotel and other lodging places	7	37	3	3	4 Personal services	6	4	A. Food, agriculture and fishery
95	Eating and drinking places	3	18	5	5	4 Personal services	12	5	A. Food, agriculture and fishery
11	Beverage	9	39	49	49	1 Non-metal	13	6	A. Food, agriculture and fishery
2	Livestock	8	16	39	39	1 Non-metal	16	7	A. Food, agriculture and fishery
12	Feeds and organic fertilizer, n.e.c.	15	20	40	40	1 Non-metal	17	8	A. Food, agriculture and fishery
10	Foods	29	40	62	62	1 Non-metal	18	9	A. Food, agriculture and fishery
5	Fisheries	13	19	37	**38**	1 Non-metal	14	10	A. Food, agriculture and fishery
1	Crop cultivation	42	44	63	63	1 Non-metal	26	11	A. Food, agriculture and fishery
3	Agricultural services	43	46	64	64	1 Non-metal	27	12	A. Food, agriculture and fishery
88	Medical service, health, social security and nursing care	4	34	4	4	4 Personal services	7	13	
27	Medicaments	46	47	65	65	1 Non-metal	28	14	
34	Glass and glass products	55	50	66	66	1 Non-metal	43	15	
89	Other public services	5	24	9	9	4 Personal services	5	16	
33	Leather, fur skins and miscellaneous leather products	14	17	45	45	1 Non-metal	15	17	B. Leather and textile products
15	Wearing apparel and other textile products	6	28	46	46	1 Non-metal	21	18	B. Leather and textile products
14	Textile products	39	30	53	53	1 Non-metal	25	19	B. Leather and textile products
20	Publishing, printing	59	52	51	51	1 Non-metal	46	20	C. Paper products
36	Pottery, china and earthenware	56	49	71	71	1 Non-metal	44	21	
98	Office supplies	2	13	42	42	1 Non-metal	22	22	C. Paper products
19	Paper products	58	51	50	50	1 Non-metal	45	23	C. Paper products
50	Household electronic and electric appliances	21	11	17	17	2 Metal	35	24	
18	Pulp, paper, paperboard, building paper	60	54	57	57	1 Non-metal	47	25	C. Paper products
63	Miscellaneous manufacturing products	51	36	47	47	1 Non-metal	34	26	
28	Final chemical products, n.e.c.	66	56	69	69	1 Non-metal	50	27	D. Plastic, rubber, coal and petro-
32	Rubber products	38	22	48	48	1 Non-metal	24	28	D. leum products
31	Plastic products	72	60	52	52	1 Non-metal	58	29	D. Plastic, rubber, coal and …
26	Synthetic fibers	40	31	54	54	1 Non-metal	30	30	D. Plastic, rubber, coal and …
25	Synthetic resins	73	62	55	55	1 Non-metal	65	31	D. Plastic, rubber, coal and …
24	Organic chemical products	80	71	56	56	1 Non-metal	69	32	D. Plastic, rubber, coal and …
21	Chemical fertilizer	61	48	58	58	1 Non-metal	64	33	D. Plastic, rubber, coal and …
30	Coal products	16	70	67	67	1 Non-metal	70	34	D. Plastic, rubber, coal and …
23	Basic organic chemical products	81	73	60	60	1 Non-metal	72	35	D. Plastic, rubber, coal and …

Table 1. Rankings of sectors based on block-triangulation (continued)

Sector	Org	G100	G095	G.mod	Pre-defined blocks	E100	E095	Block found out
29 Petroleum refinery products	90	74	68	68	1 Non-metal	73	36	D. Plastic, rubber, coal and ...
62 Precision instruments	36	6	14	14	2 Metal	29	37	E. Transportation equipments ...
22 Basic inorganic chemical products	75	65	70	70	1 Non-metal	66	38	D. Plastic, rubber, coal and ...
49 Machinery for office and service industry	17	14	13	13	2 Metal	23	39	E. Transportation equipments
53 Applied electronic equipment and electric measuring instruments	23	9	12	12	2 Metal	37	40	E. other than ships and preci-
52 Communication equipment	50	8	15	15	2 Metal	40	41	E. sion, electronic and electric
51 Electronic computing equipment and accessory equipment	19	7	20	20	2 Metal	38	42	E. appliances
54 Semiconductor devices and integrated circuits	53	43	28	28	2 Metal	41	43	E. Transportation equipments ...
55 Electronic components	54	45	29	29	2 Metal	42	44	E. Transportation equipments ...
58 Passenger motor cars	20	4	16	16	2 Metal	10	45	E. Transportation equipments ...
59 Other cars	37	12	21	21	2 Metal	11	46	E. Transportation equipments ...
57 Other electrical equipment	67	57	30	30	2 Metal	51	47	E. Transportation equipments ...
75 House rent	28	26	6	6	4 Personal services	3	48	
66 Public construction	11	3	74	74	3 Construction	53	49	F. Civil engineering
67 Other civil engineering and construction	26	2	73	73	3 Construction	54	50	F. Civil engineering
35 Cement and cement products	49	41	41	41	1 Non-metal	59	51	F. Civil engineering
65 Repair of construction	47	32	11	11	0 Others	31	52	
47 Special industrial machinery	18	5	22	22	2 Metal	19	53	G. Machinery including ship and
46 General industrial machinery	44	23	24	24	2 Metal	36	54	G. metal products, and build-
56 Heavy electrical equipment	52	29	25	25	2 Metal	39	55	G. ing construction
64 Building construction	24	1	72	72	3 Construction	32	56	G. Machinery including ship and ...
44 Metal products for construction and architecture	62	42	18	18	2 Metal	55	57	G. Machinery including ship and ...
48 Other general machines	63	35	26	26	2 Metal	52	58	G. Machinery including ship and ...
45 Other metal products	68	58	31	31	2 Metal	56	59	G. Machinery including ship and ...
43 Non-ferrous metal products	71	59	33	33	2 Metal	57	60	G. Machinery including ship and ...
42 Non-ferrous metals	76	68	35	35	2 Metal	67	61	G. Machinery including ship and ...
17 Furniture and fixtures	48	33	36	**37**	1 Non-metal	33	62	H. Wooden products
41 Other iron or steel products	69	61	32	32	2 Metal	61	63	G. Machinery including ship and ...
40 Cast and forged steel products	64	53	27	27	2 Metal	60	64	G. Machinery including ship and ...
39 Steel products	70	63	34	34	2 Metal	62	65	G. Machinery including ship and ...
60 Ships and repair of ships	35	21	23	23	2 Metal	20	66	G. Machinery including ship and ...
38 Pig iron and crude steel	74	64	38	**36**	2 Metal	63	67	G. Machinery including ship and ...
37 Other ceramic, stone and clay products	77	69	43	43	1 Non-metal	68	68	
16 Timber and wooden products	65	55	59	59	1 Non-metal	48	69	H. Wooden products
4 Forestry	86	72	61	61	1 Non-metal	49	70	H. Wooden products

Table 1. Rankings of sectors based on block-triangulation (continued)

Sector	Org	G100	G095	G.mod	Pre-defined blocks	E100	E095	Block found out
99 Activities not elsewhere classified	82	66	1	1	0 Others	71	71	
85 Public administration	84	67	10	10	4 Personal services	74	72	
61 Other transportation equipment and repair of transportation equipme	34	10	19	19	2 Metal	9	73	
86 Education	22	27	2	2	4 Personal services	1	74	
80 Freight forwarding	57	86	75	75	L 7 Business services	85	75	(kept at the bottom)
78 Water transport	83	83	83	83	L 7 Business services	83	76	(kept at the bottom)
71 Waste management service	32	77	78	78	L 6 Utilities	75	77	(kept at the bottom)
79 Air transport	10	84	82	82	L 7 Business services	84	78	(kept at the bottom)
81 Storage facility service	25	85	76	76	L 7 Business services	86	79	(kept at the bottom)
6 Metallic ores	78	75	81	81	L 5 Mining	76	80	(kept at the bottom)
7 Non-metallic ores	79	78	87	87	L 5 Mining	77	81	(kept at the bottom)
69 Gas and heat supply	27	76	84	84	L 6 Utilities	79	82	(kept at the bottom)
77 Road transport (except transport by private cars)	87	82	85	85	L 7 Business services	82	83	(kept at the bottom)
8 Coal mining	31	79	79	79	L 5 Mining	78	84	(kept at the bottom)
76 Railway transport	33	81	86	86	L 7 Business services	80	85	(kept at the bottom)
82 Services relating to transport	88	87	88	88	L 7 Business services	87	86	(kept at the bottom)
70 Water supply	45	80	80	80	L 6 Utilities	81	87	(kept at the bottom)
68 Electricity	89	88	89	89	L 6 Utilities	88	88	(kept at the bottom)
9 Crude petroleum and natural gas	91	89	90	90	L 5 Mining	89	89	(kept at the bottom)
84 Broadcasting	12	90	77	77	L 7 Business services	90	90	(kept at the bottom)
87 Research	85	91	91	91	L 7 Business services	91	91	(kept at the bottom)
91 Goods rental and leasing services	92	92	92	92	L 7 Business services	92	92	(kept at the bottom)
92 Repair of motor vehicles and machine	93	93	93	93	L 7 Business services	93	93	(kept at the bottom)
72 Commerce	94	94	94	94	L 7 Business services	94	94	(kept at the bottom)
83 Communication	95	95	95	95	L 7 Business services	95	95	(kept at the bottom)
74 Real estate agencies and rental services	96	96	96	96	L 7 Business services	96	96	(kept at the bottom)
73 Financial and insurance	97	97	97	97	L 7 Business services	97	97	(kept at the bottom)
90 Advertising, survey and information services	98	98	98	98	L 7 Business services	98	98	(kept at the bottom)
93 Other business services	99	99	99	99	L 7 Business services	99	99	(kept at the bottom)

Note: The input coefficient matrix obtained from the Japanese input-output table for the year 2000 is triangulated by the proposed methods. The column labeled ‘Org’ refers to the result based on the program (19), ‘E100’ and ‘E095’ to (25) with $\alpha = 1.0$ and 0.95, ‘G100’ and ‘G095’ to (27) with $\alpha = 1.0$ and 0.95, and ‘G.mod’ to an optimal ranking obtained by slightly modifying the one in the column labeled ‘G095’. The sectors are arranged following the ranking ‘E095’. See also Figure 1 and Tables B2 and B3 for the sequences of sectors following the rankings ‘Org’, ‘G.mod’ and ‘E095’, respectively.

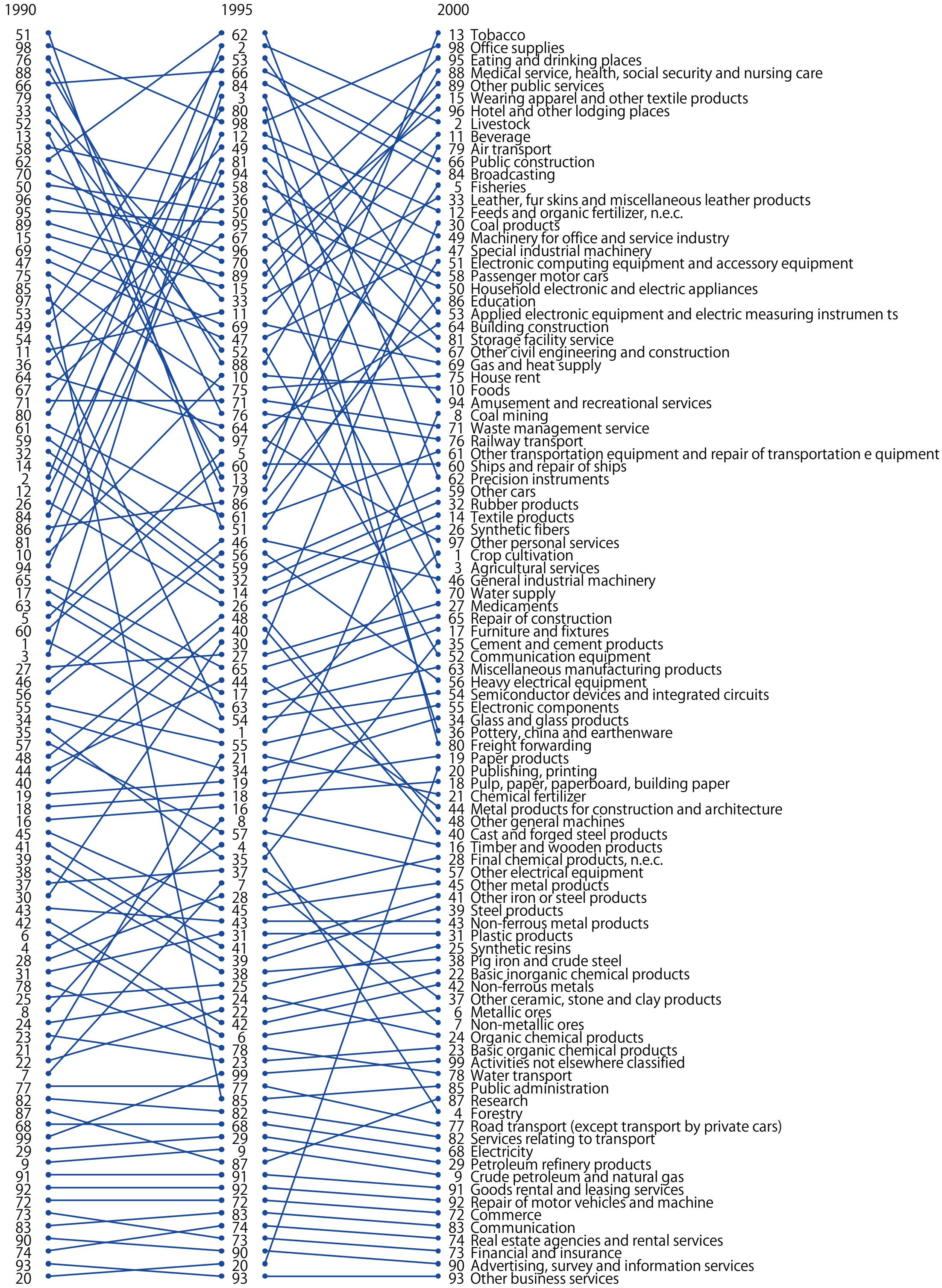

Figure 1. The rankings of sectors for the input coefficient matrices of the year 1990, 1995 and 2000 triangulated independently

Figure 2. The rankings of sectors for the input coefficient matrices of the year 1990 and 1995 triangulated to minimize the Kendall rank correlation coefficient

Figure 3. The rankings of sectors for the input coefficient matrices of the year 1995 and 2000 triangulated to minimize the Kendall rank correlation coefficient

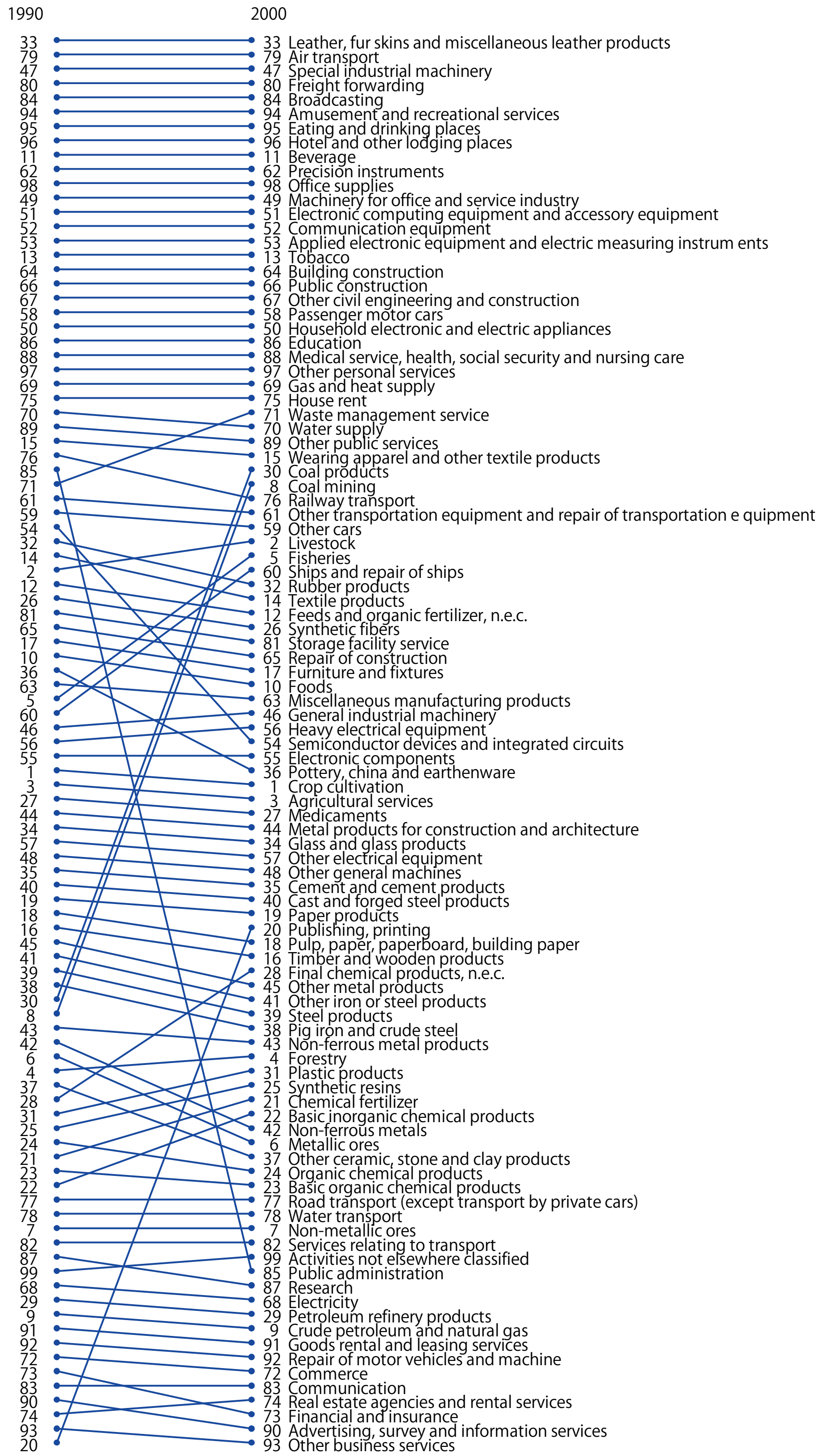

Figure 4. The rankings of sectors for the input coefficient matrices of the year 1990 and 2000 triangulated to minimize the Kendall rank correlation coefficient

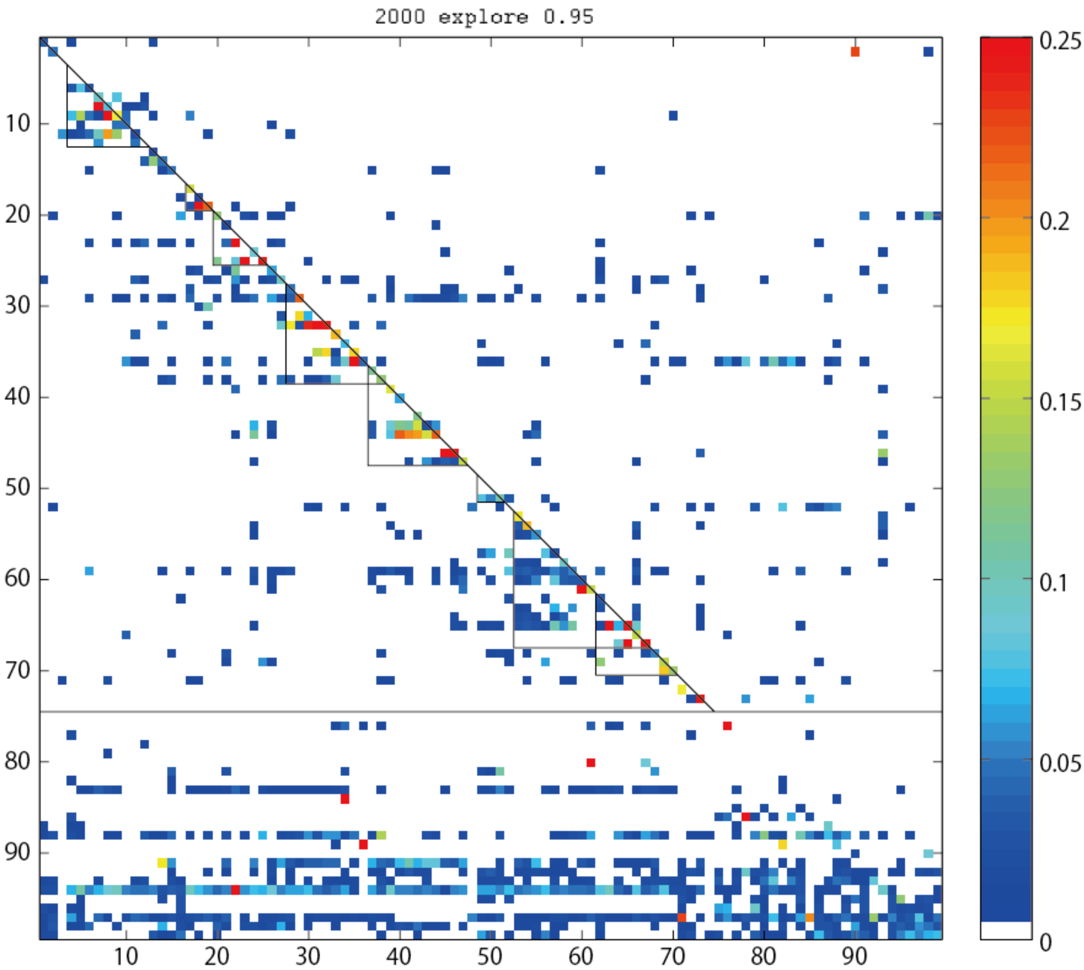

Figure 5. Block-triangulated Japanese input coefficient matrix of the year 2000.

Note: The sectors are arranged based on the best integer solution to the program (27) with $\alpha = 0.95$. Each small square refers to an element of the input coefficient matrix, whereby different colors are used to indicate its relative magnitude along the color scale on the right. The elements greater than 0.25 are displayed as if they are equal to 0.25.

Appendix A. Derivation of the system of inequalities which defines the feasible set of the triangulation problem

Let there be n industrial sectors. For ease of exposition, we define the set of natural numbers referring to these sectors as $N = \{1,2,\dots,n\}$. We then denote a permutation of n industrial sectors by $\boldsymbol{\pi} = \left(\pi(1), \pi(2), \dots, \pi(n)\right)$ and the set of all permutations of industrial sectors by Π. A permutation $\boldsymbol{\pi}$ can be represented by the corresponding permutation matrix $\mathbf{P} = \left(P_{ij}\right)$ such that $P_{ij} = 1\{i = \pi(j)\}$ $(i, j \in N)$, where $1\{\cdot\}$ is the indicator function such that $1\{Q\} = 1$ if the proposition Q is true and $1\{Q\} = 0$ otherwise. The following set is thus equivalent to Π.

$$\Pi_{\mathbf{P}} = \left\{\mathbf{P} \in \mathbb{R}^{n\times n} \middle| \mathbf{P}\mathbf{1}_n = \mathbf{1}_n,\ \mathbf{P}^{\mathrm{T}}\mathbf{1}_n = \mathbf{1}_n,\ P_{ij} \in \{0,1\}\ (i,j \in N)\right\}. \tag{A1}$$

In addition, the following set $\Pi_{\mathbf{X}}$ is also equivalent to Π.

$$\Pi_{\mathbf{X}} = \{\mathbf{X} \in \mathbb{R}^{n\times n} | \mathbf{X} = \mathbf{PLP}^{\mathrm{T}},\quad \mathbf{P} \in \Pi_{\mathbf{P}}\}, \tag{A2}$$

where $L_{ij} = 1\{i \geq j\}$ $(i, j \in N)$.

The following system of linear inequalities is known for properly defining $\Pi_{\mathbf{X}}$ in the literature (Grötschel, Jünger and Reinelt (1984, eqs. (10)–(12) on p. 1202), Chiarini, Chaovalitwongse and Pardalos (2004, eqs. (9)–(10) on p. 259)).

(F1) $X_{ii} = 1$ $(i \in N)$;

(F2) $X_{ij} + X_{ji} = 1$ $(i < j; i, j \in N)$; and

(F3) $0 \leq X_{ij} + X_{jk} - X_{ik} \leq 1$ $(i < j < k;\ i, j, k \in N)$.

This system can be derived by considering a binary relation on N as follows.

Definition 1. Let S be a set. A subset B of $S \times S$, i.e., $B \subset S \times S$ is called a binary relation on S.

Definition 2. A binary relation B on a set S is called a linear order if it is

reflexive, i.e., xBx $(x \in S)$;

anti-symmetric, i.e., $xBy, yBx \rightarrow x = y \ (x, y \in S)$;
transitive, i.e., $xBy, yBz \rightarrow xBz \ (x, y, z \in S)$; and
complete, i.e., xBy or $yBx \ (x, y \in S)$;

where xBy refers to $(x, y) \in B$.

We define a binary relation $R_{\mathbf{X}}$ on N, a given matrix $\mathbf{X}$, as follows.

$$R_{\mathbf{X}} = \{(j, i) \in N \times N \mid X_{ij} = 1\}. \tag{A3}$$

A proposition $jR_{\mathbf{X}}i$, or $(j, i) \in R_{\mathbf{X}}$, refers to the relationship that sector j precedes sector i. The binary relation $R_{\mathbf{X}}$ defined above is a linear order if $\mathbf{X}$ represents a permutation $\boldsymbol{\pi}$, i.e., $X_{ij} \ (i, j \in N)$ is given by $\mathbf{X} = \mathbf{PLP}^{\mathrm{T}}$ as in (A2), because the binary relation $\geq$ 'greater than or equal to' on N is a linear order and a permutation $\boldsymbol{\pi}$ is a bijection. Therefore, a $\{0,1\}$-matrix $\mathbf{X}$ satisfies the following conditions if and only if $\mathbf{X}$ represents a permutation $\boldsymbol{\pi}$.

(D1) reflexivity, i.e., $X_{ii} = 1 \ (i \in N)$;
(D2) anti-symmetricity, i.e., $X_{ij} = X_{ji} = 1 \rightarrow i = j \ (i, j \in N)$;
(D3) transitivity, i.e., $X_{ij} = X_{jk} = 1 \rightarrow X_{ik} = 1 \ (i, j, k \in N)$; and
(D4) completeness, i.e., $X_{ij} + X_{ji} \geq 1 \ (i, j \in N)$;

Consider the following system of inequalities:

(E1) $X_{ii} = 1 \ (i \in N)$;
(E2) $X_{ij} + X_{ji} = 1 \ (i \neq j; i, j \in N)$; and
(E3) $X_{ij} + X_{jk} - X_{ik} \leq 1 \ (i, j, k \in N)$.

Lemma 1. Let $\mathbf{X}$ be an $n \times n$ $\{0,1\}$-matrix, and $R_{\mathbf{X}}$ be the corresponding binary relation on N. Then $\mathbf{X}$ satisfies the system (E1)–(E3) if and only if $R_{\mathbf{X}}$ is a linear order, that is, $\mathbf{X}$ satisfies the system (D1)–(D4).

Proof. (if part) (E1) and (E2) hold obviously by (D1), (D2) and (D4). Now suppose (E3) is violated, that is, $X_{ij} + X_{jk} - X_{ik} > 1$ for some i, j and k. This implies

$X_{ij} + X_{jk} - 1 > X_{ik}$. This inequality further implies that $X_{ik} = 0$ if $X_{ij} = X_{jk} = 1$, which violates (D3).

(only if part) (D1), (D2) and (D4) hold obviously by (E1) and (E2). If $X_{ij} = X_{jk} = 1$ and (E3) holds, then $X_{ij} + X_{jk} - X_{ik} = 2 - X_{ik} \leq 1$. Thus $X_{ik} = 1$, which concludes.

□

In addition, it can be easily shown that the system (E1)–(E3) is equivalent to

(F1) $X_{ii} = 1$ $(i \in N)$;

(F2) $X_{ij} + X_{ji} = 1$ $(i < j; i, j \in N)$; and

(F3) $0 \leq X_{ij} + X_{jk} - X_{ik} \leq 1$ $(i < j < k;\ i, j, k \in N)$.

(E1) and (F1) are the same, and (E2) and (F2) are equivalent because the roles of the indices i and j are parallel. Note that the range of subscripts has changed.

Appendix B. Supplementary tables and figures

Table B1. Sector classification and pre-defined blocks

ID	Sector Name	ID	Block
1	Crop cultivation	1	Non-metal
2	Livestock	1	Non-metal
3	Agricultural services	1	Non-metal
4	Forestry	1	Non-metal
5	Fisheries	1	Non-metal
6	Metallic ores	L 5	Mining
7	Non-metallic ores	L 5	Mining
8	Coal mining	L 5	Mining
9	Crude petroleum and natural gas	L 5	Mining
10	Foods	1	Non-metal
11	Beverage	1	Non-metal
12	Feeds and organic fertilizer, n.e.c.	1	Non-metal
13	Tobacco	1	Non-metal
14	Textile products	1	Non-metal
15	Wearing apparel and other textile products	1	Non-metal
16	Timber and wooden products	1	Non-metal
17	Furniture and fixtures	1	Non-metal
18	Pulp, paper, paperboard, building paper	1	Non-metal
19	Paper products	1	Non-metal
20	Publishing, printing	1	Non-metal
21	Chemical fertilizer	1	Non-metal
22	Basic inorganic chemical products	1	Non-metal
23	Basic organic chemical products	1	Non-metal
24	Organic chemical products	1	Non-metal
25	Synthetic resins	1	Non-metal
26	Synthetic fibers	1	Non-metal
27	Medicaments	1	Non-metal
28	Final chemical products, n.e.c.	1	Non-metal
29	Petroleum refinery products	1	Non-metal
30	Coal products	1	Non-metal
31	Plastic products	1	Non-metal
32	Rubber products	1	Non-metal
33	Leather, fur skins and miscellaneous leather products	1	Non-metal
34	Glass and glass products	1	Non-metal
35	Cement and cement products	1	Non-metal
36	Pottery, china and earthenware	1	Non-metal
37	Other ceramic, stone and clay products	1	Non-metal
38	Pig iron and crude steel	2	Metal
39	Steel products	2	Metal
40	Cast and forged steel products	2	Metal
41	Other iron or steel products	2	Metal
42	Non-ferrous metals	2	Metal
43	Non-ferrous metal products	2	Metal
44	Metal products for construction and architecture	2	Metal
45	Other metal products	2	Metal
46	General industrial machinery	2	Metal
47	Special industrial machinery	2	Metal
48	Other general machines	2	Metal
49	Machinery for office and service industry	2	Metal
50	Household electronic and electric appliances	2	Metal
51	Electronic computing equipment and accessory equipment	2	Metal
52	Communication equipment	2	Metal
53	Applied electronic equipment and electric measuring instruments	2	Metal
54	Semiconductor devices and integrated circuits	2	Metal
55	Electronic components	2	Metal
56	Heavy electrical equipment	2	Metal
57	Other electrical equipment	2	Metal
58	Passenger motor cars	2	Metal
59	Other cars	2	Metal
60	Ships and repair of ships	2	Metal
61	Other transportation equipment and repair of transportation equipment	2	Metal
62	Precision instruments	2	Metal
63	Miscellaneous manufacturing products	1	Non-metal
64	Building construction	3	Construction
65	Repair of construction	0	Others
66	Public construction	3	Construction
67	Other civil engineering and construction	3	Construction
68	Electricity	L 6	Utilities
69	Gas and heat supply	L 6	Utilities
70	Water supply	L 6	Utilities
71	Waste management service	L 6	Utilities
72	Commerce	L 7	Business services
73	Financial and insurance	L 7	Business services
74	Real estate agencies and rental services	L 7	Business services
75	House rent	4	Personal services
76	Railway transport	L 7	Business services
77	Road transport (except transport by private cars)	L 7	Business services
78	Water transport	L 7	Business services
79	Air transport	L 7	Business services
80	Freight forwarding	L 7	Business services
81	Storage facility service	L 7	Business services
82	Services relating to transport	L 7	Business services
83	Communication	L 7	Business services
84	Broadcasting	L 7	Business services
85	Public administration	4	Personal services
86	Education	4	Personal services
87	Research	L 7	Business services
88	Medical service, health, social security and nursing care	4	Personal services
89	Other public services	4	Personal services
90	Advertising, survey and information services	L 7	Business services
91	Goods rental and leasing services	L 7	Business services
92	Repair of motor vehicles and machine	L 7	Business services
93	Other business services	L 7	Business services
94	Amusement and recreational services	4	Personal services
95	Eating and drinking places	4	Personal services
96	Hotel and other lodging places	4	Personal services
97	Other personal services	4	Personal services
98	Office supplies	1	Non-metal
99	Activities not elsewhere classified	0	Others

Note: The letter ‘L’ refers to blocks which are positioned at the bottom part of the input-output table when block-triangularity is taken into account.

Table B2. Rankings of sectors based on block-triangulation by the program (25)

Sector		G.mod	G095	G100	Org	Pre-defined blocks	
99	Activities not elsewhere classified	1	1	66	82	0	Others
86	Education	2	2	27	22	4	Personal services
96	Hotel and other lodging places	3	3	37	7	4	Personal services
88	Medical service, health, social security and nursing care	4	4	34	4	4	Personal services
95	Eating and drinking places	5	5	18	3	4	Personal services
75	House rent	6	6	26	28	4	Personal services
94	Amusement and recreational services	7	7	25	30	4	Personal services
97	Other personal services	8	8	38	41	4	Personal services
89	Other public services	9	9	24	5	4	Personal services
85	Public administration	10	10	67	84	4	Personal services
65	Repair of construction	11	11	32	47	0	Others
53	Applied electronic equipment and electric measuring instruments	12	12	9	23	2	Metal
49	Machinery for office and service industry	13	13	14	17	2	Metal
62	Precision instruments	14	14	6	36	2	Metal
52	Communication equipment	15	15	8	50	2	Metal
58	Passenger motor cars	16	16	4	20	2	Metal
50	Household electronic and electric appliances	17	17	11	21	2	Metal
44	Metal products for construction and architecture	18	18	42	62	2	Metal
61	Other transportation equipment and repair of transportation equipment	19	19	10	34	2	Metal
51	Electronic computing equipment and accessory equipment	20	20	7	19	2	Metal
59	Other cars	21	21	12	37	2	Metal
47	Special industrial machinery	22	22	5	18	2	Metal
60	Ships and repair of ships	23	23	21	35	2	Metal
46	General industrial machinery	24	24	23	44	2	Metal
56	Heavy electrical equipment	25	25	29	52	2	Metal
48	Other general machines	26	26	35	63	2	Metal
40	Cast and forged steel products	27	27	53	64	2	Metal
54	Semiconductor devices and integrated circuits	28	28	43	53	2	Metal
55	Electronic components	29	29	45	54	2	Metal
57	Other electrical equipment	30	30	57	67	2	Metal
45	Other metal products	31	31	58	68	2	Metal
41	Other iron or steel products	32	32	61	69	2	Metal
43	Non-ferrous metal products	33	33	59	71	2	Metal
39	Steel products	34	34	63	70	2	Metal
42	Non-ferrous metals	35	35	68	76	2	Metal
38	Pig iron and crude steel	**36**	38	64	74	2	Metal
17	Furniture and fixtures	**37**	36	33	48	1	Non-metal
5	Fisheries	**38**	37	19	13	1	Non-metal
2	Livestock	39	39	16	8	1	Non-metal
12	Feeds and organic fertilizer, n.e.c.	40	40	20	15	1	Non-metal
35	Cement and cement products	41	41	41	49	1	Non-metal
98	Office supplies	42	42	13	2	1	Non-metal
37	Other ceramic, stone and clay products	43	43	69	77	1	Non-metal
13	Tobacco	44	44	15	1	1	Non-metal
33	Leather, fur skins and miscellaneous leather products	45	45	17	14	1	Non-metal
15	Wearing apparel and other textile products	46	46	28	6	1	Non-metal
63	Miscellaneous manufacturing products	47	47	36	51	1	Non-metal
32	Rubber products	48	48	22	38	1	Non-metal
11	Beverage	49	49	39	9	1	Non-metal
19	Paper products	50	50	51	58	1	Non-metal

Table B2. Rankings of sectors based on block-triangulation by the program (25) (continued)

Sector	G.mod	G095	G100	Org	Pre-defined blocks
20 Publishing, printing	51	51	52	59	1 Non-metal
31 Plastic products	52	52	60	72	1 Non-metal
14 Textile products	53	53	30	39	1 Non-metal
26 Synthetic fibers	54	54	31	40	1 Non-metal
25 Synthetic resins	55	55	62	73	1 Non-metal
24 Organic chemical products	56	56	71	80	1 Non-metal
18 Pulp, paper, paperboard, building paper	57	57	54	60	1 Non-metal
21 Chemical fertilizer	58	58	48	61	1 Non-metal
16 Timber and wooden products	59	59	55	65	1 Non-metal
23 Basic organic chemical products	60	60	73	81	1 Non-metal
4 Forestry	61	61	72	86	1 Non-metal
10 Foods	62	62	40	29	1 Non-metal
1 Crop cultivation	63	63	44	42	1 Non-metal
3 Agricultural services	64	64	46	43	1 Non-metal
27 Medicaments	65	65	47	46	1 Non-metal
34 Glass and glass products	66	66	50	55	1 Non-metal
30 Coal products	67	67	70	16	1 Non-metal
29 Petroleum refinery products	68	68	74	90	1 Non-metal
28 Final chemical products, n.e.c.	69	69	56	66	1 Non-metal
22 Basic inorganic chemical products	70	70	65	75	1 Non-metal
36 Pottery, china and earthenware	71	71	49	56	1 Non-metal
64 Building construction	72	72	1	24	3 Construction
67 Other civil engineering and construction	73	73	2	26	3 Construction
66 Public construction	74	74	3	11	3 Construction
80 Freight forwarding	75	75	86	57	L 7 Business services
81 Storage facility service	76	76	85	25	L 7 Business services
84 Broadcasting	77	77	90	12	L 7 Business services
71 Waste management service	78	78	77	32	L 6 Utilities
8 Coal mining	79	79	79	31	L 5 Mining
70 Water supply	80	80	80	45	L 6 Utilities
6 Metallic ores	81	81	75	78	L 5 Mining
79 Air transport	82	82	84	10	L 7 Business services
78 Water transport	83	83	83	83	L 7 Business services
69 Gas and heat supply	84	84	76	27	L 6 Utilities
77 Road transport (except transport by private cars)	85	85	82	87	L 7 Business services
76 Railway transport	86	86	81	33	L 7 Business services
7 Non-metallic ores	87	87	78	79	L 5 Mining
82 Services relating to transport	88	88	87	88	L 7 Business services
68 Electricity	89	89	88	89	L 6 Utilities
9 Crude petroleum and natural gas	90	90	89	91	L 5 Mining
87 Research	91	91	91	85	L 7 Business services
91 Goods rental and leasing services	92	92	92	92	L 7 Business services
92 Repair of motor vehicles and machine	93	93	93	93	L 7 Business services
72 Commerce	94	94	94	94	L 7 Business services
83 Communication	95	95	95	95	L 7 Business services
74 Real estate agencies and rental services	96	96	96	96	L 7 Business services
73 Financial and insurance	97	97	97	97	L 7 Business services
90 Advertising, survey and information services	98	98	98	98	L 7 Business services
93 Other business services	99	99	99	99	L 7 Business services

Note: The input coefficient matrix obtained from the Japanese input-output table for the year 2000 is triangulated by the proposed methods. The column labeled 'Org' refers to the result based on the program (19), 'G100' and 'G095' to (25) with $\alpha = 1.0$ and 0.95, and 'G.mod' to an optimal ranking obtained by slightly modifying the one in the column labeled 'G095'.

Table B3. Rankings of sectors based on block-triangulation by the program (27)

Sector	E095	E100	Org	Block found out
97 Other personal services	1	8	41	
94 Amusement and recreational services	2	2	30	
13 Tobacco	3	4	1	
96 Hotel and other lodging places	4	6	7	A. Food, agriculture and fishery
95 Eating and drinking places	5	12	3	A. Food, agriculture and fishery
11 Beverage	6	13	9	A. Food, agriculture and fishery
2 Livestock	7	16	8	A. Food, agriculture and fishery
12 Feeds and organic fertilizer, n.e.c.	8	17	15	A. Food, agriculture and fishery
10 Foods	9	18	29	A. Food, agriculture and fishery
5 Fisheries	10	14	13	A. Food, agriculture and fishery
1 Crop cultivation	11	26	42	A. Food, agriculture and fishery
3 Agricultural services	12	27	43	A. Food, agriculture and fishery
88 Medical service, health, social security and nursing care	13	7	4	
27 Medicaments	14	28	46	
34 Glass and glass products	15	43	55	
89 Other public services	16	5	5	
33 Leather, fur skins and miscellaneous leather products	17	15	14	B. Leather and textile products
15 Wearing apparel and other textile products	18	21	6	B. Leather and textile products
14 Textile products	19	25	39	B. Leather and textile products
20 Publishing, printing	20	46	59	C. Paper products
36 Pottery, china and earthenware	21	44	56	
98 Office supplies	22	22	2	C. Paper products
19 Paper products	23	45	58	C. Paper products
50 Household electronic and electric appliances	24	35	21	
18 Pulp, paper, paperboard, building paper	25	47	60	C. Paper products
63 Miscellaneous manufacturing products	26	34	51	
28 Final chemical products, n.e.c.	27	50	66	D. Plastic, rubber, coal and petro-
32 Rubber products	28	24	38	D. leum products
31 Plastic products	29	58	72	D. Plastic, rubber, coal and …
26 Synthetic fibers	30	30	40	D. Plastic, rubber, coal and …
25 Synthetic resins	31	65	73	D. Plastic, rubber, coal and …
24 Organic chemical products	32	69	80	D. Plastic, rubber, coal and …
21 Chemical fertilizer	33	64	61	D. Plastic, rubber, coal and …
30 Coal products	34	70	16	D. Plastic, rubber, coal and …
23 Basic organic chemical products	35	72	81	D. Plastic, rubber, coal and …
29 Petroleum refinery products	36	73	90	D. Plastic, rubber, coal and …
62 Precision instruments	37	29	36	E. Transportation equipments …
22 Basic inorganic chemical products	38	66	75	D. Plastic, rubber, coal and …
49 Machinery for office and service industry	39	23	17	E. Transportation equipments
53 Applied electronic equipment and electric measuring instruments	40	37	23	E. other than ships and preci-
52 Communication equipment	41	40	50	E. sion, electronic and electric
51 Electronic computing equipment and accessory equipment	42	38	19	E. appliances
54 Semiconductor devices and integrated circuits	43	41	53	E. Transportation equipments …
55 Electronic components	44	42	54	E. Transportation equipments …
58 Passenger motor cars	45	10	20	E. Transportation equipments …
59 Other cars	46	11	37	E. Transportation equipments …
57 Other electrical equipment	47	51	67	E. Transportation equipments …
75 House rent	48	3	28	
66 Public construction	49	53	11	F. Civil engineering
67 Other civil engineering and construction	50	54	26	F. Civil engineering

Table B3. Rankings of sectors based on block-triangulation by the program (27) (continued)

Sector	E095	E100	Org	Block found out
35 Cement and cement products	51	59	49	F. Civil engineering
65 Repair of construction	52	31	47	
47 Special industrial machinery	53	19	18	G. Machinery including ship and
46 General industrial machinery	54	36	44	G. metal products, and build-
56 Heavy electrical equipment	55	39	52	G. ing construction
64 Building construction	56	32	24	G. Machinery including ship and ...
44 Metal products for construction and architecture	57	55	62	G. Machinery including ship and ...
48 Other general machines	58	52	63	G. Machinery including ship and ...
45 Other metal products	59	56	68	G. Machinery including ship and ...
43 Non-ferrous metal products	60	57	71	G. Machinery including ship and ...
42 Non-ferrous metals	61	67	76	G. Machinery including ship and ...
17 Furniture and fixtures	62	33	48	H. Wooden products
41 Other iron or steel products	63	61	69	G. Machinery including ship and ...
40 Cast and forged steel products	64	60	64	G. Machinery including ship and ...
39 Steel products	65	62	70	G. Machinery including ship and ...
60 Ships and repair of ships	66	20	35	G. Machinery including ship and ...
38 Pig iron and crude steel	67	63	74	G. Machinery including ship and ...
37 Other ceramic, stone and clay products	68	68	77	
16 Timber and wooden products	69	48	65	H. Wooden products
4 Forestry	70	49	86	H. Wooden products
99 Activities not elsewhere classified	71	71	82	
85 Public administration	72	74	84	
61 Other transportation equipment and repair of transportation equipment	73	9	34	
86 Education	74	1	22	
80 Freight forwarding	75	85	57	(kept at the bottom)
78 Water transport	76	83	83	(kept at the bottom)
71 Waste management service	77	75	32	(kept at the bottom)
79 Air transport	78	84	10	(kept at the bottom)
81 Storage facility service	79	86	25	(kept at the bottom)
6 Metallic ores	80	76	78	(kept at the bottom)
7 Non-metallic ores	81	77	79	(kept at the bottom)
69 Gas and heat supply	82	79	27	(kept at the bottom)
77 Road transport (except transport by private cars)	83	82	87	(kept at the bottom)
8 Coal mining	84	78	31	(kept at the bottom)
76 Railway transport	85	80	33	(kept at the bottom)
82 Services relating to transport	86	87	88	(kept at the bottom)
70 Water supply	87	81	45	(kept at the bottom)
68 Electricity	88	88	89	(kept at the bottom)
9 Crude petroleum and natural gas	89	89	91	(kept at the bottom)
84 Broadcasting	90	90	12	(kept at the bottom)
87 Research	91	91	85	(kept at the bottom)
91 Goods rental and leasing services	92	92	92	(kept at the bottom)
92 Repair of motor vehicles and machine	93	93	93	(kept at the bottom)
72 Commerce	94	94	94	(kept at the bottom)
83 Communication	95	95	95	(kept at the bottom)
74 Real estate agencies and rental services	96	96	96	(kept at the bottom)
73 Financial and insurance	97	97	97	(kept at the bottom)
90 Advertising, survey and information services	98	98	98	(kept at the bottom)
93 Other business services	99	99	99	(kept at the bottom)

Note: The input coefficient matrix obtained from the Japanese input-output table for the year 2000 is triangulated by the proposed methods. The column labeled 'Org' refers to the result based on the program (19), 'E100' and 'E095' to (27) with $\alpha = 1.0$ and 0.95.

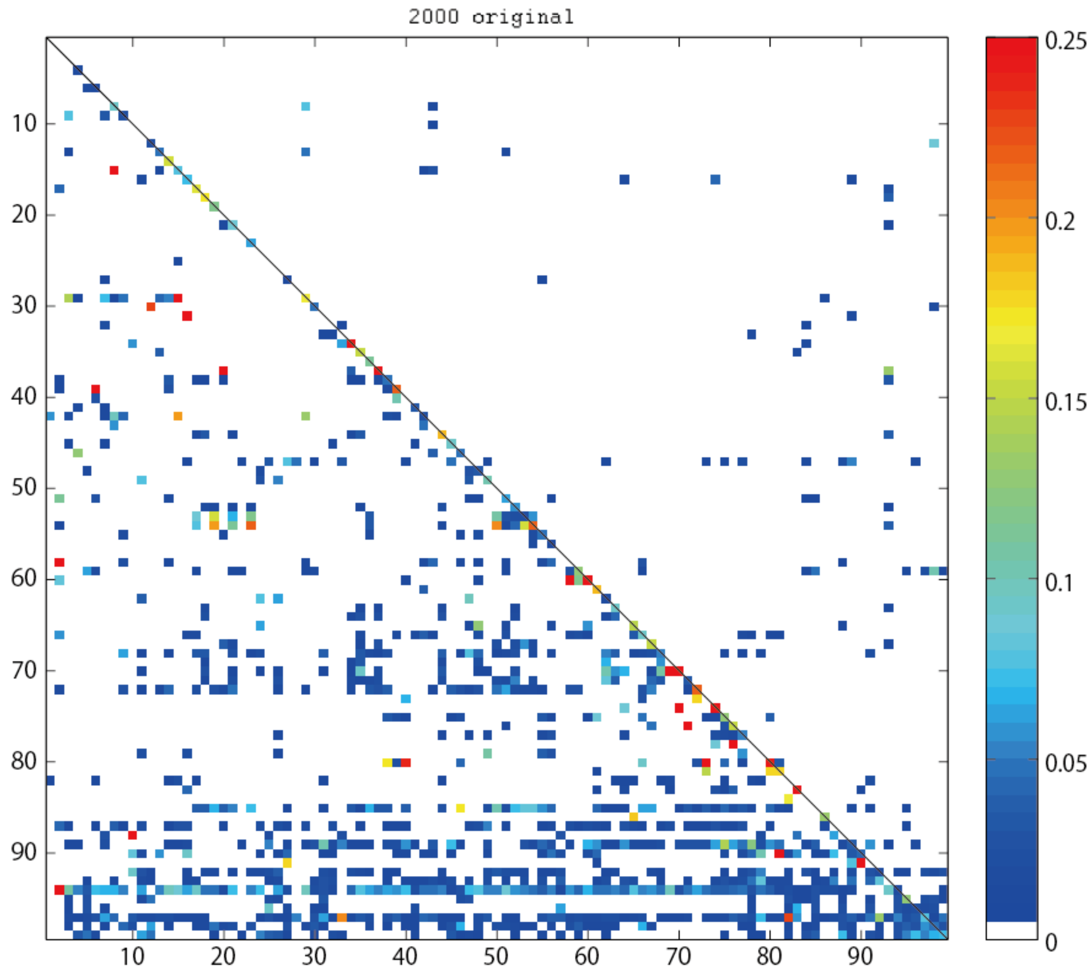

Figure B1. Triangulated Japanese input coefficient matrix of the year 2000.

Note: The sectors are arranged based on the obtained optimal solution to the program (19). Each small square refers to an element of the input coefficient matrix, whereby different colors are used to indicate its relative magnitude along the color scale on the right. The elements greater than 0.25 are displayed as if they are equal to 0.25.

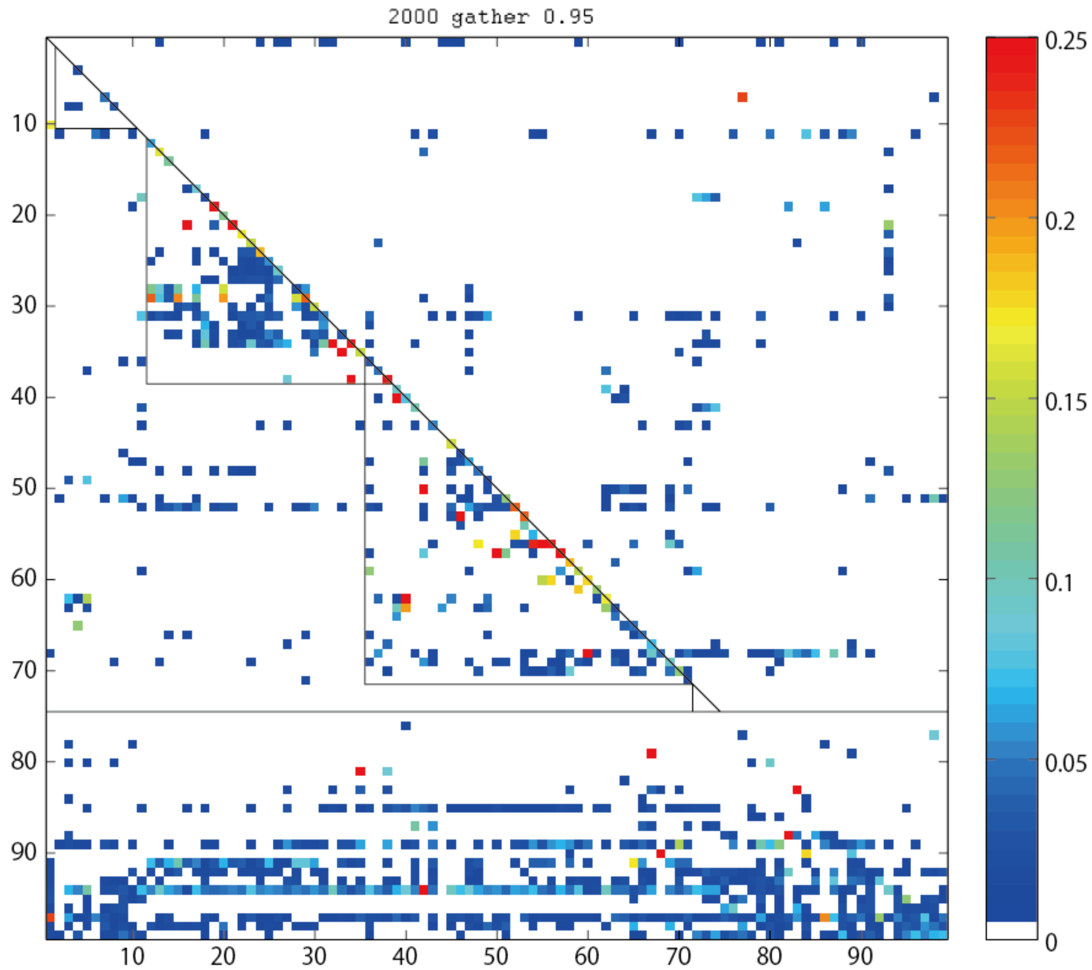

Figure B2. Block-triangulated Japanese input coefficient matrix of the year 2000.

Note: The sectors are arranged based on the best integer solution to the program (25) with $\alpha = 0.95$. See also notes to Figure B1.