Rent and Profit

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1 Introduction

The aim of this paper is to consider and discuss the relation between rent and profit in the system with fixed capital and land. In this paper, we will not discuss the difference in fertility or the difference in productivity of land. And we will focus on the uniform rent throughout industries. By using the Sraffa system, we can derive a very simple relation between rent of land and profit.

2 The Sraffa System

2.1 The Exogenously Given Quantity System

Let us explain two different notions of income and capital in the Sraffa system. We will concern only with the case of single product industries. In this section, we will assume that there is no joint production, and no land and no fixed capital. In the next section, we will consider the Sraffian system with fixed capital. Each industry produces a single commodity by using a certain quantity of labour and certain quantities of commodities as means of production. The number of industries and thus the number of products is equal to n. Let us assume that there is no heterogeneity in labour, and the amount of total labour is equal to L.

Let us denote the diagonal output matrix by $\mathbf{X}$, the quantity matrix of the produced means of production by $\mathbf{M}$, and the labout input vector by $\mathbf{L}$. They will be represented as

$$\mathbf{X}=\begin{bmatrix} X_1 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & X_n \end{bmatrix} \quad \mathbf{M}=\begin{bmatrix} M_{11} & \cdot & \cdot & M_{21} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ M_{12} & \cdot & \cdot & M_{nn} \end{bmatrix} \quad \mathbf{L}=\begin{bmatrix} L_1 \\ \cdot \\ \cdot \\ L_n \end{bmatrix}$$

From this, we will define the following:

$$\mathbf{A} = \mathbf{X}^{-1}\mathbf{M} \tag{1}$$
$$\mathbf{l}_A = \mathbf{X}^{-1}\mathbf{L} \tag{2}$$

Now let us denote the unit vector for aggregation by

$$\mathbf{e} = (1, \cdots, 1) \tag{3}$$

Then the total labour L will be given by

$$L = \mathbf{eL} \tag{4}$$

Also the output vector $\mathbf{x}$ will be defined by

$$\mathbf{x} = \mathbf{eX} \tag{5}$$

In the case of production of a surplus, the total output is divided into two components: the part which is required for the replacement, and the part of the surplus produced, which is over and above the replacement for reproduction. If we denote the actual net product vector by $\mathbf{y}$, then the vector $\mathbf{y}$ can be given by

$$\mathbf{y} = \mathbf{x}(\mathbf{I} - \mathbf{A}) \tag{6}$$

where $\mathbf{I}$ is the identity matrix. The equation (6) gives the definition of the vector of the actual net product that makes up the actual national income. The j th component of $\mathbf{y}$ represents the net product of industry j. If each industry produces a surplus, $\mathbf{y}$ will be strictly positive: $\mathbf{y} > 0$, and if there is no surplus in some industries, $\mathbf{y}$ will be semi-positive: $\mathbf{y} \geq 0$.

Now let $\mathbf{m}_y$ be the vector of the produced means of production, capital vector. Then it is defined by

$$\mathbf{m}_y = \mathbf{xA} = \mathbf{eXA} \tag{7}$$

The the relation between $\mathbf{x}, \mathbf{y}$ and $\mathbf{m}_y$ can be written as

$$\mathbf{x} = \mathbf{y} + \mathbf{m}_y \tag{8}$$

The equation (8) represents the physical relation between the actual outputs, actual net products and the products for replacement.

On the other hand, let us denote the left-hand eigenvector of the matrix $\mathbf{A}$ by $\mathbf{q}$ and let us assume the follwoing

$$\mathbf{q}\mathbf{l}_A = \mathbf{x}\mathbf{l}_{\mathbf{A}} \quad (= \mathbf{e}\mathbf{L}) \tag{9}$$

Since $1\ /(1{+}R)$ is the eigenvalue of the matrix $\mathbf{A}$, we have

$$\mathbf{q} = (1+R)\mathbf{q}\mathbf{A} \tag{10}$$

From this, we can define the standard net product vector $\mathbf{s}$ as

$$\mathbf{s} = \mathbf{q}(\mathbf{I} - \mathbf{A}) = R\mathbf{q}\mathbf{A} \tag{11}$$

The the relation between $\mathbf{q}, \mathbf{s}$ and $\mathbf{m}_S$ can be written as

$$\mathbf{q} = \mathbf{s} + \mathbf{m}_S \tag{12}$$

2.2 The Sraffian Price System

Let us denote the column vector of the commodity prices by $\mathbf{p}_A$, the rate of profits by r, which is assumed to be uniform all over the economic system, the maximum rate of profits by R. The rate of profits will take real numbers ranging from 0 to R ($0 \leq r \leq R$). Similarly, a uniform rate of wage (post factum) is assumed to be prevailing in the economy. It is indicated by w_A. Then we have the following price equation:

$$\mathbf{X}\mathbf{p}_A = (1+r)\mathbf{M}\mathbf{p}_A + w_A\mathbf{L} \tag{13}$$

The Sraffian price equation system will be rewritten as

$$\mathbf{p}_A = (1+r)\mathbf{A}\mathbf{p}_A + w_A\mathbf{l}_A \tag{14}$$

When the wage is equal to zero, the price equation (14) will reduce to

$$\mathbf{p}_A = (1+R)\mathbf{A}\mathbf{p}_A \tag{15}$$

This implies that, under the condition of $w_A = 0$, the price vector $\mathbf{p}_A$ is the right-hand eigenvector of the matrix $\mathbf{A}$ and $1\ /(1{+}R)$ is its eigenvalue.

By using the standard net product vector, we can rewrite the price vector (14) as.

$$(\mathbf{I} - \mathbf{A})\mathbf{p}_A = \frac{r}{R}R\mathbf{A}\mathbf{p}_A + w_A\ \mathbf{l}_A \tag{16}$$

Multiplying the vector $\mathbf{q}$ to the both members of this equation, we have

$$\mathbf{q}(\mathbf{I} - \mathbf{A})\mathbf{p}_A = \frac{r}{R}R\mathbf{q}\mathbf{A}\mathbf{p}_A + w_A\mathbf{l}_A \tag{17}$$

From this, we have

$$\mathbf{s}\mathbf{p}_A = \mathbf{x}\mathbf{l}_A \iff r = R(1 - w_A) \tag{18}$$

From the equation (18), we have

$$\mathbf{s}\mathbf{p}_A\ /\ \mathbf{x}\mathbf{l}_A\ = 1 \iff r = R(1 - w_A) \tag{19}$$

2.3 The Sraffian Evaluation System

In order to interpret the equality between the standard national income and the standard total labour, we will introduce the value of labour v_L into the equation $\mathbf{s}\mathbf{p}_A = \mathbf{x}\mathbf{l}_A$ as follows.

$$\mathbf{s}\mathbf{p}_A = v_L \mathbf{x}\mathbf{l}_A \tag{20}$$

From this equation we have

$$\mathbf{s}\mathbf{p}_v = \mathbf{x}\mathbf{l}_A \tag{21}$$

where $\mathbf{p}_v = \mathbf{p}_A \ / \ v_L$. This equation means that the standard income measured in terms of the quantity of labour is equal to the standard total labour. The standard condition (normalization condition) is

$$v_L = 1 \tag{22}$$

In order to show the relationship between the standard condition $v_L = 1$ and the Sraffian price system, we will show the Evaluation System. When a set of data ($\mathbf{x}$, $\mathbf{A}$, $\mathbf{l}_A$, $\mathbf{s}$, R) is given, we have the following Evaluation System

$$[\textbf{Evaluation System}] \quad \begin{aligned} \mathbf{s}\mathbf{p}_A &= v_L \mathbf{x}\mathbf{l}_A && (23)\\ \mathbf{p}_A &= (1+r)\mathbf{A}\mathbf{p}_A + w_A \mathbf{l}_A && (24) \end{aligned}$$

In this system, there are (n+1) independent equations and (n +3) unknowns(v_L, $\mathbf{p}_A$, w_A, r). Then, if the rate of profits is given exogenously, the price system will become determinate.

In the above Evaluation System, the following proposition will hold

$$\begin{aligned} v_L &= 1 && (25)\\ &\Longleftrightarrow \frac{\mathbf{s}\mathbf{p}_A}{\mathbf{x}\mathbf{l}_A} = 1 && (26)\\ &\Longleftrightarrow r = R(1 - w_A) && (27) \end{aligned}$$

The condition of $v_L = 1$ means that the value of one unit of labour is set equal to one.The reduced form of $\mathbf{p}_v$ of the Sraffa system can be represented as

$$\mathbf{p}_v = \frac{\mathbf{p}_A}{v_L} = (1 - r/R)[\mathbf{I} - (1 - r)\mathbf{A}]^{-1}\mathbf{l}_A \tag{28}$$

The right-hand member of this equation is the Reduction equation of Sraffa [1960], which is measured in terms of quantities of labour. The prices of this equation are subject to the changes in the rate of profits and the changes in the technological condition of production. But, since the value of labour embodied in the standard national income is invariant, there is no effect of the changes in the value of the chosen standard on the prices of commodities. The Sraffa's theory can be considered not as a simple theory of price determination but as a theory of value.

Under the condition of $r = R(1 - w_A)$, we have

$$\frac{\mathbf{sp}_v}{\mathbf{xl}_A} = 1 \tag{29}$$

This means that the value of the standard net product produced by means of one unit of labour becomes equal to one.

2.4 Distributive Variables

Let us consider distribution under the condition of $v_L = 1$ or $\mathbf{sp}_v \ / \ \mathbf{xl}_A \ = 1$. The profit share should be considered as the value of the standard net product produced by means of one unit of labour $\mathbf{sp}_v \ / \ \mathbf{xl}_A$.Let us denote the profit share to the share of the value of one unit of labour or to the value of the standard net product produced by means of one unit of labour by π_v. Then it will be given by

$$\pi_v = r/R \tag{30}$$

Also let us denote the profit share to the share of the value of one unit of labour or to the value of the standard net product produced by means of one unit of labour by ω_v.Then it will be defined by

$$\omega_v = w_A \ / \ v_L \tag{31}$$

And the wage share ω_v will be given by

$$\omega_v = 1 - r/R = 1 - \pi_v \tag{32}$$

From this equation, we obtain the following

$$\frac{1}{\omega_v} = 1 + \frac{\pi_v}{\omega_v} \tag{33}$$

Let us use the following notation

$$\theta_v = \frac{\pi_v}{\omega_v} \tag{34}$$

Then we have

$$\frac{1}{\omega_v} = 1 + \theta_v \tag{35}$$

It should be noted that these distributive variabls $\pi_v, \ \omega_v, \ \theta_v$ are measured under the condition of $\mathbf{sp}_v \ / \ \mathbf{xl}_A \ = 1$.

2.5 The Value of Produced Means of Production

In the Sraffa System, there will be two kind of produced means of production. One is the actual means of production. It will be defined by

$$\mathbf{m}_y = \mathbf{x}\mathbf{A} \tag{36}$$

Another is the produced means of production when the economy produces the standard net product. It will be defined by

$$\mathbf{m}_S = \mathbf{q}\mathbf{A} \tag{37}$$

This is a useful expression of the standard net product.Post-multiplying (24) by price vector $\mathbf{p}_v$, we can obtain

$$\mathbf{s}\mathbf{p}_v = R\ \mathbf{m}_S\mathbf{p}_v \tag{38}$$

If we divide both members of this equation by $\mathbf{x}\mathbf{l}_A$, we have

$$\frac{\mathbf{s}\mathbf{p}_v}{\mathbf{x}\mathbf{l}_A} = R\frac{\mathbf{m}_S\mathbf{p}_v}{\mathbf{x}\mathbf{l}_A} \tag{39}$$

Under the conditoon of $r = R(1-\omega_v)$, we have $\frac{\mathbf{s}\mathbf{p}_v}{\mathbf{x}\mathbf{l}_A} = 1$. Therefore we have

$$\frac{\mathbf{m}_S\mathbf{p}_v}{\mathbf{x}\mathbf{l}_A} = \frac{1}{R} \tag{40}$$

It is important to understand that the value of standard capital is independent of the variation of the prices and distribution. It is given simply as the inverse of R.

$$\frac{\mathbf{s}\mathbf{p}_v}{\mathbf{m}_S\mathbf{p}_v} = R \tag{41}$$

2.6 Difference between the Actual Income and the Standard Income

Let us consider the difference between the notions of actual system ($\mathbf{x}$, $\mathbf{y}$, $\mathbf{m}_y$) and those of the standard system ($\mathbf{q}$, $\mathbf{s}$, $\mathbf{m}_S$). The quantity vectors of the standard system are different from those of the actual system. It is true that, in or multiple product model, we should confronted with the aggregation problem, but the standard system of Sraffa [1960] will make things very simple and transparent. The reason why we use the standard system is that the standard national income can be considered as a proxy for the actual national income and works as a helpful reference.

When the rate of profit is equal to zero and the wage rate is equal to one, the price vector become equal to the vertically integrated labour coefficient vector. It is represented as

$$\mathbf{v}_A = [\mathbf{I} - \mathbf{A}]^{-1}\mathbf{l}_A \tag{42}$$

Then the total labour $\mathbf{xl}_A$ will be rewritten as

$$\mathbf{xl}_A = \mathbf{x}[\mathbf{I} - \mathbf{A}][\mathbf{I} - \mathbf{A}]^{-1}\mathbf{l}_A = \mathbf{yv}_A \tag{43}$$

Therefore the relation between the standard iincome ans the actual income will become

$$\mathbf{sp}_v = \mathbf{xl}_A = \mathbf{yv}_A \tag{44}$$

In order to explain this, let us take notice of the difference between the actual national income and the standard national. The social accounting corresponding to the actual national income can be obtained as

$$\mathbf{yp}_v = r\mathbf{m}_y\mathbf{p}_v + w_v\mathbf{xl}_A \tag{45}$$

On the other hand, we can obtain the social accounting corresponding to the standard national income as follows:

$$\mathbf{sp}_v = r\mathbf{m}_S\mathbf{p}_v + w_v\mathbf{xl}_A \tag{46}$$

Subtracting (46) from (45), we have

$$\mathbf{yp}_v - \mathbf{sp}_v = r(\mathbf{m}_y\mathbf{p}_v - \mathbf{m}_S\mathbf{p}_v) \tag{47}$$

This equation means that the difference between the actual national income and the standard national income is equal to the profit which can be obtained by the capitalist from the difference of two notions of capital, i.e. ($\mathbf{m}_y\mathbf{p}_v$ - $\mathbf{m}_S\mathbf{p}_v$).The meaning of this equation is the definition of profit.

$$\Delta\mathbf{\Pi}_\mathbf{y} = \mathbf{yp}_v - \mathbf{sp}_v \tag{48}$$

From (47), we can obtain

$$\mathbf{yp}_v = \mathbf{sp}_v + r(\mathbf{m}_y\mathbf{p}_v - \mathbf{m}_S\mathbf{p}_v) \tag{49}$$

This result should be important, because, though the standard system is a hypothetical construction, the wage curve obtained by using the standard system explains the same rate of profit as that of the actual system. And the value of

the standard income and the value of the standard capital is constant. Therefore we can use the standard system as a reference. It comes to that the linear wage curve of r =R(1-ω_v) can be considered as the relation of distribution of the actual economy.

2.7 Numerical Example

It will be usefull to show an numerical example of our Sraffian price system.

Numerical Example

$$\mathbf{A} = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} \qquad \mathbf{l}_A = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$$

$$\mathbf{x} = [0.8\ 1.05] \qquad R = 0.25 \qquad \omega_{v(r=0\%)} = 1 \qquad \omega_{v(r=10\%)} = 0.6 \qquad \omega_{v(r=20\%)} = 0.2$$

$$\mathbf{q} = [1 \quad 1] \qquad \mathbf{ql}_A = 1 \qquad \mathbf{xl}_A = 1$$

$$\mathbf{p}_v = \mathbf{v}_A = \begin{bmatrix} 2.75 \\ 2.25 \end{bmatrix} \quad \mathbf{p}_{v(r=10\%)} = \begin{bmatrix} 2.974359 \\ 2.025641 \end{bmatrix} \quad \mathbf{p}_{v(r=20\%)} = \begin{bmatrix} 3.210526 \\ 1.789474 \end{bmatrix}$$

$$\mathbf{s} = [0.20.2] \qquad \mathbf{sp}_{v(r=0\%)} = 1 \qquad \mathbf{sp}_{v(r=10\%)} = 1 \qquad \mathbf{sp}_{v(r=20\%)} = 1$$

$$\mathbf{m}_S = [0.8 \quad 0.8] \qquad \mathbf{m}_S\mathbf{p}_{v(r=0\%)} = 4 \qquad \mathbf{m}_S\mathbf{p}_{v(r=10\%)} = 4 \qquad \mathbf{m}_S\mathbf{p}_{v(r=20\%)} = 4$$

$$\mathbf{y} = [0.11 \quad 0.31] \qquad \mathbf{yp}_{v(r=0\%)} = 1 \qquad \mathbf{yp}_{v(r=10\%)} = 0.955 \qquad \mathbf{yp}_{v(r=20\%)} = 0.9078$$

$$\mathbf{m}_y = [0.69 \quad 0.74] \qquad \mathbf{m}_y\mathbf{p}_{v(r=0\%)} = 3.56 \qquad \mathbf{m}_y\mathbf{p}_{v(r=10\%)} = 3.55 \qquad \mathbf{m}_y\mathbf{p}_{v(r=20\%)} = 3.5395$$

The values of the standard income and the produced means of production in the standrad system are constant as the rate of profit varies, i.e. $\mathbf{sp}_v = 1, \mathbf{m}_S\mathbf{p}_v = 4$. On the contrary, the values of the actual income $\mathbf{yp}_v$ and the actual produced means of production $\mathbf{m}_y\mathbf{p}_v$ varies.

The value of the produced means of producion in the standard system measured in terms of the standard income is obtained by

$$\mathbf{m}_S\mathbf{p}_v/\mathbf{sp}_v = 4 = 1/0.25 = 1/R$$

The diference between the actual income $\mathbf{yp}_v$ and the standard income $\mathbf{sp}_v$ corresponds to the profit which comes from the diffrence between the value of actual means of production and the means of production in the standard system. If we calcualte them in the case of above numerical example, we can obtain the following

$$\begin{aligned}
\mathbf{yp}_v - \mathbf{sp}_v &= 0 && \textit{when} \quad r = 0 \\
\mathbf{yp}_v - \mathbf{sp}_v &= 0.955 - 1 = 0.1(3.55 - 4) \quad = r(\mathbf{m}_y\mathbf{p}_v - \mathbf{m}_S\mathbf{p}_v) && \textit{when} \quad r = 10\% \\
\mathbf{yp}_v - \mathbf{sp}_v &= 0.9079 - 1 = 0.2(3.5395 - 4) \quad = r(\mathbf{m}_y\mathbf{p}_v - \mathbf{m}_S\mathbf{p}_v) && \textit{when} \quad r = 20\%
\end{aligned}$$

3 A System with Fixed Capital and Land

3.1 Fixed Capital

Sraffa[1960] state :

'The interest of Joint Production does not lie so much in the familiar examples of wool and mutton, or wheat and straw, as in its being the genus of which Fixed Capital is the leading species. And it is mainly as an introduction to the subject of fixed capital that the preceding chapters devoted to the intricacies of joint products find their place'**(Sraffa (1960) p.63)**

Also Sraffa [1906] wrote,

' While the two methods give the same result in the extremely simplified case of constant efficiency to which both can be applied, the advantage of the joint- production -equation method is that it is not restricted to that case but has general validity. It will give the 'correct' answer in every case, no matter how complex, over the life of durable instrument of production, may be the pattern of falling productivity or increasing maintenance and repairs. It will, besides, make due allowance for any variation in the prices of the different materials and services required.$'$ **(Sraffa (1960) p.66).**

'In every case the price at any given age of a durable instrument of production or fixed capital asset, as it results from the equations, represents its correct book-value after depreciation. The difference between the values of the asset at two consecutive ages gives the allowance to be made for depreciation for that year. And this latter amount (for example , $M_1p_{m1}-M_2\ p_{m2}$) added to the profit at the general rate on the value of asset at the beginning of the year ($M_1p_{m1}r$) gives the annual charge for that year. This charge will in general not be constant but changing, and probably falling, with the ageing of the instrument or asset.' **(Sraffa (1960) p.66).**

We will take notice of these sentences. The annual charge is, in the end, composed of two parts: the profit at the general rate on the value of asset at the beginning of the year ($M_1p_{m1}r$) and the allowance to be made for depreciation for that year ($M_1p_{m1}-M_2\ p_{m2}$).

Let us denote the diagonal matrix of depreciation by **D.** And let us denote the diagonal quantity matrix of fixed capital by **K**,, the i th element of whcih means the phisical quantity of capital used in the i th industry. Let us denote the price or book -value vector of fixed capital by $\mathbf{p}_F$, the i th element of which

means the price or book value of fixed capital used in the i th industry. Then we can obtain the following matrixes

$$\hat{\mathbf{K}} = \mathbf{K}\mathbf{p}_F = \begin{bmatrix} K_1 p_{F1} & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & K_n p_{Fn} \end{bmatrix} \tag{50}$$

$$\mathbf{D} = \begin{bmatrix} D_1 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & D_n \end{bmatrix} \tag{51}$$

The i th element of $\hat{\mathbf{K}}$ is the book value of the fixed capital used in the i th industry, which corresponds to Sraffa's notation $M_1 p_{m1}$.The i th element of $\mathbf{D}$ is **,** the i the element of which means the physical quantity of depreciation of capital used in the i th industry, which corresponds to Sraffa's notation $(M_1 p_{m1} - M_2\ p_{m2})$.

We assume that the wage rate is cnstant though time and given exogenously. If we measure the value of capital and depreciation in terms of labour commanded, we will have the following matrixes,

$$\hat{\mathbf{K}}_w = \mathbf{K}\mathbf{p}_F / w_A = \begin{bmatrix} K_1 p_{F1}/w_A & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & K_n p_{Fn}/w_A \end{bmatrix} \tag{52}$$

$$(53)$$

$$\mathbf{D}_w = \mathbf{D}/w_A = \begin{bmatrix} D_1/w_A & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & D_n/w_A \end{bmatrix}$$

Let us use the following notations,

$$\mathbf{k}_w = \mathbf{X}^{-1}\hat{\mathbf{K}}_w \tag{54}$$

$$\mathbf{d}_w = \mathbf{X}^{-1}\hat{\mathbf{D}}_w \tag{55}$$

$\mathbf{k}_w$ means the vector of book value of fixed capital measured in terms of commanded labour of industry i. $\mathbf{d}_w$ means the vector of depreciation measured in terms of commanded labour of inductry i.

3.2 Land

If we denote the quantity of land by T_i, and the price of land by p_{Ti}, we have the matrix of the value of land as follows

$$\hat{\mathbf{T}} = \mathbf{T}\mathbf{p}_T = \begin{bmatrix} T_1 p_{T1} & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & T_n p_{Tn} \end{bmatrix} \tag{56}$$

If we denote the rent of land by $\rho_{i,}$ it will be given by

$$\rho_i = r p_{Ti} \tag{57}$$

The matrix of rent of land will become

$$\begin{bmatrix} \rho_1 T_1 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \rho_n T_n \end{bmatrix} \tag{58}$$

However, when land is introduced, we will consider a very simple case. There is no difference in fertility or no difference in productivity of land. The rent of each industry is equal throughout industries.

$$\rho_1 = \cdots = \rho_n \tag{59}$$

We assume that the value of land is measured in termsof commonded labour, which is calculated by the actual wage rate w_A. Moreover, we also assume that the following matrix can be composed from the above rent matrix.

$$\rho \mathbf{T}_w = \rho \begin{bmatrix} T_{w1} & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & T_{wn} \end{bmatrix}$$

where T_{wi}, means the value of land measured in termsof commonded labour. It will be convenient to define the following vector.

$$\mathbf{t}_w = \mathbf{X}^{-1} \hat{\mathbf{T}}_w$$

3.3 The Price System with profit for fixed capital and rent for land

By using the these matrixes, the price equation system with fixed capital and depreciation will be represented by

$$\mathbf{X}\mathbf{p}_A = (1+r)\mathbf{M}\mathbf{p}_A + w_A \mathbf{L} + r\ w_A \hat{\mathbf{K}}_w + w_A \mathbf{D}_w + +\boldsymbol{\rho} w_A \mathbf{T}_w \tag{60}$$

The price system will be represented by

$$\mathbf{p}_A = (1+r)\mathbf{A}\mathbf{p}_A + w_A \mathbf{l}_A + r\ w_A \mathbf{k}_w + w_A \mathbf{d}_w + \boldsymbol{\rho} w_A \mathbf{t}_w \tag{61}$$

From this, we will have

$$
\begin{aligned}
\mathbf{p}_A &= w_A[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{l}_A \\
&\quad +rw_A[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\ \mathbf{k}_w \\
&\quad +w_A[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{d}_w \\
&\quad +\boldsymbol{\rho} w_A[\mathbf{I}-(1+r)\mathbf{A}]_w^{-1}\mathbf{t}_w \\
&= w_A\ \{[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{l}_A \\
&\qquad +r[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\ \mathbf{k}_w \\
&\qquad +[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{d}_w \\
&\qquad +\boldsymbol{\rho}[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{t}_w\} \\
&= \frac{1}{\omega_v}\ w_A\ \omega_v\ \{[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{l}_A \\
&\qquad\qquad +r[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\ \mathbf{k}_w \\
&\qquad\qquad +[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{d}_w \\
&\qquad\qquad +\rho[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{t}_w\}
\end{aligned}
\tag{62}
$$

Therefore, from (62), we have

$$
\begin{aligned}
\mathbf{p}_A = (1+\theta_v)\ w_A\ & \{\omega_v[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{l}_A \\
& +r\omega_v[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\ \mathbf{k}_w \\
& +\omega_v[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{d}_w \\
& +\rho\omega_v[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\boldsymbol{\tau}_w\}
\end{aligned}
\tag{63}
$$

As is seen in (28), under the condition of $r = R(1-\omega_v)$, the first term of the right member can be replaced by the Sraffian price equation :

$$
\mathbf{p}_v = \omega_v\ [\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{l}_A \tag{64}
$$

The prices $\mathbf{p}_v$ will represent the fraction of total labour $\mathbf{x}\mathbf{l}_A$ to produce the standard net product $\mathbf{s}$, because we have $\mathbf{s}\mathbf{p}_v = \mathbf{x}\mathbf{l}_A$ under the condition of $r = R(1-\omega_v)$.

In the same way, let us define the price equations as

$$
\begin{aligned}
\mathbf{p}_k &= \omega_v[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{k}_w \qquad (65)\\
\mathbf{p}_d &= \omega_v[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{d}_w \qquad (66)\\
\mathbf{p}_t &= \omega_v[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{t}_w \qquad (67)
\end{aligned}
$$

then, from (65)-(67), we will have

$$
\mathbf{p}_A = (1+\theta_v)\ w_A\ (\mathbf{p}_v + r\mathbf{p}_k + \mathbf{p}_d + \rho\mathbf{p}_t) \tag{68}
$$

3.4 Labour Productivity

The value of standard income will be represented by

$$\mathbf{sp}_A = (1+\theta_v)\ w_A\ (\mathbf{sp}_v + r\mathbf{sp}_k + \mathbf{sp}_d + \rho\mathbf{sp}_t) \tag{69}$$

Under the condition of $r = R(1-\omega_v)$, we have $\mathbf{sp}_v = \mathbf{xl}_A,\ \mathbf{sp}_k = \mathbf{qk}_w, \mathbf{sp}_d = \mathbf{qd}_w$.

If we divide the equation (69), then we have the per capita standard income as

$$\frac{\mathbf{sp}_A}{\mathbf{xl}_A} = (1+\theta_v)\ w_A\ \left(\frac{\mathbf{sp}_v}{\mathbf{xl}_A} + r\frac{\mathbf{sp}_k}{\mathbf{xl}_A} + \frac{\mathbf{sp}_d}{\mathbf{sp}_k}\frac{\mathbf{sp}_k}{\mathbf{xl}_A} + \rho\frac{\mathbf{sp}_t}{\mathbf{xl}_A}\right) \tag{70}$$

Under the condition of $r = R(1-\omega_v)$, we have

$$\frac{\mathbf{sp}_v}{\mathbf{xl}_A} = 1 \tag{71}$$

This means that the value of the standard net product produced by means of one unit of labour is set equal to one.

$$\kappa = \frac{\mathbf{sp}_k}{\mathbf{xl}_A} \tag{72}$$

$$\delta = \frac{\mathbf{sp}_d}{\mathbf{sp}_k} \tag{73}$$

$$\boldsymbol{\tau} = \frac{\mathbf{sp}_t}{\mathbf{xl}_A} \tag{74}$$

If the wage rate w_A and the quantity system are given exogenously, these ratios are given numbers. κ means the capital -labour ratio measured when the fixed capital is measured in terms of labour commanded, and when the standard net product is produced. δ means the average or social rate of depreciation when the standard net product is produced.

The labour productivity will become

$$\frac{\mathbf{sp}_A}{\mathbf{xl}_A} = (1+\theta_v)\ w_A\ \{1 + (r+\delta)\ \kappa + \rho\boldsymbol{\tau}\} \tag{75}$$

The term $\{1+(r+\delta)\ \kappa+\rho\boldsymbol{\tau}\}$ in the right member means the labour productivity measured in terms of labour.

3.5 The Wage Share

Now let us consider distribution when fixed capital si considered. From (32)(75),

the wage share $\omega_S = w_A\ \mathbf{xl}_A\ /\ \mathbf{sp}_A$ will become

$$\omega_S = \frac{w_A\ \mathbf{xl}_A}{\mathbf{sp}_A} = \frac{\omega_v}{1+(r+\delta)\ \kappa+\rho\boldsymbol{\tau}} = \frac{1-r/R}{1+(r+\delta)\ \kappa+\rho\boldsymbol{\tau}} \tag{76}$$

As is seen in (41), the maximum rate of profit R is the ratio between the standard income and the means of production. But in this section we are considering the system with fixed capital. When we calculate the maximum rate of profit, we should take the exsistence of fixed capital into consideration. In order to take the fixed capital into account, let us consider the following ratio

$$\chi = \frac{\mathbf{m}_S \mathbf{p}_v}{\mathbf{m}_S \mathbf{p}_v + \mathbf{s}\mathbf{p}_k} \tag{77}$$

This is the ratio between the value of produced means of production and the total value of capital (the value of produced means of production + the value of fixed capital). Then the maximum rate of profit (R′) should be

$$R' = R\chi \tag{78}$$

Also the wage share to the value of $\mathbf{sp}_A$ should be rewritten by

$$\omega_S = \frac{1 - r/R\chi}{1 + (r+\delta)\ \kappa + \rho\boldsymbol{\tau}} \tag{79}$$

It should be noted that the wage share ω_S is given exgogenously becuase the wage measured in terms of money is given exogenously and the quantity system is also given exogenously.

3.6 Imaginary Relation between Rent and Profit Rate

Since the wage rate w_A and the quantity system are given exogenously, the wage share ω_S , the maximum rate of profit R, the ratios $\chi, \kappa, \delta, \tau$ are considered as given numbers. Therefore from (79) we can obtain the following equation :

$$\rho = -\frac{1}{\tau}\left(\frac{1}{\omega_S R\chi} + \kappa\right)\ r\ + \frac{1}{\tau}\left(\frac{1}{\omega_S} - 1 - \delta\kappa\right) \tag{80}$$

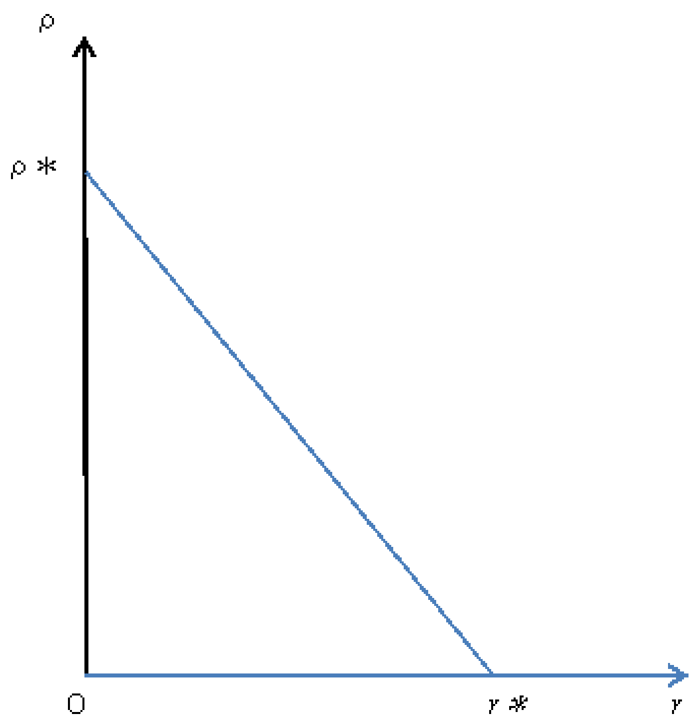
This equation gives the imaginary relation between rent and profit. If the rate of profit is equal to 0, then the rent of land will become

$$\rho^*\ = \frac{1}{\tau}\left(\frac{1}{\omega_S} - 1 - \delta\kappa\right)$$

On the contrary, if the rent of land is equal to 0, the profit will become

$$r^*\ = \left(\frac{1}{\omega_S} - 1 - \delta\kappa\right)\ /\left(\frac{1}{\omega_S R\chi} + \kappa\right) \tag{81}$$

From the above, when the wage rate and the quantity system are given, we can show the graph of the imaginary relation between ρ and r as follows.


ρ
ρ *
O
γ *
γ

4 Conclusion

In the system with fixed capital and land, we derived the following results
1) the labour productivity measured in terms of labour can be given by

$$1 + (r + \delta)\ \kappa + \rho\boldsymbol{\tau}$$

2) the wage share can be given by very simple equation of

$$\omega_S = (1 - (r/R\chi)\ /\ \{1 + (r + \delta)\ \kappa + \rho\boldsymbol{\tau}\}$$

The wage share to the standard income is given exogenously in our model.
3) when the wage rate and the quantity system are given, the imaginary relation between rent and profit rate

$$\rho = -\frac{1}{\tau}\left(\frac{1}{\omega_S R\chi} + \kappa\right)\ r\ + \frac{1}{\tau}\left(\frac{1}{\omega_S} - 1 - \delta\kappa\right)$$

The relation between ρ and r becomes linear.

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