An operational, nonlinear input-output system*

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Abstract: We develop a scale-dependent nonlinear input-output model which is a practical alternative to the conventional linear counterpart. The model contemplates the possibility of different assumptions on returns-to-scale and is calibrated in a simple manner that closely resembles the usual technical coefficient calibration procedure. Multiplier calculations under this nonlinear version offer interval estimates that provide information on the effectiveness and variability of demand-driven induced changes in equilibrium magnitudes. In addition, and unlike linear multipliers, the nonlinear model allows us to distinguish between physical and cost effects, the reason being that the traditional dichotomy between the price and quantity equations no longer holds in the nonlinear model version. Preliminary analysis will use archetype interindustry data for US, China and Brazil. Here in this version we show results for a simplified small economy.

Keywords: nonlinear input-output, scale-dependent input-output, nonlinear multipliers.

JEL: C67, D57, 022, R15

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INTRODUCTION

There is a glaring contrast between the theoretical advances in nonlinear input-output (NIO) theory and the surprisingly scarce list of applications in the empirical literature. This divorce cannot be attributed to the computational requirements for solving nonlinear models. With today's specialized software computation should not be a decisive issue. The question probably lies on the informational requirements needed for the implementation of NIO models, particularly on sectoral response elasticities. As Lahiri (1983) acutely points out, empirical estimation of NIO models is nearly impossible—too many parameters for too few available observations. The same type of problematic data requirement situation is also common for the specification of computable general equilibrium models (CGE) but this has not stopped practitioners at all (see Dervis *et al*, 1982, Mansur and Whalley, 1984). CGE modeling and research has become a very important area for policy analysis and evaluation and this has been possible, in part, thanks to the adoption of operational assumptions on behavior and the use of calibration techniques. We believe practical implementation of NIO models is equally possible once we accept (i) some specific behavioral rules in the definition of production activities and (ii) are able to use observed empirical data for the parameterization of production processes.

The theory of NIO models has been concerned with establishing theorems that prove existence and uniqueness of solutions for a nonlinear version of the Leontief quantity equation. Under quite general conditions, but all of them sharing a modified system productivity assumption, existence and uniqueness can be proved. Sandberg (1973), Chander (1983), Fujimoto (1986), Szidarovsky (1989), Herrero and Silva (1991), among others, provide the necessary theoretical background for NIO logical consistency. In a NIO model technical coefficients are not taken as fixed. Their variability can be attributed to many different factors (technical innovation, input substitution, productivity changes, non-homogeneity, etc.) as Rose (1983) very clearly explains in his review and assessment paper. Theorists need not concern themselves with these possibilities but applied economists should at least explore them and consider how to sensibly incorporate them. The nonlinearities we consider in this paper are of the scale-dependent type, i.e. changes in total output need not be proportional to changes in total inputs but still a unique production mix is all that is available to firms. Isoquants are L-shaped but the isoquant map is not necessarily homothetic. Price-induced nonlinearities in coefficients due to smooth input substitution, as dealt with in Tokutsu (1989) or Sancho (2010), are not considered here. West and Gamage (2001), in turn, is one of the few empirical examples of using a nonlinear assumption although restricted to the households' income account, where average coefficients are substituted by marginal ones. Zhao *et al* (2006) introduce a Cobb-Douglas production function for defining the interindustry technical coefficients but since their model does not contemplate any price behavior whatsoever, the selection of the input mix is very much based on some *ad-hoc* assumptions—such as maintaining total output constant when substitution takes place in some sector. This has little if any economic justification.

The paper follows this organization. Section 2 discusses the characteristics of the proposed nonlinear input-output model. In Section 3 we implement the model using interindustry data for a simplified fictitious economy. Section 4 summarizes.

2. NONLINEAR INPUT-OUTPUT.

2.1 *Review of the conventional linear model.*

Interindustry data provide a detailed multisectoral depiction of the revenue-expenditure-output macroeconomic identities. Consider an economy composed of *n* distinct productive sectors indexed as *i*, *j*=1, 2,..., *n*. In the period when data is assembled, identified here by super index 0, the following identities representing the circular flow of income hold true for all *j*=1, 2,..., *n*:

$$\sum_{i=1}^{n} p_i^0 \cdot x_{ij}^0 + p_v^0 \cdot v_j^0 = \sum_{i=1}^{n} p_j^0 \cdot x_{ji}^0 + p_j^0 \cdot f_j^0 = p_j^0 \cdot x_j^0 \qquad (1)$$

In expression (1) the left-hand side collects total expenditure in intermediate purchases and value-added acquisition incurred by sector *j* to carry out the production of its output x_j^0, the middle part is total revenue accruing to sector *j* from the sale of its output x_j^0 to other sectors –as intermediate demand– and to final demanders. Finally, the right-hand side of the expression is the value of total production x_j^0 obtained in sector *j*. Expression (1) can therefore be seen as a sort of sectoral budget constraint. Interindustry data, however, is expressed in value and the distinction between physical magnitudes $(x_{ij}^0, x_j^0, v_j^0, f_j^0)$ and prices (p_j^0, p_v^0) is not usually available. We can take observed transaction values as if they were physical magnitudes and in doing so we redefine units in such a way that every one of the new units has a worth of 1 dollar. In other words, we use new prices p_j=1 for goods and p_v=1 for value added such that $p_i \cdot x_{ij} = p_i^0 \cdot x_{ij}^0$, $p_v \cdot v_j = p_v^0 \cdot v_j^0$, and $p_j \cdot f_j = p_j^0 \cdot f_j^0$.

With this implicit normalization it is customary in interindustry analysis to omit the presence of prices in the balance identities in (1) for the base year. Contrary to the tradition, we will keep them explicit for reasons that will become clear shortly. Thus (1) becomes:

$$\sum_{i=1}^{n} p_i \cdot x_{ij} + p_v \cdot v_j = p_j \cdot x_j \tag{1a}$$

from the expenditure perspective and:

$$\sum_{i=1}^{n} p_j \cdot x_{ji} + p_j \cdot f_j = p_j \cdot x_j \tag{1b}$$

from the perspective of revenue; notice that since only price p_j is involved here, it can be eliminated altogether from (1b) if so desired. However written, expressions (1) are nothing but observed data. The standard IO model adopts the assumption that input-output ratios and value-added ratios are constant; in other words, it takes output as proportional to inputs by way of assuming nonnegative technical coefficients defined by $a_{ij} = x_{ij} / x_j$ and $\upsilon_j = v_j / x_j$. They are assumed to be unique and independent of the scale of production. Substituting these coefficients in (1a) and simplifying yields:

$$\sum_{i=1}^{n} p_i \cdot a_{ij} + p_v \cdot \upsilon_j = p_j \tag{2a}$$

which, using the obvious vector-matrix notational translation, can be expressed and solved as:

$$\mathbf{p}' = \mathbf{p}' \cdot \mathbf{A} + p_v \cdot \boldsymbol{\upsilon}' = p_v \cdot \boldsymbol{\upsilon}' \cdot (\mathbf{I} - \mathbf{A})^{-1} \tag{3a}$$

provided matrix $\mathbf{A}$, with $[\mathbf{A}]_{ij} = a_{ij}$, is productive and the value-added price p_v is taken as *numéraire*. Similarly, substituting in expression (1b) and eliminating now the irrelevant price p_j gives:

$$\sum_{i=1}^{n} a_{ji} \cdot x_i + f_j = x_j \tag{2b}$$

In matrix terms we would obtain:

$$\mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{f} = \left(\mathbf{I} - \mathbf{A}\right)^{-1} \cdot \mathbf{f} \tag{3b}$$

The linear IO model in (3a) and (3b) is composed of two sets of equations, one for prices and one for quantities, which solve independently of each other. For a given technology matrix **A**, cost covering prices $\mathbf{p}'$ depend only on the value-added technical coefficient vector $\boldsymbol{\upsilon}'$, and output levels $\mathbf{x}$ on external final demand $\mathbf{f}$. This is the well-known dichotomy between prices and quantities in the conventional IO model and it is a property that derives from the linearity assumption in the technology.

2.2 *A nonlinear input-output system.*

Let us assume now that output and inputs are no longer related though a linear relationship. Instead we posit a simple, nonlinear relationship along the technically efficient locus such as:

$$x_j = \alpha_{ij} \cdot x_{ij}^{\beta_{ij}} = \eta_j \cdot v_j^{\beta_{vj}} \tag{4}$$

For $\alpha_{ij}, \beta_{ij}, \beta_{vj} > 0$ this corresponds to a monotonically increasing production function, i.e. more output can only be obtained applying more inputs, where coefficients β_{ij} and β_{vj} can be quickly seen to be output-to-input, or scale, elasticities. Additionally, when $\beta_{ij} < 1$ (>1) the production coordinate linking output x_j with inputs x_{ij} can be seen to present decreasing (increasing) returns to scale denoted by DRS (IRS). Notice that for the particular case when $\beta_{ij} = \beta_{vj} = 1$, for all i and j, expression (4) reverts to the standard input-output linear assumption (with $a_{ij} = (\alpha_{ij})^{-1}$ and $\upsilon_j = (\eta_j)^{-1}$). In the linear IO model all such elasticities are therefore implicitly assumed to be unitary. For those unitary values and the observed benchmark data set (x_j, x_{ij}, v_j) all technical coefficients for goods and value-added are accordingly calibrated. Nothing of course precludes using another, non unitary, set of values for the output-to-input elasticities and proceed likewise with the calibration of all required parameters.

We illustrate these ideas in Figure 1 where we can visualize the structure of a possible nonlinear production map for a 2 good economy, with production of good 1 following these hypothetical technological relationships: $x_1 = \alpha_{11} \cdot x_{11}^{\beta_{11}}$ and $x_1 = \alpha_{21} \cdot x_{21}^{\beta_{21}}$ and $\beta_{11} = 0.5$, $\beta_{21} = 1.25$,

$\alpha_{11} = \alpha_{21} = 1$. We can see that the map is nonlinear on two counts. First, returns-to-scale are not constant as the non-unitary output elasticities show, and second the map is non-homothetic as the efficiency path is nonlinear due to the fact that $\beta_{11} / \beta_{21} \neq 1$. Notice that the two depicted *L*-shaped isoquants conform to the perfect complementarity property of input-output economics but the efficient input mix is itself variable, depending now on the scale of production.

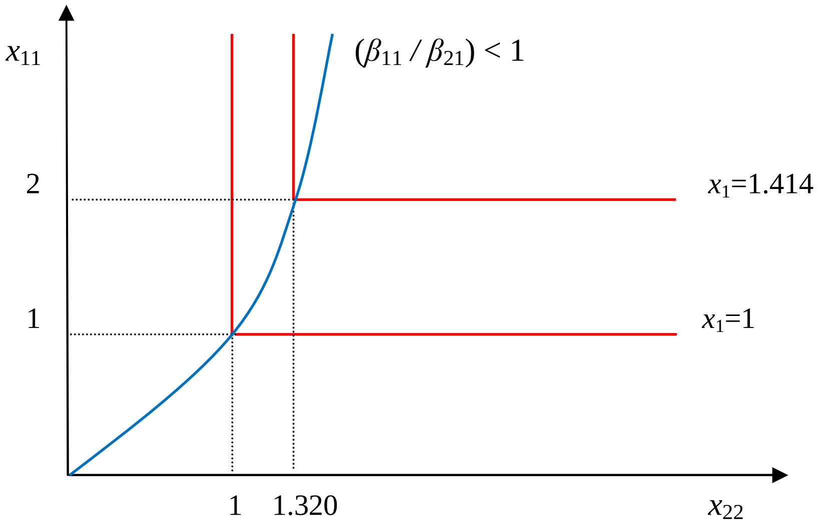

Figure 1: Non-homothetic isoquant map.

The derivation of the complete nonlinear input-output system follows readily from assumption (4). For the quantity equation we start from the balance expression (1b) which we rewrite here as:

$$\sum_{i=1}^{n} x_{ji} + f_j = x_j \tag{5}$$

From (4) we find:

$$x_{ji} = \left(\frac{x_i}{\alpha_{ji}} \right)^{\frac{1}{\beta_{ji}}} = \left(a_{ji} \right)^{\frac{1}{\beta_{ji}}} \cdot \left(x_i \right)^{\frac{1}{\beta_{ji}}} = \left(a_{ji} \right)^{\frac{1}{\beta_{ji}}} \cdot \left(x_i \right)^{\frac{1-\beta_{ji}}{\beta_{ji}}} \cdot x_i \tag{6}$$

Substituting (6) in expression (5) and using matrix notation we would find:

$$\mathbf{x} = \mathbf{A(x)} \cdot \mathbf{x} + \mathbf{f} \tag{7}$$

with the elements of matrix **A(x)** being:

$$\left[\mathbf{A(x)}\right]_{ji} = \left(a_{ji}\right)^{\frac{1}{\beta_{ji}}} \cdot \left(x_i\right)^{\frac{1-\beta_{ji}}{\beta_{ji}}} \tag{8}$$

Notice that unitary output elasticities everywhere would yield back the standard technical coefficients of the linear input-output matrix **A**. Under quite general conditions, as we mentioned above, nonlinear equations such as the one in (7) have been proved to have a unique non-negative solution **x** for any possible non-negative vector **f.** The requirements for solvability include the following set of assumptions: (ass_1) the vector function $\mathbf{A(x)} \cdot \mathbf{x}$ is non-decreasing (i.e. more output **x** requires more intermediate inputs $\mathbf{A(x)} \cdot \mathbf{x}$), (ass_2) continuity of $\mathbf{A(x)} \cdot \mathbf{x}$**,** and (ass_3) a productivity condition guaranteeing that expression (7) holds true for some pair (**f**, **x**). The function **A(x)** in expression (8) can be seen to satisfy (a_1) and (a_2). Assumption (a_3), on the other hand, is always satisfied in the case of an empirically implemented model by the base year solution. Thus equation (7) is in principle solvable, and for any other vector of final demand $\mathbf{f} \geq \mathbf{0}$ there will always be a unique production plan $\mathbf{x} \geq \mathbf{0}$ compatible with the nonlinear quantity equation (7).

The fact that the quantity equation is scale-dependent has another far-reaching implication, namely, that cost covering prices are no longer independent of quantities. With non-constant returns to scale, unitary costs are not constant either and their level depends on the actual production level. Despite the focus of the theoretical literature on the solvability of the quantity equation (7), the economic system as a whole has another component that needs to be factored in if overall balance, as described in expressions (1), is to be maintained after a change in final demand takes place. Plugging in assumption (4) into expression (1) and remembering that $\upsilon_j = (\eta_j)^{-1}$ we would obtain:

$$p_j \cdot x_j = \sum_{i=1}^{n} p_i \cdot \left(a_{ij}\right)^{\frac{1}{\beta_{ij}}} \cdot \left(x_j\right)^{\frac{1}{\beta_{ij}}} + p_v \cdot \left(\upsilon_j\right)^{\frac{1}{\beta_{vj}}} \cdot \left(x_j\right)^{\frac{1}{\beta_{ij}}} \tag{9}$$

Simplifying:

$$p_j = \sum_{i=1}^{n} p_i \cdot \left(a_{ij}\right)^{\frac{1}{\beta_{ij}}} \cdot \left(x_j\right)^{\frac{1-\beta_{ij}}{\beta_{ij}}} + p_v \cdot \left(\upsilon_j\right)^{\frac{1}{\beta_{vj}}} \cdot \left(x_j\right)^{\frac{1-\beta_{vj}}{\beta_{vj}}} \tag{10}$$

Recalling expression (8) and defining in a like manner the vector of value-added marginal coefficients as:

$$\left[\boldsymbol{\upsilon}'(\mathbf{x})\right]_j = \left(\upsilon_j\right)^{\frac{1}{\beta_{vj}}} \cdot \left(x_j\right)^{\frac{1-\beta_{vj}}{\beta_{vj}}} \tag{11}$$

we would obtain the scale-dependent system of prices:

$$\mathbf{p}' = \mathbf{p}' \cdot \mathbf{A}(\mathbf{x}) + \mathbf{p}_v \cdot \boldsymbol{\upsilon}'(\mathbf{x}) \tag{12}$$

The system of prices depends now on quantities and the traditional—and very convenient in computing terms—separation of prices and quantities no longer holds. As expected, with unitary output elasticities everywhere, the price equation in (12) reverts to the standard price equation of the linear model in (3a). The nonlinear, scale-dependent input-output system must therefore include both equations for quantities (7) and prices (12) for the system to be complete and all magnitudes to be in balance after external shocks are absorbed within the system. Prices in (12) can be interpreted as shadow prices, i.e. prices supporting the efficient production plans stemming from (7), or as accounting prices, i.e. prices that guarantee the sectoral and economy-wide balance relationships between total costs and total resources. They cannot and should not be interpreted as market prices since no demand behaviour is incorporated in this type of models.

2.3. Calibration issues.

The numerical implementation of the nonlinear input-output system in (7) and (12) requires that all coefficients a_{ij} and β_{ij} introduced in the production function (4) be available prior to computation. In the linear model things are particularly easy since $\beta_{ij} = 1$ for all i and j. In this case, the a_{ij} coefficients can be calibrated from (4) by using the observations for output (x_j) and inputs (x_{ij}) read from the base year data:

$$a_{ij} = (\alpha_{ij})^{-1} = \frac{x_{ij}}{x_j} \tag{13}$$

These coefficients are the slope of the linear rays going through the origin as we can see in the fictional example in Figure 2. This is the consequence of taking all scale elasticities to be unitary. In the general case, when the relationship is not linear, we need to adjust the value of the a_{ij} coefficients to the non-unitary scale elasticities so that for the pair of parameters (a_{ij}, β_{ij}) and base year data (x_j, x_{ij}) expression (4) is upheld. In this case:

$$a_{ij} = (\alpha_{ij})^{-1} = \frac{x_{ij}^{\beta_{ij}}}{x_j} \qquad (14)$$

In the linear example of Figure 2 we find that the technical coefficient is given by $a_{21} = x_{21} / x_1 = 8/10 = 0.8$ (or $\alpha_{21} = 1.25$). Figure 3 illustrates the nonlinear situation for the same base year data and, in this DRS case with an elasticity value of $\beta_{21} = 0.6$, the calibrated coefficient turns out to be $a_{21} = x_{21}^{\beta_{21}} / x_1 = 8^{0.6} / 10 = 0.348$ (or $\alpha_{21} = 2.872$).

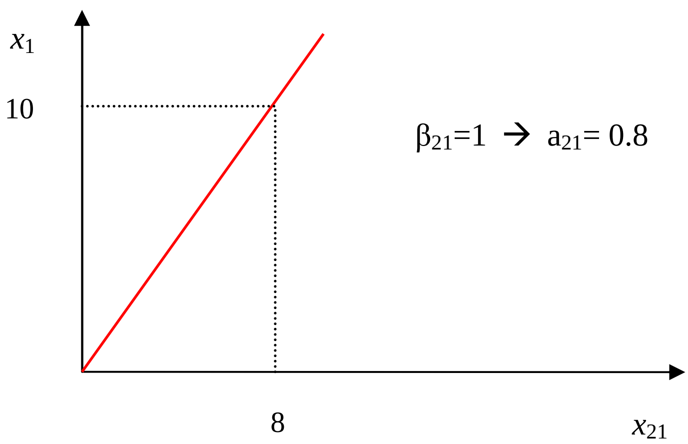

Figure 2: Calibration to linear.

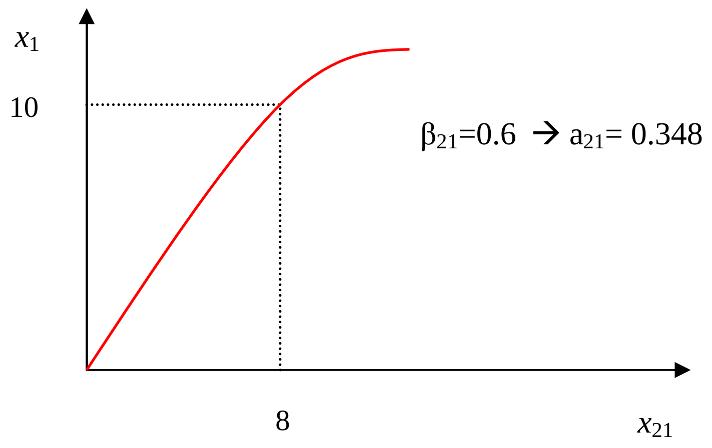

Figure 3: Calibration to nonlinear.

Notice the procedure is exactly the same for the linear and the nonlinear case, conditional in both cases to the elasticity values being adopted. In the linear case they are unitary everywhere in the production function but this set of values is just but one of the many possible ones in parameter space and is therefore as *ad-hoc* as any other selectable set. In probability terms the linear set is of course highly unlikely. The justification for linearity should, in any case, rest on economic grounds. The empirical evidence in favour of universal constant-returns-to scale, however, is not conclusive. Based on engineering evidence assembled for the US manufacturing sectors, Jorgenson (1972) reported that CRS seem to be prevalent, at least from some minimal optimal size plant on. Dragonette (1983), in contrast, used time-series data on US manufacturing too but concluded using concentration indicators that CRS seem to be more the exception than the rule, with a majority of DRS industries. Chirinko and Fazzari (1994), on their part, checked for market power using a time-series database for 11 manufacturing US sectors. Since market power is theoretically incompatible with CRS, this estimate provides indirect evidence for the presence of non constant returns to scale. He concludes that market power, hence IRS, are present in a majority of those industries. This contradictory evidence is for manufacturing industries alone, and no similar information seems to be available for non-manufactures, such as the services sectors.

2.4. Computing issues.

The complete input-output system includes the two nonlinear equations (7) and (13) and both need to be solved. The solution to (7) provides output levels **x** that satisfy final demand **f** and are compatible with the production mapping **A(x)**. The solution to (13) guarantees cost covering prices compatible with the dual cost functions. Considered together (7) and (13) yield the required economy-wide balance between uses and resources in expressions (1). For any possible level of final demand **f,** the solution **x(f)**, **p(x(f)**) is obtained using the GAMS (General Algebraic Modeling System) software[1]. The system of equations is transformed into a dummy nonlinear optimization program using a fake objective function with no relation to the endogenous variables. The nonlinear program includes as constraints (7) and (13). Since GAMS looks first for a feasible solution to the constraints and only after finding it tries to increase the objective function while keeping feasibility, the process converges swiftly. The feasible solution to the nonlinear program exists and is unique, as implied by the theoretical literature. The nonlinear solver is seeded with the benchmark data values and is run to obtain a first solution which of course coincides with the benchmark values. Once the first feasible solution is found, GAMS tries to find a better feasible one but, being the unique feasible solution, there is no room for improvement and the process quickly stops. This first run provides an initial basis for the subsequent, non-benchmark runs and facilitates convergence.

These additional runs simulate sequential changes in final demand and are solved using the loop facility in GAMS. For instance, final demand in good 1 is assumed to change by one unit and a new nonlinear input-output balance is found. Then final demand for 1 is reset to the benchmark level and next a unitary change in demand for good 2 is considered. The process is repeated for the rest of goods. This allows calculation of multiplier effects induced by changes in final demand for all goods both in quantities and in prices, a distinct and novel possibility not present in the standard linear model, where prices are not responsive to changes in quantities.

The fact that quantities and prices change has an interesting implication for the measurement of output multipliers. In the linear model, since prices do not change when final demand changes, the total multiplier value can be easily calculated by column summation. After all, units have been chosen to the effect that physical and value magnitudes coincide. With constant prices, aggregation is simple and adding up quantities is permissible. In the NIO system, however, prices are scale-dependent too and in the new economy-wide balance both quantities and prices will have changed. This requires the use of index numbers to isolate quantity effects from price effects.

[1] See Brooke *et al* (1992).

3. NUMERICAL RESULTS.

Given the absence of firm evidence on returns-to-scale, the variety of possible sectoral situations, and the lack of data for economy-wide estimation of returns-to-scale, the second best option is to estimate economic effects using a set of distinct global scenarios which try to encompass reasonable alternatives on output-to-input elasticities. For instance, a central scenario with CRS can be accompanied by two scenarios with DRS and IRS, providing interval estimates of effects quite richer than the usual and point-estimate typical of the linear case. To this effect we use fictitious interindustry data to calibrate all the needed coefficients. The data includes a disaggregation of 3 distinct sectors, which is sufficient for our expository purposes here.

Three sets of elasticities are assumed. The first one adopts the universal unitary elasticity values specific to the linear model (CRS), i.e. $\beta_{ij} = \beta_{vi} = 1$. We then modify those values downward (DRS) and upward (IRS) by 10 percent and proceed to recalibrate the needed two sets of a_{ij} coefficients for matrix **A(x).** The model is then solved for unitary sequential changes in final demand. This allows us to obtain estimates of the quantity multiplier matrices for the three scenarios on returns to scale and estimates of the induced cost changes under the two non-constant returns to scale cases. To simplify the exposition and save space, we present summary results instead of the complete matrix results, which are of course available from the authors. We calculate total multipliers as aggregators of matrix columns using quantity Paasche indices. These numbers inform of the effects on total physical output in response to unitary changes in final demand and they are similar to the usual presentation in the linear interindustry model. We also evaluate aggregate price responses to changes in final demand, in this case using weighted cost elasticities so as to neutralize the size effect.

The effects on all output levels of a changed in the final demand in a specific good are captured by the matrix **M** of partial derivatives:

$$[\mathbf{M}]_{ij} = m_{ij} = \frac{\partial x_i}{\partial f_j} \tag{15}$$

Similarly a matrix **C** of price changes can be calculated for each possible change in final demand:

$$[\mathbf{C}]_{ij} = c_{ij} = \frac{\partial p_i}{\partial f_j} \qquad (16)$$

Once matrices **M** and **C** are calculated under all scenarios we derive the appropriate summary results, which we reproduce in the Tables in the Appendix.

4. CONCLUDING REMARKS.

We have shown that nonlinear input-output models of the scale dependent type can be easily implemented using standard interindustry data. The numerical implementation is possible thanks to the existence theorems provided by the theoretical literature. The determination of the nonlinear equilibrium requires the use of standard computing techniques. We use here a nonlinear programming algorithm as backbone solver. The algorithm computes both quantities and prices simultaneously since the classical dichotomy no longer holds and both quantities and prices are now scale dependent. We perturb the initial equilibrium to generate multiplier matrices in several scenarios.

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APPENDIX

Input-output table:

	1	2	3	D	V	X
1	10	15	25	50	0	100
2	40	20	0	40	0	100
3	30	25	35	10	0	100
D	0	0	0	0	0	0
V	20	40	40	0	0	0
X	100	100	100	0	0	0

Parameter BETA = 0.8

```
----    153 PARAMETER mulx  multiplier matrix

             1           2           3

1      2.3501      1.1319      1.3063
2      1.5673      2.0882      0.8711
3      2.4386      1.9155      3.1337


----    153 PARAMETER muxp  Paasche multipliers

1 6.3562,    2 5.1355,    3 5.3119


----    155 PARAMETER elas  elasticity of cost to final demand

             1           2           3

1      0.0894      0.0519      0.0144
2      0.0667      0.0513      0.0109
3      0.0813      0.0494      0.0176
```

PARAMETER BETA = 1.2

```
----    153 PARAMETER mulx  multiplier matrix

            1           2           3

1      1.3010      0.2908      0.3826
2      0.5203      1.3163      0.1530
3      0.6121      0.4897      1.5917


----    153 PARAMETER muxp  Paasche multipliers

1 2.4333,    2 2.0967,    3 2.1271


----    155 PARAMETER elas  elasticity of cost to final demand

            1           2           3

1     -0.0247     -0.0128     -0.0038
2     -0.0155     -0.0165     -0.0028
3     -0.0174     -0.0099     -0.0056
```

PARAMETER BETA = 1 (standard Leontief)

```
----    153 PARAMETER mulx  multiplier matrix

            1           2           3

1      1.5116      0.4651      0.5814
2      0.7558      1.4826      0.2907
3      0.9884      0.7849      1.9186


----    153 PARAMETER muxp  Paasche multipliers

1 3.2558,    2 2.7326,    3 2.7907
```

NO COST EFFECTS