# Block structural path analysis in a multiregional input-output system: An environmental application to Asia Pacific region

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## 1. Introduction

The increasing level of anthropogenic carbon dioxide (CO2) emissions and the related reduction mechanisms are important topics in the sustainability debate. Current national GHG inventories account for "territorial emissions", which do not take account of trade and embodied emissions.

Since the early 1990s, input-output analysis has been widely used for environmental accounting, such as carbon footprint accounting and the calculations of embodied emissions. However, still little is known about the relationship between supply chain and CO2 emission.

In this context, the purpose of this paper is to decompose multiregional economic complex into regional blocks based on hierarchical structure of regional interactions. Similar to the method of Structural Path Analysis (SPA), we adopt the so called "Block Structural Path Analysis (hereafter Block SPA)" based on multiregional input-output (hereafter IO) model. This method is then extended to environmental analysis for $CO_2$ emissions. Results from this study can help: 1) unveil by what paths and how strong regions interact with each other; 2) attribute aggregate environmental responsibility to those associated with paths connecting regional interactions.

## 2. Methodology

Block SPA, introduced by Sonis, M. and Hewings, G.J.D.(1998), is a decomposition method of the Leontief inverse matrix in multiregional IO model. The transaction matrix is partitioned into region blocks and the Leontief inverse matrix is decomposed upon blocks by means of matrix algebra techniques.

The 2000 Asian multi-region input-output analysis. (hereafter MRIO)[3)] is applied, in which 10 countries/region are included and there are 24 sectors in each country/region.

### *2.1 Basic multiregional IO model and environmental application*

The basic IO model is as follows:

$$X = AX + F + E \tag{1}$$

with $X$: total output vector; $A$: transaction coefficient matrix; $F$: final demand matrix; $E$: export vector.

We obtain the following equation:

$$X = [I - A]^{-1}[F + E] = B[F + E] \tag{2}$$

$[I - A]^{-1}$ is Leontief inverse matrix notated as $B$. To calculate embedded environmental loads $EL$, we first pre-multiply $B$ by environmental index matrix $E$ (a diagonal matrix of regional environmental load per unit sectoral output) (see Eq.(3)).

$$EL = E[I - A]^{-1}F = EBF \tag{3}$$

### *2.2 Block Structural Path Analysis*

To conduct Block SPA, the multi-region input-output model is partitioned into regional blocks. We use a two-region model for illustration as follows.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (I - A_{11}) & -A_{12} \\ -A_{21} & (I - A_{22}) \end{bmatrix}^{-1} \left( \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} B_{11}^{(2)} & B_{12}^{(2)} \\ B_{21}^{(2)} & B_{22}^{(2)} \end{bmatrix} \left( \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right) \tag{4}$$

With xr: output vector of Region r (r=1,2); $A_{rs}$ (r,s =1,2):matrix of interregional transaction coefficients; $f_{rs}$: final demand in Region r that is supplied by Region s; $e_r$: export from region r to rest of the World(ROW); $B_{rs}^{(2)}$:block matrix in the two-region Leontief inverse matrix and the superscript represents number of regions(i.e. two regions in this case).

The purpose of Block SPA is to decompose the Leontief block inverse matrix B into a combination of block matrix

The partitioned Leontief inverse matrix $B$ could be solved by Gaussian elimination.

$$
\mathrm{B} = \begin{bmatrix} B_{11}^{(2)} & B_{12}^{(2)} \\ B_{21}^{(2)} & B_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} B_{11}^{(2)} & B_1 A_{12} B_{22}^{(2)} \\ B_2 A_{21} B_{11}^{(2)} & B_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} B_{11}^{(2)} & B_{11}^{(2)} A_{12} B_2 \\ B_{22}^{(2)} A_{21} B_1 & B_{22}^{(2)} \end{bmatrix} \tag{5}
$$

with $B_1 = \left[I - A_{11}\right]^{-1}$; $B_2 = \left[I - A_{22}\right]^{-1}$

$$
B_{11}^{(2)} = (I - A_{11} - A_{12} B_2 A_{21})^{-1}, \; B_{22}^{(2)} = (I - A_{22} - A_{21} B_1 A_{12})^{-1} \tag{6}
$$

$$
B_{12}^{(2)} = B_1 A_{12} B_{22}^{(2)} = B_{11}^{(2)} A_{12} B_2 , \;\; B_{21}^{(2)} = B_2 A_{21} B_{11}^{(2)} = B_{22}^{(2)} A_{21} B_1 \tag{7}
$$

The hierarchy of bilateral linkages could be depicted by the following chart (Fig. 2). $B_{11}^{(2)}$, the so-called "extended Leontief multiplier" of Region 1, indicates self-influence of the region. It indicates i) direct self-influence of Rgeion 1 through the intra-regional transaction matrix, $A_{11}$; and ii) indirect self-influence of Region 1 via Region 2 built upon the path $A_{12}B_2A_{21}$. A similar explanation is held for $B_{22}^{(2)}$. $B_{22}^{(2)}$ shows the mechanism of transferring influences from Region 1 to Region 2, which consists of the block matrix of the Leontief inverse of Region 1 (i.e. $B_1$) and the block matrix of the extended Leontief multiplier of Region 2 (i.e. $B_2$) via inter-regional transaction from Region 1 to 2 (i.e. $A_{12}$). A similar explanation can be given to $B_{21}^{(2)}$.

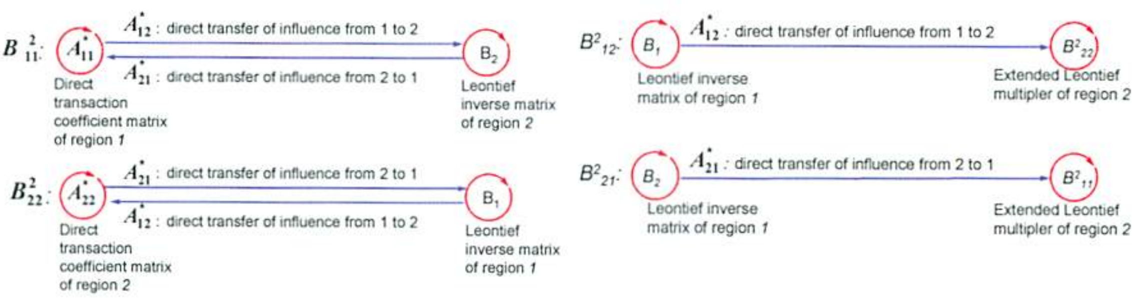



Fig.2 Hierarchy of two-regional interactions

The extrapolation of the two-region model to a three-region model is presented in Appendix A. The proof of the recursive form from (n-1) regions to n regions can be referred to in Sonis ans Hewings(1988). In this paper, we focus on the hierarchical structure of transferring influences among region , i.e. $B_{rs}^{(n)}(r, s = 1, \cdots, n \; and \; r \neq s)$, block matrix on the off-diagonal of B. The decomposition of $B_{rs}^{(10)}$ in MRIO is presented as follows.

$$B_{rs}^{(10)} = B_{rr}^{(10)} A_{rs}^{(10)} B_{ss}^{(9)}$$

$$= B_{rr}^{(10)} \left\{ \begin{array}{c} A_{rs} + \sum_{\substack{s_1=1 \\ s_1 \neq r,s}}^{10} A_{rs_1} B_{s_1 s_1}^{(8)}(r,s) A_{s_1 s} + \sum_{\substack{s_1=1 \\ s_1 \neq r,s}}^{10} \sum_{\substack{s_2=1 \\ s_2 \neq r,s,s_1}}^{10} A_{rs_1} B_{s_1 s_1}^{(8)}(r,s) A_{s_1 s_2} B_{s_2 s_2}^{(7)}(r,s,s_1) A_{s_2 s} \\ + \cdots + \sum_{\substack{s_1=1 \\ s_1 \neq r,s}}^{10} \sum_{\substack{s_2=1 \\ s_2 \neq r,s,s_1}}^{10} \sum_{\substack{s_3=1 \\ s_3 \neq r, \\ s,s_1,s_2}}^{10} \sum_{\substack{s_4=1 \\ s_4 \neq r, \\ s,s_1, \\ s_2,s_3}}^{10} \sum_{\substack{s_5=1 \\ s_5 \neq r, \\ s,s_1, \\ s_2,s_3, \\ s_4}}^{10} \sum_{\substack{s_6=1 \\ s_6 \neq r, \\ s,s_1, \\ s_2,s_3, \\ s_4,s_5}}^{10} \sum_{\substack{s_7=1 \\ s_7 \neq r, \\ s,s_1, \\ s_2,s_3, \\ s_4,s_5, \\ s_6}}^{10} \sum_{\substack{s_8=1 \\ s_8 \neq r, \\ s,s_1, \\ s_2,s_3, \\ s_4,s_5, \\ s_6,s_7}}^{10} \begin{bmatrix} A_{rs_1} B_{s_1 s_1}^{(8)}(r,s) A_{s_1 s} \\ B_{s_2 s_2}^{(7)}(r,s,s_1) A_{s_2 s_3} \\ B_{s_3 s_3}^{(6)}(r,s,s_1,s_2) A_{s_2 s_3} \\ B_{s_4 s_4}^{(5)}(r,s,s_1,s_2,s_3) A_{s_3 s_4} \\ B_{s_5 s_5}^{(4)}(r,s,s_1,s_2,s_3,s_4) A_{s_4 s_5} \\ B_{s_6 s_6}^{(3)}(r,s,s_1,s_2,s_3,s_4,s_5) A_{s_5 s_6} \\ B_{s_7 s_7}^{(2)}(r,s,s_1,s_2,s_3,s_4,s_5,s_6) A_{s_6 s_7} \\ B_{s_8 s_8}^{(1)}(r,s,s_1,s_2,s_3,s_4,s_5,s_6,s_7) A_{s_7 s} \end{bmatrix} \end{array} \right\} B_{ss}^{(9)}(r) \quad (8)$$

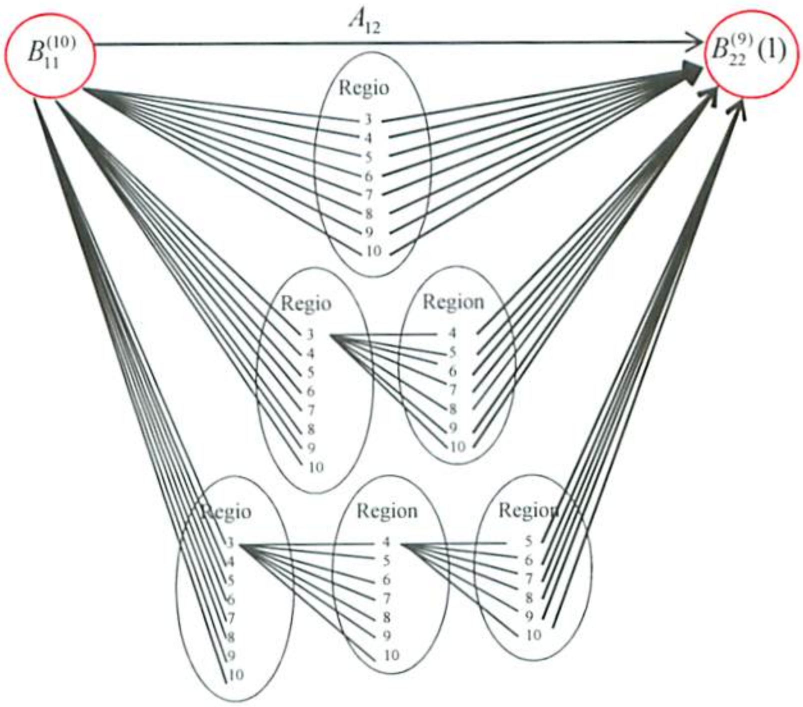
with $B_{rs}^{(10)}$: block matrix on the off-diagonal of B, representing the transfer of influences from Region r to s; $B_{rr}^{(10)}$:block matrix on the diagonal of B, indicating self-influences of Region r; $B_{ss}^{(9)}$:matrix on diagonal of the nine-region partial Leontief inverse matrix, which is calculated from $[I - A(r)]^{-1}$ and A(r) is a nine-region partial matrix of A derived by deleting block row r and block column r from A;

The hierarchal structure of $B_{12}^{(10)}$ is illustrated in Fig.1 as an example for how influences are transferred among regions. $B_{11}^{(10)}$ and $B_{22}^{(9)}(1)$ are two pillar block matrices upon which $B_{11}^{(10)}$ is built. $B_{11}^{(10)}$ indicates the pushing power of Region 1 and $B_{22}^{(9)}(1)$ represents the aggregate pulling power of Region 2. $A_{12}^{(10)}$ the link connecting these two block matrices, indicates the strength of influence transfer. $B_{12}^{(10)}$ , a product of $B_{11}^{(10)}$ , $A_{12}^{(10)}$ and $B_{22}^{(9)}(1)$ , is influenced by the three blocks simultaneously.

### *2.3 Environmental Application of Block Structural Path Analysis*

The extension of Block SPA to environmental analysis is conducted by pre-multiplying B by the environmental coefficient matrix $\hat{C}(z)$ to obtain the environmental multiplier matrix (Equation 3) in a partitioned fashion. For the calculation of carbon intensity, we use GTAP-E Database Version 6.

$$\widehat{C}(z)B = \begin{bmatrix} \widehat{C_1}(z)B_{11}^{(10)} & \cdots & \widehat{C_1}(z)B_{11}^{(10)} \\ \vdots & \ddots & \vdots \\ \widehat{C_{10}}(z)B_{101}^{(10)} & \cdots & \widehat{C_{10}}(z)B_{1010}^{(10)} \end{bmatrix} \quad (9)$$

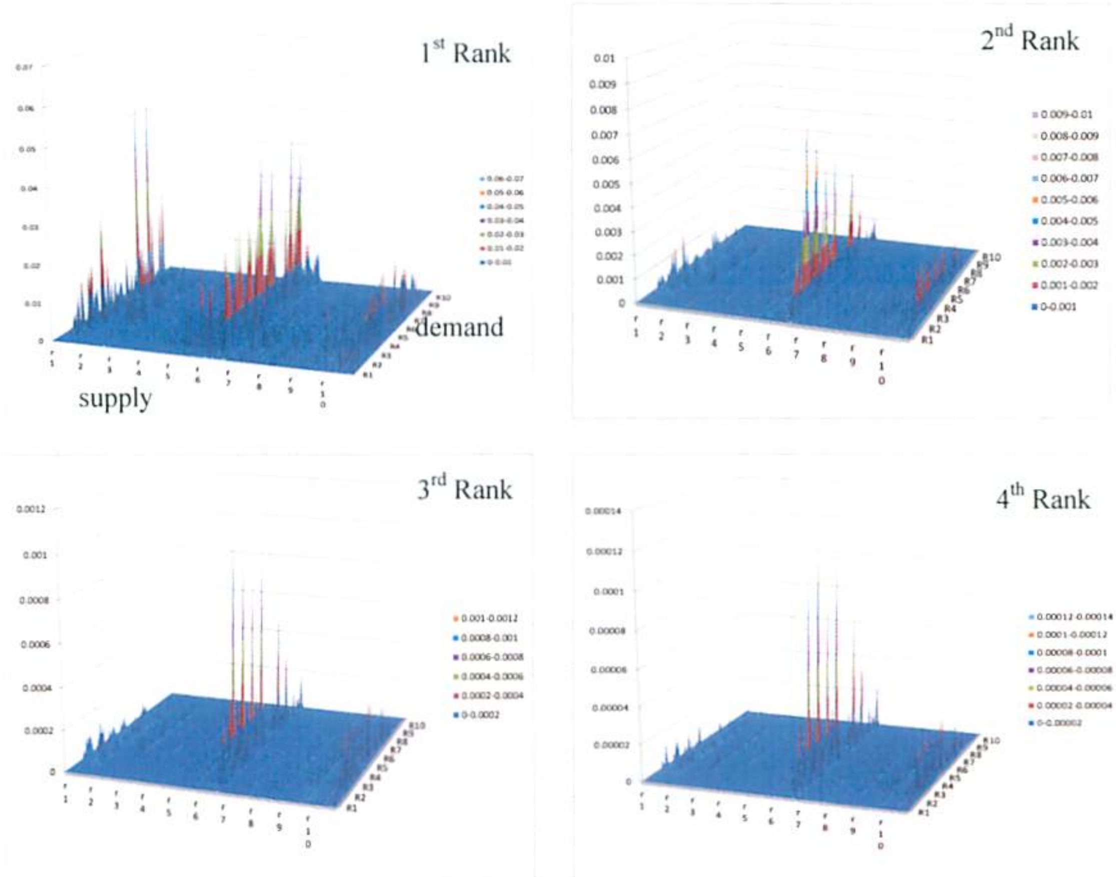



| **Hierarchy and components** | **No. of paths** |
|---|---|
| First rank $B_{11}^{(10)} A_{12} B_{22}^{(9)}$ | 1 |
| Second rank $B_{11}^{(10)} \left\{ \sum_{s_1=3}^{8} A_{1s_1} B_{s_1 s_1}^{(6)}(1,2) A_{s_1 2} \right\} B_{22}^{(9)}$ | $C_8^1 = 8$ |
| Third rank $B_{11}^{(10)} \left\{ \sum_{s_1=3}^{8} \left( \sum_{\substack{s_2=3 \\ s_2 \neq s_1}}^{8} \begin{matrix} A_{1s_1} B_{s_1 s_1}^{(6)}(1,2) A_{s_1 s_2} \\ B_{s_2 s_2}^{(5)}(1,2,s_1) A_{s_1 2} \end{matrix} \right) \right\} B_{22}^{(9)}$ | $C_8^1 C_7^1 = 56$ |
| Forth rank $B_{11}^{(10)} \left\{ \sum_{s_1=3}^{8} \left( \sum_{\substack{s_2=3 \\ s_2 \neq s_1}}^{8} \sum_{\substack{s_3=3 \\ s_3 \neq s_1, \\ s_2}}^{8} \begin{matrix} A_{1s_1} B_{s_1 s_1}^{(6)}(1,2) A_{s_1 s_2} \\ B_{s_2 s_2}^{(5)}(1,2,s_1) A_{s_2 s_3} \\ B_{s_3 s_3}^{(4)}(1,2,s_1,s_2) A_{s_3 2} \end{matrix} \right) \right\} B_{22}^{(9)}$ | $C_8^1 C_7^1 C_6^1 = 336$ |
| ⋮ | ⋮ |
| Total paths | $1 + C_8^1 + C_8^1 C_7^1 + C_8^1 C_7^1 C_6^1 + C_8^1 C_7^1 C_6^1 C_5^1$ $C_8^1 C_7^1 C_6^1 C_5^1 C_4^1 + C_8^1 C_7^1 C_6^1 C_5^1 C_4^1 C_3^1$ $= 20{,}161$ |

Fig.1 Causal links of $B_{12}^{(10)}$ in ten-region economic complex

## 3. Results

We conduct the Block SPA for each $B_{rs}^{(10)}$ ($r \neq s$, $r,s = 1, \cdots, 10$) in the MRIO model at both sectoral base and global intensity base ($\sum_r \sum_s B_{rs}^{10}$).Figre 2. 1st, 2nd ,3rd and 4th rank of the decompositions of $\hat{C}(z)B_{rs}^{(10)}$ ..




Figre 2. $1^{st}$, $2^{nd}$ ,$3^{rd}$ and $4^{th}$ rank of the decompositions of $\hat{C}(z)B_{rs}^{(10)}$

(Note) No of x and y axis is country code. 1:Indonesia, 2:Malaysia, 3:Philipine, 4:Singapore, 5:Thailand 6:China, 7:Taiwan, 8:Korea, :9:Japan, 10:USA

Results show that linkages in the first, second and third rank are generally stronger but tend to be neglectable after third rank. And China (region6) has very strong back ward effect.

Figure 3 shows the amount of induced $CO_2$ emission by rank and Figure 4 shows share of induce $CO_2$ emission for total amount of emission.

China' s $CO_2$ emission in these four ranks is about 160 million t and it is rather large compared with other countries. The second largest induced emission country is USA, and it' s induced emission is 49 million t.

Regarding with share of induced $CO_2$ emission for total emission, Malaysia, Indonesia and Taiwan are relatively larger than other countries. It is about 24% in Malaysia, 19% in Indonesia and 18% in Taiwan. And in these countries, share of induced CO2 emission by second rank for total emission also is relatively larger than other countries. On the other hand, share of induced CO2 emission for total emission is 1% in USA, 4% in Japan and 6% in China.

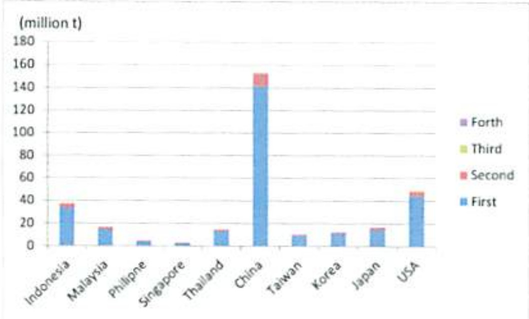



Figure 3. Amount of induced CO2 emission by rank

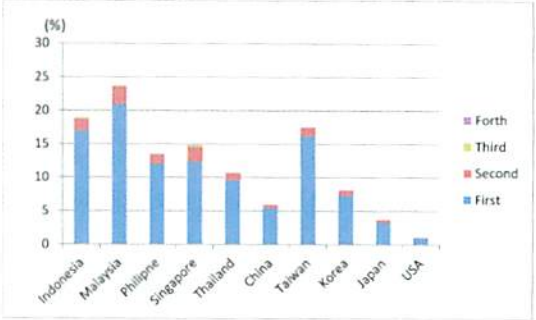



Figure 4. Share of induced CO2 emission for total emission

## 4. Concluding remarks

This paper provides a regional-level decomposition method of the Leontief inverse matrix in multiregional IO framework. It helps establish the hierarchy of regional interdependency and quantify how strong the influences and their propagations are. This method is then applied to analyze regional interactions for both economic and environmental purposes. Results from economic analysis help find out which region plays what role and by what interactions in a multiregional economic complex. Environmental extension of this method is a novel one, by which embedded environmental responsibilities are attributed to each economic interaction among regions in an economic complex.

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Appendix A

Consider the transaction coefficient matrix A in a partitioned from in a three-region MRIO system:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad \text{(A1)}$$

With Ars (r,s = 1,2,3) : transaction coefficient block matrix from region r to region s.
The corresponding Leontief inverse block matrix is as follows:

$$B = (I-A)^{-1} \begin{bmatrix} B_{11}^{(3)} & B_{12}^{(3)} & B_{13}^{(3)} \\ B_{21}^{(3)} & B_{22}^{(3)} & B_{23}^{(3)} \\ B_{31}^{(3)} & B_{32}^{(3)} & B_{33}^{(3)} \end{bmatrix} \quad \text{(A2)}$$

The superscript of $B_{rs}^{(3)}$ represents number of regions( i.e. three regions in this case).
To make use of the formulae derived from the two-region model (see Equations 6, 7 and 8), we define three partial block matrix of A.

$$\mathrm{A}(1) = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}, \ \mathrm{A}(2) = \begin{bmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{bmatrix}, \ \mathrm{A}(3) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{(A3)}$$

With A(1): the partial block of A which includes the transaction coefficients of Region 2 and 3; A(2): the partial block of A which includes the transaction coefficients of Region 1 and 3; and A(3) partial block matrix of A which includes the transaction coefficients of Regions 1 and 2.

Based on Equation 6, the corresponding Leontief inverse block matrix of Equation B3 are calculated as follows:

$$\mathrm{B}(1) = [I - A(1)]^{-1} = \begin{bmatrix} B_{22}^{(2)}(1) & B_{23}^{(2)}(1) \\ B_{32}^{(2)}(1) & B_{33}^{(2)}(1) \end{bmatrix}$$

$$= \begin{bmatrix} B_{22}^{(2)}(1) & B_{23}^{(2)}(1)A_{23}B_3 \\ B_{32}^{(2)}(1)A_{32}B_2 & B_{33}^{(2)}(1) \end{bmatrix} = \begin{bmatrix} B_{22}^{(2)}(1) & B_2A_{23}B_{33}^{(2)}(1) \\ B_3A_{32}B_{22}^{(2)}(1) & B_{33}^{(2)}(1) \end{bmatrix} \quad \text{(A4-1)}$$

$$\mathrm{B}(2) = [I - A(2)]^{-1} = \begin{bmatrix} B_{11}^{(2)}(2) & B_{13}^{(2)}(2) \\ B_{31}^{(2)}(2) & B_{33}^{(2)}(2) \end{bmatrix} \quad \text{(A4-2)}$$

$$= \begin{bmatrix} B_{11}^{(2)}(2) & B_{23}^{(2)}(2)A_{23}B_3 \\ B_{33}^{(2)}(3)A_{31}B_1 & B_{33}^{(2)}(2) \end{bmatrix} = \begin{bmatrix} B_{11}^{(2)}(2) & B_1A_{13}B_{33}^{(2)}(2) \\ B_3A_{31}B_{11}^{(2)}(2) & B_{33}^{(2)}(2) \end{bmatrix}$$

$$\mathrm{B}(3) = [I - A(3)]^{-1} = \begin{bmatrix} B_{11}^{(2)}(3) & B_{12}^{(2)}(3) \\ B_{21}^{(2)}(3) & B_{22}^{(2)}(3) \end{bmatrix} \quad \text{(A4-3)}$$

$$= \begin{bmatrix} B_{11}^{(2)}(3) & B_{11}^{(2)}(3)A_{12}B_2 \\ B_{22}^{(2)}(3)A_{21}B_1 & B_{22}^{(2)}(3) \end{bmatrix} = \begin{bmatrix} B_{11}^{(2)}(3) & B_1A_{12}B_{22}^{(2)}(3) \\ B_3A_{21}B_{11}^{(2)}(3) & B_{22}^{(2)}(3) \end{bmatrix}$$

Where $B_1 = (I - A_{11})^{-1}, B_2 = (I - A_{22})^{-1}$ , $B_3 = (I - A_{33})^{-1}$

First, we partition A in the following way which is different from Equation B1:

$$\mathrm{A} = \begin{bmatrix} A_{11} & (A_{12}\ A_{13}) \\ \begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} & A(1) \end{bmatrix} \quad \text{(A5)}$$

Based on Equation 6 and Equation B2, the Leontief inverse block matrix calculated from Equation B5

is:

$$\mathrm{B} = (I - A)^{-1} = \begin{bmatrix} I - A_{11} & -(A_{12}\ A_{13}) \\ -\begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} & I - A(1) \end{bmatrix}^{-1} \quad \text{(A6)}$$

$$= \begin{bmatrix} B_{11}^{(3)} & B_{11}^{(3)}(A_{12}\,A_{13})B(1) \\ \bar{B}^{(3)}\begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix}B_1 & \bar{B}^{(3)}(1) \end{bmatrix}$$

Our focus is on calculating $B_{11}^{(3)}$ and $B_{11}^{(3)}(A_{12}\,A_{13})B(1)$ based on Equation 7.

$$B_{11}^{(3)} = \left[I - A_{11} - (A_{12}\,A_{13})B(1)\begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix}\right]^{-1} \quad \text{(A7)}$$

$$= \left[I - A_{11} - (A_{12}\,A_{13})\begin{pmatrix} B_{22}^{(2)}(1) & B_2A_{23}B_3 \\ B_3A_{32}B_2 & B_{33}^{(2)}(1) \end{pmatrix}\begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix}\right]^{-1}$$

$$= \left[I - A_{11} - A_{12}B_{22}^{(2)}(1)A_{21} - A_{13}B_{33}^{(2)}(1)A_{32}B_2A_{21} - A_{12}B_{22}^{(2)}(1)A_{23}B_3A_{31} - A_{13}B_{33}^{(2)}(1)A_{31}\right]^{-1}$$

$$= \left[I - A_{11} - A_{12}B_{22}^{(2)}(1)(A_{21} + A_{23}B_3A_{31}) - A_{13}B_{33}^{(2)}(1)(A_{31} + A_{32}B_2A_{21})\right]^{-1}$$

$$= \left[I - A_{11} - A_{12}B_{22}^{(2)}(1)A_{21}^{3} - A_{13}B_{33}^{(2)}(1)A_{31}^{3}\right]^{-1}$$

Where $A_{21}^{3} = A_{21} + A_{23}B_3A_{31}$, $A_{31}^{3} = A_{31} + A_{32}B_2A_{21}$.

$$B_{11}^{(3)}(A_{12}\ A_{13})B(1) = B_{11}^{(3)}(A_{12}\ A_{13})\begin{pmatrix} B_{22}^{(2)}(1) & B_2A_{23}B_3 \\ B_3A_{32}B_2 & B_{33}^{(2)}(1) \end{pmatrix} \tag{A8}$$

$$= \left[\left(B_{11}^{(3)}A_{12}B_{22}^{(2)}(1) + B_{11}^{(3)}A_{13}B_{33}^{(2)}(1)A_{32}B_2\right)\ \left(B_{11}^{(3)}A_{12}B_{22}^{(2)}(1)A_{23}B_3 + B_{11}^{(3)}A_{13}B_{33}^{(2)}(1)\right)\right]$$

$$= \left[\left(B_{11}^{(3)}A_{12}B_{22}^{(2)}(1) + B_{11}^{(3)}A_{13}B_3A_{32}B_{22}^{(2)}(1)\right)\ \left(B_{11}^{(3)}A_{12}B_2A_{23}B_{33}^{(2)}(1) + B_{11}^{(3)}A_{13}B_{33}^{(2)}(1)\right)\right]$$

$$= \left[\left(B_{11}^{(3)}(A_{12} + A_{13}B_3A_{32})B_{22}^{(2)}(1)\right)\ \left(B_{11}^{(3)}(A_{13} + A_{12}B_2A_{23})B_{33}^{(2)}(1)\right)\right]$$

$$= \left[B_{11}^{(3)}A_{12}^{3}B_{22}^{(2)}(1)\ \ B_{11}^{(3)}A_{13}^{3}B_{33}^{(2)}(1)\right]$$

Where $A_{12}^3 = A_{12} + A_{13}B_3A_{32}$, $A_{13}^3 = A_{13} + A_{12}B_2A_{23}$.

Therefore we obtain the formulae for $B_{12}^{(3)}$ and $B_{13}^{(3)}$ as follows.

$$B_{12}^{(3)} = B_{11}^{(3)}A_{12}^3B_{22}^{(2)}(1),\ B_{13}^{(3)} = B_{11}^{(3)}A_{13}^3B_{33}^{(2)}(1) \tag{A9}$$

Next, we partition A in other two ways as follows:

$$\mathrm{A} = \begin{bmatrix} A_{22} & (A_{21}\ A_{23}) \\ \begin{pmatrix} A_{12} \\ A_{32} \end{pmatrix} & A(2) \end{bmatrix} \tag{A10}$$

$$\mathrm{A} = \begin{bmatrix} A_{33} & (A_{31}\ A_{32}) \\ \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix} & A(3) \end{bmatrix} \tag{A11}$$

Similarly to Equations B7, B8 and B9, we can obtain the following formulae:

$$B_{22}^{(3)} = \left[I - A_{22} - A_{21}B_{11}^{(2)}(2)A_{12}^3 - A_{23}B_{33}^{(2)}(2)A_{32}^3\right]^{-1} \tag{A12}$$

$$B_{33}^{(3)} = \left[I - A_{33} - A_{31}B_{11}^{(2)}(3)A_{13}^3 - A_{32}B_{22}^{(2)}(3)A_{23}^3\right]^{-1} \tag{A13}$$

$$B_{21}^{(3)} = B_{22}^{(3)}A_{21}^3B_{11}^{(2)}(2),\ B_{23}^{(3)} = B_{22}^{(3)}A_{23}^3B_{33}^{(2)}(2) \tag{A14}$$

$$B_{31}^{(3)} = B_{33}^{(3)}A_{31}^3B_{11}^{(2)}(3),\ B_{32}^{(3)} = B_{33}^{(3)}A_{32}^3B_{22}^{(2)}(3) \tag{A15}$$

Where $A_{23}^3 = A_{23} + A_{21}B_1A_{13}$, $A_{32}^3 = A_{32} + A_{31}B_1A_{12}$.

$$B = (I - A)^{-1} = \begin{bmatrix} B_{11}^{(3)} & B_{11}^{(3)} A_{12}^{3} B_{22}^{(2)}(1) & B_{11}^{(3)} A_{13}^{3} B_{33}^{(2)}(1) \\ B_{22}^{(3)} A_{21}^{3} B_{11}^{(2)}(2) & B_{22}^{(3)} & B_{33}^{(3)} A_{32}^{3} B_{22}^{(2)}(3) \\ B_{33}^{(3)} A_{31}^{3} B_{11}^{(2)}(3) & B_{33}^{(3)} A_{32}^{3} B_{22}^{(2)}(3) & B_{33}^{(3)} \end{bmatrix} \quad \text{(A16)}$$

More general formulae are as follows:

$$A_{ij}^{3} = A_{ij} + A_{is} B_s A_{sj} \quad \mathrm{i, j, s} = 1,2,3; \mathrm{i} \neq \mathrm{j}, \mathrm{i} \neq \mathrm{s}, \mathrm{j} \neq \mathrm{s} \quad \text{(A17)}$$

$$B_{ii}^{(3)} = \left[ I - A_{ii} - A_{ij} B_{jj}^{(2)}(i) A_{ji}^{3} - A_{is} B_{ss}^{(2)}(2) A_{si}^{3} \right]^{-1}, \ \mathrm{i, j, s} = 1,2,3; \mathrm{i} \neq \mathrm{j}, \mathrm{i} \neq \mathrm{s}, \mathrm{j} \neq \mathrm{s} \quad \text{(A18)}$$

A similar extrapolation can be conducted for a four-region based on Equation A16, B17 and B18. And so on for the extrapolation of n-region MRIO system based on the formulae derived from a (n-1)-region MRIO framework. For the derivation of the recursive generalization for a n-region model, see Sonis and Hewings (1998).