General Equilibrium Theory and Increasing Returns: an Alternative to the Survival Assumption

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Abstract

Existence results for equilibria in economies under increasing returns to scale, fixed costs, or showing more general types of non convexity in the production sector, strongly rest on a crucial condition, known as the survival Assumption. This assumption is unsatisfactory in the sense that it poses a condition on the set of production equilibria, an endogenous variable. We propose here conditions on the firms characteristics, notably on the firms' pricing behavior, under which the equilibrium existence can be proved.

Key Words: General Equilibrium Theory, Increasing Returns, Survival Assumption.

JEL Classification: C62, C67, D21, D51

1 Introduction

The presence of increasing returns to scale, of fixed costs, and, more generally the lack in convexity properties in the production sector, are recognized as failure factors of the competitive mechanism. Walras (1874) first proposed that the non convex firms should be set to follow an average cost pricing behavior. Later, the theory of marginal cost pricing has been developed, with the works of Pigou (1932), Lange (1936-1937), Lerner (1936), Hotelling (1938), and later of Allais (1953). The marginal cost principle suppose that: (i) firms

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minimize their costs at given prices, and (ii) they set their selling prices equal to marginal cost. Since Guesnerie (1975), this theory has been formulated in the abstract framework of general equilibrium theory, and many results were proposed, addressing the existence problem of marginal cost pricing equilibria and their optimality. In parallel, an other formulation was developed, with the emergence of existence results for general pricing rules with bounded losses. For a global survey of this kind of results, see the special issue of the *Journal of Mathematical Economics* on General Equilibrium and Increasing Returns, edited by Cornet in 1988 [see Cornet (1988)], Brown's chapter in *Handbook of Mathematical Economics* [Brown (1991)], or Villar (1996).

All these results make a crucial assumption, the *survival assumption*. The purpose of this hypothesis is to solve the consumers' aggregate survival problem. In the classic Arrow-Debreu framework, it is sufficient to assume that the total initial endowments vector lies in the interior of the aggregate consumption set. In models incorporating increasing returns, where firms may exhibit deficits, this kind of assumption is no longer enough. We then need a condition typically stated on *production equilibria*. In an economy with finite numbers ℓ of commodities and n of firms, we denote by Y_j the production set of firm j, $1 \leq j \leq n$, and by ϕ_j its pricing rule, which associates with any production plan $y_j \in Y_j$ a set $\phi_j(y_j)$ of price systems acceptable at y_j, according to the pricing behavior of firm j. Firm j is thus said to be at equilibrium for a pair (p, y_j) if the price system p is acceptable for the firm j at $y_j \in Y_j$, i.e. if $p \in \phi_j(y_j)$. A production equilibrium is then a state $(p, (y_j))$ of the economy in which the price system p is acceptable for every firm, given the production allocation (y_j). Bonnisseau and Cornet (1991) propose the following version of the survival assumption: at production equilibrium, if a positive amount of a reference commodity bundle (which can be seen as a specie) is added to the total initial endowments, so that the production equilibrium becomes attainable, then the total wealth in the economy is above the consumers aggregate subsistence level. This kind of assumption is unsatisfactory in the sense that it poses a condition on the set of production equilibria, an endogenous variable of the model. It would be worth preferable to state a condition on the primitive data of the economy, notably on the firms characteristics.

The aim of this article is to propose conditions on the firms individual pricing behavior under which the aggregate survival is ensured. These conditions are expressed in a framework where the commodity set is parted, a posteriori, into inputs and outputs. At every production plan y, we call an input every commodity h satisfying $y^h < 0$. We thus mean by output any commodity not being engaged in the production process y. Our first condition expresses a willing to forward the commodities sophistication. We shall suppose that the commodity set can be parted into classes representing a grading according to their sophistication. Commodities that are useless in the production of any other commodity, usually called final outputs, are the most sophisticated commodities, and the more production transformations are needed to get a commodity h, the more it is sophisticated. This assumption forwarding the commodities sophistication says that, for every firm j, given a production plan y_j, the acceptable price systems $p \in \phi_j(y_j)$ are such that if the set of the outputs associated with a positive price is nonempty, then one of these outputs is more sophisticated than each input associated with a positive price. The second condition can be seen as a "token damages" insurance: for a given production plan $y_j \in Y_j$, any price system which associates at least one input, but to no output, with a positive price can't be acceptable for firm j. In other words, for a given production plan, a firm doesn't accept price systems involving costs but no incomings.

In the particular case of marginal pricing, Vohra (1992) also proposes conditions on the production sets which are easily interpretable and which imply for the survival assumption to be satisfied. Indeed, he points out, by considering a single consumer, single producer and two commodities economy, that there cannot be a marginal pricing equilibrium when the producer possibility curve is tangent to the "output" axis, i.e. to the axis corresponding to the produced commodity. Such a tangency implies for the marginal rate at which the input is transformed into output to be infinite. To obtain the existence of a marginal pricing equilibrium, marginal returns, though they are increasing, must be finite, hence bounded. Vohra (1992) formalizes this condition on marginal returns in a model where inputs and outputs are distinguished *a priori*. This distinction can be seen as a succinct ranking according to the commodities sophistication, and the bounded marginal assumption imply a kind of token damages insurance. We shall propose a formulation of this bounded marginal returns condition in our framework, where inputs and outputs are distinguished *a posteriori*, and show that the token damages condition is indeed a consequence of the bounded marginal returns condition.

2 The general equilibrium model and the survival assumption

We consider an economy with positive finite numbers ℓ of commodities, m of consumers and n of firms. We take $\mathbb{R}^\ell$ for commodity space, and we consider normalized price vectors in S, the unit simplex of $\mathbb{R}^\ell$.

The technological possibilities of firm j ($j = 1, \ldots, n$) are represented by a subset $Y_j \subset \mathbb{R}^\ell$ satisfying the following assumption:

Assumption (P) *For every j, Y_j is a nonempty, closed subset of $\mathbb{R}^\ell$ such that $Y_j - \mathbb{R}^\ell_+ \subset Y_j$ and $0 \in Y_j$.*

Note that, under Assumption (P), for every j, the boundary ∂Y_j of the production set Y_j exactly coincides with the set of (weakly) efficient production plans of firm j. For every j, the behavior of firm j is described by its *pricing rule* ϕ_j, a correspondence which associates with each (weakly) efficient production plan y_j a subset $\phi_j(y_j)$ of acceptable price vectors. This formalization is compatible with various behaviors considered in the economic literature, notably with the average pricing and the marginal pricing behaviors. Given an individually (weakly) efficient production allocation $(y_j) \in \prod_{j=1}^n \partial Y_j$, if all the firms regard as acceptable a price vector $p \in S$, i.e. if $p \in \varphi_j(y_j)$ for every j, then we say that the collection $(p,(y_j))$ is a *production equilibrium*. We shall denote by PE the set of production equilibria of the economy, that is:

$$PE = \left\{ (p,(y_j)) \in S \times \textstyle\prod_{j=1}^n \partial Y_j \middle| p \in \phi_j(y_j) \text{ for every } j \right\}.$$

An individually (weakly) efficient production allocation $(y_j) \in \prod_{j=1}^n \partial Y_j$ is said to be t*-attainable* if it becomes attainable when t units of the reference commodity bundle $e = (1,\ldots,1) \in \mathbb{R}^\ell$ are injected in the economy. For every $t \in \mathbb{R}_+$, we let A_t be the set of t-attainable individually (weakly) efficient production allocations:

$$A_t = \left\{ (y_j) \in \textstyle\prod_{j=1}^n \partial Y_j \;\middle|\; \sum_{j=1}^n y_j + \omega + te \in X + \mathbb{R}_+^\ell \right\}.$$

The consumption side of the economy is standard and we shall not dwell on its description, recalling that our focus is on the production sector. We let $X \subset \mathbb{R}^\ell$ be the aggregate consumption set, and $\omega \in \mathbb{R}^\ell$ be the vector of total initial endowments. We make the following assumption on the consumption side:

Assumption (C*) $X = \mathbb{R}_+^\ell$.

Existence results of equilibria under increasing returns rest on the Survival Assumption. We shall consider in this paper the version of this assumption given in Bonnisseau and Cornet (1991).

Assumption (WSA) *For every $(p,(y_j)) \in PE$ and for every $t \geq 0$, if $(y_j) \in A_t$, then $p \cdot (\sum_{j=1}^n y_j + \omega + t\,e) > \inf p \cdot X$.*

Assumption (WSA) is a subsistence condition in the sense that it ensures that, at any production equilibrium, if a positive amount of the reference commodity bundle e is injected in the economy so that the production equilibrium becomes attainable, then the total wealth is above the consumers aggregate subsistence level. Note that when the firms follow loss-free pricing rules [the profit maximizing behavior is a loss-free pricing rule under Assumption (P) if the production sets are convex and allow for inaction], Assumption (WSA) is satisfied under the following classical survival assumption:

Assumption (S) $\omega \in \operatorname{int} X$.

3 An alternative to the survival assumption

It is quite justified to wonder about the appropriateness of this Assumption (WSA), or more precisely if it is satisfactory, since it poses a condition on an endogenous set of the economy, the set of production equilibria. We thus propose in this section to derive Assumption (WSA) from Assumption (S) and two further conditions on the primitive data of the economy.

Before stating these assumptions, which are conditions on the firms pricing rules, we shall introduce several notations. For every j and every (weakly) efficient production plan $y_j \in \partial Y_j$, we let $I(y_j)$ be the set of y_j*-inputs*, that is the subset of commodities h ($h = 1, \dots, \ell$) satisfying $y_j^h < 0$, and $O(y_j) = \{1, \dots, \ell\} \setminus I(y_j)$ the set of y_j*-outputs*. For any $H \subset \{1, \dots, \ell\}$, we shall use the notation z^H to refer to the coordinates of the vector $z \in \mathbb{R}^\ell$ corresponding to the commodities in H. For a given partition $(H_1, \dots, H_u)$ of the commodity set $\{1, \dots, \ell\}$, for every commodity h we let $\nu(h)$ be the only subscript satisfying $h \in H_{\nu(h)}$, and for every pair $(p, z) \in S \times \mathbb{R}^\ell$ we let

$$\hat{I}(p, z) = \left\{ h \;\middle|\; 1 \leq h \leq \ell,\ p^h > 0 \text{ and } z^h < 0 \right\},$$

and:

$$\eta(p, z) = \begin{cases} \max \nu(\hat{I}(p, z)) & \text{if } \hat{I}(p, z) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let us see the partition $(H_1, \dots, H_u)$ as a ranking of the commodities according to their sophistication: the more production transformations are needed to get a commodity, the more this commodity is sophisticated.The most sophisticated commodities may be seen as consumption goods, also called final outputs. For every commodity h, $\nu(h)$ then represents its sophistication degree. For any pair $(p, z) \in S \times \mathbb{R}^\ell$, $\hat{I}(p, z)$ is the set of z-inputs associated with a positive price, and $\zeta(p, z)$ is the highest degree of sophistication of the commodities in $\hat{I}(p, z)$.

The first condition expresses a willing to forward the commodities sophistication. It says that there exists a partition of the commodity set, which can be seen as a ranking of the commodities according to their sophistication, and that, for every firm j and every (weakly) efficient production plan $y_j \in \partial Y_j$, if an admissible price system $p \in \phi_j(y_j)$ associates at least one output with a positive price, then one of these outputs is in an upper class than each of the inputs associated with a positive price.

Assumption (PH) *There exists a partition* $(H_1, \dots, H_u)$ *of the commodity set* $\{1, \dots, \ell\}$ *such that, for every* j*, every* $y_j \in \partial Y_j$ *and every* $q_j \in \phi_j(y_j)$*, if* $\{h \in O(y_j) \mid q_j^h > 0\} \neq \emptyset$*, then there exists* $\mu > \eta(q_j, y_j)$ *such that* $\{h \in O(y_j) \mid q_j^h > 0\} \cap H_\mu \neq \emptyset$*.*

This partition of the commodity set may be seen as a ranking according to the commodities completion: the more transformations are needed to produce a commodity, the more completed this commodity is. A commodity in the first class H_1 is either free or like raw materials, i.e. it does not derive from any transformation, unless the commodities from which it derives are free. A commodity in the last class H_u is either free or like consumption goods (also called "final outputs"), i.e. it is useless in any production process, unless the commodities deriving from it are free.

Our token damages assumption is stated as follows:

Assumption (TD) *For every j, for every $y_j \in \partial Y_j$ and for every $q_j \in \phi_j(y_j)$, if $\{h \in I(y_j) \mid q_j^h > 0\} \neq \emptyset$, then $\{h \in O(y_j) \mid q_j^h > 0\} \neq \emptyset$.*

This means that, for a given (weakly) efficient production plan $y_j \in \partial Y_j$, any price system which associates at least one input, but no output, with a positive price can't be admissible for firm j. In other words, no firm admits any price system involving positive costs for a given production plan, unless it is ensured to be awarded "token damages", i.e. a non negative incoming, whatever its amount.

The principal motivation of this note is not to give an existence result. Our main result is the following: it states that the Assumption (WSA) is satisfied under the classical survival assumption and our conditions of commodities hierarchy and of token damages. Thanks to Proposition 1, we can get the existence of equilibria by substituting Assumption (WSA) for Assumptions (S), (PH) and (TD) and by adding Assumption (C*) in the Theorem 4 of Bonnisseau and Cornet (1991).

Proposition 1 *Assumption (WSA) is satisfied if Assumptions (C*), (S), (PH) and (TD) hold.*

Proof: Suppose that Proposition 1 is false. Then there exists $(p, (y_j)) \in PE$ and $t \geq 0$ such that $\sum_{j=1}^n y_j + \omega + t\,e \in X$ and $p \cdot (\sum_{j=1}^n y_j + \omega + t\,e) = \inf p \cdot X$. Under Assumption (C*), this is equivalent to:

$$\textstyle\sum_{j=1}^n y_j + \omega + t\,e \in \mathbb{R}^\ell_+ \text{ and } p \cdot (\sum_{j=1}^n y_j + \omega + t\,e) = 0,$$

hence $(\sum_{j=1}^n y_j)^h + \omega^h + t = 0$ for every commodity h such that $p^h > 0$. Since $t \geq 0$ and $\omega \in \mathrm{int}X$ from Assumption (S), we get:

$$\text{for every } h, \text{ if } p^h > 0 \text{ then } (\textstyle\sum_{j=1}^n y_j)^h < 0 \cdot \tag{1}$$

Suppose there exists $h \in H_u$ such that $p^h > 0$. Then $(\sum_{j=1}^n y_j)^h < 0$ from (1) and there exists j such that $y_j^h < 0$. Hence, $\eta(p, y_j) = u$, and $\{h \in I(y_j) \mid p^h > 0\} \neq \emptyset$ implies that $\{h \in O(y_j) \mid p^h > 0\} \neq \emptyset$ from Assumption (TD). From Assumption (PH), there exists μ satisfying $\eta(p, y_j) < \mu \leq u$, and this contradicts the fact that $\eta(p, y_j) = u$. We thus have $p^h = 0$ for every $h \in H_u$.

Since $p \in S$, there exists $\nu^0 \neq u$ and $h^0 \in H_{\nu^0}$ such that $p^{h^0} > 0$. From (1), we get $(\sum_{j=1}^n y_j)^{h^0} < 0$. This implies that there exists j^0 such that $y_{j^0}^{h^0} < 0$, hence $h^0 \in I(y_{j^0})$. Assumption (TD) then implies that $\{h \in O(y_{j^0}) \mid p^h > 0\} \neq \emptyset$, and there exists $\nu^1 > \nu^0$ and $h^1 \in O(y_{j^0}) \cap H_{\nu^1}$ such that $p^{h^1} > 0$ from Assumption (PH). By iterating this line of argument, we find a commodity $h \in H_u$ such that $p^h > 0$, which contradicts (1) and ends the proof. □

4 The particular case of marginal pricing

We shall consider in this section the particular case of marginal pricing, derived from the marginal cost principle: firm j is said to follow the *marginal pricing rule* if $\phi_j(y_j) = MP_j(y_j) := N_{Y_j}(y_j) \cap S$ for every $y_j \in \partial Y_j$, where $N_{Y_j}(y_j)$ denotes the Clarke's normal cone to Y_j at y_j.

When all the firms in the economy follow the marginal pricing rule, we can be more precise and give a condition on the shape of the production sets under which our token damages condition is satisfied. This more primitive condition is a condition of *bounded marginal returns*. Indeed, in a single consumer, single firm and two commodities economy, the Survival Assumption (WSA) may not be satisfied if the firm's possibility curve is tangent to the "output" axis, i.e. to the axis corresponding to the produced commodity. Such a tangency implies for the marginal rate at which the input is transformed into output to be infinite. We show that, even if they are increasing, marginal returns must be bounded for the survival assumption to be satisfied.

The bounded marginal returns condition says that, for every firm and corresponding to every production plan, if an input is actually used, then the rate at which it is used to produce an output must be finite. The marginal rate of substitution is generally seen as a normal vector. Since we don't make any smoothness condition on production set, we won't consider normal vectors but generalized gradients, and formalize the bounded marginal returns condition using the Clarke's tangent cone. If Y is a nonempty subset of $\mathbb{R}^\ell$ and $y \in \mathrm{cl}Y$, the closure of Y, then the Clarke's tangent cone to Y at y (the negative polar cone of the Clarke's normal cone $N_Y(y)$ to Y at y) is defined by:

$$T_Y(y) = \left\{ v \in \mathbb{R}^\ell \,\middle|\, \begin{array}{l} \forall (y^\nu) \subset Y, (y^\nu) \to y, \ \forall (t^\nu) \subset \mathbb{R}_+^*, (t^\nu) \to 0, \ \exists (v^\nu) \subset \mathbb{R}^\ell : \\ (v^\nu) \to v \text{ and } y^\nu + t^\nu v^\nu \in Y \text{ for } \nu \text{ large enough} \end{array} \right\}.$$

Assumption (BMR') *For every j and every $y_j \in \partial Y_j$, there exists $z \in T_{Y_j}(y_j)$ such that $z^{I(y_j)} \gg 0$.*

By duality, bounded marginal returns give a condition on marginal prices, our token damages condition.

Proposition 2 *If $\phi_j = MP_j$ for every j, then Assumption (TD) is satisfied under Assumptions (BMR') and (P).*

Proof: Suppose that Proposition 2 is false. Then there exists a firm j, a (weakly) efficient production plan $y_j \in \partial Y_j$ and an admissible price vector $q_j \in \phi_j(y_j) = MR_j(y_j)$ such that $\{h \in I(y_j) \mid q_j^h > 0\} \neq \emptyset$ and $\{h \in O(y_j) \mid q_j^h > 0\} = \emptyset$. From Assumption (BMR'), there exists $z \in T_{Y_j}(y_j)$ such that $z^{I(y_j)} \gg 0$. From the fact that $I(y_j) \neq \emptyset$, the possibility of inaction and the free disposal property [Assumption (P)], we deduce that $O(y_j) \neq \emptyset$, hence $\{h \in O(y_j) \mid q_j^h > 0\} = \emptyset$ imply $q_j^{O(y_j)} = 0$. Thus, we have $q_j \cdot z = q_j^{I(y_j)} \cdot z^{I(y_j)}$. Since $q_j \in S$ and $z^{I(y_j)} \gg 0$, we have $q_j^h z^h \geq 0$ for every $h \in I(y_j)$. From the fact that $\{h \in I(y_j) \mid q_j^h > 0\} \neq \emptyset$, we then deduce that $q_j^{I(y_j)} \cdot z^{I(y_j)} > 0$. But this implies that $q_j \cdot z > 0$ and contradicts the fact that $q_j \in N_{Y_j}(y_j)$. $\square$

Let us finally discuss the relationship between Assumption (BMR') and Assumption (BMR) in Vohra (1992). To begin, we must recall that Vohra's condition is formalized in a model where inputs and outputs are distinguished a priori. Vohra supposes that the commodity set is divided into two classes, the upper class being the one of "final" outputs, i.e., commodities that are never used as an input by any firm. This can be seen as a hierarchical condition and, actually, Assumption (PH) can be seen as a consequence of Assumption (BMR) [see Lemma 3.2 in Vohra (1992)] under this a priori distinction. The problem with this distinction of the commodities is that it implicitly suppose (together with bounded marginal returns) that every firm is able to produce a final output at any production plan [Lemma 3.2 in Vohra (1992)]. This rules out the possibility for a firm to be an intermediate producer, in the sense that it only produces commodities that will be inputs for others firms. This fundamental difference in the distinction between commodities is the reason why our Assumption (BMR') is not directly comparable to Assumption (BMR) in Vohra (1992), whereas they both express the same idea for the firms.

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