A THEORETICAL INDICATOR OF GENERAL INTERDEPENDENCE IN SOME LINEAR SYSTEMS : APPLICATION TO THE FRENCH REGIONAL CASE

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Many fields of economic theory can be represented by linear models. When the matrix of the coefficients of the linear system has certain properties, the system can be divided into subsets. The aim of this paper is to define a unique and general measure of interdependence between two or more parts of a linear system, such that it can be applied at every level: from the elementary variables of the system to the larger subsets of those variables. The recurrence relation allows us to provide the logical articulations linking "partial" interdependencies between subsets for any partition of the system and the "general" interdependence between its elementary components. This approach is illustrated by a comparison of two French tables (6 large regions and 6 large sectors, and 21 regions and 12 sectors respectively), each relating to two different years (1982 and 1992). In the last part of the paper, we suggest some calculus and methods that might be of use to regional policymakers.

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This paper investigates the question of general interdependence, which has been tackled time and again in the literature. Our main aim is to provide a good measure of general interdependence for a simultaneous linear equations system where the values of the variables are determined all together. An intuitive example could be given by the following system:

$$\begin{cases} 0{,}98x - 0{,}02y = 0{,}94 \\ -0{,}01x + 0{,}95y = 1{,}89 \end{cases}$$

the solution of which is obviously ($x = 1; y = 2$). Both equations are of course interdependent, but in fact it is the first equation that determines the x value and the second the y value. This is due to the relative values of the coefficients and to the terms of the second member. Thus we can say that in this system, a weak degree of general interdependence prevails.

Our aim is to develop a method that would produce a global indicator of the degree of interdependence between the variables in a system of linear simultaneous equations (Miller, 1986; Hewings, 2002). That might be quite simple if we limited ourselves to fixing some obvious conditions: lower bound (for example 0) if there is no interdependence at all, upper bound (for example 1) if there is maximum interdependence (this maximum remains to be defined), intuitive property of monotony of the indicator as a function of interdependence… The task becomes more difficult if we add one or more constraints: the indicator must be suitable for measuring interdependence between two or more subsets of the linear system as well as between two or more elementary variables. For simplicity's sake, we focus on the paradigmatic case of a generalized input-output system such as the Defourny-Thorbecke (1984) system. This limitation is justified in so far as Leontief's work constitutes the first operational representation of the Walrasian general equilibrium.

The main intuition of Leontief equations is derived from the Walrasian assumption of constant production coefficients[1]. According to Leontief's assumption, well corroborated by the accounting measures, the technical coefficients remain quite constant. However, even if they are regarded as variables, that would be a mere technical sophistication if their variations, taken part by part, remain linearizable in a dynamic perspective. The point is still the analysis of interdependence between linear variables.

Thus the core of an input-output system is not linked mainly to the invariability of the coefficients, since the effects of linearizable variations can be dealt with by the Bode formula (1945) or any other way of endogenizing the variations (see for example Guzzi, 2000). The most important thing is that a Leontief matrix is productive. Technically, it means that if *(I-A)* represents the technical coefficients matrix, *(X)* the output column-vector and *(F)* the final demand vector, a necessary and sufficient condition of consistency of the static system *(I-A)* . *(X)* = *(F)* (that is to say the existence of a single and significant solution of this equation) is that all the main minors of matrix *(I-A)* are positive. This theorem is known as the Hawkins-Simon condition (1949).

Our concern here is with the properties of general interdependence in linear productive systems only. Anti-productive or consumptive systems are ignored. Having clarified these points, we now attempt to analyze and measure the level of interdependence in linear productive systems.

The first section analyzes the impact of partial feedbacks on interdependence. Section 2 suggests some methods for reducing the complexity of the representative graph of an input-output matrix. Thereafter, section 3 provides the theoretical relations linking the partial interdependencies between parts of a matrix to the general interdependence between the variables. Lastly, starting from a reshaping of an inter-industrial and inter-regional table (see methodological appendix), section 4 applies the previous methods and measures to the regional connections in the French economy.

1. Measures of the Impact of Partial Feedbacks on Interdependence

Formal presentation

[1] In his last works, Léon Walras considered the production coefficients as variables, a point of view shared by Vilfredo Pareto.

Following Walras, Pareto, Leontief (1986), Arrow and Hahn (1971), let us call general interdependence the interdependence between all the elementary variables of the system. If *Q* is a partition of the Leontief matrix into square submatrices, we may call "partial" interdependence the interdependence between two or more parts (i.e. submatrices) defined by the partition *Q*. The general interdependence is naturally linked with the partial interdependencies. The former is strongly connected with the latter but those connections look complex. The paper aims to define a function of the terms of the system in order to produce a good indicator of the general interdependence in this system and to find a simple relation between indicators of interdependence of the submatrices and the global indicator of general interdependence. Optimally, the partial interdependencies and the general interdependence could be expressed in the same way.

We begin by defining the system by its Leontief matrix: let a vertex (e.g. a sector *p* in a region *r*) delivering goods to another vertex (e.g. a sector *q* in a region *s*) be denoted by *i* and *j* respectively. Thus the assumption that the matrix is a productive one can be written as follows: in the system of all the regional sectors $(I\text{-}A)\ .\ (X) = (F)$, where $A = [a_{ij}]$ and where (X) is the output vector and (F) the final demand vector, each vertex j has a positive or nil value added : $\sum_i a_{ij} \leq 1$. The general interdependence between the elements of set (X) is a complex result of the architecture of matrix $(I\text{-}A)$.

A trite solution appears when matrix $(I\text{-}A)$ is a block-diagonal one:

$$(I\text{-}A) = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

We may consider that, in this case, the general interdependence between variables is reduced to the internal interdependencies between the variables of *A*, *B* and *C* respectively.

This deliberately simple example corroborates the recursivity condition:

- the measure of general interdependence will be robust only if it is independent from the possible partitions of the matrix;
- there are some components of partial interdependencies that contribute to general interdependence: this being so, a measure of interdependence must make it possible to distinguish between partial interdependencies and general interdependence.

For this purpose, Influence Graph theory can be very useful, not in order to draw a representation of the system but to get specific properties of matrices based on mathematical graph theory (Harary et al., 1966; Diestel, 1997), which do not appear merely using matricial calculus (Maybee et al., 1989; Horn and Johnson, 1991).

The Influence Graph of the I-O structure

The influence graph of the I-O structure is defined as follows:
1) each regional sector *i* is represented by a vertex *i*;
2) each exchange X_{ij} is represented by an arc; all the arcs are oriented in the same direction: either from the supply to the demand or from the demand to the supply;

3) the arcs of the influence graph are valuated by the technical coefficient[2] a_{ij} and the loops are valuated by the coefficients $\ell_i = 1 - a_{ii}$

The theorem of loops and circuits

We denote as "Hamiltonian" any set of circuits and loops where each vertex of the influence graph belongs once and only once to the set. Further, we call G_h the hth Hamiltonian set of circuits and loops. These Hamiltonian sets are often called Hamiltonian partial graphs (HPGs); in the hth HPG G_h, the number of (oriented) circuits of two or more arcs is denoted by c_h; the "signed product" of the coefficients of all its loops and arcs is called the value of the set G_h and is denoted by V_h. Let v_h denote the product of the coefficients of all the loops and circuits of HPG G_h. In accordance with loops and circuits theorem (Lantner, 2001), the value V_h is given by :

$$V_h = (-1)^{ch} \cdot v_h \tag{1}$$

The determinant D of the Leontief matrix may be written:

$$D = |I - A| = |I - T| = \sum_h V_h \tag{2}$$

The theorem of the partition

Let us divide the vertices of the initial graph of influence into a certain number of parts. Some of the HPGs have no circuit belonging to two different subgraphs of the partition. Let us denote them by G_d (for "disconnected" HPGs). We will call "connected" and denote by G_c the HPGs having at least one circuit linking the vertices of two (or more) subgraphs of the partition.

As an HPG must be either disconnected or connected, the theorem of the loops and circuits can be written as follows:

$$D = \sum_h V_h = \sum_d V_d + \sum_c V_c \tag{3}$$

The sum $\sum V_d$ is the product of the determinants D_p of the submatrices defined by the partition itself. The relation becomes:

$$D = \prod_p D_p + \sum_c V_c \tag{4}$$

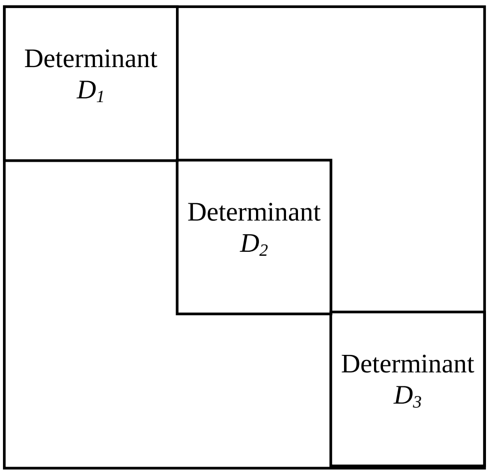
As circuits lead to a decrease in the value of the determinant, it is easy to prove that $\sum V_c$ is negative or equal to zero. The product is :

[2] We could have chosen an alternate presentation with the trade coefficients t_{ij}: $\sum_j t_{ij} \leq 1$ $(\forall i)$. The arcs of the influence graph would be valuated by the trade coefficients t_{ij} and the coefficients of the loops would remain unchanged.

$$D \leq \prod_{p} D_p \tag{5}$$

THEOREM. However the Leontief matrix is divided into square submatrices, *the determinant of the Leontief matrix is smaller than the product of the determinants of the submatrices.* The difference is a measure of the interdependence between the submatrices (due to the linkage terms between them).

Another way to obtain the proof would be to note that, in any case, the determinant D of a Leontief matrix is always smaller or equal to the product of the diagonal terms of the matrix.



The determinant D of the matrix $(I\text{-}A)$ is smaller than the product $D_1{\times}D_2{\times}D_3$ except if there is no circuit linking the vertices of two or more of the three subgraphs defined by the partition.

Fig. 1. *The Partition Theorem*

An immediate consequence of the partition theorem is that, for a partition Q dividing an input-output matrix into square submatrices the determinants of which are D_p:

$$D \leq \prod_{p} D_p \tag{6}$$

If there is at least one circuit connecting two (or more) subgraphs of the associate graph of influence, we get:

$$D < \prod D_p \tag{7}$$

Let the interdependence between the parts p of the partition Q be denoted by I_Q. It can be written:

$$I_Q = - \sum V_c \ \ \text{(where } \sum V_c \text{ is negative or equal to zero)} \tag{8}$$

Another expression of the partition theorem is that D can be written as a function of the D_p and of a non-negative "linkage" or interdependence term denoted by I_Q:

$$D = \prod_{p \in Q} D_p - I_Q \tag{9}$$

The example of the interdependence between two parts of a partition

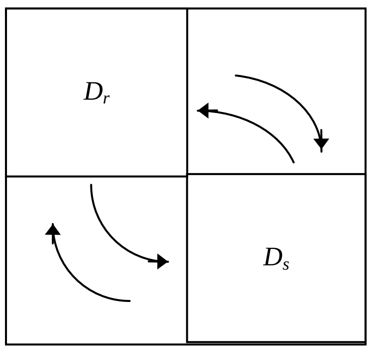
One of the advantages of the theorem of the partition is that it can be used for any partition: between sectors, regions, some sectors in some regions... and hence reinvigorates analysis of structural decomposition (Dietzenbacher, 1988; Hewings et al., 1999) and, more generally, the problem of disaggregation (Malinvaud, 1955; Miller and Blair, 1985; Leontief, 1986; de Mesnard and Dietzenbacher, 1995; Miller, 1999...). Here, we would like to focus on a partition between n sectors and m regions.

Let us assume now that, in this partition, region *r* and region *s* are joined to form a single region called *u* with $u = r \cup s$. Applying the theorem of the partition, we get:

$$D_u = D_r \times D_s - I_u \qquad (10)$$

where I_u is the interdependence term resulting from the different circuits between region *r* and region *s*.

The determinant D_u is lower than the product $D_r \times D_s$ only if there are some circuits connecting sectors in region *r* with sectors in region *s*.



$$D_u = D_r \times D_s - I_u$$

Fig. 2. *The Aggregation of Two Parts of the Partition*

2. To Reduce the Complexity of an Input-Output Matrix

There are two main ways of reducing the complexity of an input-output structure.

Strongly connected components

The first method is to use a concept of Graph theory called strong connectivity: a subgraph or component of a directed graph is said to be strongly connected if for every pair of distinct vertices *i* and *j*, there is a path from *i* to *j* as well as from *j* to *i*.

Consider an influence graph whose arcs represent the (directed) flows between a group of poles (regions, sectors, regional sectors...). If the graph is strongly connected, then a change in each pole has an influence on every other pole: two distinct vertices *i* and *j* are always interdependent.

Let us call maximally strongly connected (MSC) subgraph *g* of a directed graph *G* a subgraph made up of *m* vertices of the graph *G* (and both their loops and the arcs joining them) in such a way that is impossible to retain the property of strong connectivity if any new vertex of the graph *G* is added to the subgraph *g*.

Let us then define the skeleton *S* of the graph *G* as the minimally reduced graph made up of vertices *v*, *w*, ... (and both their incident loops ℓ_v, ℓ_w, ... and arcs a_{vw}) in such a way that:

- every MSC subgraph of *G* is represented by one and only one vertex with an incident loop. The loop represents all the loops and circuits of the corresponding subgraph;
- all the paths from any vertex of *G* belonging to the MSC subgraph *g* (supposed to be represented in the skeleton by vertex *v*) to any other vertex of *G* belonging to another MSC subgraph *h* (supposed to be represented in the skeleton by vertex *w*) are reduced to only one arc from *v* to *w*.

As *g* and *h* are MSC subgraphs of the graph *G*, if there is an arc from *v* to *w*, then there is no arc from *w* to *v*.

If the coefficients of the loops ℓ_v, ℓ_w,... and arcs a_{vw} are properly calculated, the skeleton of the graph *G* gives a good qualitative and quantitative image of the general orientation of the connections between its vertices (and between the corresponding poles of the structure represented by the graph *G*).

For example, consider a structure made up of two MSC components: *g*, with four poles, and *h*, with two poles. Its skeleton is made up of two vertices *v* and *w*, with their attached loops, and one arc, as shown in Figure 3:

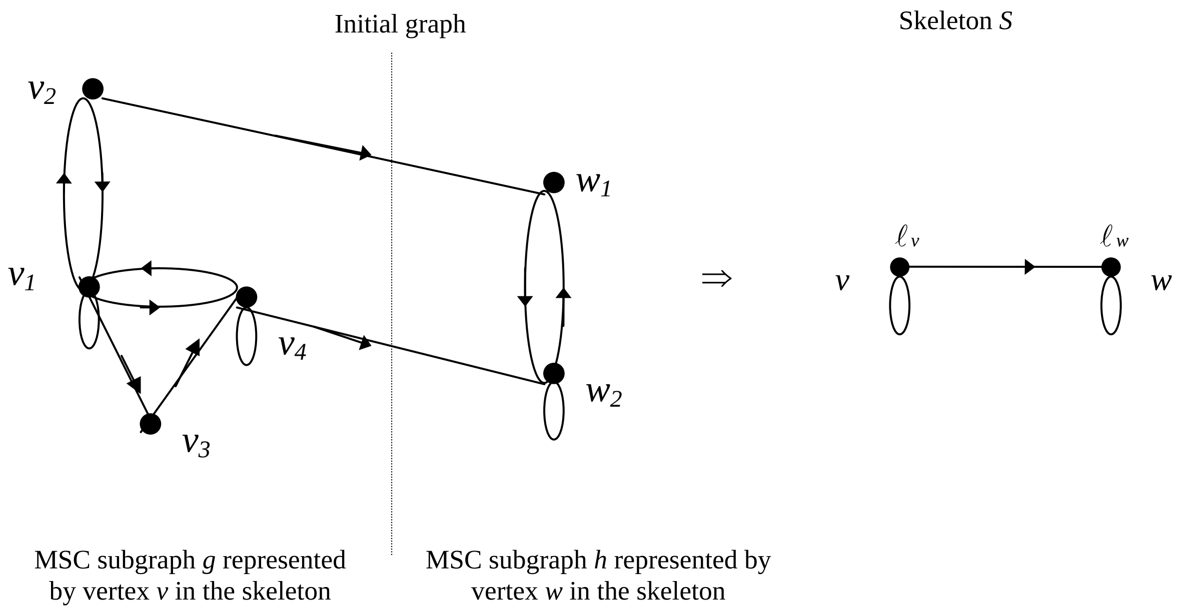


Fig. 3. *From the Initial Graph to its Skeleton*

In order to define the skeleton *S* of a directed graph *G*, it is necessary to remove some vertices of *G*: in the example given in Figure 3, three of the four vertices, v_1, v_2, v_3, v_4, of the MSC subgraph *g* and one of the two vertices w_1, w_2 of the MSC subgraph *h* have to be removed.

Using a recurrence method, that removal leads to the following theorem. In order to express it, consider a directed graph *G* made of vertices and valuated loops and arcs. Then assume that graph *G* is reduced to another graph *R* by removing some of its vertices under the condition that all the quantitative effects between the left vertices remain strictly unchanged.

Let the removed vertices be denoted by r, and the two determinants of the Leontief matrices associated with the initial graph G and the reduced graph R be denoted by D and D_R respectively.

A theorem of the aggregation of Input-Output structures

The determinant D_R of the reduced structure is equal to the initial determinant D of the Leontief matrix divided by the product of the values ℓ_r of the loops incident with the removed vertices:

$$D_R = D \,/\, \prod_r \ell_r \qquad (11)$$

As $\ell_r \leq 1$ $(\forall r)$, it comes: $D_R \geq D$.

More generally, every aggregation of the vertices (which represent the poles of the input-output structure) leads to a well-determined increase in the value of the determinant of the Leontief matrix.

This means that the value of the determinant of the Leontief matrix is not invariant as the level of aggregation changes. That value is modified by changing the scale of aggregation.

As a partition of the structure is an aggregation, this last property must be taken into account when the partitions of the structure are analyzed.

Partitions of the Leontief matrix

The second way to reduce the complexity of the structure is to define directly a partition of the poles (regions, sectors, regional sectors...) of the structure.

It is clear that in a structure with at least three poles, each pole can be associated with each other one in the same component of one or more partition(s).

We have just used the concept of skeleton of the graph to reduce the complexity of the structure: if every pole belongs to a MSC subgraph, the division of the structure into MSC components is a particular case of partition.

Going back to the general case of partitions may be helpful in analyzing how general interdependence is connected with partial interdependencies.

In any Leontief structure it is possible to define the general interdependence between all the poles; if the structure is divided into various substructures by a partition, it is possible to define the partial interdependencies both between some substructures of the partition and between the poles of every substructure of that partition.

What we intend to do is to express the general interdependence between all the elements as a function of the partial interdependencies between the parts p of a partition Q of the Leontief structure.

3. General Interdependence and Partial Interdependencies

Relations between different levels of interdependence

Assume that a Leontief table represents the sales by every sector p of every region r to every sector q of every region s; n denotes the total number of sectors and m the total number of regions: $p,q \in (1,2, ..., n)$ and $r,s \in (1,2, ..., m)$.

We are interested in the general interdependence between all the regional sectors, and also in the interdependence between the m regions of the table.

If we intend to analyze the problem at an upper level, we may group together some of the n small sectors in order to form $\overline{n}$ large sectors, with of course: $\overline{n} < n$.

In the same way, the number m of small regions can be reduced to a number $\overline{m}$ of large regions, with: $\overline{m} < m$.

These changes in the partition of the matrix raise two questions.

- Firstly, how can the general interdependence between all the regional sectors be defined?
- And secondly, how is this general interdependence linked to the partial interdependencies at different levels of the aggregation of the Leontief table? For example, what are the interdependencies between the elements of the $\overline{m}$ large regions (the $\overline{n}$ large sectors respectively) and the elements of the m small regions (the n small sectors respectively)?

We will see that the influence graph theory may be useful to provide a method to solve that problem (and any combination of the previous problems): a general solution and a recursive theorem will be established after.

General interdependence between the variables of a Leontief table

Identification of the general interdependence

Let us return to the usual relation: *(I-A) . (X) = (F)*, where *(X)* is a set of variables (e.g. outputs of regional sectors) and *(F)* the final demand vector.

The general interdependence between all the variables depends upon all the circuits of the influence graph associated to the Leontief matrix *(I-A)*. But each one of these circuits must be taken into account with an appropriate multiplier.

One of the easiest way to find such multipliers is to express the determinant D of the Leontief matrix, which is closely connected with the general interdependence, as the sum of the values of the HPGs.

Consider the vertices of the influence graph (which here represent the regional sectors) as the smallest parts of the most disaggregated partition of the Leontief structure. They are connected by all the circuits (two arcs or more) belonging to the influence graph. In order to obtain a measure of the general interdependence between all the vertices, three elementary conditions are imposed:

- the measure reaches its lower bound (zero, if possible) when there is no interdependence at all between even any two vertices;
- the measure reaches its upper bound (one, if possible) when the intuitive general interdependence is maximal;
- the measure is a growing monotonous function of the general interdependence: the more intense the circuits are, the higher the measure of general interdependence is (the derivative of the production of all other vertices are maximum when the production of one vertex changes).

It should be noted that the determinant of the structure is lower than the product of the “mini-determinants” of all the vertices: ℓ_1, …, ℓ_i, $\ell_{n\times m}$. Let us call GI the difference:

$$D = \prod_{i=1}^{n\times m} \ell_i - GI \tag{12}$$

The lower bound of GI is equal to zero. GI is nil if there is not even one circuit of two arcs (or more) in the whole graph of the structure (that means no interdependence at all). The more numerous and intense the circuits are, the higher the term GI is. The upper bound of GI is reached if ℓ_i is equal to one[3] ($\forall i$) and if the determinant D is equal to zero. In this (very

[3] This is the case when the intra-consumptions of the regional sectors i are equal to zero. This means that the entire production of this regional sector is sent to the other ones (this case corresponds to the maximum interaction (interdependence) of this regional sector with the other ones).

special) case of a perfectly circular structure[4], the whole structure is reduced to only one very intensive circuit connecting all the vertices and GI is equal to one.

This leads us to consider the perfectly circular structure as the most interdependent one. It is obviously the structure in which the interdependence between vertices is the most global, the most general.

If there is more than one circuit, the global circuit is less intensive, and other, shorter circuits limit the diffusion of general interdependence effects.

That is the reason why we may consider GI as a good measure of the general interdependence. Another way to prove it would be to assume that the $n \times m$ vertices have loops but are not connected at all: no connection, no interdependence. And the determinant of the whole matrix would be equal to:

$$D = \prod_{i=1}^{n \times m} \ell_i \tag{13}$$

By definition, we call general interdependence between the variables *(X)* of a Leontief system, the difference:

$$GI = \prod \ell_i - D \tag{14}$$

in which: $0 < GI < 1$ and $\ell_i = 1 - a_{ii} = i^{th}$ diagonal term of the Leontief matrix.

THEOREM. *A good indicator of the general interdependence in a Leontief system is given by the difference between the product of the diagonal terms of the Leontief matrix and the value of its determinant.*

Consistency between the indicator of the general interdependence and the theorem of the circuit

Let us give an example of the way a circuit C_j affects the value of the determinant D and, at the same time, give an alternate proof of one of our key theorems: "*the more intensive the circuits, the lower the value of the determinant D*".

With the notations of Appendix, we can say that the more intensive a circuit C_j, the greater the value of $\Pi_j D_{\bar{J}}$. The term $\Pi_j D_{\bar{J}}$ is the sum of the values of all the HPGs including the circuit C_j. As GI is the sum of the "connected" HPGs (which here means all the HPGs except the set of loops ℓ_i), an increase in the value of Π_j leads to an increase in the value of GI and, thus, to a decrease in the value of the determinant D. One consequence is that the definition of the general interdependence is consistent with the theorem of the circuit.

[4] In the perfectly circular structure, the final demand is nil and $a_{i-1,1} = 1$ ($\forall i$, $i = 2, 3, \ldots, n \times m$) and $a_{n \times m,1} = 1$.

Relations between partial interdependencies and general interdependence

Relations for one level of disaggregation

Let us assume that the set of $n \times m = N$ variables (e.g. outputs of the regional sectors represented by the vertices of the influence graph) is divided into parts p by a partition Q.
From relation (12), we can write:

$$\prod_{i=1}^{N} \ell_i = D + GI \tag{15}$$

According to section 1, the determinant D_p of each part p can be considered as the sum of the values of disconnected Hamiltonian graphs and connected Hamiltonian partial graphs of the substructure p:

$$D_p = \sum_{d} V_d + \sum_{c} V_c = \prod_{j \in p} \ell_j - I_p \tag{16}$$

Hence:

$$\forall p \in Q: \prod_{j \in p} \ell_i = I_p + D_p \tag{17}$$

Applying this latest relation to all the parts p of Q, we get:

$$\prod_{i=1}^{N} \ell_i = \prod_{p \in Q} (I_p + D_p) \tag{18}$$

This means that, with only one level of disaggregation, the general interdependence between all the elements of the structure is a function of all the partial interdependencies between the elements of each part p of the partition Q. The relation is not additive (as might have been expected) but multiplicative:

$$GI = \prod \ell_i - D = \prod_{p \in Q} (I_p + D_p) - D \tag{19}$$

It can be written more simply as follows:

$$GI + D = \prod_{p \in Q} (I_p + D_p) \tag{20}$$

Recurrence relation

Let us group together the variables X_i (or vertices i) and let the unit or subset made up of a certain number of grouped variables (1, 2, …, i, …, i_u) be denoted by u: we assume that unit u includes all the arcs linking 1, 2, …, i, …, i_u. We get:

$$D_u = \prod_{i \in u} \ell_i - I_u \tag{21}$$

The different units *u* can be grouped again in order to make up substructures *s*:

$$D_s = \prod_{u \in s} D_u - I_s = \prod_{u \in s} \left(\prod_{i \in u} \ell_i - I_u \right) - I_s \qquad (22)$$

Then the substructures *s* can be grouped together in order to make up parts *p* :

$$D_p = \prod_{s \in p} D_s - I_p = \prod_{s \in p} \left[\prod_{u} \left(\prod_{i} \ell_i - I_u \right) - I_s \right] - I_p \qquad (23)$$

The determinant *D* of the matrix made up of the parts *p* can be written:

$$D = \prod_{p \in Q} D_p - I_Q = \prod_{p} \left[\prod_{s} \left[\prod_{u} \left(\prod_{i} \ell_i - I_u \right) - I_s \right] - I_p \right] - I_Q \qquad (24)$$

A measure of the interdependence between parts *p*, made up of substructures *s*, which are themselves made up of units *u* formed by vertices or variables *I*, may be written as follows:

$$I_Q = \prod_{p} \left[\prod_{s} \left[\prod_{u} \left(\prod_{i} \ell_i - I_u \right) - I_s \right] - I_p \right] - D \qquad (25)$$

According to the last relation, we get the following theorem.
THEOREM. Added to the determinant *D*, the general interdependence GI between all the variables of a Leontief system is equal to the product, for all the parts *p* (of a unique partition *Q* of the variables), of the sums of the interdependence and determinant of each part *p* of the partition *Q*.
It is possible to choose a level of disaggregation. For example, the regional sectors could be denoted by *i*, the regions by *u*, the states by *s*, the countries by *p* and a set of countries (such as NAFTA, MERCOSUR or the European Union) by *Q*.
The recursive relation established between the different levels of aggregation may look complex because it is multiplicative. However, it is quite simple and easy to use. It allows us to measure the weights of the different interconnections, to specify their most important levels and to link them altogether in order to understand how and at what levels general interdependence works.

4. Regional Connections in the French Economy

Computation of the determinants

Using the sectoral regional and national value added data for the period 1975-1992 and the two input-output tables[5] for 1982 and 1992 provided by the French national statistical institute INSEE, we calculate the determinant of trade coefficients (knowing that it is

[5] The proportion of the value added in output being assumed to be similar between all the sectors *i* at national level, under the assumptions of spatial homogeneity and isotropy (no protected market and no inefficient firm. It corresponds to a construction of multi-regional input-output tables based on the well known biproportional method (de Mesnard, 1990; Boomsma and Oosterhaven, 1992...).

identical to the determinant of technical coefficients). The total output is equal to X_{ir} obtained from the following formula:

$$X_{ir} = Y_{ir}/Y_i \times X_i \tag{26}$$

where all the terms are known.

The trade coefficients are:

$$t_{ir,js} = X_{ir,js}/X_{ir} \tag{27}$$

Thus the determinant is $D = |I - T|$, since the intra-consumptions are conventionally cancelled, that is to say the diagonal terms of $(I - T)$ are equal to one. In order to compute regional determinants we must first increment to each line the index j from 1 to n.

Once the determinant D and the determinants D_1, D_2, ... D_m are obtained, we will have:

$$D = \prod D_k - I \qquad \text{with } 0 \le D \le 1;\ 0 \le D_k \le 1 \text{ and } 0 \le L \le 1. \tag{28}$$

where I is the term of complex linkage bound to regional circuits (general interdependence).

Results

Using the value added of interregional exchanges as a basis (Sonis et al., 1996; Lahr and Dietzenbacher, 2001), computation of the determinants leads to slightly different results. The results give a first approximation of the main changes in the French regions and the nation as a whole during the 1980s.

Table 1
Regional and inter-regional interdependencies obtained from value added for a high degree of aggregation (6 regions and 6 sectors)

Determinants	Global D	GI	Autarky	IDF	PB	NE	W	SW	SE
1982	0.1246	0.0689	0.8066	0.6714	0.7536	0.7946	0.7978	0.7976	0.7561
1992	0.1862	0.0609	0.7529	0.7196	0.7841	0.8268	0.8076	0.8479	0.7734
Difference (%)	+49.5	-11.9	-4.67	+7.19	+4.05	+4.05	+1.22	+6.32	+2.28

Where the six regions are respectively : IDF = Ile-de-France, PB = Paris Basin, NE = North-East, W = West, SW = South-West and SE = South-East.

Starting from the principle that a strong determinant reflects a low density of internal relationships (between sectors), it can be noted that the value of the global determinant is relatively high (indicating a weak degree of interdependence between regions) and, particularly, that it has increased by almost 50% over the past decade (for a high degree of spatial and sectoral aggregation). However this tendency is less clear for a low degree of aggregation (the increase is about 15%) where the values of the determinant are considerably lower (above 30%) and less subject to change (Table 2).

At the same time, the decrease in general interdependence, as the indicator of autarky, between regions is highlighted by the value of the interdependence term, whose trend is downwards (this time the phenomenon is more marked at a low degree of disaggregation). It should be noted that the main difference between the two levels of analysis is the evolution of the product of the loops (or the degree of autarky) between the two dates: a pronounced decrease in the first case (above 5%) and perfect stability in the second one.

Table 2
Regional and inter-regional interdependencies obtained from value added at a low degree of aggregation (21 regions and 12 sectors)

Years	1992	1982
Global *D*	0.0955	0.0828
GI	0.0411	0.0535
Autarky	0.8633	0.8637
Ile-de-France	0.5997	0.5871
Champagne-Ardenne	0.9445	0.9488
Picardy	0.9394	0.9391
Upper Normandy	0.9285	0.9386
Centre	0.9201	0.9202
Burgundy	0.9484	0.9489
Brittany	0.9224	0.9205
Lower Normandy	0.9535	0.9559
Pays-de-Loire	0.9067	0.8985
Poitou-Charentes	0.9532	0.952
Aquitaine	0.9106	0.9148
Limousin	0.9811	0.9788
Auvergne	0.9601	0.9604
Midi-Pyrenees	0.9348	0.9310
North-Pas-de-Calais	0.8938	0.8907
Lorraine	0.9357	0.9290
Alsace	0.9392	0.9398
Franche-Comte	0.9623	0.9668
Rhone-Alps	0.8187	0.8169
Languedoc-Roussillon	0.9484	0.9545
Provence-Alps-French Riviera-Corsica	0.8860	0.9002

As for the regional determinants, the general trend in both tables 1 and 2 is confirmed: the highest degree of interconnectedness is found for Ile-de-France (the second one, the Rhone-Alps region, is a long way behind). At this level, the increase is particularly great for the South-West and the IDF during the period. This observation should be qualified for a low level of aggregation where the shifts are small. Examination of the partial interdependencies is more revealing. Annex 2 gives the results of local interdependencies taken two by two and three by three (for 6 large regions and 6 large sectors, and for 21 small regions and 12 medium sectors), especially for two incentive cases involving the IDF and the Paris Basin (the area around the capital), and the IDF and the South-East. Overall, the partial interdependencies are reduced whereas the determinants go up by 10% (two by two) and slightly more than 10% (three by three).

In the first case (IDF-PB), the local interdependencies are weaker. This reflects a tendency towards the segmentation of the territory of France. At a low level of aggregation, the evolution is confirmed: the regions are getting more and more independent, especially the North-East which is "disconnected" (Appendix, table 3). Yet, the core region (IDF) is more connected to the other regions even if this old established fact (French polarization) is less clear for the end of the period. Within the Paris Basin, the main partners are Upper Normandy and Centre.

In the second case (IDF-SE), the main findings are an increase in the determinants ($D_{IDF,\ SE}$, $D_{IDF,W,SE}$, $D_{IDF,SW,SE}$ and $D_{IDF,NE,SE,}$) and a decrease in the partial interdependencies ($I_{IDF,SE}$, $I_{IDF,W,SE}$, $I_{IDF,SW,SE}$ and $I_{IDF,NE,SE}$; $I_{IDF,PRO}$ and $I_{IDF,LR}$). This suggests a structural change in the exchange networks of both those large regions: their coefficients of mutual exchanges are becoming weaker but their external exchanges are increasing (tendency to autarky with regard to the other French regions). However it should be noted that IDF maintains preferential relationships with Provence (part of the South-East), as a strong partial interdependence of 0.48% testified (Appendix, Table 4).

5. Concluding Comments

Several theorems, including some new ones, have been proved and may be particularly useful in realizing various structural decompositions. They draw on the properties of matrices, determinants and graphs and underline the possibilities for evaluating:
- the internal interdependence between the variables of any submatrice (sector, region, regional sector…);
- the interdependence between some submatrices themselves;
- and the general interdependence between all the variables in the system.

Applied to the inter-industrial and interregional French relationships, the mathematical model provides interesting results about regional evolutions during the period 1982-1992. It shows especially the key position of the Ile-de-France region and the tendency towards a decreasing of general interdependence between the French regions. This suggests a trend towards greater autonomy and hence a less cohesive system.

As many simultaneous linear equations systems have the same nature as a Leontief system, such a system need not necessarily be an inter-industrial exchanges system. It could be a relational system bring together areas, regions, nations, industries, firms, organizations, social groups or even individuals. Consequently, the nature of the flows inside the system could be very different and might include goods, services, financial data, information, sociological or psychological "influences" and so on.

Appendix

The theorem of the circuit

Each circuit of the graph of influence leads to a decrease in the value of the determinant D.

Proof: Let us call C_j one of the circuits of the graph of influence and J the set of vertices belonging to C_j. $\bar{J}$ is the set of all the (other) vertices which do not belong to C_j. When the determinant D is written as the sum of the values V_h of the Hamiltonian partial graphs (HPGs):

$$D = \sum_h V_h \tag{29}$$

it is possible to divide this set of HPGs into two subsets: those which do not include the circuit C_j and those which include it. We need to consider the second ones only in order to ascertain the contribution of the circuit C_j to the expression of the determinant D. This contribution is made by all the HPGs including circuit C_j. It is given by the sum of the values of those HPGs. As he product of the coefficients of the arcs of circuit C_j is denoted by Π_j, the contribution of circuit C_j to the value of determinant D is equal to:

$$(-1)\Pi_j m_j \tag{30}$$

where m_j is the multiplier of the circuit. This multiplier is the sum of the values of all the HPGs of the subgraph the vertices of which belong to $\bar{J}$. Using the theorem of the loops and circuits, we see immediately that this sum is equal to the determinant $D_{\bar{J}}$ of this subgraph.

Therefore, the part of the value of the determinant D explained by the circuit C_j taken separately is:

$$(-1)\ \Pi_j\ D_{\bar{J}} \tag{31}$$

As the product Π_j is positive and the determinant of any subgraph of the graph of influence is positive, we get the theorem.

Methodological Clarifications

Sectoral nomenclature

The sectoral decomposition (productive system) identifies 6 large sectors and 12 medium sectors:
1- Farming, forestry and fisheries;
2 - Energy: oil products, natural gas, electricity, gas, water;
3 - Manufacturing: meat and dairy products, other food products, ores and ferrous metals, ores and non ferrous metals, building materials, miscellaneous minerals, glass, basic chemicals, smelting works, metal works, paper, cardboard, rubber, transformation of plastics; mechanical engineering, professional electric and electronic equipments, motor vehicles, shipping, aircraft and arms; pharmaceuticals, textile and clothing industries, leather and shoe industries, timber and wood industries, printing and publishing, miscellaneous industries. Thus manufacturing is further divided into four subsectors;
4 - Engineering: building, civil and agricultural;
5 - Market services: trade; automobile trade and repair services, hotels, catering; transport, telecommunications and mail ; business services, marketable services to private individuals ; housing rentals and leasing, insurance, financial services. Thus market services are further divided into four subsectors;
6 - Non-market services.

Regional nomenclature

The spatial decomposition (France) identifies six large regions and 21 small regions:
1 - Ile-de-France [IDF] ;
2 - the Paris Basin [PB]: Champagne-Ardenne, Picardy, Upper Normandy, Centre and Burgundy ;
3 - the West [W]: Brittany, Lower Normandy, Pays-de-Loire and Poitou-Charentes ;
4 - the South-West [SW]: Aquitaine, Limousin, Auvergne and Midi-Pyrenees ;
5 - the North-East [NE]: North-Pas-de-Calais, Lorraine, Alsace and Franche-Comte ;
6 - and the South-East [SE]: Rhone-Alps, Provence-Alps-French Riviera-Corsica and Languedoc-Roussillon.

Results

Tables 3

Partial Interdependencies (PI) at a High Degree of Aggregation (6 Regions and 6 Sectors)

Two by two	*D*	PI	*D*	PI
Year	**1992**		**1982**	
IDF - PB	0.553	**0.011**	0.491	**0.015**
IDF - W	0.585	**0.01**	0.519	**0.014**
IDF - SW	0.572	**0.009**	0.524	**0.011**
IDF - NE	0.603	**0.008**	0.520	**0.015**
IDF - SE	0.544	**0.013**	0.491	**0.017**
PB - W	0.639	**0.009**	0.587	**0.012**
PB - SW	0.622	**0.011**	0.588	**0.014**
PB - NE	0.657	**0.008**	0.589	**0.012**
PB - SE	0.596	**0.011**	0.556	**0.013**
W - SW	0.66	**0.008**	0.624	**0.010**
W - NE	0.695	**0.006**	0.624	**0.009**
W - SE	0.631	**0.009**	0.589	**0.011**
SW - NE	0.677	**0.008**	0.627	**0.010**
SW - SE	0.615	**0.010**	0.592	**0.011**
NE - SE	0.648	**0.007**	0.592	**0.011**

Three by three	*D*	PI	*D*	PI
Year	**1992**		**1982**	
IDF - PB - W	0.442	**0.025**	0.370	**0.032**
IDF - PB - SW	0.431	**0.024**	0.373	**0.03**
IDF - PB - NE	0.457	**0.022**	0.371	**0.032**
IDF - PB - SE	0.409	**0.027**	0.348	**0.034**
IDF - W - SW	0.458	**0.022**	0.398	**0.028**
IDF - W - NE	0.484	**0.020**	0.394	**0.031**
IDF - W - SE	0.434	**0.026**	0.370	**0.033**
IDF - SW - NE	0.473	**0.02**	0.399	**0.029**
IDF - SW - SE	0.425	**0.025**	0.374	**0.031**
IDF - NE - SE	0.449	**0.023**	0.370	**0.034**
PB - W - SW	0.500	**0.024**	0.448	**0.029**
PB - W – NE	0.530	**0.020**	0.451	**0.027**
PB - W – SE	0.478	**0.023**	0.423	**0.030**
PB - SW - NE	0.514	**0.023**	0.451	**0.029**
PB - SW – SE	0.464	**0.026**	0.424	**0.031**
PB - NE - SE	0.492	**0.022**	0.425	**0.029**
W - SW - NE	0.547	**0.019**	0.481	**0.024**
W - SW - SE	0.494	**0.022**	0.453	**0.026**

SW - NE - SE	0.523	**0.019**	0.453	**0.026**

Table 4

Partial Interdependencies (PI) between IDF and Some Specific Regions (12 Sectors)

Two by two	*D*	PI	*D*	PI
Year	**1992**		**1982**	
IDF - CA	0.5645	**0.0019**	0.5625	**0.0023**
IDF - PIC	0.5611	**0.0023**	0.5591	**0.0026**
IDF - HN	0.5537	**0.0032**	0.5461	**0.0052**
IDF - CEN	0.5486	**0.0032**	0.5482	**0.0038**
IDF - BUR	0.5668	**0.002**	0.5643	**0.0024**
IDF- RA	0.5495	**0.0028**	0.5428	**0.0037**
IDF - PARC	0.5667	**0.0021**	0.5661	**0.0026**
IDF - LR	0.5266	**0.0048**	0.5266	**0.0064**

The five regions of the Paris Basin are: Champagne-Ardenne [CA], Picardy [PIC], Upper Normandy [HN], center [CEN] and Burgundy [BUR], and the three regions of the South-East are: Rhone-Alps [RA], Provence-Alps-French Riviera-Corsica [PARC] and Languedoc-Roussillon [LR].

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