Switching of Techniques along the Wage-Profit Frontier: a More General Framework

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Abstract

It is known that in input-output models with a discrete spectrum of techniques, the quantity of each commodity employed as input as well as the value of capital per worker display discontinuities in correspondence of every switching point; in this paper I give a general proof that these discontinuities does not disappear—like in neoclassical models—as long as the number of techniques becomes higher and higher, flukes aside. From the proof of this result it is argued the greater generality of input-output models in depicting technology in theoretical production models: the usual 'small' or infinitesimal variations in proportions between production factors turn out to be, in this context, a very particular and untenable way of representing the phenomenon of the switching of techniques.

1 Introduction

The idea that switchings of techniques can be represented by 'small' or continuous variations in the proportions in which inputs are employed is commonly widespread in economics analysis. Also in those cases where a finite number of techniques is available this property is normally thought to be restored by assuming that the number of techniques becomes higher and higher, till to smooth the contour of production set. In Bellino (1993) some counterexamples disprove this conjecture.[1] In the present work I provide a general proof of the fact that the

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[1] See also Rosser (1991, Chapter 8).

increase of the number of known techniques does not support the convergence towards a production set with a smooth contour, except than in a very restricted class of cases. Thus Bellino's (1993) counterexamples, far from constituting 'pathological' cases, describe the general case for production systems.

2 Description of the economy

Consider a system with I single-product constant returns to scale industries. Each commodity m is thus produced by a specific industry, by using commodities and labour. For the generic industry one unit of commodity i is produced by a single method of production, represented by a $(M+1, 1)$ semi-positive column vector $\begin{bmatrix}\mathbf{a}_i \\ \ell_i\end{bmatrix}$. The technique for the whole system is thus represented by a $M+1 \times M$ matrix,

$$\begin{bmatrix}\mathbf{A} \\ \boldsymbol{\ell}^T\end{bmatrix} = \begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_M \\ \ell_1 & \cdots & \ell_M\end{bmatrix}.$$

The price equations of such a system is:[2]

$$\mathbf{p}^T = (1+r)\mathbf{p}^T\mathbf{A} + w\boldsymbol{\ell}^T, \tag{1a}$$

$$\mathbf{p}^T\mathbf{b} = 1, \tag{1b}$$

where $\mathbf{p}$ is the $(M, 1)$ price vector, r is the rate of profit, w is the wage rate and $\mathbf{b}$ is a $(M, 1)$ vector representing the commodity bundle used as numeraire; apex ^{T} indicates the transposition operator.

System (1) has $M+1$ equations in $M+2$ unknowns, $\mathbf{p}$, w and r. It can be solved with respect to $\mathbf{p}$ and w once r is given:

$$w = \frac{1}{\boldsymbol{\ell}^T[\mathbf{I} - (1+r)\mathbf{A}]^{-1}\mathbf{b}} =: w(r) \tag{2a}$$

$$\mathbf{p}^T = w(r)\boldsymbol{\ell}^T[\mathbf{I} - (1+r)\mathbf{A}]^{-1} =: \mathbf{p}^T(r). \tag{2b}$$

Let $R := 1/\lambda^*(\mathbf{A}) - 1$, where $\lambda^*(\mathbf{A})$ is the dominant eigenvalue of $\mathbf{A}$. If $\lambda^*(\mathbf{A}) < 1$ (this condition can be interpreted as a viability condition for technique

[2]System 1 is easily referable to Sraffa's (1960) framework, but the results here contained can be extended to all input-output systems and to activity analysis models. Thus it is worth to recall that some variants of systems 1 were studied—although with different purposes—by Morishima (1958), (1959) and by Solow (1959).

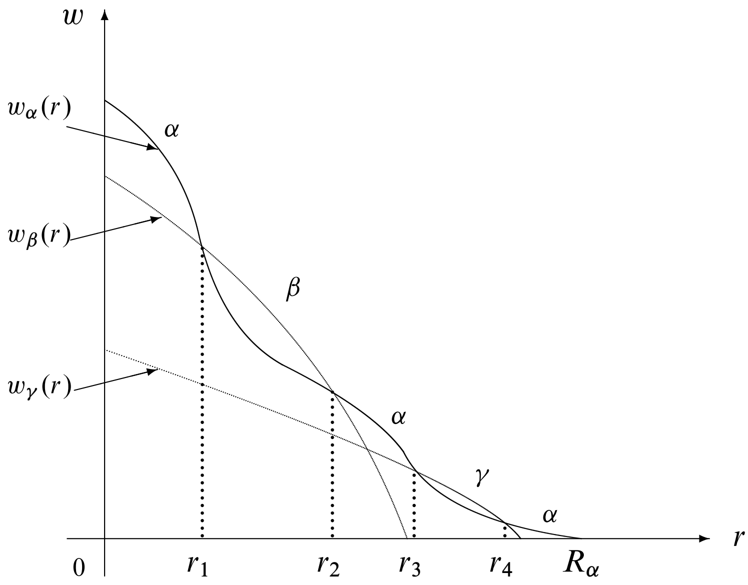

Figure 1: Technological frontier: a *discrete* spectrum of techniques

A) it can be proved that for $0 \leq r < R$ we have $w(r) \geq 0$, $\mathbf{p}(r) \geq \mathbf{o}$ and $w'(r) \leq 0$.

If we relax the assumption that just one method is available for producing the various commodities the technology of the system comes to be constituted by a set of matrices,

$$\begin{bmatrix} \mathbf{A}^{(\alpha)} \\ \boldsymbol{\ell}^{(\alpha)T} \end{bmatrix}, \begin{bmatrix} \mathbf{A}^{(\beta)} \\ \boldsymbol{\ell}^{(\beta)T} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{A}^{(\omega)} \\ \boldsymbol{\ell}^{(\omega)T} \end{bmatrix},$$

each matrix representing a particular constant return to scale technique. It has been shown firstly by Levhari (1965, pp. 99–102), then by Garegnani (1970, Sections 7 and 8), by Pasinetti (1977, Sections 4 and 5) and by Kurz and Salvadori (1994, Chapter 5) that for any given profit rate the adoption of the cheaper method in each industry will permit to yield the highest wage rate for the whole system. This will bring the system on its *technological frontier*, that is, the outer envelope of the $w^{(\iota)}(r)$ relationships, $\iota = \alpha, \beta, \ldots, \omega$. This technological frontier is defined by:

$$w = W(r) := \max_{\mathbf{A}^{(\iota)}, \boldsymbol{\ell}^{(\iota)}} \frac{1}{\boldsymbol{\ell}^{(\iota)T}[\mathbf{I} - (1+r)\mathbf{A}^{(\iota)}]^{-1}\mathbf{b}}.$$

Let

$$\begin{bmatrix} \mathbf{A}(r) \\ \boldsymbol{\ell}^T(r) \end{bmatrix} := \arg\max_{\mathbf{A}^{(\iota)},\boldsymbol{\ell}^{(\iota)}} \frac{1}{\boldsymbol{\ell}^{(\iota)T}[\mathbf{I}-(1+r)\mathbf{A}^{(\iota)}]^{-1}\mathbf{b}}$$

the most profitable technique when the profit rate is r and

$$\boldsymbol{p}^T(r) := W(r)\boldsymbol{\ell}^T(r)[\mathbf{I}-(1+r)\mathbf{A}(r)]^{-1}$$

the price system prevailing when in correspondence of each level of r is adopted the most profitable technique.[3]

With reference to the graph of Figure 1 technique α is adopted for $0 \leq r < r_1$, for $r_2 < r < r_3$ and for $r_4 < r \leq R_\alpha$; technique β is adopted for $_1 < r < r_2$ and technique γ is adopted for $r_3 < r < r_4$. r_1, r_2, r_3 and r_4 are *switching points*. At each switching point we have a *jump*, a discontinuity, in technical coefficients of at least one industry, $\begin{bmatrix} \mathbf{a}_i(r) \\ \ell_i(r) \end{bmatrix}$, and, by consequence, in the capital-labour ratio of the system, $k(r) = \boldsymbol{p}^T(r)\mathbf{A}(r)\mathbf{q}/\boldsymbol{\ell}^T(r)\mathbf{q}$, where $\mathbf{q}$ represents the vector of gross product of the system.

A question may arise: does these discontinuities appear due to the supposition of a *discrete* spectrum of techniques and tend to disappear if we suppose that the number of techniques becomes higher and higher, till to obtain a *continuous* spectrum of techniques?

In the literature on linear activity analysis as well as in several subsequent contributions this result has been outlined in different forms[4] The intuitive idea may be caught, for example, by considering three elementary processes for producing one unit of a given commodity, α, β and γ; processes, which are constant returns to scale and additive, employ two inputs, a and ℓ. Hence the unitary isoquant may be depicted as the broken line in Figure 2.a, which is composed by the three elementary processes and by their convex combinations. If a fourth efficient process, δ, should become available it should place itself as in Figure 2.b; a fifth process

[3]Note the difference between symbol $\mathbf{p}(r)$, used to indicate the solution of system (1), that is, the price system generated by a given technique for any level of the rate of profit, and symbol $\boldsymbol{p}(r)$, used to indicate the price system prevailing when, in correspondence of the various levels of r, the most profitable technique is adopted; according to what proved by Levhari (1965, pp. 99–102), by Garegnani (1970, Sections 7 and 8), by Pasinetti (1977, Sections 4 and 5) and by Kurz and Salvadori (1994, Chapter 5), $\boldsymbol{p}(r) = \min_\iota \mathbf{p}^{(\iota)}(r)$, where $\mathbf{p}^{(\iota)}(r)$ is the price system prevailing for any given r when technique adopted is ι.

[4]See, for example, Dorfman, Samuelson, and Solow (1958, pp.248–253), Solow (1967, p.39), Hahn (1982, Section IV).

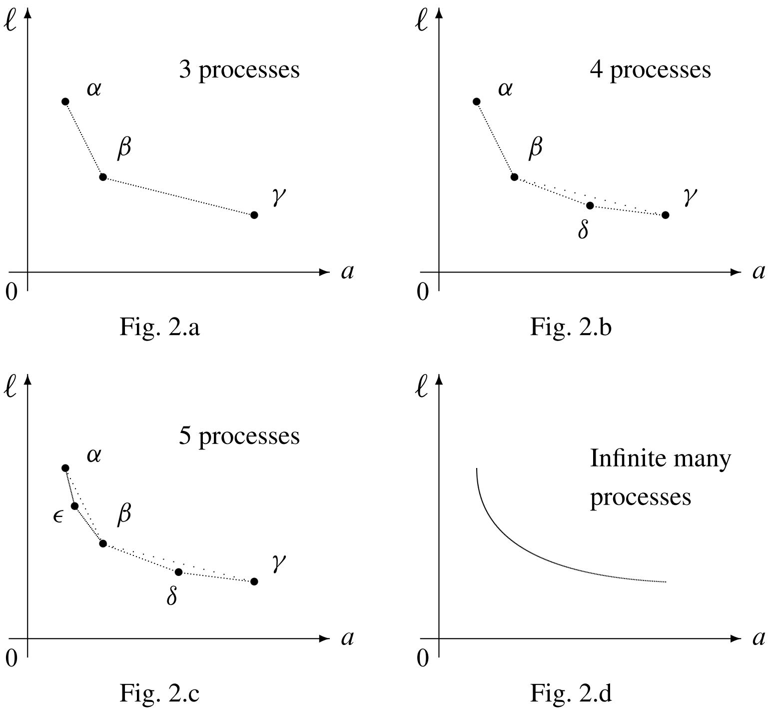

Figure 2: From a discrete to a continuum spectrum of techniques

as in Figure 2.c, etc. If the number of elementary processes becomes higher and higher the broken line tends to become a smooth curve as in Figure 2.d.

With such a smooth isoquant it is evident that any change, however small it is, in the relative price of inputs entails a switch towards another process, which employs the two inputs in a proportion *slightly* varied in favour of the input that has fallen. Here *small* variations in the relative price of inputs entail *small* variations in the proportion in which they are employed.

3 A Neoclassical continuous spectrum of techniques

We can made the above reasoning rigorous. Suppose that each commodity is produced according the following production function:

$$y_i = F_i(x_{1i}, \cdots, x_{Mi}, L_i),$$

where y_i is tue quantity produced of commodity i and x_{mi} and L_i are the quantity of commodity m and of labour employed to produce commodity i, $i, m = 1, \dots, I = M$. F_i is twice continuously differentiable and homogeneous of first degree. Thus

$$F_i\left(\frac{x_{1i}}{y_i}, \cdots, \frac{x_{Mi}}{y_i}, \frac{L_i}{y_i}\right) = 1 \quad \Leftrightarrow \quad F_i(\mathbf{a}_i, \ell_i) = 1.$$

is the unitary isoquant. For each level of r the cost-minimizing technique of industry i is the solution of:

$$\min_{\mathbf{a}_i, \ell_i} (1+r)\mathbf{p}^T \mathbf{a}_i + w\ell_i \qquad \text{s. v.} \qquad F(\mathbf{a}_i, \ell_i) = 1, \tag{3}$$

where, in analogy with price system (1), means of production are supposed to be paid at the beginning, while labour at the end of the production process. The first order conditions of problems (3) for the whole system are:

$$F_i(\mathbf{a}_i, \ell_i) = 1 \qquad [M \text{ eqns.}] \tag{4a}$$

$$\frac{\frac{\partial F_i}{\partial a_{mi}}(\mathbf{a}_i, \ell_i)}{\frac{\partial F_i}{\partial a_{\mu i}}(\mathbf{a}_i, \ell_i)} = \frac{p_m}{p_\mu} \qquad [M^2 - M \text{ eqns.}] \tag{4b}$$

$$\frac{\frac{\partial F_i}{\partial a_{mi}}(\mathbf{a}_i, \ell_i)}{\frac{\partial F_i}{\partial \ell_i}(\mathbf{a}_i, \ell_i)} = \frac{p_m}{w} \qquad [M \text{ eqns.}] \tag{4c}$$

$$m, \mu, i = 1, \dots, M = I, m \neq \mu,$$

where p_m and w satisfy the following zero-extra-profit conditions:

$$\mathbf{p}^T = w\boldsymbol{\ell}^T[\mathbf{I} - (1+r)\mathbf{A}]^{-1} \qquad [M \text{ eqns.}] \tag{2b}$$

$$w = \frac{1}{\boldsymbol{\ell}^T[\mathbf{I} - (1+r)\mathbf{A}]^{-1}\mathbf{b}} \qquad [1 \text{ eqn.}] \tag{2a}$$

Systems (4) and (2) are constituted by M^2+2M+1 equations in M^2+2M+1 unknowns: M^2 physical technical coefficients ($\mathbf{A}$), M labour coefficients ($\boldsymbol{\ell}$), M prices ($\mathbf{p}$) and the wage rate (w).

For any $r \in [0, R)$, with $R := \max_{\mathbf{A}}[1/\lambda(\mathbf{A}) - 1]$ system (4)-(2) has an economically meaningful solution, under some appropriate conditions for F_i:

$$\boldsymbol{p}(r), w(r), \mathbf{A}(r), \boldsymbol{\ell}(r).$$

It holds the following

Proposition 1. $\mathbf{A}(r)$ *and* $\boldsymbol{\ell}(r)$ *are* continuous *functions of* r.

Proof. (Sketch) It is an application of the theorem of the maximum.

□

4 A non-Neoclassical continuous spectrum of techniques

That of the previous section is the framework normally used in Neoclassical analysis to represent the problem of cost minimization in the case of technologies with infinitely many techniques. But this is *not* the only way to depict a continuous spectrum of techniques. Bellino (1993) gave some counterexamples to disprove the idea that continuous switchings of techniques along the technological frontier entail continuous variations of technical input coefficients and of the value of capital per worker. It is however possible to show that these counterexamples are not particular or pathological cases. On the contrary it will be proved here that there exist a robust family of cases where the transition to continuity along technological frontier *does not entail* continuity in input requirements, i.e. in functions $\mathbf{A}(r)$ and $\boldsymbol{\ell}(r)$.

Consider a technological frontier $W(r)$ along which techniques switches in a continuous way, that is, any change in r, however small, entails a switch to another technique that yields the highest wage rate for any given profit rate. Suppose, by simplicity, that $W(r)$ si differentiable. A given technique $\begin{bmatrix} \mathbf{A} \\ \boldsymbol{\ell}^T \end{bmatrix}$ appears as a singleton along the $W(r)$ frontier at $r = \bar{r}$ if and only if the wage-profit relationship is tangent to the technological frontier at $r = \bar{r}$. This happens if and only if

$$\begin{aligned} w_{\mathbf{A},\boldsymbol{\ell}}(\bar{r}) &= W(\bar{r}) \\ w'_{\mathbf{A},\boldsymbol{\ell}}(\bar{r}) &= W'(\bar{r}), \end{aligned}$$

that is, if and only if,[5]

$$\frac{1}{\boldsymbol{\ell}^T[\mathbf{I}-(1+\bar{r})\mathbf{A}]^{-1}\mathbf{b}} = W(\bar{r}) \tag{5a}$$

$$\frac{\boldsymbol{\ell}^T[\mathbf{I}-(1+\bar{r})\mathbf{A}]^{-1}\mathbf{A}[\mathbf{I}-(1+\bar{r})\mathbf{A}]^{-1}\mathbf{b}}{\{\boldsymbol{\ell}^T[\mathbf{I}-(1+\bar{r})\mathbf{A}]^{-1}\mathbf{b}\}^2} = W'(\bar{r}). \tag{5b}$$

Conditions (5) are just 2 equations in M^2+M unknowns, $\mathbf{A}$ and $\boldsymbol{\ell}$. There are thus M^2+M-2 degrees of freedom to chose the coefficients $\mathbf{A}$ and $\boldsymbol{\ell}$ in order that a given technique appears as a singleton on the technological frontier at a particular level of r. Suppose that at $r=\bar{r}$ $\begin{bmatrix}\bar{\mathbf{A}}\\ \bar{\boldsymbol{\ell}}^T\end{bmatrix}$ is the most profitable technique. An infinitesimal change in r, from $\bar{r}$ to $\bar{r}+\mathrm{d}r$, will induce a change in some coefficients in $\mathbf{A}$ and $\boldsymbol{\ell}$ which *needs not* to be infinitesimal. That is, 'small' variations along the $W(r)$ frontier can entail *whatever* variation of $\mathbf{A}$ and $\boldsymbol{\ell}^T$. These considerations prove the following

Proposition 2. *In a multi-sectoral production model with a continuum spectrum of techniques a change of technique induced by an infinitesimal change of r is represented by a variation of technical coefficients* $\mathbf{A}$ *and* $\boldsymbol{\ell}$ *which, in general, is discontinuous.*

5 Concluding remarks

Compare the frameworks of sections 3 (the Neoclassical continuum spectrum of techniques) and 4 (the non-Neoclassical continuum spectrum of techniques). In section 4 we have a production system which is minimal in relation to the assumptions on technology: it is assumed only that production processes are constant returns to scale (in order to avoid to study the consequences of the change of the composition of the final product as income distribution is varied); no assumption have been put on matrices representing the various techniques that appear on the technological frontier as the rate of profit is varied, except that they are viable. The choice process has been described at the system level but, as recalled, it is possible to show that such a choice entails the minimum cost for each industry; thus the choice of technique is based on a *rational* criterion.

[5]For the obtainment of the expression of $w'(r)$ see Kurz and Salvadori (1994, p. 99).

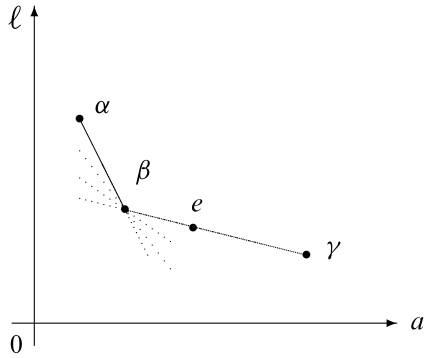

Fig. 3.a

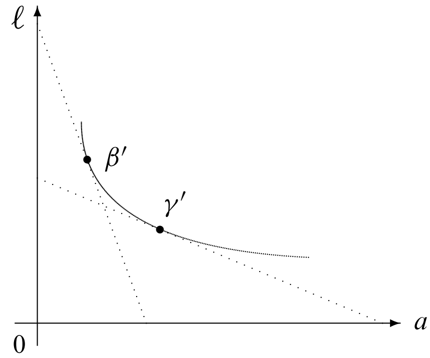

Fig. 3.b

Figure 3: Indeterminacy versus determinacy of Neoclassical distribution theory

The result contained in Proposition 2 of Section 4 shows that the assumptions that permit to describe the contour of the set of the relevant techniques by a *continuous* surface (i.e. continuity and convexity of production function or the closeness, additivity and convexity of production sets) confine the analysis to a very restricted set of cases. Ant this conclusion holds also when we have numerous or even infinitely many techniques. In other words the usual representation of technology is normally built on unnecessary restrictive assumptions.

There is another relevant implication of proposition 2: continuity, together with the appropriate differentiability conditions, permit to make 'determinate' the theory of distribution, or the theory of determination of input prices. Consider, for example, the isoquant of Figure 3.a: if an industry employs input as in point e (a convex combination of techniques β and γ) the relative input price is determined univocally by the slope of the segment joining techniques β and γ; if the industry employs input as in point β the relative input price is indeterminate: it can be any value between the slopes of segments α-β and β-γ. If isoquants are differentiable, like in Figure 3.b, relative input price (and thus income distribution) is univocally determined everywhere: in β' it is equal to the slope of the straight line passing through β'; in γ' is the straight line passing through γ', and so on.

These seem to be obsolete arguments nowadays, but—I would say—they have been explicitly dealt with some decades ago (see, for example, Solow (1967, p. 39) or Burmeister (1984, Sect. V))) and are always in the backward of mind of every mainstream economist. Proposition 2 proves that also this result is build on unnecessary restrictive assumptions. Moreover also the very notion of marginal productivity of one input, as well as of capital, appears to be definable in a very restrictive subset of cases.

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