Dynamic input-output and capital

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Abstract. The equilibrium prices of the closed dynamic input output model are interpreted as quantities of stratified produced capital, human capital included. The proper representation of the production process requires that the model can cope with long, but finite, productive periods, long gestation periods as well as long production periods. It is also necessary to take the inventories into account. Generalizations of the dynamic input-output model with at least these some of these properties are discussed and the one that meets all the requirements is pointed out. The interpretation of the equilibrium prices as quantities of stratified capital is based on an iterative method of solving the price equations.

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1. Introduction

Capital is, as everyone knows, a concept plagued with theoretical controversies as well as practical problems of measurement. This paper is not intended to solve either. Instead, it focuses on two things. The first one is the treatment of long productive periods and long gestation periods in the dynamic input output model with a uniform growth rate. The second one is the interpretation of the equilibrium prices of the dynamic input-output model as quantities of stratified capital. The first topic serves to show that the equilibrium prices interpreted in the second one do exist.

There is a long tradition in economics to make a distinction between fixed capital and circulating capital. Circulating capital is thought to contribute to the annual output and after that, so to say, "disappear". Fixed capital contributes to the output during several time units. It also takes time to install capital equipment in a plant. Long productive periods are not a problem, in a dynamic input-output model, in the case of geometric decline in efficiency and infinite lifetime of capital goods.

Johansen (1978) and Åberg & Persson (1981) have however introduced dynamic input-output models, with a uniform rate of growth, to deal with both long, but finite, productive periods and long gestation periods of fixed capital even in the case of non-geometric efficiency decline. The latter model is the more general one since it allows any pattern of efficiency decline of capital equipment. Lager (1997) has demonstrated that both these models, under some assumptions, appear as special cases of his version of the Sraffa-von Neumann type of model.

Both the Johansen model and the Åberg & Persson model follow the tradition of making the distinction between circulating and fixed capital. By capital they mean fixed capital. The existence of inventories is actually not taken into account. However in real economies the existence of inventories is an essential feature of the production process. Quite obviously inventories are not kept for fun but because it is not possible to organize the production process without them. Considerable amounts of products are in fact tied up in inventories. E.g. in 1981 the value of inventories held by business sector in the U.S. was about one fifth of the value of all produced tangible assets hold by the sector (See Eisner, 1989).

In Aulin-Ahmavaara (1987) a dynamic input-output model with long productive periods of fixed capital that covers also inventories was introduced. Differently from the Johansen and

Åberg & Persson models, it didn't cover long gestation periods of capacity within the utilizing unit but it did cover long production periods within the producing unit. Besides, in it the replacement requirements and net additions to capacity were separated. In Aulin-Ahmavaara (1990) the differences and similarities between these three models were discussed and a general model covering all of them was introduced.

The novelty of the present paper is the interpretation of the equilibrium prices of the closed dynamic input-output model as stratified capital. It is something that I came by when doing something else. I found this point of view rather amusing and wanted to share it with others. It actually was the reason why I wanted to write this paper.

It should be noticed that, in this paper, only produced asset (other than R&D capital) are treated as capital. R&D capital is different from the rest of produced assets in the sense that its services are not meant to be used as inputs in the production process itself but rather as means to change the production technology. Human capital however is included.

2. Open dynamic input-output model

A discrete version of the open dynamic input-output model, without technical change, can be written as follows

$$\mathbf{x}_t - \mathbf{A}_1\mathbf{x}_t - \mathbf{A}_2\mathbf{x}_t = \mathbf{B}_1(\mathbf{x}_{t+1} - \mathbf{x}_t) + \mathbf{B}_2(\mathbf{x}_{t+1} - \mathbf{x}_\mathbf{t}) + \mathbf{z}_\mathbf{t}$$

where

$\mathbf{A}_1 = \lfloor a_{ij} \rfloor$ is the matrix of input (flow) coefficients for those inputs from sector i that are used in the production process of sector j for a period shorter than the time unit of the model (normally a year);

$\mathbf{B}_1 = \lfloor b_{ij} \rfloor$ is the matrix of stock coefficients representing the quantity of the product of industry i that is tied up in the production process of the industry j for less than a year (coefficients of inventories);

$\mathbf{A}_2 = \lfloor a_{ij} \rfloor$ is the matrix of coefficients of replacement that indicate the quantity of the product of the industry i contributing to the production process of the industry j at least for a year that has to be replaced in the stocks of the industry j in order to keep the productive capital of the of type i at the level of the year t and

$\mathbf{B}_2 = \lfloor b_{ij} \rfloor$ is the matrix of stock coefficients representing the quantity of the product of sector i tied up in the production process of the industry j at least for a year (coefficients of fixed capital).

In this dynamic input-output model fixed capital and intermediate inputs are, in principle, treated symmetrically. The only difference is that inputs in the form of fixed capital contribute to the production process for longer than a year. All the stocks, either in the form of fixed capital or of inventories, are raised at the level needed during a time unit by the beginning of that unit.

Since every industry (or product) has, in this model, only one production technology, the stocks of fixed capital of different vintages have to be measured in efficiency units, or new machine equivalents. But they still are, at least in principle, physical quantities. The coefficients of replacement are in fixed proportion to output only if geometric decline in efficiency is assumed (See e.g. Brody, 1970 and Jorgenson et. al. 1987).

In a dynamic input-output model also inventories have to be treated as capital stocks. The inputs are tied up in the production process from the moment they enter the stocks of a unit using them to the moment that all the outputs to which they have contributed have left the stocks of that unit. It is essential for the description of the production process of the entire economy that all the products always are somewhere (for further discussion, see Aulin-Ahmavaara, 1990). It is also true that "the distinction between raw materials and short-lived [fixed] capital is arbitrary, and it can become troublesome if the length of the time period is changed." (Domar, 1961).

3. Closed dynamic input-output models with long productive and gestation periods

A closed version of the dynamic input-output model for an economy growing at a constant rate λ, also called the Brody model (see e.g. Steenge, 1990) can be written in the following form

$$\overline{\mathbf{x}} - \mathbf{A}_1\overline{\mathbf{x}} - \mathbf{A}(\lambda)_2\,\overline{\mathbf{x}} = \lambda\mathbf{B}_1\overline{\mathbf{x}} + \lambda\mathbf{B}(\lambda)_2\,\overline{\mathbf{x}}\,, \qquad (1)$$

where $\overline{\mathbf{x}}$ is the vector of output proportions.

For the model to be really closed it obviously has to include the production processes of different kinds of human capital and human time (or labor) as well. Aulin-Ahmavaara (1991 and 1997) give a detailed description of the way, in which these processes can be represented in the closed dynamic input-output model.

For human capital geometric decline in efficiency is however very hard to accept. Fortunately in the case of balanced growth the assumption of geometric pattern is not needed. In the general case coefficients of replacement of fixed capital depend on the rate of growth. Also the stock coefficients relating to fixed capital can depend on the rate growth provided that it takes time to assemble the units of fixed capital within the plant, where they will be used.

The exact form of the equations of the model (1) can be derived from a model introduced by Åberg and Persson (1981) with the following typical equation:

$$x_i - \sum_{j=1}^{m} a_{ij} x_j - \sum_{j=m+1}^{n} \sum_{\theta=1}^{T_j} \left[\frac{(1+\lambda)^{\theta}}{\sum_{p=1}^{P_j} w_{jp} (1+\lambda)^{-(p-1)}} \right] b_{ij\theta} x_j = 0 . \qquad (2)$$

where

x_i is the output of sector i ;

$j = 1, \dots, m$ for sectors, in which the input from sector i contributes for less than a year;

$j = m+1, \dots, n$ for sectors, in which the input from sector i contributes at least for year;

a_{ij} are the input coefficients for the first group inputs;

$b_{ij\theta}$ is the amount of product i required per unit of increase in the output of sector j at time $t+\theta$;

T_j is the maximal gestation period of capacity (the minimal is one time unit, since the capacity has to be there at the beginning of the time unit when the increase in the output takes place);

w_{jp} is the contribution to the capacity of sector j by capacity installed at time $t-p$

P_j is the number of productive years of units of capital goods in sector j .

Åberg and Persson (1981) have shown that under certain conditions there is a solution of the equation (2) with a uniquely determined growth rate λ and output proportions $\bar{\mathbf{x}}$.

Differently from the original equation of Åberg and Persson (1981) the inputs from sector i have in the equation (2) been divided into two classes, those contributing to the production of the receiving sector for less than a year and those contributing a least for a year (fixed capital). This follows from the fact that the third term represents both replacement and net additions to capacity. There cannot be two separate input flows from one sector to another. By letting $T_j = P_j = 1$ it is easy to see that the second and third terms cannot refer to the same utilising sector.

As a description of a real world production process there are still a few shortcomings in the equations (2). First there are no coefficients relating to inventories. This, too, becomes obvious by selecting $T_j = P_j = 1$ in the third term. The necessity to take inventories into account was already discussed in the previous section.

Secondly, all the capital goods used in a sector are assumed to have the same pattern of efficiency decline. Considering that capital goods consist of items as different as computers and buildings this is not a very plausible assumption.

Thirdly, for some of the products it takes longer than a year to get finished. This problem relating to long production periods can however be easily solved by treating the intermediate phases of the product as different products (Aulin-Ahmavaara, 1990).

To correct the model for the first and second shortcomings and also to separate the inputs needed to replace the existing capital goods from those needed to expand production the equations (2) can be rewritten as follows:

$$x_i - \sum_{j=1}^{m} a_{ij} x_j - \lambda \sum_{j=1}^{m} b_{ij} x_j -$$
$$\sum_{j=m+1}^{n} \sum_{\theta=1}^{T} \left[\frac{(1+\lambda)^{\theta}}{\sum_{p=1}^{P_{ij}} w_{ijp} (1+\lambda)^{-(p-1)}} - \lambda (1+\lambda)^{\theta-1} \right] b_{ij\theta} x_j - \lambda \sum_{j=m+1}^{n} \sum_{\theta=1}^{T} (1+\lambda)^{\theta-1} b_{ij\theta} x_j = 0. \tag{3}$$

The term with square brackets in (3) shows the replacement requirements of products of sector i per unit of output in sectors j while the last term represents the quantities of the output of the sector i required to raise the level of the output of the sectors j.

The third term represents the addition to inventories. It could have been included in the last term. This has not been done since it is not possible to include coefficients of inventories in the last term on the left-hand side of equations 2. (This too can be easily verified by choosing $T_j = P_j = 1$ in equations 2.)

The possibility of different patterns of efficiency decline has been introduced by the coefficients w_{ijp}. Following the proof given by Åberg and Persson (1981) it is possible to show (see Aulin-Ahmavaara, 1990) that the model represented by the equations (3) also has a solution with the uniquely determined growth rate λ and output proportions $\bar{\mathbf{x}}$.

Lager (1997) has demonstrated that the model of equations (2), under some assumptions, appears as special cases of his version of the Sraffa-von Neumann type of model. But this does not cover the model that includes the coefficients of inventories represented by the equations (3). In a von Neumann type of model the inputs are fed into process at the beginning of the time unit and the outputs come out from it at the end of the time unit. And it seems to me that there is no way to take, in that type of a model, into account the fact that some of the inputs are fed into the processes and some of the outputs come out of them at intervals shorter than the time unit. With any time unit longer than the singleton interval this however is likely to happen.

With an infinitesimally short time unit it would indeed be possible to think that all the inputs are fed into the process at the beginning of each time unit and that all the outputs appear from their production processes at the end of the time unit. But this of course would mean that there should be an infinite number of processes, since intermediate phases of products as well as capital goods at different ages should be treated as separate products. In any case the shorter the time unit the bigger the number of processes. Another problem with a short time unit is that more and more inputs are fed into the production process in batches, which are not used up during the time unit and will actually contribute the output during several time units.

There is the problem of selecting the correct time unit in the dynamic input-output model as well. But in any case it is possible to take into account the fact that some of the inputs are fed into the production processes and some of the outputs come out of them at intervals shorter than the time unit. The shorter these intervals are the smaller are the stock coefficients (coefficients of inventory) for these inputs. (For a more detailed discussion see Aulin-Ahmavaara, 1990.)

4. The price equations of the closed model

Price proportions, as well as the normal rate of profit are uniquely determined (see Brody, 1970, Åberg & Persson, 1981 and Aulin-Ahmavaara, 1990) by the technological matrices in the dual system:

$$\overline{\mathbf{p}} = \overline{\mathbf{p}}\mathbf{A}_1 + \overline{\mathbf{p}}\mathbf{A}_2(\lambda) + \lambda\overline{\mathbf{p}}\mathbf{B}_1 + \lambda\overline{\mathbf{p}}\mathbf{B}_2(\lambda)\,. \tag{4}$$

where $\overline{\mathbf{p}}$ is the vector of price proportions. In the equilibrium the normal rate of profit, uniquely determined by price equations, and the balanced rate of growth, determined by the quantity equations, are equal. Therefore they are both here denoted by λ.

The typical price equation can be written as follows:

$$\begin{aligned} p_j &= \sum_{i=1}^{m} a_{ij} p_i + \lambda \sum_{i=1}^{m} b_{ij} p_i - \\ &\sum_{i=m+1}^{n} \sum_{\theta=1}^{T} \left[\frac{(1+\lambda)^{\theta}}{\sum_{p=1}^{P_{ij}} w_{ijp} (1+\lambda)^{-(p-1)}} - \lambda (1+\lambda)^{\theta-1} \right] b_{ij\theta} p_i - \lambda \sum_{i=m+1}^{n} \sum_{\theta=1}^{T} (1+\lambda)^{\theta-1} b_{ij\theta} p_i \end{aligned} \tag{5}$$

For inputs with a productive period that does not exceed the time unit the first term on the right-hand side of the equations (5) gives simply the cost of intermediate inputs used up in the production process. The second term represents the return to capital in the form of inventories.

The forth term represents return to fixed capital. The third term, with square brackets, represents the value of the fixed capital that has to be replaced during a time unit provided that value of fixed capital is to be kept at the level of the previous year. In other words it represents depreciation. Together the third and forth terms are equal to the Hall & Jorgenson (1967) rental price (or user cost) of capital in the absence of asset revaluation and taxes.

5. Prices as stratified capital

Next consider the following iterative method of calculating the solution of the price equations. Since $\mathbf{B}(\mathbf{I}-\mathbf{A})^{-1}$, denoted below by $\mathbf{D}$, is a positive Frobenius matrix we have (see Tsukui & Murakami, 1979 and Aulin-Ahmavaara, 1987) for any semipositive row vector $\mathbf{s}$

$$\lim_{t\to\infty}(\mathbf{s}\mathbf{D}^t)\left(\sum_j\left[\mathbf{s}\mathbf{D}^t\right]_j\right)^{-1}=\overline{\mathbf{p}}\left(\sum_j\overline{p}_j\right)^{-1}=\overline{\mathbf{p}}, \tag{6}$$

The matrix $\mathbf{D}$ can of course be developed as follows

$$\mathbf{D}=\mathbf{B}(\mathbf{I}-\mathbf{A})^{-1}=\mathbf{BI}+\mathbf{BA}+\mathbf{BA}^{2}+\ldots. \tag{7}$$

The columns of the first term of the series in equation 7 represent the quantities of different types capital (both fixed capital and inventories) needed per unit of output of the sector represented by the column. The second term represents direct capital stocks needed for the direct input flows, replacement of fixed included while the third term represents the direct capital stocks for direct input flows, replacement of fixed capital included, to direct input flows etc. This is the first layer of capital. It should be noted that this is closed model, which includes human capital.

Obviously also

$$\mathbf{DxD}=\left[\mathbf{B}(\mathbf{I}-\mathbf{A})^{-1}\right]\mathbf{D}=\mathbf{BID}+\mathbf{BAD}+\mathbf{BA}^{2}\mathbf{D}+\ldots.$$

where the first term represents the direct capital stocks needed for producing the first layer of capital stocks etc. This is the second layer of capital and so on ad infinitum. Thus only the matrices $\mathbf{A}$ and $\mathbf{B}$ based on physical quantities are needed. But they must also include the coefficients relating to human capital and human time.

The fact that the coefficients depend on the rate of growth requires in fact alternating iterations to solve the equations 1. The first one to solve the rate of growth for given values of the coefficients (e.g. those with zero rate of growth) and then calculating the coefficients again etc. (see Aulin-Ahmavaara 1987). But this does not change the interpretation.

6. Summary

This paper ends up suggesting an interpretation of the equilibrium prices of closed dynamic input output model as quantities of stratified capital. For this a model from which the prices can be solved is needed. The representation of the production process requires that the model can cope with long, but finite, productive periods, long gestation periods as well as long production periods. Since human capital is included geometric decline in efficiency does not seem plausible. A version of the dynamic input-output model introduced by Åberg and Persson (1981) allows any pattern of efficiency decline. Their model is completed to include inventories and its representation is changed to show the depreciation and the return to capital separately. An iterative method of calculating the empirical solution of this model is utilised to interpret the equilibrium prices as quantities of stratified capital.

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