Comparing and Expanding SDA and INA Techniques Applied to Physical Flows in the Economy

by Rutger Hoekstra and Jeroen van den Bergh[1]

Department of Spatial Economics
Faculty of Economics and Econometrics
Free University
De Boelelaan 1105
1081 HV Amsterdam
The Netherlands
rhoekstra@econ.vu.nl, jbergh@econ.vu.nl

ABSTRACT

Structural decomposition analysis (SDA) and index number analysis (INA) are methods that decompose (economic) relationships into determinant sources. The effects of technological, demand and structural changes on physical flows can therefore be analyzed. A large number of environmental problems are related to material flows generated by economy. The first aim of the paper is to compare the SDA and INA methods. The fundamental difference is that SDA is based on input-output data while INA uses only the aggregate sector data. SDA and INA have developed fairly autonomously and different application practices are used. INA, for example, has developed a more sophisticated set of decomposition indices. The second aim of this paper is to transfer these INA insights to the SDA setting. The methods are subsequently evaluated using a numerical example.

[1] The authors would like to thank Erik Dietzenbacher and Marcel Boumans for their comments on earlier versions of the paper.

1. INTRODUCTION

To analyze and understand historical changes in economic or environmental indicators it is useful to assess the driving forces or determinants that underlie these changes. Two techniques for decomposing variable changes into their determinant effects are structural decomposition analysis (SDA) and index number analysis (INA) methods.[2] Both methods allow researchers to assess the influence of economic growth, sectoral shifts and technical change on the indicator changes. SDA has been used to study issues such as energy consumption, environmental emissions, economic output, value added, and employment. INA has been applied mainly to energy and energy related emissions.

The methods differ with regard to the model and data used and therefore also leads to different determinants being distinguished. SDA is based on the input-output model of the economy, which includes data on the intersector deliveries. INA, on the other hand, only uses aggregate sector information. Apart from this fundamental difference some application distinctions also exist. The INA literature have used a wider range of indicators types and decomposition indices. The aims of this paper are twofold: To give an overview of the differences and similarities in the 2 streams of decomposition and to introduce the mathematical intricacies of the INA indices to the SDA context. A hypothetical numerical example is set up to further examine differences between the various INA and SDA methods. The example is an extension of Ang (1999).

The two methods will focus on "physical flows" driven by sectors and the economy as a whole. Physical flows is a catch-all phrase to capture all the material inputs (such as energy, metals, plastics) or outputs (such as CO_2 emissions, acidifying emissions). Many physical flows have a direct relation to environmental problems. The decomposition techniques discussed in this paper are, however, not exclusive to physical flow analysis. The decomposed equations use an intensity vector of the material intensity per sector, i.e. material throughput per unit output. Similar vectors could be used for analyses such as labor supply or value added development.

The structure of the paper is as follows. Section 2 gives a general comparison of the SDA and INA fields. Section 3 discusses decomposition indicators, methods and indices that are used in the INA literature. In section 4 these INA decomposition techniques are translated to the SDA setting. Section 5 is devoted to the analysis of a numerical example in which all the decomposition methods are calculated. In section 6 the characteristics of the indices are discussed to facilitate index choice. Section 6 concludes.

---

[2] This paper uses the terms adopted by Rose and Casler (1996). SDA is sometimes also referred to as input-output structural decomposition analysis while INA is sometimes simply called decomposition analysis or energy decomposition analysis.

2. SDA AND INA COMPARED

Fundamental Differences

In their review of SDA, Rose and Casler (1996) briefly compare SDA and INA methods. It is noted that if input-output information was added to INA it “might actually generalize to IO SDA”. A comprehensive comparison of the two methods is however lacking in the literature.

The primary difference is the economic data and model that are used (see table 1 for the data and appendix 1 for the variables explanations). SDA uses input-output data in the form of technical coefficients ($A_{ij}=Z_{ij}/x_j$) and the final demand (y_j) per sector (Miller and Blair, 1985). INA on the other hand uses the output per sector (x_i) for the economic decomposition. The material intensity ($r_i=m_i/x_i$), a measure of the sector’s material use (m_i) per unit output, is used as a determinant in both methods as the link to the physical flows.

Table 1. Data used in INA and SDA

Monetary Accounts					Physical Accounts
	Sector 1	Sector 2	Final Demand	Output	Material Use
Sector 1	Z_{11}	Z_{11}	y_1	x_1	m_1
Sector 2	Z_{11}	Z_{11}	y_2	x_2	m_2

An advantage of the INA method is therefore the lower data requirement. However, the extra data used in SDA give more detailed decompositions of the economic structure than INA. The input-output model also includes indirect demand effects of direct demand. Direct demand for the products of goods from one sector also leads to increases in demand for other sectors because they supply inputs to that sector. This is known as indirect demand. The INA model does not use the input-output model and is therefore only capable of assessing the impact of the direct effects.

A second advantage of the input-output model is that the columns of the technical coefficient matrix may be regarded as a production function. This description of the technology of the economy makes decomposition of substitution and efficiency effects possible. Such effects can not be distinguished in the INA framework.

Differences in the Applications

Some of the differences of SDA and INA cannot be ascribed to the fundamental differences mentioned above. One of these differences is the range of policy issues that have been analyzed. INA has almost exclusively been used for the analysis of energy use and energy related emissions. SDA has been applied to a wider range of issues including energy use, CO_2-emissions, labor requirements, value added and economic output. Another contrast is the shorter time-steps in INA applications (annual decomposition are common). SDA applications usually use 5-10 year time-steps. The reason for this difference is probably that input-output data is that many countries do not produce input-output tables on an annual basis.

The largest difference in the applications is that INA has developed a greater degree of methodological sophistication. The decomposition studies have developed a greater amount of indicator types and decomposition indices than the SDA literature.

Figure 2 depicts the different decomposition approaches that have been applied in the INA literature. It is mainly based on the summary paper by Ang (1999).

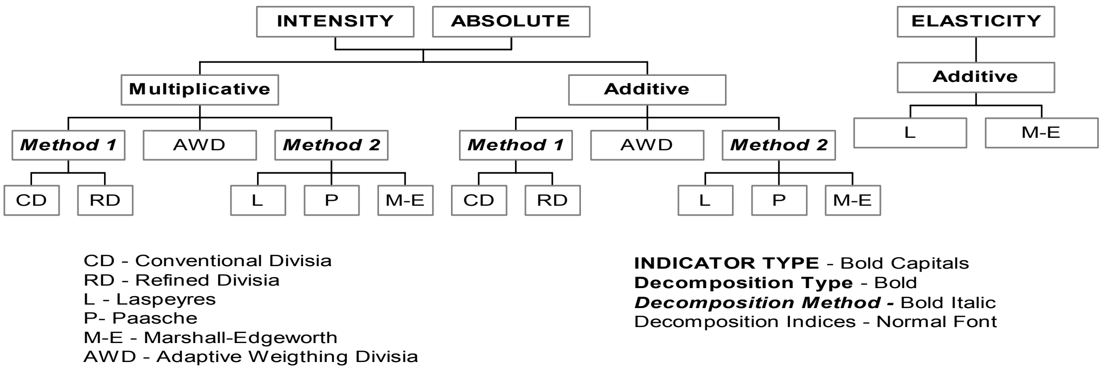

Figure 1. Decomposition methods applied in the INA literature

Figure 1 shows that there are 3 *indicator types*[3] (absolute, intensity and elasticity) that have been decomposed. The energy consumption or material use would be an example of absolute indicators, while the intensity measure records the amount of material used per unit output (or value added). The third indicator is the output elasticity of energy that measures the relative change of in energy use compared to the monetary output. The SDA literature is almost exclusively focussed on the absolute indicator type (the only exception known to the authors is Dietzenbacher, Hoen and Los (2000)).

The intensity and absolute indicators may be decomposed by the multiplicative and additive *decomposition types*. There seems to be no reason to prefer either of these methods and the INA literature uses both types. SDA applications generally use the additive decomposition (exceptions being Dietzenbacher, Hoen and Los (2000, Han and Lakshmanan (1994)).

The INA literature has yielded 2 *decomposition methods* that may be applied to either the multiplicative or additive decompositions. This distinction is based on the growth rate (method 1) or difference (method 2) decomposition of the determinants.[4] Parametric forms of these methods were introduced in Liu, Ang and Ong (1992)

Finally there are a number of *decomposition indices*: conventional Divisia and refined Divisia (method 1); Laspeyres, Paasche and Marshall-Edgeworth (method 2); and the adaptive weighting Divisia index (combination of method 1 and 2).

Figure 1 shows 26 possible decomposition results that can be obtained from the same dataset. A researcher first needs to decide which indicator they would like to investigate. Thereafter an index should be chosen by selecting the decomposition type, method and index.

---

[3] In Ang (1999) uses the terms "energy intensity approach", "energy consumption approach" and "energy elasticity apporach". Since the aim of the paper is to give an overview of SDA and INA in a wider context, these energy related names have been replaced by more general terms.

[4] The terms "method 1 and 2" are based on Liu, Ang and Ong (1992) which proposed the "parametric Divisia index method 1 and 2". Since these terms would not enable classification of the refined Divisia index, the classification has been broadened by excluding "parametric Divisia".

In the SDA literature the indices that are used are more limited. Decomposition of intensity indicator is rare and multiplicative decompositions are also rarely done. The standard SDA application uses the additive decomposition of an absolute indicator. Laspeyres and Marshal-Edgeworth indices are used while the Divisia indices have not, as far as we know, been used.

The SDA literature has, however, yielded one approach that is not used in INA applications. Dietzenbacher and Los (1998) note that a variable with *n* determinants can be decomposed in *n!* ways, assuming that the decomposition has no residual and that each determinant can either use base or terminal year weights. In other words the magnitude of the effect of a determinant d_1 could be calculated by $\Delta d_1 \cdot d_2^t \cdot d_3^{t-1} \cdot d_4^{t-1} \cdot d_5^t$ or $\Delta d_1 \cdot d_2^{t-1} \cdot d_3^t \cdot d_4^{t-1} \cdot d_5^t$ or any other combination of the determinants and weights. As you can see the assumption of consistent weights for all determinants (as in INA) is discarded. The average of all the possible combinations is then taken as the actual determinant effect.

3. OVERVIEW OF INA INDICES

Indicator and Decomposition Types

The indicator types (absolute, intensity and elasticity) combined with the decomposition types (multiplicative or additive decomposition) result in 5 combinations.[5]

To illustrate the different indicator and decomposition types assume a functional relationship $m=f(d_1,\dots d_c)$ where *m* is an absolute indicator of material use by an economy and $d_1,\dots,d_c$ are its determinants. Similarly the material intensity ($r=m/x$) (where *x* is the output) may be dependent on determinants $g_1,\dots,g_c$ in the function $r=f(g_1,\dots,g_c)$. The following equations illustrate the 5 combinations.

Table 2. Decomposition indicators and types

Indicator	**Type**	**Equation**	
Intensity	Multiplicative	$\frac{r^t}{r^{t-1}} = (\text{effect g}_1)\times(\text{effect g}_2)\times\dots\times(\text{effect g}_c)$	(1)
Intensity	Additive	$r^t - r^{t-1} = (\text{effect g}_1)+(\text{effect g}_2)+\dots+(\text{effect g}_c)$	(2)
Absolute	Additive	$m^t - m^{t-1} = (\text{effect d}_1)+(\text{effect d}_2)+\dots+(\text{effect d}_c)$	(3)
Absolute	Multiplicative	$\frac{m^t}{m^{t-1}} = (\text{effect d}_1)\times(\text{effect d}_2)\times\dots\times(\text{effect d}_c)$	(4)
Elasticity	Additive	$\frac{\frac{\Delta m}{m}}{\frac{\Delta x}{x}} = \frac{\frac{(\text{effect d}_1)}{m}}{\frac{\Delta x}{x}} + \frac{\frac{(\text{effect d}_2)}{m}}{\frac{\Delta x}{x}} + \dots + \frac{\frac{(\text{effect d}_c)}{m}}{\frac{\Delta x}{x}}$	(5)

These 3 indicator types and 2 decomposition methods are the starting point for the decompositions. As has been noted the INA literature uses all 5 of these

[5] Remember that elasticity is only decomposed additively in the INA literature.

combinations. SDA focuses almost exclusively on the absolute-additive decomposition.

Decomposition Indices

Figure 1 displayed the wide variety of indices that have been applied in INA. Index theory has a long history that is mainly focussed on the development of price and quantity indices in economics. Fisher (1922) in "The Making of Index Numbers", provided the most influential contribution to the development of index number theory by comparing and discussing hundreds of different indices. Despite Fisher's initial claim to an "ideal" index[6], the index number literature has concluded that there is no single index that is satisfies all beneficial properties.

Vogt (1978) introduced the idea that each index describes a path between two discrete time points. The index therefore acts as an approximation of the actual continual time path. There are an infinite number of integral paths and therefore an infinite range of possible indices.

One method for selecting index numbers is based on the axiomatic approach. This approach analyses the properties of the indices by testing their properties (for an overview see Vogt and Barta, 1997). The "commensurability test", for example, finds whether the index is invariant to the units used. Clearly if the results change simply because you convert your data from kilograms to tonnes the index is of minor use. Another test is the "time reversal test" which checks if an index for which the time 0 and 1 are reversed, gives the reciprocal value. Some of the indices that are often used in SDA and INA, such as the Laspeyres and Paasche, fail this test.[7]

4. DERIVATION OF SDA INDICES

In this section, indices for the multiplicative decomposition of intensity and the additive decomposition of absolute and elasticity indicators are derived for the SDA setting. The intensity (additive) and absolute (multiplicative) are not derived because their derivation is very similar. The equivalent INA equation can be found in appendix 2 and the symbols used are given in appendix 1. The following indices are derived:

Multiplicative decomposition of an intensity indicator

1. Method 1 (Conventional Divisia, Refined Divisia[8])
2. Method 2 (Laspeyres, Marshall-Edgeworth, Paasche)
3. Method 1 and 2 combined (Adaptive Weighting Divisia index)

Additive decomposition of an absolute indicator

4. Parametric Divisia Index Method 1 (Conventional Divisia, Refined Divisia)

---

[6] Fisher was so convinced that his index was perfect that when he found that his index did not satisfy the so-called "circular test", he concluded that it was the test which was flawed (see Vogt and Barta (1997)).

[7] For example if a quantity changes from 80 to 100 the Laspeyres index indicates that this is a 25% increase, while if the variable changes from 100 to 80 it gives a 20% decrease. The Laspeyres index therefore fails the time reversal test.

[8] This is a special case of method 1 in which the equation is also expressed in growth terms but instead of the arithmetic mean (which is used for the Conventional Divisia index) the logarithmic mean between two time periods is used.

5. Parametric Divisia Index Method 2 (Laspeyres, Marshall-Edgeworth, Paasche)
6. Method 1 and 2 combined (Adaptive Weighting Divisia index)

Additive decomposition of an elasticity indicator

7. Laspeyres, Marshall-Edgeworth, Paasche

Multiplicative decomposition of an intensity indicator

In the following section the derivation of the multiplicative decomposition of the material intensity is given. In an economy with *n* sectors the material intensity *r* of the economy is given by the following equation:

$$r = \frac{m}{x} = \frac{\sum_i \sum_j L_{ij} \cdot y_j \cdot r_i}{x} \tag{6}$$

Where *i* and *j* range from sectors 1 to *n*. The variables that do not have subscripts are economy-wide values. The national material intensity *r* is therefore equal to the ratio of the total material use and the total output of the country in question. The product of the Leontief inverse L_{ij} and the final demand y_j is the input-output model specification for sector level output (x_i). Differentiating with respect to time and dividing both sides by the *r* leaves:

$$\hat{r} = \sum_i \sum_j \left(\frac{dL_{ij}}{dt}\right) \cdot \left(\frac{y_j \cdot r_i}{m}\right) + \sum_i \sum_j \left(\frac{dy_j}{dt}\right) \cdot \left(\frac{L_{ij} \cdot r_i}{m}\right) + \sum_i \sum_j \left(\frac{dr_i}{dt}\right) \cdot \left(\frac{L_{ij} \cdot y_j}{m}\right) - \left(\frac{1}{x}\right) \cdot \left(\frac{dx}{dt}\right) \tag{7}$$

Where relative growth rates are indicated by a hat, e.g. $\hat{r} = \frac{dr}{dt} \Big/ r$.

The equation could however also be rewritten entirely in relative growth terms:

$$\hat{r} = \sum_i \sum_j \hat{L}_{ij} \cdot w_{ij} + \sum_i \sum_j \hat{y}_j \cdot w_{ij} + \sum_i \sum_j \hat{r}_i \cdot w_{ij} - \hat{x} \tag{8}$$

Where $w_{ij}=m_{ij}/m$ is a weight function and m_{ij} ($=L_{ij} \cdot y_j \cdot r_i$) is the material throughput generated in sector *i* due to the final demand of sector *j*. Equations 7 and 8 can both be used as a basis for the decomposition. If equation 7 is adopted it is referred to as a method 2 decomposition because the determinants are expressed in terms of differences, while equation 8 (method 1) expresses the determinant change in terms of growth rates. Upon integration of the left and right hand side over discrete period *t-1* to *t*, both methods result in the same general decomposition form:

$$D_{I(m)} = \frac{r^t}{r^{t-1}} = D_{I(m)}^{ltf} \cdot D_{I(m)}^{fd} \cdot D_{I(m)}^{\text{int}} \cdot D_{I(m)}^{pdn} \cdot D_{I(m)}^{rsd} \tag{9}$$

The *D* variables are the decomposition effects for which the superscripts indicate the determinant.[9] The subscript will be used to indicate the indictor and type of

[9] *ltf* - technology (Leontief matrix) effect, *fd* – final demand effect, *int* – material intensity effect and *pdn* – the production effect. The *rsd* superscript stands for the residual effect.

decomposition that is used. In this case it is the multiplicative decomposition of intenisty (*I(m)*). The total change in the indicator (no superscript) is given on the left hand side of the equation. All decomposition subscripts and superscripts may be found in appendix 1.

The above decomposition results in 4 determinant effects and a residual. Further decomposition of the ($n*n$) elements of the Leontief matrix[10] and the n elements of the final demand and material intensity vectors is however possible.

Parametric Method 1

By integrating both sides of equation 8 the following Liu, Ang and Ong (1992) found the following parametric specification.[11]

$$
\begin{aligned}
D_{I(m)1}^{ltf} &= \exp\left[\sum_i \sum_j \ln\left(\frac{L_{ij}^t}{L_{ij}^{t-1}}\right) \cdot \left[w_{ij}^{t-1} + \alpha_{ij}^{ltf} \cdot \Delta w_{ij}\right]\right] \\
D_{I(m)1}^{fd} &= \exp\left[\sum_i \sum_j \ln\left(\frac{y_j^t}{y_j^{t-1}}\right) \cdot \left[w_{ij}^{t-1} + \alpha_{ij}^{fd} \cdot \Delta w_{ij}\right]\right] \\
D_{I(m)1}^{\text{int}} &= \exp\left[\sum_i \sum_j \ln\left(\frac{r_i^t}{r_i^{t-1}}\right) \cdot \left[w_{ij}^{t-1} + \alpha_{ij}^{\text{int}} \cdot \Delta w_{ij}\right]\right] \\
D_{I(m)1}^{pdn} &= \exp\left[-\ln\left(\frac{x^t}{x^{t-1}}\right)\right]
\end{aligned}
\qquad (10)
$$

A special case of the parametric Divisia index method 1 is the conventional Divisia index where $\alpha_{ij}^{ltf} = \alpha_j^{fd} = \alpha_i^{int} = 0.5$.

Non-Parametric Method 1

The refined Divisia index is a non-parametric case of method 1 where instead of the arithmetic mean (as in the conventional Divisia index) the normalized logarithmic mean is taken of the weights. It is also classified under as "method 1" because it expresses the determinants in growth rates as a basis for its decomposition[12].

---

[10] One could look at more specific technology effects by grouping technical coefficients (see Rose and Casler (1996) for details).

[11] Although mathematical details are not given in Liu, Ang and Ong (1992) it is assumed that the parametric specifications is based on the integral form of the mean value theorem. Assuming $L_{ij}(t)$ and $w_{ij}(t)$ are continuous on [t-1,t] and $L_{ij} > 0$ on (t-1,t). Then there is some point c between t-1 and t such that

$$\sum_i \sum_j \int_{t-1}^{t} L_{ij}(\tau) \cdot w_{ij}(\tau) d\tau = \sum_i \sum_j w_{ij}(c) \cdot \int_{t-1}^{t} L_{ij}(\tau) d\tau$$

The point c also be written in the following parametric from:

$$w_{ij}(c) = w_{ij}^{t-1} + \alpha_{ij}^{ltf} \cdot \Delta w_{ij}$$

[12] Although never applied in the INA literature, the "refined" methodology could, it seems, be applied to method 2.

$$
\begin{aligned}
D_{I(m)R}^{ltf} &= \exp\left[\sum_i\sum_j \ln\left(\frac{L_{ij}^t}{L_{ij}^{t-1}}\right)\cdot\frac{\Delta w_{ij}}{\ln\left(w_{ij}^t / w_{ij}^{t-1}\right)} \Big/ \sum_i\sum_j \frac{\Delta w_{ij}}{\ln\left(w_{ij}^t / w_{ij}^{t-1}\right)}\right] \\
D_{I(m)R}^{fd} &= \exp\left[\sum_i\sum_j \ln\left(\frac{y_j^t}{y_j^{t-1}}\right)\cdot\frac{\Delta w_{ij}}{\ln\left(w_{ij}^t / w_{ij}^{t-1}\right)} \Big/ \sum_i\sum_j \frac{\Delta w_{ij}}{\ln\left(w_{ij}^t / w_{ij}^{t-1}\right)}\right] \\
D_{I(m)R}^{\text{int}} &= \exp\left[\sum_i\sum_j \ln\left(\frac{r_i^t}{r_i^{t-1}}\right)\cdot\frac{\Delta w_{ij}}{\ln\left(w_{ij}^t / w_{ij}^{t-1}\right)} \Big/ \sum_i\sum_j \frac{\Delta w_{ij}}{\ln\left(w_{ij}^t / w_{ij}^{t-1}\right)}\right] \\
D_{I(m)R}^{pdn} &= \exp\left[-\ln\left(\frac{x^t}{x^{t-1}}\right)\right]
\end{aligned}
\tag{11}
$$

Parametric Method 2

The difference between method 1 and method 2 is the specification into growth or difference terms respectively. Method 2 decomposition is based on integration of both sides of equation 7.

$$
\begin{aligned}
D_{I(m)2}^{ltf} &= \exp\left[\sum_i\sum_j \Delta L_{ij}\cdot\left[\left(\frac{y_j^{t-1}\cdot r_i^{t-1}}{m^{t-1}}\right)+\alpha_{ij}^{ltf}\cdot\Delta\left(\frac{y_j\cdot r_i}{m}\right)\right]\right] \\
D_{I(m)2}^{fd} &= \exp\left[\sum_i\sum_j \Delta y_j\cdot\left[\left(\frac{L_{ij}^{t-1}\cdot r_i^{t-1}}{m^{t-1}}\right)+\alpha_{ij}^{fd}\cdot\Delta\left(\frac{L_{ij}\cdot r_i}{m}\right)\right]\right] \\
D_{I(m)2}^{\text{int}} &= \exp\left[\sum_i\sum_j \Delta r_i\cdot\left[\left(\frac{L_{ij}^{t-1}\cdot y_j^{t-1}}{m^{t-1}}\right)+\alpha_{ij}^{\text{int}}\cdot\Delta\left(\frac{L_{ij}\cdot y_j}{m}\right)\right]\right] \\
D_{I(m)2}^{pdn} &= \exp\left[-\Delta x\cdot\left[\left(\frac{1}{x^{t-1}}\right)+\alpha^{pdn}\cdot\Delta\left(\frac{1}{x}\right)\right]\right]
\end{aligned}
\tag{12}
$$

Three special cases of parametric Divisia method 2 are Laspeyres ($\alpha_{ij}^{ltf}=\alpha_{ij}^{fd}=\alpha_{ij}^{int}=\alpha^{pdn}=0$), Paasche ($\alpha_{ij}^{ltf}=\alpha_{ij}^{fd}=\alpha_{ij}^{int}=\alpha^{pdn}=1$) and Marshall-Edgeworth ($\alpha_{ij}^{ltf}=\alpha_{ij}^{fd}=\alpha_{ij}^{int}=\alpha^{pdn}=0.5$).

Parametric Methods 1 and 2 Combined

The adaptive weighting Divsia index provides a way of finding the α-terms in a non-arbitrary way by assuming that the decomposition results are the same for method 1 and 2. The assumptions of the adaptive weighting Divisia index are as follows:

$$
\begin{aligned}
D_{I(m)1}^{lft} &= D_{i(m)2}^{lft} &\qquad \alpha_{ij}^{lft}(\text{method 1}) &= \alpha_{ij}^{lft}(\text{method 2}) \\
D_{I(m)1}^{fd} &= D_{I(m)2}^{fd} &\qquad \alpha_{ij}^{fd}(\text{method 1}) &= \alpha_{ij}^{fd}(\text{method 2}) \\
D_{I(m)1}^{\text{int}} &= D_{I(m)2}^{\text{int}} &\qquad \alpha_{ij}^{\text{int}}(\text{method 1}) &= \alpha_{ij}^{\text{int}}(\text{method 2}) \\
D_{I(m)1}^{pdn} &= D_{I(m)2}^{pdn} & &
\end{aligned}
\tag{13}
$$

The equation equality lead to the following unique values for the parameters.

$$\alpha_{ij}^{lft} = \frac{\Delta L_{ij} \cdot \left(\frac{y_j^{t-1} \cdot r_i^{t-1}}{m^{t-1}} \right) - \ln\left(\frac{L_{ij}^t}{L_{ij}^{t-1}} \right) \cdot w_{ij}^{t-1}}{\Delta w_{ij} \cdot \ln\left(\frac{L_{ij}^t}{L_{ij}^{t-1}} \right) - \Delta\left(\frac{y_j \cdot r_i}{m} \right) \cdot \Delta L_{ij}}$$

$$\alpha_{ij}^{fd} = \frac{\Delta y_j \cdot \left(\frac{L_{ij}^{t-1} \cdot r_i^{t-1}}{m^{t-1}} \right) - \ln\left(\frac{y_j^t}{y_j^{t-1}} \right) \cdot w_{ij}^{t-1}}{\Delta w_{ij} \cdot \ln\left(\frac{y_j^t}{y_j^{t-1}} \right) - \Delta\left(\frac{L_{ij} \cdot r_i}{m} \right) \cdot \Delta y_j} \tag{14}$$

$$\alpha_{ij}^{\text{int}} = \frac{\Delta r_i \cdot \left(\frac{L_{ij}^{t-1} \cdot y_j^{t-1}}{m^{t-1}} \right) - \ln\left(\frac{r_i^t}{r_i^{t-1}} \right) \cdot w_{ij}^{t-1}}{\Delta w_{ij} \cdot \ln\left(\frac{r_i^t}{r_i^{t-1}} \right) - \Delta\left(\frac{L_{ij} \cdot y_j}{m} \right) \cdot \Delta r_i}$$

$$\alpha^{pdn} = \frac{\ln\left(\frac{x^{t-1}}{x^t} \right) - \left(\frac{\Delta x}{x^{t-1}} \right)}{\Delta x \cdot \Delta\left(\frac{1}{x} \right)}$$

These values can be used an inputs to the method 1 or 2 decomposition equations to obtain the associated decomposition results.

Additive decomposition of an absolute indicator

In this section an absolute indicator of material use will be decomposed by means of additive decomposition. To avoid repetition the derivation will be less detailed than those of the previous section. The base equation of material use in SDA is:

$$m = \sum_i \sum_j L_{ij} \cdot y_j \cdot r_i \tag{15}$$

Differentiating with respect to time gives:

$$\left(\frac{dm}{dt} \right) = \sum_i \sum_j \left(\frac{dL_{ij}}{dt} \right) \cdot \left(y_j \cdot r_i \right) + \sum_i \sum_j \left(\frac{dy_j}{dt} \right) \cdot \left(L_{ij} \cdot r_i \right) + \sum_i \sum_j \left(\frac{dr_i}{dt} \right) \cdot \left(L_{ij} \cdot y_j \right) \tag{16}$$

Which can also be rewritten in terms of relative growth rates:

$$\left(\frac{dm}{dt} \right) = \sum_i \sum_j \hat{L}_{ij} \cdot m_{ij} + \sum_i \sum_j \hat{y}_j \cdot m_{ij} + \sum_i \sum_j \hat{r}_i \cdot m_{ij} \tag{17}$$

Integrating both sides of equation 17 leads to a method 1 decomposition while equation 16 yields a method 2 specification. Integrating both sides of these two equation over the discrete time period *t-1* and *t* leads to the general decomposition form:

$$D_{A(a)} = \Delta m = D^{ltf}_{A(a)} + D^{fd}_{A(a)} + D^{\text{int}}_{A(a)} + D^{rsd}_{A(a)} \tag{18}$$

Parametric Method 1

$$\begin{aligned} D^{ltf}_{A(a)1} &= \sum_i \sum_j \ln\left(\frac{L^t_{ij}}{L^{t-1}_{ij}}\right) \cdot \left[m^{t-1}_{ij} + \alpha^{ltf}_{ij} \cdot \Delta m_{ij}\right] \\ D^{fd}_{A(a)1} &= \sum_i \sum_j \ln\left(\frac{y^t_j}{y^{t-1}_j}\right) \cdot \left[m^{t-1}_{ij} + \alpha^{fd}_{ij} \cdot \Delta m_{ij}\right] \\ D^{\text{int}}_{A(a)1} &= \sum_i \sum_j \ln\left(\frac{r^t_i}{r^{t-1}_i}\right) \cdot \left[m^{t-1}_{ij} + \alpha^{\text{int}}_{ij} \cdot \Delta m_{ij}\right] \end{aligned} \tag{19}$$

Note that compared to the multiplicative decomposition of material intensity, there is no production effect in this case. The conventional Divisia index is given by setting the α-values to 0.5.

Non-Parametric Method 1

The following are the equations for the refined Divisia index decomposition.

$$\begin{aligned} D^{ltf}_{A(a)R} &= \sum_i \sum_j \ln\left(\frac{L^t_{ij}}{L^{t-1}_{ij}}\right) \cdot \frac{\Delta m_{ij}}{\ln\left(m^t_{ij} / m^{t-1}_{ij}\right)} \\ D^{fd}_{A(a)R} &= \sum_i \sum_j \ln\left(\frac{y^t_j}{y^{t-1}_j}\right) \cdot \frac{\Delta m_{ij}}{\ln\left(m^t_{ij} / m^{t-1}_{ij}\right)} \\ D^{\text{int}}_{A(a)R} &= \sum_i \sum_j \ln\left(\frac{r^t_i}{r^{t-1}_i}\right) \cdot \frac{\Delta m_{ij}}{\ln\left(m^t_{ij} / m^{t-1}_{ij}\right)} \end{aligned} \tag{20}$$

Parametric Method 2

$$\begin{aligned} D^{ltf}_{A(a)2} &= \sum_i \sum_j \Delta L_{ij} \cdot \left[\left(y^{t-1}_j \cdot r^{t-1}_i\right) + \alpha^{ltf}_{ij} \cdot \Delta\left(y_j \cdot r_i\right)\right] \\ D^{fd}_{A(a)2} &= \sum_i \sum_j \Delta y_j \cdot \left[\left(L^{t-1}_{ij} \cdot r^{t-1}_i\right) + \alpha^{fd}_{ij} \cdot \Delta\left(L_{ij} \cdot r_i\right)\right] \\ D^{\text{int}}_{A(a)2} &= \sum_i \sum_j \Delta r_i \cdot \left[\left(L^{t-1}_{ij} \cdot y^{t-1}_j\right) + \alpha^{\text{int}}_{ij} \cdot \Delta\left(L_{ij} \cdot y_j\right)\right] \end{aligned} \tag{21}$$

The Laspeyres, Marshall-Edgeworth and Paasche and are produced if the α-parameters are set to 0, 0.5 and 1 respectively.

Parametric Methods 1 and 2 Combined

The adaptive weighting Divisia index result from the same assumptions (equation 13) as the intensity case except for those that are specifically focussed on the production effect.

$$\alpha_{ij}^{ltf} = \frac{\Delta L_{ij} \cdot \left(y_j^{t-1} \cdot r_i^{t-1}\right) - \ln\left(\frac{L_{ij}^t}{L_{ij}^{t-1}}\right) \cdot m_{ij}^{t-1}}{\Delta m_{ij} \cdot \ln\left(\frac{L_{ij}^t}{L_{ij}^{t-1}}\right) - \Delta\left(y_j \cdot r_i\right) \cdot \Delta L_{ij}}$$
$$\alpha_{ij}^{fd} = \frac{\Delta y_j \cdot \left(L_{ij}^{t-1} \cdot r_i^{t-1}\right) - \ln\left(\frac{y_j^t}{y_j^{t-1}}\right) \cdot m_{ij}^{t-1}}{\Delta m_{ij} \cdot \ln\left(\frac{y_j^t}{y_j^{t-1}}\right) - \Delta\left(L_{ij} \cdot r_i\right) \cdot \Delta y_j} \qquad (22)$$
$$\alpha_{ij}^{\text{int}} = \frac{\Delta r_i \cdot \left(L_{ij}^{t-1} \cdot y_j^{t-1}\right) - \ln\left(\frac{r_i^t}{r_i^{t-1}}\right) \cdot m_{ij}^{t-1}}{\Delta m_{ij} \cdot \ln\left(\frac{r_i^t}{r_i^{t-1}}\right) - \Delta\left(L_{ij} \cdot y_j\right) \cdot \Delta r_i}$$

Additive decomposition of an elasticity indicator

The output elasticity[13] of materials use can be found by replacing the results of the additive decomposition of material use into the equation for the output elasticity:

$$D_{E(a)} = \frac{\Delta m}{m} \Big/ \frac{\Delta x}{x} = \left(\frac{D_{A(a)}^{ltf}}{m} \Big/ \frac{\Delta x}{x}\right) + \left(\frac{D_{A(a)}^{fd}}{m} \Big/ \frac{\Delta x}{x}\right) + \left(\frac{D_{A(a)}^{\text{int}}}{m} \Big/ \frac{\Delta x}{x}\right) + \left(\frac{D_{A(a)}^{rsd}}{m} \Big/ \frac{\Delta x}{x}\right) \qquad (23)$$

Laspeyres, Marshall-Edgeworth and Paasche

In Ang and Lee (1996) the Laspeyres and Marshall-Edgeworth version of this decomposition are given.

$$D_{E(a)}^{ltf} = \left(\frac{D_{A(a)2}^{ltf}(\alpha^{ltf})}{m^{t-1} + \alpha^{ltf} \cdot \Delta m}\right) \Big/ \left(\frac{\Delta x}{x^{t-1} + \alpha^{ltf} \cdot \Delta x}\right) \qquad (24)$$

[13] Ang and Lee (1996) also refer to it as the “energy coefficient” in their study of energy issues.

$$D_{E(a)}^{fd} = \left(\frac{D_{A(a)2}^{fd}(\alpha^{fd})}{m^{t-1} + \alpha^{fd} \cdot \Delta m} \right) \Bigg/ \left(\frac{\Delta x}{x^{t-1} + \alpha^{fd} \cdot \Delta x} \right)$$

$$D_{E(a)}^{\text{int}} = \left(\frac{D_{A(a)2}^{\text{int}}(\alpha^{\text{int}})}{m^{t-1} + \alpha^{\text{int}} \cdot \Delta m} \right) \Bigg/ \left(\frac{\Delta x}{x^{t-1} + \alpha^{\text{int}} \cdot \Delta x} \right)$$

To obtain Laspeyres and Marshall-Edgeworth indices the α-parameters are set to 0 and 0.5 respectively. Paasche, although not implemented in Ang and Lee (1996) could be obtained by setting the α-parameters to 1. The equation shows that the decomposition results that come from the additive decomposition of material use are also dependent on the parameter values. I.e. If you are calculating the Laspeyres weighted determinant effect on elasticity then the additive Laspeyres decomposition results should be used as input.

5. NUMERICAL EXAMPLE

In this section a hypothetical numerical example used in Ang (1999) is expanded to study the differences in SDA and INA (Table 2). The bold information is from Ang (1999) while the input-output information (normal font) has been added for the SDA decomposition. All information is in monetary units, except for the material use (in brackets) which is in physical units. The results of all the decomposition approaches are given in table 4.

Table 3. Example in year t-1 and year t

Year t-1	**Sector 1**	**Sector 2**	**y**	**x (m)**
Sector 1	3	2	5	**10 (30)**
Sector 2	5	20	15	**40 (20)**
W	2	18		
Total	**10**	**40**		

Year t	**Sector 1**	**Sector 2**	**y**	**x (m)**
Sector 1	8	2	10	**20 (40)**
Sector 2	10	30	20	**60 (24)**
W	2	28		
Total	**20**	**60**		

Table 4. Results for the SDA and INA decomposition approach

MULTIPLICATIVE DECOMPOSITION OF AN INTENSITY INDICATOR								
Method/Index	**SDA/INA**	**Determinant effects**						
		Total	**Production**	**Structure**	**Leontief**	**Final Demand**	**Intensity**	**Residual**
Method 1								
Conventional Divisia	INA	0.8		1.118			0.716	1.000
	SDA	0.8	0.625		1.050	1.701	0.716	1.002
Refined Divisia	INA	0.8		1.118			0.716	1.000
	SDA	0.8	0.625		1.051	1.702	0.716	1.000
Method 2								
Laspeyres	INA	0.8		1.133			0.756	0.934
	SDA	0.8	0.549		1.042	1.998	0.756	0.926
Marshall-Edgeworth	INA	0.8		1.119			0.710	1.008
	SDA	0.8	0.614		1.051	1.740	0.710	1.004
Paasche	INA	0.8		1.105			0.666	1.087
	SDA	0.8	0.687		1.059	1.515	0.666	1.089
Method 1 & 2 combined								
Adaptive Weighting Divisia Index	INA	0.8		1.118			0.716	0.999
	SDA	0.8	0.625		1.052	1.700	0.721	0.993

ADDITIVE DECOMPOSITION OF AN ABSOLUTE INDICATOR								
Method/Index	**SDA/INA**	**Determinant effects**						
		Total	**Production**	**Structure**	**Leontief**	**Final Demand**	**Intensity**	**Residual**
Method 1								
Conventional Divisia	INA	14	26.8	6.4			-19.1	-0.1
	SDA	14			2.9	30.4	-19.1	-0.3
Refined Divisia	INA	14	26.7	6.3			-19.0	0.0
	SDA	14			2.9	30.0	-18.9	0.0
Method 2								
Laspeyres	INA	14	30.0	6.3			-14.0	-8.3
	SDA	14			2.1	34.6	-14.0	-8.7
Marshall-Edgeworth	INA	14	27.0	6.3			-20.0	0.7
	SDA	14			2.9	30.6	-20.0	0.5
Paasche	INA	14	24.0	6.4			-26.0	9.6
	SDA	14			3.7	26.6	-26.0	9.7
Method 1 & 2 combined								
Adaptive Weighting Divisia Index	INA	14	26.9	6.3			-18.5	-0.8
	SDA	14			2.8	30.4	-18.5	-0.7

ADDITIVE DECOMPOSITION OF AN ABSOLUTE INDICATOR								
Method/Index	**SDA/INA**	**Determinant effects**						
		Total	**Production**	**Structure**	**Leontief**	**Final Demand**	**Intensity**	**Residual**
Laspeyres	INA	0.47	1.00	0.21			-0.47	-0.28
	SDA	0.47			0.07	1.15	-0.47	-0.29
Marshall-Edgeworth	INA	0.53	1.03	0.24			-0.76	0.03
	SDA	0.53			0.11	1.16	-0.76	0.02
Paasche	INA	0.58	1.00	0.27			-1.08	0.40
	SDA	0.58			0.15	1.11	-1.08	0.41

In the multiplicative decomposition of material intensity INA leads to 2 separate determinant effects while SDA distinguishes 4 effects. In the additive decomposition of material use and elasticity both methods distinguish 3 effects. These are the top tier effects but each of these, except the production effect, are composed of sub-effects. The Leontief (n^2 sub-effects)[14] and the structure, final demand and intensity effects (all n effects) can therefore be further decomposed. The multiplicative decomposition of INA therefore has $2n$ sub-effects as opposed to n^2+2n+1 in SDA. In the additive decompositions of material use and elasticity the difference is $2n+1$ versus n^2+2n. It is not surprising that SDA distinguishes more sub-effects since it also uses more data.

This paper has thusfar not discussed the interpretations of the determinant effects. The *production effect* measures the effect of the changes in the overall output level of the economy on the indicator in question. As table 4 shows if total output grows it has a diminishing effect on the intensity indicator because it is taken up in the denominator of this SDA effect. In the case of an additive INA decomposition of absolute indicators, rising output obviously has a positive effect on the total material use. The *structure effect* indicates the effect of a shift in the relative shares of output on the indicator. The *Leontief effect* indicates the effect of the changes in the Leontief coefficients. Since this matrix is actually derived from the technical coefficients matrix it is actually a measure of the change in technology of the economy. Another technological effect is the *intensity effect* which assesses the effect of change in the material intensity values in each sector. Lastly, the *final demand effect* estimates the change in the indicator that can be ascribed to the shift in the final demand for products from each sector.

Table 4 shows that each of the decomposition indices produces different results. An important component of each approach is the residual effects. The table shows that Laspeyres and Paasche generally have substantial residuals that are large components of the total decomposition. The Marshall-Edgeworth, conventional Divisia, and adaptive weighting Divisia indices have lower residual effects. The refined Divisia index, however, has no residual.

The range of results for each determinant effect can be quite large. Invariably the Laspeyres and Paasche indices provide the two extremes of the range with the other indices somewhere close to the center of this range.

6. APPROACH SELECTION

INA or SDA?

This paper has shown that research into historical data can provide valuable information about the importance of specific determinants. The question of the preferred method has however not been addressed.

SDA has the main advantage of including indirect effects of demand changes and therefore giving a feel for the interactions that exist in an economy. The indirect effects are often substantial and in environmental analyses these effects may be very important. Although a sector may be very energy-extensive it could require a lot of inputs that required large amounts of energy. In the input-output model this indirect energy is included in the analysis. The drawback of the Leontief model is its assumption of constant technical coefficients. No scale effects or substitution is

[14] Remember that n is the number of sectors

therefore present in the standard input-output model. Nevertheless the input-output model is widely used and the inclusion of the indirect effect is a major advantage of SDA over INA.

Overall SDA decompositions provide more detailed information about the effects of the determinants. Clearly this is related to the fact that more data is used in the input-output setup. The decomposition of material intensity yields 4 effects while the INA distinguishes 2 effects. If these effects are further decomposed into their component parts then INA ($2n$) has far fewer sub-effects than SDA (n^2+2n+1). In the absolute decomposition the difference in the number of sub-effects is $2n+1$ versus n^2+2n.[15] The technical coefficient matrix (represented in the Leontief inverse) with its n^2 elements in fact holds information about the technological input requirements of all sectors. SDA therefore has the opportunity for further decomposition of this technology effect.

It may be concluded that if input-output data is available then SDA should be preferred over INA. It provides more details about the way economic changes affect materials use, intensity or elasticity. In cases where the data availability is low and short time-step analysis is of interest, INA is more likely to be feasible. However it should be noted that short time steps do have a disadvantage (whether for SDA or INA) in that the changes that are found may be short-term effects that do not necessarily imply an irreversible shift in the economic structure. Long-term decompositions therefore give a more accurate depiction of the determinant changes.

Intensity, Absolute or Elasticity?

Clearly the choice of decomposition indicator depends on the issue under investigation. Changes in the output elasticity will be of interest to people who are researching how material use reacts to output increases. Ang and Lee (1996) also introduced a method of using the elasticity indicator to project energy use towards the future.

The absolute indicator is best when it is important to investigate the actual quantity of material use. The intensity indicator looks at the use of material relative to the output and is therefore better suited to questions of material productivity. Ang (1999) argues that the choice between absolute and intensity measures is a matter of ease of presentation and interpretation. Intensity indicators are more easily graphed because they are often indices that are close to one. Absolute decompositions are however easier to interpret by non-specialists.

An added selection criteria (in the INA case) is that if the time period is long or the output growth has been large, the production effect may dominate in the absolute indicator decomposition. If the researcher is interested in investigations of structural change, the intensity approach is therefore preferred.

Multiplicative or additive?

There seems to be no reason to prefer either of these decomposition types except for perhaps the ease of presentation and interpretation argument.

[15] INA and SDA both have the same material intensity effect. It is only in the economic portion of the decomposition that they differ.

Method 1 or Method 2?

Again there is very little reason to prefer expressing the determinant in terms of relative growth rates (method 1) or differences terms (method 2). Perhaps a minor argument is that the method 1 formulas are easier to present because they all have equal weights.

Which Index?

The larger issue that has not been addressed in this paper is the issue of index selection. All indices have certain properties and should therefore be chosen according to the demands set. Table 5 summarizes the properties of the indices discussed in this paper (this summary of properties is largely based on Ang (1999).

Table 5. Properties of the indices in this paper

	Parametric	**Residual**	**Zero value problem**
Laspeyres	yes	large	no
Marshall-Edgeworth	yes	moderate	no
Paasche	yes	large	no
Conventional Divisia	yes	moderate	yes
Refined Divisia	no	none	no (see explanation)
Adaptive Weighting Divisia	no	moderate	yes

The first issue is whether the index is a special case of a parametric form. As section 4 showed, the Laspeyres, Marshall-Edgeworth and Paasche indices are special case of parametric Divisia method 2. The conventional Divisia index is a special case of parametric Divisia method 1. This means that the author has to choose the parameter values. The non-parametric indices do not require an extra parameter assumption and the result is therefore purely data driven. This does not, however, make the non-parametric theoretically superior since the assumptions that have to be made about the index so that it leads to a unique value are just as arbitrary as choosing a parameter value.

The value of the residual is very important because when you are interested in the importance of determinant effects then large residuals defeat the purpose. Laspeyres and Paasche score worst on this criterion. Marshall-Edgeworth, conventional Divisia and the adaptive weighting Divisia index score better but Ang (1999) notes that if changes in the data are drastic the residuals may deteriorate. The refined Divisia index leads to perfect decompositions of the indicator change.

The third criterion in table 5 is the issue of zero values in the data set. This is a particularly important issue for SDA because detailed input-output data nearly always have zero values in the table because many sectors do not have intersectoral deliveries. Certain material types may also not be used in the base or terminal year leading to zero value in the decomposition. The decomposition methods based on method 2 approaches are based on difference terms and therefore have no problem. The conventional refined Divisia do however have a problem if the terminal year is zero (natural logarithm of zero is minus infinity) or the base year (division by zero gives plus infinity). It is common practice to replace the zero value by a small value δ to solve this problem. Ang and Choi (1997) show that the refined Divisia index shows converging decomposition results as δ approaches zero but that this is not the case for the conventional Divisia index.

Ang (1999) adds some observations about the interpretability of some of the indices. The Laspeyres index is deemed to be easy to understand because weighting by the base year is often done. Weighting using parameter values of 0.5 has the advantage of treating time symmetrically. The Paasche index both decomposition and forecasting can be done with reference to the same year.

Overall it can be said that the Laspeyres and Marshall-Edgeworth indices that are commonplace in SDA do not measure up well against some of the indices that come from the INA literature. Particularly the refined Divisia index has some very appealing properties. In particular it is the only index that does not generate a residual.

7. CONCLUSIONS

A number of conclusions may be drawn from this paper.

1. SDA and INA are closely related decomposition methods. Fundamentally the only difference is the use of input-output data in SDA.
2. The sophisticated decomposition methods developed in the INA literature are transferable to the SDA framework (section 4).
3. Many of the indices in use in INA have superior properties over the Laspeyres and Marshall-Edgeworth indices that are common in SDA.
4. INA and SDA researchers should be more aware of the literature in both fields to fully benefit from the methodological advances in decomposition techniques.

Further research would include comparing the index system developed in Dietzenbacher and Los (1998) and applying it to INA. The results for the additive decomposition of intensity and the multiplicative decomposition of absolute indicators should also be discussed.

APPENDIX 1. VARIABLES

The superscript t always defines the time at which the variable is taken.

Country-level

m^t – Total material use (tons)

x^t – Total output (☒)

r^t – Material intensity of the economy (tons/☒) (=*m/x*)

n – Number of sector in the economy

Sector-level

Z_{ij}^t – The intersectoral deliveries of goods or services of sector *i* to sector *j* (☒)

A_{ij}^t – Technical coefficient matrix. The amount of input from sector *i* required per unit output of sector *j*.

L_{ij}^t – Leontief inverse.The direct and indirect effect on the output of sector *i* per unit change of final demand of sector *j* (=$(I - A_{ij}^t)^{-1}$).

m_{ij}^t – Material use by sector *i* due to demand for products from sector *j* (tonnes).

m_i^t – Material use by sector *i* (tonnes).

w_{ij}^t – Material use weights. Material use in sector *i* due to demand for products from sector *j* as a proportion of the total material use (tonnes) (=m_{ij}/m).

w_i^t – Material use weights. Material use in sector *i* as a proportion of the total material use (tonnes) (=m_i/m).

x_i^t – Output of sector *i* (☒)

y_j^t – Final demand of sector *j* (☒)

r_i^t – Material intensity of sector *i* (tons/☒) (=m_i/x_i)

s_i^t – Output share. Sector *i* output as a proportion of total output (=x_i/x)

Subscripts and Superscripts of parameters α_{ij}

Superscripts:

- *str* – Structural effect
- *int* – Intensity effect
- *pdn* – production effect
- *ltf* – Leontief effect
- *fd* – final demand effect

Subscripts and Superscripts of decomposition variable D

(If it does not have a superscript it is equal to the total change of the indicator)

Superscripts:

- *str* – Structural effect
- *int* – Intensity effect
- *pdn* – production effect
- *ltf* – Leontief effect
- *fd* – final demand effect
- *rsd* – residual effect

1st Subscript (Indicator type)

I – Intensity

A – Absolute

E – Elasticity

2nd Subscript (Decomposition type)

(a) – Additive

(m) – Multiplicative

3rd Subscript (Decomposition index)

1 – Parametric Divisia Index Method 1

R – Refined Divisia Index (Non-Parametric Method 1)

2 – Parametric Divisia Index Method 2

A – Adaptive Weighting Divisia Index

APPENDIX 2. INA DECOMPOSITION FORMULAS

This appendix presents the parametric specifications of the different decomposition forms in INA (see also Ang (1999)).

Multiplicative decomposition of an intensity indicator

Parametric Method 1

$$D_{I(m)1}^{str} = \exp\left[\sum_i \ln\left(\frac{s_i^t}{s_i^{t-1}}\right)\cdot\left[w_i^{t-1} + \alpha_i^{str}\cdot\Delta w_i\right]\right]$$

$$D_{I(m)1}^{int} = \exp\left[\sum_i \ln\left(\frac{r_i^t}{r_i^{t-1}}\right)\cdot\left[w_i^{t-1} + \alpha_i^{int}\cdot\Delta w_i\right]\right]$$

Non-Parametric Method 1

$$D_{I(m)R}^{str} = \exp\left[\sum_i \ln\left(\frac{s_i^t}{s_i^{t-1}}\right)\cdot\left(\frac{\Delta w_i}{\ln\left(w_i^t / w_i^{t-1}\right)}\middle/\sum_i \frac{\Delta w_i}{\ln\left(w_i^t / w_i^{t-1}\right)}\right)\right]$$

$$D_{I(m)R}^{int} = \exp\left[\sum_i \ln\left(\frac{r_i^t}{r_i^{t-1}}\right)\cdot\left(\frac{\Delta w_i}{\ln\left(w_i^t / w_i^{t-1}\right)}\middle/\sum_i \frac{\Delta w_i}{\ln\left(w_i^t / w_i^{t-1}\right)}\right)\right]$$

Parametric Method 2

$$D_{I(m)2}^{str} = \exp\left[\sum_i \Delta s_i \cdot\left[\frac{r_i^{t-1}}{r^{t-1}} + \alpha_i^{str}\cdot\Delta\left(\frac{r_i}{r}\right)\right]\right]$$

$$D_{I(m)2}^{int} = \exp\left[\sum_i \Delta r_i \cdot\left[\frac{x_i^{t-1}}{r^{t-1}} + \alpha_i^{int}\cdot\Delta\left(\frac{x_i}{r}\right)\right]\right]$$

Parametric Methods 1 and 2 Combined

$$\alpha_i^{str} = \frac{\Delta s_i \cdot \left(\frac{r_i^{t-1}}{r^{t-1}}\right) - \ln\left(\frac{s_i^t}{s_i^{t-1}}\right) \cdot w_i^{t-1}}{\Delta w_i \cdot \ln\left(\frac{s_i^t}{s_i^{t-1}}\right) - \Delta\left(\frac{r_i}{r}\right) \cdot \Delta s_i}$$

$$\alpha_i^{\text{int}} = \frac{\Delta r_i \cdot \left(\frac{x_i^{t-1}}{r^{t-1}}\right) - \ln\left(\frac{r_i^t}{r_i^{t-1}}\right) \cdot w_i^{t-1}}{\Delta w_i \cdot \ln\left(\frac{r_i^t}{r_i^{t-1}}\right) - \Delta\left(\frac{x_i}{r}\right) \cdot \Delta r_i}$$

Additive decomposition of an absolute indicator

Parametric Method 1

$$D_{A(a)1}^{str} = \sum_i \ln\left(\frac{s_i^t}{s_i^{t-1}}\right) \cdot \left[m_i^{t-1} + \alpha_i^{str} \cdot \Delta m_i\right]$$

$$D_{A(a)1}^{\text{int}} = \sum_i \ln\left(\frac{r_i^t}{r_i^{t-1}}\right) \cdot \left[m_i^{t-1} + \alpha_i^{\text{int}} \cdot \Delta m_i\right]$$

$$D_{A(a)1}^{pdn} = \sum_i \ln\left(\frac{x^t}{x^{t-1}}\right) \cdot \left[m_i^{t-1} + \alpha_i^{pdn} \cdot \Delta m_i\right]$$

Non-Parametric Method 1

$$D_{A(a)R}^{str} = \sum_i \ln\left(\frac{s_i^t}{s_i^{t-1}}\right) \cdot \left[\frac{\Delta m_i}{\ln\left(m_i^t / m_i^{t-1}\right)}\right]$$

$$D_{A(a)R}^{\text{int}} = \sum_i \ln\left(\frac{r_i^t}{r_i^{t-1}}\right) \cdot \left[\frac{\Delta m_i}{\ln\left(m_i^t / m_i^{t-1}\right)}\right]$$

$$D_{A(a)R}^{pdn} = \sum_i \ln\left(\frac{x^t}{x^{t-1}}\right) \cdot \left[\frac{\Delta m_i}{\ln\left(m_i^t / m_i^{t-1}\right)}\right]$$

Parametric Method 2

$$D_{A(a)2}^{str} = \sum_i \Delta s_i \cdot \left[r_i^{t-1} \cdot x^{t-1} + \alpha_i^{str} \cdot \Delta\left(r_i \cdot x\right)\right]$$

$$D_{A(a)2}^{\text{int}} = \sum_i \Delta r_i \cdot \left[s_i^{t-1} \cdot x^{t-1} + \alpha_i^{\text{int}} \cdot \Delta\left(s_i \cdot x\right)\right]$$

$$D_{A(a)2}^{pdn} = \sum_i \Delta x \cdot \left[s_i^{t-1} \cdot r_i^{t-1} + \alpha_i^{pdn} \cdot \Delta\left(s_i \cdot r_i\right)\right]$$

Parametric Methods 1 and 2 Combined

$$\alpha_i^{str} = \frac{\Delta s_i \cdot \left(r_i^{t-1} \cdot x^{t-1}\right) - \ln\left(\frac{s_i^t}{s_i^{t-1}}\right) \cdot m_i^{t-1}}{\Delta m_i \cdot \ln\left(\frac{s_i^t}{s_i^{t-1}}\right) - \Delta\left(r_i \cdot x\right) \cdot \Delta s_i}$$

$$\alpha_i^{\text{int}} = \frac{\Delta r_i \cdot \left(s_i^{t-1} \cdot x^{t-1}\right) - \ln\left(\frac{r_i^t}{r_i^{t-1}}\right) \cdot m_i^{t-1}}{\Delta m_i \cdot \ln\left(\frac{r_i^t}{r_i^{t-1}}\right) - \Delta\left(s_i \cdot x\right) \cdot \Delta r_i}$$

$$\alpha_i^{pdn} = \frac{\Delta x \cdot \left(r_i^{t-1} \cdot s_i^{t-1}\right) - \ln\left(\frac{x^t}{x^{t-1}}\right) \cdot m_i^{t-1}}{\Delta m_i \cdot \ln\left(\frac{x^t}{x^{t-1}}\right) - \Delta\left(r_i \cdot s_i\right) \cdot \Delta x}$$

Additive decomposition of an elasticity indicator

$$D_{E(a)}^{str} = \left(\frac{D_{A(a)2}^{str}(\alpha^{str})}{m^{t-1} + \alpha^{str} \cdot \Delta m}\right) \Bigg/ \left(\frac{\Delta x}{x^{t-1} + \alpha^{str} \cdot \Delta x}\right)$$

$$D_{E(a)}^{\text{int}} = \left(\frac{D_{A(a)2}^{\text{int}}(\alpha^{\text{int}})}{m^{t-1} + \alpha^{\text{int}} \cdot \Delta m}\right) \Bigg/ \left(\frac{\Delta x}{x^{t-1} + \alpha^{\text{int}} \cdot \Delta x}\right)$$

$$D_{E(a)}^{pdn} = \left(\frac{D_{A(a)2}^{pdn}(\alpha^{pdn})}{m^{t-1} + \alpha^{pdn} \cdot \Delta m}\right) \Bigg/ \left(\frac{\Delta x}{x^{t-1} + \alpha^{pdn} \cdot \Delta x}\right)$$

If the α-values are set to 0, 0.5 and 1, the Laspeyres, Marshall-Edgeworth and Paasche indices are produced respectively.

REFERENCES

-Ang, B.W., 1999. Decomposition Methodology in Energy Demand and Environmental Analysis. In: *Handbook of Environmental and Resource Economics.* (ed.) J.C.J.M van den Bergh. Edward Elgar, Cheltenham.

-Ang, B.W. and K.H. Choi, 1997. Decomposition of Aggregate Energy and Gas Emissions Intenisties for Industry: A Refined Divisia ndex Method. *The Energy Journal* **18**(3) pp. 59-73

-Ang, B.W. and P.W. Lee, 1996. Decomposition of Industrial Energy Consumption: The Energy Coefficient Approach. *Energy Economics* **18** pp. 129-143

-Dietzenbacher, E., A. Hoen and B. Los, 2000. Labor Productivity in Western Europe 1975-1985: An Intercountry, Interindustry analysis. *Journal of Regional Science* **(forthcoming)** pp.

-Dietzenbacher, E. and B. Los, 1998. Structural Decomposition Techniques: Sense and Sensitivity. *Economic Systems Research* **10**(4) pp. 307-323

-Fisher, I., 1922. *The Making of Index Numbers: A Study of their Varieties. Tests and Reliability*. Augutus M. Kelly, New York.

-Han, X. and T.K. Lakshmanan, 1994. Structural Changes and Energy Consumption in the Japanese Economy 1975-85: An Input-Output Analysis. *The Energy Journal* **15**(3) pp. 165-188

-Liu, X.Q., B.W. Ang and H.L. Ong, 1992. The Application of the Divisia Index to the Decomposition of Changes in Industrial Energy Consumption. *The Energy Journal* **13**(4) pp. 161-177

-Miller, R.E. and P.D. Blair, 1985. *Input-Output Analysis: Foundations and Extensions*. Prentice-Hall, Englewood-Cliffs, New Jersey.

-Rose, A. and S. Casler, 1996. Input-Output Structural Decomposition Analysis: A Critical Appraisal. *Economic Systems Research* **8**(1) pp. 33-62

-Vogt, A., 1978. Divisia Indices on Different Paths. In: *Theory and applications of economic indices.* (eds.) W. Eichhorn, R. Henn, O.Opitz and R.W. Shepard. Physica-Verlag, Wurzburg.

-Vogt, A. and J. Barta, 1997. *The Making of Tests of Index Numbers: Mathematical Methods of Descriptive Statistics*. Springer-Verlag, Heidelberg.