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Bounds and Sensitivity of Growth Rates in an Input Output-Model with Consumption

Abstract

For the dynamic input output-model with consumption functions realistic sufficient conditions for growth equilibria are given as well as sharp bounds for their number and the growth rates. When investment sectors are aggregated formulae for the growth rate and its change with respect to perturbations in the underlying matrices are derived without computing eigenvectors. For the general model the differential of the growth rate is given and compared with a result for the model without consumption functions. It is shown that its sensitivity with respect to changes (or errors) in the matrices is extremely affected by consumption. Two methods for approximating growth equilibria are suggested and interpreted economically.

Key words: Growth equilibrium, input output-model, consumption, sensitivity analysis.

Field of designation: growth equilibrium

Bounds and Sensitivity of Growth Rates in an Input Output-Model with Consumption

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1 Introduction

In this paper, the dynamic input output-model (see Leontief (1953)) extended by consumption functions will be studied as a balanced growth model. In case of proportional equilibrium growth at a constant rate γ, Leontief's model reads

$$(I - A - \gamma B)x = 0, \quad \gamma > 0, x \geq 0 \tag{1}$$

where $A \geq 0$ and $B \geq 0$ denote $n \times n$ matrices of current input and capital coefficients, respectively, and x is a semi-positive $n \times 1$ vector of outputs.
In the literature, the usual approach transforms the dynamic input output-model into

$$((I - A)B^{-1} - \gamma I)x = 0, \quad \gamma > 0x \geq 0 \tag{2}$$

which yields the growth rates as eigenvalues of $(I - A)B^{-1}$.
This approach, however, has two deficiencies. Firstly, B usually has zero rows because several sectors do not produce capital goods. Therefore, B is not invertible unless the model is reduced by suitable aggregation into one with a non-singular matrix B'. Secondly, B'^{-1} is supposed to have entries with different sign so that the theorems of Frobenius and Perron cannot be applied.
The problems with B'^{-1} are circumvented by considering

$$(I - \gamma L(A)B)x = 0, \quad \gamma > 0, x \geq 0 \tag{3}$$

which is equivalent to the original model under the usual assumption $\lambda_1(A) < 1$ which implies that Leontief inverse $L(A) := (I - A)^{-1}$ exists and is nonnegative. $\lambda_1(A)$ denotes the Frobenius root (dominant root) of A.
When linear sectoral consumption functions depending on national income are introduced, the previous model of balanced growth is modified as follows

$$(I - A - cv - \gamma B)x = 0, \quad \gamma > 0, x \geq 0 \tag{4}$$

where c and v, respectively, are semipositive $n \times 1$ and $1 \times n$ vectors of marginal propensities to consume and value added coefficients, respectively (see, for instance, Schumann (1975), Holub/Schnabl (1994)). It is assumed that $\sum c_i < 1$; the other variables are value terms. Therefore, in the extended model A has to be replaced by $(A + cv)$. The Leontief inverse $L(A + cv)$ equals $L(A) \cdot L(C)$ where $C = ce$ because of $v = e(I - A)$; e denotes a $1 \times n$ row vector of ones.
The growth rates of model (4) to be examined in the following are reciprocal eigenvalues of $W := L(A)L(C)B$ with associated semi-positive eigenvectors x which are equilibrium outputs. Therefore, growth rates will be studied in the following by considering the eigenvalues of the "growth matrix" W or preferably of $\tilde{W} = BL(A)L(C)$ which has the same characteristic roots

but different eigenvectors Bx. The latter matrix facilitates the investigation since it has as many zero rows as B.
From the economic point of view the element $d_{ij} = \sum_\nu b_{i\nu} l_{\nu j}$ of $D := BL(A)$ gives the total increase in capital output of sector i induced via augmented current input needs in the whole economy per additional unit of final demand for commodities of sector j. The matrix D may be called accelerator matrix since it links net investment (necessary for extending capacities) to final demand via production increases due to the Leontief multiplier. D is the multisectoral analogon of Harrod's accelerator b which gives investment per unit of additional income (or final demand)(cf. Harrod (1939)).

2 Existence of Equilibria and Bounds for Growth Rates

A growth equilibrium (γ,x) with $\gamma > 0$, $x \geq 0$ exists if and only if the growth matrix W or $\tilde{W} = BL(A)L(C)$ has a positive Frobenius root λ_1, since W has a corresponding eigenvector $x^1 \geq 0$. It is well-known that for a $n \times n$ matrix $M \geq 0$

$$\min M_{.j} \leq \lambda_1(M) \leq \max M_{.j} \quad \text{and} \quad \lambda_1(M) = \sum M_{.j} x_j^1 \tag{5}$$

where $M_{.j}$ denotes the j-th column sum of M and x^1 is normed by $\sum x_i^1 = 1$. A corresponding statement holds for the row sums. The assertion can be extended to other possible eigenvalues $\lambda_i > 0$ of W with associated eigenvectors $x^i \geq 0$ (these eigenvalues will be denoted by λ_i^+), but not for the row sums (cf. Kogelschatz (1977, p. 80)).
So, column sums of W and $\tilde{W}$ are evaluated where

$$L(C) = (I - ce)^{-1} = (I + \frac{1}{s} ce) = \frac{1}{s}(sI + ce) \tag{6}$$

with $s := 1 - \sum c_k$ is helpful; the latter matrix is the transpose of a stochastic matrix. The column sums of $\tilde{W}$ have a simple form which leads further

$$\tilde{W}_{.j} = D_{.j} + \delta \quad \text{for every j with} \quad \delta = \frac{1}{s} \sum D_{.k} c_k = \frac{1}{s} eDc. \tag{7}$$

A similar statement for W does not hold. The additive constant δ captures total investment induced via Keynesian and Leontief multiplier process per unit of final demand. Hence, $\tilde{W}_{.j} = \frac{1}{s}(sD_{.j} + \sum_k c_k D_{.k})$ is a weighted mean of the $D_{.k}$ multiplied by $\frac{1}{s}$ and δ is $\frac{1-s}{s}$ times a weighted mean of the $D_{.k}$ with weights $\frac{c_k}{1-s}$.
Therefore,

$$\frac{1-s}{s} \min D_{.k} \leq \delta \leq \frac{1-s}{s} \max D_{.k} \tag{8}$$

and

$$\frac{1}{s}\min D_{.k} \leq \tilde{W}_{.j} \leq \frac{1}{s}\max D_{.k} \quad \text{for every j} \tag{9}$$

and

$$\lambda_1(\tilde{W}) \geq \lambda_i^+(\tilde{W}) \geq \delta. \tag{10}$$

If B has k non-zero rows $j \in K = \{1, ..., k\}$ then $\tilde{W}$ has at most k positive eigenvalues which can be computed by a $k \times k$ matrix $\tilde{W}_k$. The eigenvectors $Bx^{(i)} \geq 0$ of $\tilde{W}$ associated with λ_i^+ have at most k positive components which may be chosen adding up to one. Considering (5) for this case yields λ_i^+ as a weighted mean of $\tilde{W}_{.j} = D_{.j} + \delta$ for $j \in K$ only. If, in addition, products of these k investment sectors were not consumed then in (8) min and max could be taken over $j \notin K$. The latter assumption is not made for the following. Hence,

$$\frac{1}{s}\min D_{.j} \leq \delta + \min_{j\in K} D_{.j} \leq \lambda_i^+(\tilde{W}) \leq \delta + \max_{j\in K} D_{.j} \leq \frac{1}{s}\max D_{.j}. \tag{11}$$

The constant δ is positive if and only if there is a $c_j > 0$ with $D_{.j} > 0$, i.e., if there is a consumption good j which induces investment in some sector i. Min $D_{.j}$ is positive if min $B_{.j} > 0$ since $D \geq B$ due to $L(A) \geq I$.
Summarizing we have

Proposition 1

a) Balanced growth in model (4) is possible

  i) if any production for consumption requires investment or

  ii) if every investment sector needs capital input.

b) The number of equilibrium growth rates γ_i is restricted by the number of sectors producing capital goods.

c) Bounds for the equilibrium growth rates γ_i are

$$\tfrac{s}{\max D_{.j}} \leq \tfrac{1}{\delta+\max_{j\in K} D_{.j}} \leq \gamma_i \leq \tfrac{1}{\delta+\min_{j\in K} D_{.j}} \leq \min\left(\tfrac{1}{\delta}, \tfrac{s}{\min D_{.j}}\right) \tag{12}$$

provided $\delta = eDc/s > 0$ and $\min D_{.j} > 0$, respectively.

The sufficient conditions for equilibrium growth are no restriction in reality. From a theoretical point of view the first is fullfilled if $\tilde{W}$ is not completely decomposable into two groups of sectors: investment and consumption industries. However, the conditions are not necessary since $\lambda_1(D) > 0$ already implies $\lambda_1(\tilde{W}) > 0$ because of $\tilde{W} \geq D \geq B$. For instance, since $\lambda_1(M) \geq m_{ii}$ holds, $d_{ii} > 0$ for some $i \in K$ suffices which means that some investment

sector needs its own product or final demand for sector i induces (in)direct investment in i. Similar conditions were derived by Kogelschatz (1992, p. 10) in a different approach.
The outer bounds in (12) are plausible from Harrod's finding $\gamma = \frac{s}{b}$ since the $D_{.j}$ are sectoral accelerators or capital coefficients b_j. The sharper inner bounds are mainly determined by δ which blows up a weighted mean of $D_{.j}$ by the Keynesian multiplier and hence is comparatively large.
The inner bounds in (11) and (12) cannot be improved since they coincide for the case of equal column sums $D_{.j}$ for all $j \in K$. If they are not equal then strict inequalities hold due to (5) for λ_1 of irreducible W which implies $x > 0$ and for $\tilde{W}$, hence, $\tilde{x}_j > 0$ for $j \in K$.
Also the number of equilibrium growth rates in b) cannot be reduced as the following example shows. Let A, B be diagonal matrices, $c_i = 0$ for $i \geq 2$, such that $\tilde{w}_{ii} > 0$ are different for $i \in K$. Then there may be k growth equilibria $(1/\tilde{w}_{ii}, x^{(i)})$. Evidently, this example can be extended to upper triangular matrices A and B which corresponds to complete decomposability of the economy. In this case, the number of equilibria is restricted by the number of $b_{ii} > 0$ which implies $\tilde{w}_{ii} > 0$; in particular, no growth equilibrium exists if here B (and hence $\tilde{W}$) has only zeroes in the diagonal.

3 Sensitivity of Growth Rates

For $k = 1$ and $k = 2$ explicit formulae for the growth rates can be given. Obviously, one or two sectors producing investment commodities can be achieved by aggregation which is assumed to have been performed before, since aggregation usually affects growth rates (cf.,e.g., Dietzenbacher (1991, p. 248f)). The other sectors produce consumption goods.
For $k = 1$ we get for the eigenvalue

$$\begin{aligned} \lambda_1(\tilde{W}) &= d_{11} + \delta = d_{11} + \frac{1}{s}\sum d_{1j}c_j \\ &= \sum b_{1j}l_{j1} + \frac{1}{s}\sum_j \sum_\nu b_{1j}l_{j\nu}c_\nu \\ &= \sum_j b_{1j}\sum_\nu l_{j\nu}(\delta_{1\nu} + \frac{c_\nu}{s}) \qquad (13) \end{aligned}$$

or

$$\lambda_1(\tilde{W}) = eBL(A)L(C)e_1 \qquad (14)$$

where e_1 denotes the first unit column vector which is a right eigenvector x of $\tilde{W}$ associated with λ_1; the vector e may be replaced by a left eigenvector y with first component chosen as one. More generally, (14) holds for all k, if e_1 and e are replaced by eigenvectors x and y normalized by $yx = 1$. But,

the advantage of considering $k = 1$ and $k = 2$ lies in the fact that x and y need not be computed.
From $d\lambda_1(\tilde{W})$ follows the change in the associated growth rate $\gamma_1 = \frac{1}{\lambda_1(W)}$ by

$$d\gamma_1(\tilde{W}) = -(\frac{1}{\lambda_1(\tilde{W})})^2 d\lambda_1(\tilde{W}). \tag{15}$$

If investment sectors are aggregated ($k = 1$) then according to (14)

$$d\lambda_1(\tilde{W}) = e[dBL(A)L(C) + BdL(A)L(C) + BL(A)dL(C)]e_1 \tag{16}$$

with $dL(A) = L(A)dAL(A)$ and $dL(C)$ analogously which due to its special form (6) can be evaluated as

$$dL(C) = \frac{1}{s}L(C)(dc)e = \frac{1}{s}\left(\frac{1-ds}{s}c + dc\right)e \tag{17}$$

with $ds = 1 - edc$. Hence,

$$\begin{aligned} d\lambda_1(\tilde{W}) &= e[(dB + BL(A)dA)L(A)L(C) \\ &\quad +BL(A)L(C)dCL(C)]e_1 \\ &= e[(dB + BL(A)dA)L(A)\left(e_1 + \frac{1}{s}c\right) \\ &\quad +BL(A)\frac{1}{s}\left(\frac{1-ds}{s}c + dc\right)] \\ &= e[(dB + DdA)L(A)\left(e_1 + \frac{1}{s}c\right) \\ &\quad +D\frac{1}{s}\left(\frac{1-ds}{s}c + dc\right)]. \end{aligned} \tag{18}$$

For the special case $n = 1$ this can be simplified as follows

$$d\lambda_1(\tilde{W}) = \frac{1}{s}\left[dB(1-A)^{-1} + B(1-A)^{-1}\left((1-A)^{-1}dA + \frac{1}{s}dc\right)\right]. \tag{19}$$

Furthermore, by (14)

$$d\lambda_1 = \lambda_1(\frac{1}{B}dB + (1-A)^{-1}dA + \frac{1}{s}dC)$$

and with (15)

$$d\gamma_1 = -\gamma_1(\frac{1}{B}dB + (1-A)^{-1}dA + (1-C)^{-1}dC) \tag{20}$$

which coincides with $d\gamma$ for Harrod's growth rate $\gamma = \frac{s}{b} = \frac{s}{B/(1-A)}$ since Harrod's accelerator b corresponds to $D = BL(A)$.
The more general case with k investment sectors can be treated by a theorem on the differential of a simple eigenvalue (cf. Chatelin (1993, p. 151), Deif (1992, p. 225) similarly) which states

$$d\lambda_1(M) = ydMx \tag{21}$$

with normalized eigenvectors ($yx = 1$), which is possible since $yx \neq 0$ for a simple eigenvalue (cf. Horn/Johnson (1985, p. 371)). For the complicated case of an eigenvalue with algebraic multiplicity ≥ 2 and the sensitivity of eigenvector x which will not be treated here see Chatelin (1993) and Deif (1992).
The sensitivity analysis is performed for the dominant eigenvalue $\lambda_1(W)$ which is known to be a simple one for an irreducible matrix W (cf. Horn/Johnson (1985, p. 508)).It also suffices if $(A + B + cv)$ is irreducible as a look at model (4) shows, where for $\gamma > 0$ the pair $(\lambda = 1, x)$ is an eigensystem of $(A + \gamma B + cv)$. For reducible W the results can be extended to other simple eigenvalues λ_i^+ provided that the pertubation preserves semipositivity of eigenvector $x^{(i)}$ which even for small changes is not guaranteed, because of possible zero components. However, for W irreducible $x^1 > 0$ is maintained if W remains irreducible.
Application of statement (21) to $\tilde{W}$ which has the same (simple) eigenvalues, although irreducibility may no longer hold, yields

$$d\lambda_1(\tilde{W}) = \tilde{y}[(dB + BL(A)dA)L(A)L(C) + BL(A)L(C)dCL(C)]Bx \tag{22}$$

where $\tilde{u}Bx = \tilde{u}\tilde{x} = 1$ with $\tilde{u} = uB$ and $\tilde{x} = Bx$ denoting left and right hand eigenvector of $\tilde{W}$. If there are k investment sectors only $k \times k$ submatrices of the three matrices in brackets with k non-zero rows affect $d\lambda_1$ since Bx has only k non-zero components. In particular, $d\lambda_1$ and $d\gamma_1$ are not directly influenced by changes of the capital coefficients in consumption industries. An indirect impact may result from changed depreciation if captured in dA.
From

$$\lambda_1 x = L(A)L(C)Bx, \quad BL(A)L(C)Bx = \lambda_1 Bx, \quad \lambda_1\tilde{u} = \tilde{u}BL(A)L(C) \tag{23}$$

with $\lambda_1 = \lambda_1(W) = \lambda_1(\tilde{W})$ it follows that

$$\begin{aligned} d\lambda_1 &= \lambda_1\tilde{u}[dB + BL(A)dA + dCL(C)B]x && (24)\\ &= \tilde{u}[(dB + BL(A)dA)\lambda_1 + \lambda_1^2 dC(I-A)]x \\ &= \tilde{u}[\lambda_1 dB + \lambda_1^2((I-C)dA + dC(I-A))]x. && (25) \end{aligned}$$

With $d\gamma_1 = -(\lambda_1)^{-2}d\lambda_1$ we obtain:

Proposition 1 *If the growth matrix W has a simple eigenvalue $\lambda > 0$ with associated eigenvectors $x \geq 0, u$ normalized by $ux = 1$ then a change of dW*

affects the growth rate γ — provided $\lambda + d\lambda > 0, x + dx \geq 0$ — as follows

$$\begin{aligned} d\gamma &= -\tilde{u}[\gamma dB + (I-C)dA + dC(I-A)]x & (26) \\ &= -\gamma\tilde{u}[dB + B(I-A)^{-1}dA + dC(I-C)^{-1}B]x & (27) \end{aligned}$$

where $dC(I-C)^{-1} = \frac{1}{s}(dc)e = \frac{1}{s}dC$ by (6). If B is invertible then

$$d\gamma = -\gamma\tilde{u}[(dB + B(I-A)^{-1}dA)B^{-1} + dC(I-C)^{-1}]\tilde{x} \quad (28)$$

where $\tilde{x} := Bx, \tilde{u} := uB$ are corresponding eigenvectors of $\tilde{W}$.
In particular, the assumptions of the proposition are fulfilled (with $x, u > 0$) if either W or $(A + B + cv)$ is irreducible before and after perturbation.

Obviously, (27) also reveals the relative change of γ. The differential of the growth rate γ in (26) depends on dampening interactions between A and C. This does not hold for the absolute and relative change of γ in (27), where A and C are connected with B (and its column sums, resp.). If B is invertible then according to (28) the three matrices are nearly separated where the changes in A and C are augmented by their Leontief inverses the latter resulting in Keynes' multiplier and the former undergoing a similarity transformation with B.
For Harrod's model these results can be simplified because of $\tilde{u}x = \frac{1}{B}\tilde{u}\tilde{x} = \frac{1}{B}$ as follows

$$\begin{aligned} d\gamma &= -\frac{1}{B}(\gamma dB + (1-C)dA + (1-A)dC) & (29) \\ &= -\gamma(\frac{1}{B}dB + (1-A)^{-1}dA + (1-C)^{-1}dC) & (30) \end{aligned}$$

in accordance with (20).
A result similar to (26) was given by Brody (1970, p. 127f) for model (1). By a first order approximation he obtained

$$d\gamma \approx -\frac{1}{uBx}u(\gamma dB + dA)x \quad (31)$$

from $\gamma = u(I-A)x/uBx$.
Introducing consumption functions extremely changes the relative importance of changes (or errors) in A and B as a look at the fully aggregated model shows. A realistic order of empirical values is given by $\gamma \approx .03$, $C \approx .9$, $A \approx .6$. For the given figures Brody's result suggests that changes in A have ≈ 30 times the effect on $d\gamma$ as those in B of the same amount. Brody appreciates the extreme relation of impacts as very useful since measurement of the stock matrix B causes severe difficulties. Still nowadays data for B are hardly available, not even for Germany. However, with consumption functions the factor 30 is changed to ≈ 3 according to (29). Neglecting consumption overestimates the influence of changes in current input on the

growth rate by the factor $1/s$. The greatest impact in the three matrices on the growth rate is expected to come from C since the value added coefficient usually is greater than the propensity to save and the growth rate. Furthermore, in contrast to A and B a single entry of C cannot vary alone since all entries of a row are the same. Therefore, perturbations in the three matrices should be compared row by row. For equal relative changes of the entries dB becomes more influential since on average $B \gg A, C$. For the relative change of γ a similar order of impacts holds as (30) shows.
Whereas proposition 2 is applicable for small changes the method suggested at the beginning of this section may also be useful for greater changes — provided $\gamma + \Delta\gamma > 0, x + \Delta x \geq 0$ — because the vectors e and e_1 can still be taken and $\Delta L(A)$ and $\Delta L(C)$ can be determined in many interesting cases such as row changes.

4 Approximation of Growth Equilibrium

It is assumed that $\lambda_1 < 1$ and that $(A + B + Cv)$ is irreducible. Let $\lambda_1 > 0$ denote the dominant eigenvalue of $\tilde{W}$ (and W) and z (and x resp.) a right eigenvector of $\tilde{W}$ (and W) associated with λ_1 where $z \geq 0$ and $x \geq 0$ are normed such that $\sum z_i = 1$ and $\sum x_i = 1$. Since Bx is an eigenvector of $\tilde{W}$ $z = \frac{Bx}{eBx}$ is a normed one.

4.1 Approximation by successive iteration

It follows from $\lambda_1 z = \tilde{W} z$ with $\tilde{W} = D(I + \frac{1}{s}ce)$ that

$$\lambda_1 z = \frac{1}{s} D(sz + c) \tag{32}$$

where the components of vector $y = (sz + c)$ add up to one. Hence, z_i is a weighted mean of the elements of row i of $(s\lambda_1)^{-1}D$ with weights y_j independent of i. Furthermore,

$$(\lambda_1 I - D)z = \lambda_1(I - \gamma_1 D)z = \frac{1}{s} Dc \tag{33}$$

Since $\lambda_1(D) < \lambda(\tilde{W}) =: \lambda_1$ because of $D < \tilde{W}$ for $c \neq 0$, it follows that $\lambda_1^{-1} D$ has a dominant eigenvalue smaller than one. Hence, $(E - \lambda_1^{-1} D)^{-1}$ exists, is non-negative and equals the Neumann series:

$$\begin{aligned} z &= \frac{1}{s}\gamma_1(I - \gamma_1 D)^{-1} Dc \\ &= \frac{1}{s}\gamma_1 D(I + \gamma_1 D + (\gamma_1 D)^2 + \ldots)c \\ &= \frac{1}{s}\gamma_1 D(I - \gamma_1 D)^{-1} c \\ &= s^{-1}[(I - \gamma_1 D)^{-1} - I]c \end{aligned} \tag{34}$$

Here is helpful if there are $k < n$ investment sectors because B, D and the matrix in the last brackets have only k non-zero rows. Therefore, it suffices to consider a $k \times k$ matrix D_k in order to determine the k possible positive components of z. Substituting the last expression for z in the vector of weights yields normed equilibrium income

$$y = sz + c = (I - \gamma_1 D)^{-1} c \tag{35}$$

From the equilibrium investment vector z it is possible to determine equilibrium output vector x by $z = Bx$ if B is invertible. However, B is supposed to have rows of zeroes if there are sectors not producing for investment. But, this difficulty can be coped with by looking at (34) and (35). Inserting $D = BL(A)$ shows

$$\begin{aligned} x &= \alpha \frac{1}{s} \gamma_1 L(A)(I - \gamma_1 D)^{-1} c = \alpha \frac{1}{s} \gamma_1 (I - \gamma_1 L(A) B)^{-1} L(A) c \\ &= \alpha \frac{1}{s} \gamma_1 L(A)(I + \gamma_1 D + (\gamma_1 D)^2 + \ldots) c = \alpha L(A) y \end{aligned} \tag{36}$$

where α is a dispensable norming factor such that the following vector adds up to one.

The last equation reveals that equilibrium output relations are determined by $\check{x} = L(A)c$ which would be the output vector of the static input output-model if c were final demand (per unit of income) premultiplied by the matrix $(I - \gamma_1 W)^{-1}$ which is the Leontief inverse of the growth matrix $W = L(A)B$ multiplied by the growth rate γ_1.

The Leontief inverse $(I - \gamma_1 D)^{-1}$ represents the multiplier process of investment (accelerator process) of an economy growing at rate γ_1. $(I - \gamma_1 D)^{-1} \frac{1}{s} c$ represents equilibrium income vector y induced per unit of final demand.

In order to examine the influence of $\gamma_1 D$ it is useful to go back to (34).

Let u be a left eigenvector of $\tilde{W}$, then

$$\begin{aligned} uDL(C) &= \lambda_1 u \quad or \\ u \gamma_1 D &= u(I - ce) \end{aligned} \tag{37}$$

Hence, for an aggregated model $\lambda_1^{-1} D$ corresponds to s. Inserting (37) into equation (33), premultiplied by u, yields with regard to $\sum z_i = 1$

$$\begin{aligned} \lambda_1 ucez &= \frac{1}{s} uDc \\ \lambda_1 &= \frac{\frac{1}{s} uDc}{uc} \end{aligned} \tag{38}$$

A first proxy of u is e which is an exact left eigenvector of the matrix $E := \frac{1}{s} Dce$ differing from $\tilde{W}$ by D. Hence, a first proxy of λ_1 results as

$$\tilde{\lambda_1} = \frac{1}{s} eD \frac{c}{ec} \tag{39}$$

and of the growth rate γ_1 by the reciprocal value. Hence, γ_1 comes close to the ratio of the propensity to save and a weighted mean of sectoral accelerators eD with relative propensities to consume as weights. This interpretation may be regarded as an obvious extension of Harrod's finding. Indeed, for a fully aggregated model both formulae coincide. A better proxy of u is given by the column sums of $\tilde{W}$

$$\tilde{u} = eD(I + \frac{1}{s}ce) = e\tilde{W} \tag{40}$$

which would be an exact left eigenvector of $((\tilde{W}e^T)(e\tilde{W}))$. Inserting this proxy into (38) yields a better proxy of λ_1:

$$\begin{aligned}
\tilde{\lambda_1} &= \frac{eD(I + \frac{1}{s}ce)Dc}{seD(I + \frac{1}{s}ce)c} \\
&= \frac{eD(I + \frac{1}{s}ce)Dc}{eD(s + ec)c} \\
&= \frac{1}{eDc}\left(\frac{(eDc)^2}{s} + eD^2c\right) \\
&= \frac{1}{s}eD\left(c + s\frac{1}{eDc}Dc\right) = \delta + \frac{eD^2c}{s\delta} \qquad (41)
\end{aligned}$$

The vector in the last brackets adds up to one, hence it is a weight vector. The interpretation is not as evident as with the first proxy. However, Dc/eDc is the relative investment induced by c and a proxy of a normed eigenvector z of $\tilde{W}$ which stands for equilibrium investment. Since a share s of equilibrium income y is invested the weight vector can be interpreted as a proxy of equilibrium income vector y. Therefore, the growth rate γ_1 again is a ratio of s and a weighted mean of sectoral accelerators eD with improved weights given by approximate equilibrium income y. The proxy (41) of λ_1 coincides with applying the differential formula to E with eigenvectors e and $\frac{Dc}{eDc}$ where $\triangle\tilde{W} = D$.

Finally, equilibrium vector y is given by

$$y = (I - A)x = \beta\frac{1}{s}(I - \gamma_1 D)^{-1}c = \beta\frac{1}{s}(I + \gamma_1 D + \gamma_1^2 D^2 + \cdots)c \tag{42}$$

or simply by the normed vector $y = sz + c$ from (35) or as eigenvector of $L(C)DL(A)$ corresponding to λ_1. Hence, $(I - \gamma_1 D)^{-1}$ may be called dynamic income multiplier matrix which is applied to initial final demand (or income) vector of c or to consumption augmented by the Keynesian income multiplier to $\frac{1}{s}c$. Both processes seem to be separable, however, γ_1 depends on D and c.

Hence, equilibrium output results from $x = L(A)y$ which means that x_i is a weighted mean of row i of $L(A)$ with weights y_j independent of i. Similarly,

z_i is a weighted mean of row i of D with the same weights. Finally, y_i can be regarded as a weighted mean of row i of $(I-\gamma_1 D)^{-1}$ or approximately of $I+\gamma_1 D$ with weight vector c. Therefore, similar rows in these 3 matrices induce similar eigenvector components.
The easy computation procedure runs as follows:
Start with a proxy of λ_1 by (41) or (39), then compute a proxy of z by (34) and immediately of y and x by (35) and (36). An iteration is possible by defining a new $\tilde{\lambda}_1 := s^{-1}eD\tilde{y}$ according to (32).

4.2 Approximation by a matrix of rank one

In an alternative approach to approximate λ_1 and associated eigenvectors we start with a proxy of $\tilde{W}$ by a matrix of rank one constructed from its vectors r (and q) of row sums (and column sums):

$$\hat{\tilde{W}} = \frac{(\tilde{W}e^T)(e\tilde{W})}{\sigma} = \frac{rq^T}{\sigma} \quad \text{with} \quad \sigma := e\tilde{W}e^T = er = q^T e^T \tag{43}$$

Note that e has been defined as a row vector, other row vectors are indicated by a superscript T. For this matrix of rank one the exact dominant eigenvalue is simply given by

$$\hat{\lambda}_1 = \frac{(e\tilde{W})(\tilde{W}e^T)}{\sigma} = \frac{q^T r}{er} \tag{44}$$

and corresponding right and left eigenvectors are

$$\hat{z} = \frac{(\tilde{W}e^T)}{\sigma} = \frac{r}{er}, \quad \hat{u} = (e\tilde{W}) = q^T, \hat{u}' := \frac{\sigma}{q^T r}\hat{u}, \tag{45}$$

where $\hat{z}$ adds up to one and, furthermore, $\hat{u}'\hat{z} = 1$ is achieved.
Obviously, $\hat{\lambda}_1$ is a weighted mean of the column (row) sums of $\tilde{W}$ with the relative row (column) sums as weights.
Since $\tilde{W} = D + \frac{1}{s}Dce$, where the second matrix has identical column sums, it can be expected that, in particular, $\hat{u}$ is a good proxy of u and, therefore, also $\hat{\lambda}_1$ of λ_1.
Row and column sums of $\tilde{W}$ can be expressed by those of D:

$$\hat{u} = e\tilde{W} = \tfrac{1}{s}eDce + eD = \delta e + eD = e(\delta I + D) \tag{46}$$

$$\sigma\hat{z} = \tilde{W}e^T = \tfrac{n}{s}Dc + De^T = D(\tfrac{n}{s}c + e^T) = nD(\tfrac{1}{s}c + \tfrac{e^T}{n})$$

$$\hat{z} = \tfrac{\tilde{W}e^T}{e\tilde{W}e^T} = \tfrac{D(c+\frac{s}{n}e^T)}{eD(c+\frac{s}{n}e^T)} = \tfrac{D\hat{y}}{eD\hat{y}} \tag{47}$$

where the term in brackets is a first proxy of the normed equilibrium income vector y if investment were uniformly distributed. Therefore,

$$\begin{aligned}\hat{\lambda_1} &= \delta + eD\frac{D(c+\frac{s}{n}e^T)}{eD(c+\frac{s}{n}e^T)} = \delta + eD\hat{z} \\ &= \frac{1}{s}eD\left[c + sD\frac{c+\frac{s}{n}e^T}{eD(c+\frac{s}{n}e^T)}\right] = \frac{1}{s}eD(c+s\frac{D\hat{y}}{eD\hat{y}}). \qquad (48)\end{aligned}$$

However, a better proxy of λ_1 can be achieved if $\frac{1}{n}e^T$ is replaced by first round normed investment $\frac{Dc}{eDc}$:

$$\hat{\lambda_1}' = \frac{1}{s}eD\left[c + sD\frac{c+\frac{sDc}{eDc}}{eD(c+\frac{sDc}{eDc})}\right] = \frac{1}{s}eD\left[c + s\frac{D\hat{y}'}{eD\hat{y}'}\right] = \frac{1}{s}eD(c+s\hat{z}') \qquad (49)$$

Both proxies of λ_1 have the form $\frac{1}{s}$ times a weighted mean of sectoral accelerators eD with normed sectoral income proxies as weights. Hence, the growth rate γ_1 is approximated by the ratio of saving rate s and an income weighted mean of sectoral accelators in analogy with Harrod's result.
An improved proxy of λ_1 can be obtained by proposition 2 on the variation of a simple eigenvalue λ of a perturbed matrix M which states the following first order approximation

$$\lambda_1(M+\Delta M) = \lambda_1(M) + u\Delta Mx$$

where u and x are left and right eigenvectors of M such that $ux = 1$. For a matrix of rank one with $\lambda_1 > 0$, λ_1 is a simple eigenvalue since it has the eigenvalue 0 with multiplicity $(n-1)$.
Hence, for $M = \hat{\tilde{W}}$ and $\Delta M = \tilde{W} - \hat{\tilde{W}}$: $\tilde{\lambda_1} := \hat{\lambda_1} + \Delta\hat{\lambda_1}$ with $\Delta\hat{\lambda_1} := \hat{u}'(\tilde{W} - \hat{\tilde{W}})\hat{z}$.
Hence, with $\hat{u}'\hat{z} = 1$ and $\hat{u}'\hat{\tilde{W}}\hat{z} = \hat{\lambda_1}$ by (43), (44)

$$\tilde{\lambda_1} = \hat{u}'\tilde{W}\hat{z} = \frac{e\tilde{W}^3e^T}{e\tilde{W}^2e^T} = \frac{q^T\tilde{W}r}{q^Tr} = \frac{e(\delta I + D)D(I+\frac{1}{s}ce)r}{e(\delta I + D)r} \qquad (50)$$

where $\delta := \frac{1}{s}eDc$. Further iterations are possible.
This proxy of λ which can be easily computed seems to show a very good performance. Only one matrix product, D^2, has to be calculated. Furthermore,

$$\begin{aligned}\tilde{\lambda_1} &= \frac{e(\delta I + eD)D(r+\frac{1}{s}c)}{e(\delta I + D)r} = \frac{1}{s}\frac{e(\delta I + eD)D(c+sr)}{e(\delta I + D)r} \\ &= \frac{e(\delta I + eD)\hat{z}^1}{e(\delta I + D)\hat{z}^0} = \frac{q^T\hat{z}^1}{q^T\hat{z}^0}, \qquad (51)\end{aligned}$$

where $\hat{z}^0 := r = \frac{D\hat{u}}{eD\hat{u}}$ and $\hat{z}^1 := D\frac{(c+s\hat{z}_0)}{s} = D\hat{u}^1$, respectively, denote the initial investment vector for $\hat{u} := c + s\frac{e^T}{n}$ and the first period investment vector (induced by the Keynesian multiplier and the accelerator process). The last ratio shows that $\tilde{\lambda}_1$ may be interpreted as a Laspeyres index of (sectoral) investment weighted by sectoral accelerators q^T. This index for subsequent periods is expected to converge to λ_1. An improvement of $\tilde{\lambda}_1$ can be achieved by starting with $\hat{z}'^0 := D\hat{u}' = D(c + s\frac{Dc}{eDc})$ instead of $\hat{z}^0 = D\hat{u}$.

Also the differential of the eigenvectors can be computed according to the quoted literature. Furthermore, there exist better rank one approximations of $\tilde{W}$ which, however, need more computation and reveal less interpretation.

5 Application

The following example of a dynamic input output-model is taken from Holub/Schnabl (1994, p. 439 f.).

$$A = \begin{pmatrix} .2 & .25 & .25 \\ .2 & 0 & .5 \\ .2 & .5 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ .5 & 0 & 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} .4 \\ .2 \\ .2 \end{pmatrix}$$

Evidently, $(A + B + C)$ is an irreducible matrix. Hence, there is only one equilibrium (λ, x) with $x > 0$. Computations yield $\lambda = 27.4653$, $\gamma = .0364$, $x = (.3977, .3024, .2999)$, $z = (.7053, .1778, .1169)$, $y = (.5411, .2356, .2234)$ and $u = (.3114, .3332, .3554)$ as left eigenvector of $\tilde{W}$. Here vectors are normed by components adding up to one. By proposition 1 the bounds for the growth rate $\gamma = .0364$ and their absolute and relative errors are evaluated from $eD = (4.1\bar{6}, 6.08\bar{3}, 8.08\bar{3})$ and $\delta = 22.5$ as:

$$\begin{array}{rrcl} & .0247 \le .0327 & \le \gamma \le & .0375 \le \min(.0\bar{4}, .048) \\ \text{error} & -.0117 \le -.0037 & \le 0 \le & .009 \le \min(.069, .105) \\ \text{rel. error} & -.32 \le -.102 & \le 0 \le & .03 \le \min(.221, .3187) \end{array}$$

The length of the interval for γ by the outer bounds is .0233 whereas that by the inner bounds is reduced to .0048 or 20% of the former one.
Furthermore, the absolute relative error of the outer and inner bound decreases from .32 to .10 (or .065 for the mean of the two inner bounds). This is a considerable improvement by a quick computation.
The further advantage of the inner bounds that $\min D_{.j}$ and $\max D_{.j}$ are to be taken over only k investment sectors is not made use of in this example because here $n = k$.
A change in the positive entries of row two of the three matrices by .02 each results according to proposition 2 in a change of $d\gamma = -.012$.
If instead of equal absolute changes now equal relative changes are considered dB gets more influential since its entries are roughly 5 times greater than those of the other matrices.
In order to approximate λ by proposition 2 start with the simple matrix $E := \frac{1}{s}Dce$ with eigenvalue $\delta = 22.5$, left and right eigenvector $u(E) =$

$(1,1,1)$ and $z(E) = \frac{Dc}{eDc} = (.7\bar{1}, .1\bar{7}, .\bar{1})$. Applying the formula for the differential of λ_1 to $\Delta\tilde{W} = \tilde{W} - E = D$ yields $\tilde{\lambda}_1 = 27.4426$. Although $\triangle\tilde{W}$ is not small the result is good because the eigenvectors are rather stable. A twofold reason can be seen in E which has equal columns and constitutes the dominating part of $\tilde{W}$ since it is roughly $(1-s)/s$ times greater than D.

The first suggested method from section 4 with start value $\lambda_1^0 = 28.125$ by (39) has the following approximations $\tilde{\lambda}_1 = 27.4622$, $\tilde{z}_1 = (.7061, .1779, .1161)$ when starting with first order approximation $(I + \tilde{\lambda}_1^{-1} D_k)$ of $(I - \tilde{\lambda}_1^{-1} D_k)^{-1}$ and beginning with its second order approximation $\tilde{\lambda}_1 = 27.4643$, $\tilde{z}_1 = (.7056, .1778, .1166)$. Proxies for the vectors x and y are obtained by $\tilde{x} = (.3974, .3026, .3001)$ and $\tilde{y} = (.5401, .2359, .2239)$. Note that $\tilde{x}$ and $\tilde{y}$ now denote approximations of output x and income y. Further improvements of these good proxies can be achieved by iteration or by computing the Leontief inverse $(I - \tilde{\lambda}_1^{-1} D_k)^{-1}$ or by a better start value.

The approximation of $\tilde{W}$ by a matrix $\hat{\tilde{W}}$ of rank one yields $\hat{\lambda}_1 = 27.4273$ with right and left eigenvectors $\hat{z} = (.7146, .1786, .1068)$ and $\hat{u} = (.3107, .3330, .3563)$ by (44). A first approximation results in $\tilde{\lambda}_1 = 27.4668$ by (50) which coincides with applying the differential formula to $\Delta\tilde{W} = \tilde{W} - \hat{\tilde{W}}$.

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