THE CONCEPT OF SECTOR

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Abstract

We generalize the concept of industry, which stems from the analysis of single-product economies, to that of sector. The sector concept can be applied to economies with or without joint-product processes and pure capital goods. A 'sectoral economy' is an economy characterized by the 'super-adjustment' property: any strictly viable subset of methods can adapt itself to an arbitrary final demand. Given a few additional assumptions, the competitive prices are minimal in a sectoral economy, so that the subset of competitive methods is uniquely defined and the non-substitution property holds.

Keywords: linear models of production, joint production, fixed capital, non-substitution

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1. INTRODUCTION

In linear models of production in which a choice can be made between the available processes, it is customary to partition the set of processes into different industries. The notion of 'industry' originated in the study of single-product economies: an industry is the subset of methods which produce the same commodity. There are as many industries as commodities produced by the economy, and every single-product process belongs to one and only one industry. The concept plays an important role in the demonstration of the non-substitution result: in a single-product economy the competitive technique consists of exactly one process from each industry, and is independent of final demand (Arrow, 1951; Georgescu-Roegen, 1951; Samuelson, 1951; cf. Ten Raa, 1995, for a generalized version). In a joint production framework, however, the industry approach fails. Although there are special cases in which the notion of industry may be stretched somewhat (e.g. the case in which every process admits a 'dominant product'), the study of joint production economies clearly requires a similar but more general concept. In this paper the term 'sector' refers to this generalized 'industry' concept.

Already in 1972 Johansen noticed the difficulties of extending the definition to joint-product systems, but nevertheless thought the notion of sector useful:

> "There are now many conceivable bases for the definition of sectors: Similarities in output structure, similarities in input structure (or at least some common main inputs), similarities in some technological processes, organizational and/or geographical distinctions and perhaps also other criteria. The following considerations are valid regardless of how the sectors are defined, but the nonsubstitution assertion is most likely to hold when there are similarities in output structure for possible activities within a

> sector whereas output structures are markedly different as between sectors." (Johansen, 1972, p. 390)

More recently, Herrero & Villar (1988, p. 148) also adopted the sector approach. In contrast to Johansen, they did not specify how sectors are defined. They assume from the outset that the number of sectors is equal to the number of commodities and that any viable set of processes consists of one process from each sector. We will show that once this assumption is made, nothing much has to be added to arrive at non-substitution results.

Our approach concentrates upon physical relationships (the paper can be seen as a follow-up on Bidard & Erreygers, 1998a, 1998b). We first look at the combinations of processes which are strictly viable, i.e. capable of satisfying *some* positive demand vector. We then identify the sets of methods which can satisfy *any* positive demand vector; these sets have the 'adjustment property'. Economies in which *all* strictly viable sets of processes have the adjustment property are characterized as economies having the 'super-adjustment property'. The main result of the paper deals with the relationship between sectors and the super-adjustment property, which constitutes the core of the non-substitution property.

We follow Johansen in the sense that, in principle, we are willing to consider any criterion to partition the set of processes into sectors. Unlike Herrero & Villar we do not specify that the number of sectors has to be equal to the number of commodities. In the presence of pure capital goods, for instance, it is not certain how many commodities will effectively be produced, and it is therefore difficult to know from the outset how many sectors are 'indispensable'. What really matters is not so much how the sectors are defined or how many there are, but rather whether there exists at least one sector-classification which satisfies a crucial condition. If this is the case - we will say the economy is then a 'sectoral economy' -

the system has the super-adjustment property. A few additional conditions ensure that the economy also possesses the non-substitution property.

We begin with a formal presentation of the model (section 2) and a definition of the notions of adjustment, super-adjustment and sector (section 3). The main obstacles to the sectoral approach are the presence of joint-product processes and that of pure capital goods. We briefly review results obtained in the literature on joint production without pure capital goods (section 4) and in fixed capital models (section 5). Inspired by the non-transferability axiom with respect to 'machines', which is part and parcel of many fixed capital models, we will distinguish two types of pure capital goods, 'proper' and 'general' ones. This will allow us to give a definition of a 'sectoral economy' which is applicable to a broad range of economies (with and without joint production, with and without pure capital goods), and to establish a firm link between sectoral economies and the super-adjustment property (section 6). Finally, section 7 presents two related results on the non-substitution property.

2. GOODS AND PROCESSES

Consider an economy with n produced goods and one primary factor (labour), in which the m available methods (or processes) of production and/or disposal admit constant returns to scale. Process i is described by the $(n\times 1)$-vector A_i of material inputs, the scalar l_i of input of labour, and the $(n\times 1)$-vector B_i of material outputs. For all processes the material inputs vector and the labour input are nonnegative. For production processes the material outputs vector is semipositive, while for disposal processes it is zero. To say that good j can be disposed of freely, is equivalent to saying that there exists a disposal process such that the material inputs vector has only one positive element (corresponding to good j) and the labour input is zero. The economy as a whole is represented as $(A, l) \rightarrow B$, where A is the input matrix, l the labour vector, and B the output matrix; A and B have dimensions $(n\times m)$ and l dimension $(1\times m)$. The activity levels of the m available processes are represented by the semipositive $(m\times 1)$-vector y. Given the activity vector y, the net product produced by the economy is equal to $(B - A)y$.

The produced goods are either final goods or pure capital goods. A final good is a commodity for which consumption demand is normally positive, although it can exceptionally be equal to zero. (Apart from being used for consumption, a final good may also be used as an input.) By contrast, the final demand for pure capital goods is identically zero: they are purely intermediate goods, like fertilizers and fixed capital. The n produced goods are therefore partitioned into two groups: a group of f final goods and a group of k pure capital goods, with $f + k = n$, $f > 0$ and $k \geq 0$. Following this division, we rearrange the input and output matrices as:

$$A = \begin{bmatrix} A^F \\ A^K \end{bmatrix}, \qquad B = \begin{bmatrix} B^F \\ B^K \end{bmatrix} \tag{1}$$

where A^F and B^F refer to the final goods, and A^K and B^K to the pure capital goods. Final demand is represented by the $(n\times 1)$-vector d, which is likewise partitioned as:

$$d = \begin{bmatrix} d^F \\ d^K \end{bmatrix} = \begin{bmatrix} d^F \\ 0 \end{bmatrix} \tag{2}$$

The demand set *D*, to which *d* belongs, is $D = R_{+}^{f} \times \{0_k\}$. Note that *D* is a closed convex cone in R_{+}^{n}. We will sometimes consider demand vectors which are located in the relative interior of *D*, defined as $ri(D) = R_{++}^{f} \times \{0_k\}$.

Definition 1. The economy is *strictly viable* if it can produce some vector in the relative interior *ri*(*D*), i.e. if there exists a semipositive vector *y* such that $(B - A)y \in ri(D)$. The corresponding activity levels *y* are called *strictly feasible*.

Most of the time we will concentrate upon subsets of the set of *m* available methods. Let *H* be a subset of *h* methods, with $h \leq m$. The (*h*×*n*)-matrices A_H and B_H describe the input and output vectors of the processes in *H*, and the (*h*×1)-vector y_H the activity levels of the processes in *H*. The set *H* is capable of satisfying demand $\bar{d} \in D$ if there exists a nonnegative activity vector $\bar{y}_H$ such that:

$$(B_H - A_H)\bar{y}_H = \bar{d} \tag{3}$$

Definition 2. The subset of methods *H* is *strictly viable* if it can produce some vector in the relative interior *ri*(*D*).

A set *H* may contain more methods than necessary to obtain a net output in *ri*(*D*). If not, we say that *H* is minimal[1]:

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1 With respect to vectors and matrices '$x \geqq 0$' stands for nonnegativity, '$x \geq 0$' for semipositivity, and '$x > 0$' for positivity.

Definition 3. The set of methods H is called *minimal* if it is strictly viable and if the following holds:

$$\{y_H \geqq 0, \quad (B_H - A_H)y_H \in ri(D)\} \quad \Rightarrow \quad y_H > 0 \tag{4}$$

Clearly, a strictly viable set of methods is either minimal itself, or contains at least one minimal subset.

3. ADJUSTMENT AND SECTORS

We start from very general notions of classification and sectors:

Definition 4. A *classification* is a partition of the set of processes into σ nonempty subsets $S_\alpha, S_\beta, \ldots, S_\sigma$, called *sectors*.

Given a classification, let A_ι and B_ι be the submatrices of A and B made of the processes belonging to sector S_ι, and y_ι the activity levels of the processes belonging to sector S_ι (ι = α, …, σ). The net output produced by sector S_ι is equal to $(B_\iota - A_\iota)y_\iota$. Sector S_ι is *operated* if $y_\iota \neq 0$. The integer $s(y)$, with $s(y) \leq \sigma$, denotes the number of operated sectors.

The notion of sector implied by Definition 4 imposes only a very weak structure upon the data. For instance, the number of sectors σ remains unspecified: it lies between 1 (if the whole economy is considered as one giant sector) and m (if every process is considered as a single sector). Our main purpose in this paper is to associate the existence of a specific kind of sectoral classification to certain properties of the economy. The properties we are interested in here are those of adjustment and super-adjustment:

Definition 5. A set of processes has the *adjustment property* if it can meet any demand in *D* by an appropriate choice of its activity levels.

Definition 6. An economy has the *super-adjustment property* if it is strictly viable and every strictly viable subset of processes has the adjustment property.

A set of processes which contains a strictly viable subset is strictly viable itself. Therefore an equivalent definition is:

Definition 6*. An economy has the *super-adjustment property* if it is strictly viable and every minimal subset of processes has the adjustment property.

The main theorem of the paper is a general result on super-adjustment: we show that an economy has the super-adjustment if (and only if) there exists a classification which identifies the economy as a 'sectoral economy'. The result allows both for joint production and for the presence of pure capital goods.

Before we present the theorem, we review in the next section results which have been obtained in more restrictive cases. In section 4 we deal with the case in which goods are produced without pure capital goods; the main complication which occurs there is that of joint production. In section 5 we abandon the 'no pure capital goods' assumption and look at fixed capital models.

4. JOINT PRODUCTION SYSTEMS

In this section we suppose there are no pure capital goods (*n* = *f*; *k* = 0). Since all goods are final goods, the set of demands is $D = R_+^f$. To begin with, let us assume that all processes are single-product processes. Because every method produces one final good only, it is natural to classify processes according to the nature of their output. Hence we distinguish *n* = *f* different 'industries', with industry I_j consisting of all processes producing commodity *j* (*j* = 1, ..., *n*). In such a single-product economy, no set of processes can be strictly viable unless it takes at least one process from each industry; therefore, a strictly viable set consists of at least *n* processes. If it has more than *n* processes, it always contains at least one minimal subset of exactly *n* processes (see e.g. Gale, 1951, p. 297). Let *H* be a minimal set of *n* processes, one from each industry. Its input matrix is a square matrix A_H, while the corresponding output matrix B_H can, after a suitable rearrangement of columns and an appropriate definition of the units of measurement, be expressed as the identity matrix *I*. The strict viability hypothesis implies that equality:

$$(I - A_H)\bar{y}_H = \bar{d} \qquad (5)$$

holds for some positive vector $\bar{d}$ and some semipositive activity vector $\bar{y}_H$. Equality (5) itself implies the semipositivity of the Leontief inverse $L_H = (I - A_H)^{-1}$. Following a change in final demand from $\bar{d}$ to *d*, the activity levels $y_H = L_H d$ meet the new requirements. Therefore any minimal single-product system has the adjustment property. This means that a single-product economy with no pure capital goods has the super-adjustment property.

Next, let us assume that there is joint production: there exists at least one process that produces several final goods. In the single-product case, any strictly viable set of n processes is automatically minimal. This property does not hold anymore in the joint-product case, because a set of less than n processes can be strictly viable. If such a set of n' $(1 \leq n' < n)$ processes really exists, the dimension of its feasible production set is n' only, which means that the production cannot be adjusted to an arbitrary demand vector in R_+^n: the set of processes does not have the adjustment property.

This means that the super-adjustment property only holds under special circumstances. One such case occurs when every process admits a 'dominant product': for any strictly viable set of n methods H, matrix $(B_H - A_H)$ then has positive diagonal coefficients and nonpositive off-diagonal coefficients, so that the generalized Leontief matrix $L_H = (B_H - A_H)^{-1}$ is semipositive. The processes can be classified into n sectors according to the nature of their 'dominant product'. But this configuration is both exceptional and too restrictive.

The super-adjustment property appears to be closely linked to the minimal number of processes which must be operated, as illustrated by the following statement, a corollary of the general result on 'sectoral economies' which we present later in the paper. Suppose there exists a classification into $\sigma \geq n$ sectors such that every minimal subset takes processes from exactly n different sectors; then the economy has the super-adjustment property. In the single-product case, the n industries constitute a classification with exactly $\sigma = n$ sectors. In the joint-product case, however, it is not always possible to define a classification with exactly n sectors, even if the economy has the super-adjustment property. Here is an example:

Example 1. Consider the following economy, with $n = f = 3$ and $k = 0$:

$$A = \begin{bmatrix} 19 & 0 & 0 & 5 & 6 \\ 0 & 20 & 0 & 0 & 24 \\ 0 & 0 & 3 & 4 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 13 & 6 & 0 & 0 \\ 13 & 0 & 7 & 10 & 0 \\ 6 & 7 & 0 & 0 & 10 \end{bmatrix} \qquad l > 0 \qquad (6)$$

Let $\{h, i, j\}$ be the set made of processes h, i and j. The economy admits four minimal subsets: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 5\}$ and $\{3, 4, 5\}$. They all have the adjustment property, and so the economy as a whole has the super-adjustment property. It is, however, impossible to construct a classification with $n = 3$ sectors such that every minimal subset takes processes from each of these sectors. The minimal number of sectors is equal to four, as witnessed by the following classification: $S_\alpha = \{1\}$, $S_\beta = \{2, 5\}$, $S_\gamma = \{3\}$, $S_\delta = \{4\}$.

5. FIXED CAPITAL MODELS

As soon as final goods and pure capital goods coexist, additional complications occur. This can be illustrated by the results obtained in fixed capital theory. Although production with fixed capital belongs to the genus of joint production, it is specific in the sense that the total net output of fixed capital goods, like that of all pure capital goods, is equal to zero. Consider the example of a tractor used in the production of corn: a t-year-old tractor enters the production process as one of the inputs, and at the end of the annual cycle corn is produced simultaneously with a $(t+1)$-year-old tractor. The output consists of two goods, a final and a pure capital good. However, if the tractor is not used elsewhere, the net product of the 'corn industry' calculated over the lifetime of the tractor has only one positive component, viz. corn, as in single production. This is why fixed capital models in the neoclassical (Stiglitz, 1970) or neo-Ricardian (Sraffa, 1960) tradition usually rely on the following assumptions:

- Every process produces one final good at most (‘industry’ assumption).
- Every fixed capital good is ‘internal’ to an industry (non-transferability between industries).

Under these assumptions the super-adjustment property will hold; if one of them is violated, however, the property might be lost.

The requirement of ‘internal’ fixed capital goods can be relaxed slightly. For a given classification of processes, pure capital good *j* is called internal if it is used and produced in only one sector, i.e. if:

$$(B_\lambda^j - A_\lambda^j) \neq 0 \quad \Rightarrow \quad \forall \iota = \alpha, \dots, \sigma, \iota \neq \lambda, \quad (B_\iota^j - A_\iota^j) = 0 \tag{7}$$

We replace the assumption that all fixed capital goods are internal by the weaker condition that they are ‘proper’:

Definition 7. For a given classification of processes, pure capital good *j* is called *proper* if for all feasible activity vectors there is a zero net product of good *j* in every sector:

$$\{y \geqq 0, (B-A)y \in D\} \quad \Rightarrow \quad \forall \iota = \alpha, \dots, \sigma, \quad (B_\iota^j - A_\iota^j) y_\iota = 0 \tag{8}$$

Internal capital goods are always proper: for any feasible activity vector, the net product of an internal capital good is zero at the economy level, and since it is used in only one sector, its net product is zero in every sector as well. Hence the criterion of Definition 7 is satisfied. Example 2 shows that the concept of proper capital goods is more general than that of internal capital goods.

Example 2. Let there be two machines *M* and *N* of various ages. The following data only refer to the part of the processes relative to the fixed capital goods:

Sector S_α:	$0 \rightarrow 1\ M_0$	Sector S_β:	$0 \rightarrow 1\ N_0$
	$1\ M_0 \rightarrow 1\ M_1$		$1\ N_0 \rightarrow 1\ M_0 + 1\ N_1$
	$1\ M_1 \rightarrow 1\ M_2$		$1\ M_0 + 1\ N_1 \rightarrow 1\ M_1 + 1\ N_2$
	$1\ M_2 \rightarrow 0$		$1\ M_1 + 1\ N_2 \rightarrow 0$

The *N*-machines are internal to sector S_β, which means they are proper. Moreover, a zero net product of machines (N_0, N_1, N_2) at the level of the economy requires that the activity levels of the four processes within sector S_β are equal. Then the net output of machines (M_0, M_1, M_2) in sector S_β is also zero. Hence, although the *M*-machines are not internal to any sector, they have the property mentioned in Definition 7 and, therefore, are proper.

6. A GENERAL MODEL

We now examine a more general model with both joint production and pure capital goods. First we distinguish two types of pure capital goods: the 'proper' ones, which have been defined in Definition 7, and the 'general' ones. The general capital goods are simply all the pure capital goods (if any) which, under the given classification, are not proper. We stress that the characterization of a pure capital good as 'proper' or 'general' depends upon the classification; pure capital good *j* may be 'proper' under classification *X*, but 'general' under classification *Y*. Given the distinction between the two kinds of pure capital goods, we can rearrange the rows of matrices A and B so as to put the rows referring to the general capital goods (A^G, B^G) and those referring to the proper capital goods (A^P, B^P) together:

$$A = \begin{bmatrix} A^F \\ A^K \end{bmatrix} = \begin{bmatrix} A^F \\ A^G \\ A^P \end{bmatrix}, \qquad B = \begin{bmatrix} B^F \\ B^K \end{bmatrix} = \begin{bmatrix} B^F \\ B^G \\ B^P \end{bmatrix} \tag{9}$$

Given a subset of methods *H*, let k_H be the number of pure capital goods used and/or produced by the processes in *H*. These capital goods may not be independent, in the sense that the net production of some pure capital goods is automatically zero when it is the case for the others. It can be relevant to know the number of *independent* pure capital goods in set *H*:

Definition 8. Given a set of methods *H*, the number k_H^* of *independent pure capital goods* is defined as:

$$k_H^* = rk\left[B_H^K - A_H^K\right] \tag{10}$$

Likewise we can define the number of *independent* general capital goods in set *H*:

Definition 9. Given a set of methods *H* and a classification, the number g_H^* of *independent general capital goods* is defined as:

$$g_H^* = rk\left[B_H^G - A_H^G\right] \tag{11}$$

We can now introduce the idea of a sectoral economy by combining the previous notions[2]:

2 This definition of a ‘sectoral economy’ differs from the one given in Bidard & Erreygers (1998b, p. 438).

Definition 10. Let there be a strictly viable economy with f final goods and k pure capital goods. It is called *sectoral* if there exists a classification such that, for any minimal set of processes H and any strictly feasible activity vector y_H, the number of operated sectors $s(y_H)$ is equal to the sum of the number of final goods f and the number of independent general capital goods g_H^*:

$$\{y_H \geqq 0, \quad (B_H - A_H)y_H \in ri(D)\} \quad \Rightarrow \quad s(y_H) = f + g_H^* \tag{12}$$

Consider a strictly viable single-product economy without pure capital goods (g_H^* = 0). Let the industries be sectors. This is a sectoral economy, because any strictly feasible and minimal activity vector is composed of exactly one process from each sector. It also has the super-adjustment property. The coexistence of the two properties is not a coincidence and we will demonstrate that they are intimately connected. First we prove two lemmas.

Lemma 1. Let H be a minimal set of methods. Then the following implications hold:

$$(B_H - A_H)z_H \in D \quad \Rightarrow \quad z_H \geqq 0 \tag{13}$$

$$(B_H - A_H)z_H = 0 \quad \Rightarrow \quad z_H = 0 \tag{14}$$

Proof. Assume that $(B_H - A_H)z_H \in D$, with z_H having some negative component. Since H is strictly viable there exists some $\bar{y}_H > 0$ such that $(B_H - A_H)\bar{y}_H \in ri(D)$. For any positive scalar α, we have $(B_H - A_H)(\bar{y}_H + \alpha z_H) \in ri(D)$. However there exists a positive scalar α such that the activity vector $\bar{y}_H + \alpha z_H$ is semipositive with some zero component, which is contradictory with the minimality of H. Therefore implication (13) holds. In particular, given that $0 \in D$, if we have $(B_H - A_H)z_H = 0$, then both $z_H \geqq 0$ and $-z_H \geqq 0$, hence $z_H = 0$. ■

Lemma 2. Let H be a minimal set of methods. Then H has the adjustment property if and only if the number h of processes in H is equal to the number of final goods plus the number of independent pure capital goods, i.e. if and only if $h = f + k_H^*$.

Proof. If $k_H^* < k_H$, let $\left[\bar{B}_H^K - \bar{A}_H^K\right]$ be a matrix composed of k_H^* independent rows of $\left[B_H^K - A_H^K\right]$, and then define matrix $\left[\bar{B}_H - \bar{A}_H\right]$ as:

$$\left[\bar{B}_H - \bar{A}_H\right] = \begin{bmatrix} B_H^F - A_H^F \\ \bar{B}_H^K - \bar{A}_H^K \end{bmatrix} \tag{15}$$

(If $k_H^* = k_H$, we proceed with the original matrix.) H has the adjustment property if for each $\bar{d} \in \bar{D}$, where $\bar{D}$ represents the demand set after elimination of the dependent pure capital goods, there exists a nonnegative activity vector $\bar{y}_H$ such that $\left[\bar{B}_H - \bar{A}_H\right]\bar{y}_H = \bar{d}$.

Suppose first that the number of processes of H is equal to $f + k_H^*$. Then $\left[\bar{B}_H - \bar{A}_H\right]$ is a square matrix which, according to implication (14), has full rank. Therefore the matrix is surjective, which implies that equation $\left[\bar{B}_H - \bar{A}_H\right]\bar{y}_H = \bar{d}$ has a solution $\bar{y}_H$ for any $\bar{d} \in \bar{D}$. According to implication (13), vector $\bar{y}_H$ is nonnegative. In other words, H has the adjustment property.

Conversely, suppose that H has the adjustment property. Let us rearrange the columns of $\left[\bar{B}_H - \bar{A}_H\right]$ in such a way that:

$$\left[\bar{B}_H - \bar{A}_H\right] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{16}$$

where C_{22} is an invertible $(k_H^* \times k_H^*)$-matrix (this is always possible because the k_H^* last rows of the matrix are independent). Then we can write:

$$\left[\bar{B}_H - \bar{A}_H\right]\bar{y}_H = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = \begin{bmatrix} \bar{d}^F \\ 0 \end{bmatrix} = \bar{d} \tag{17}$$

It follows that $\bar{y}_2 = -\left(C_{22}\right)^{-1} C_{21}\bar{y}_1$, and therefore that:

$$\left[C_{11} - C_{12}\left(C_{22}\right)^{-1} C_{21}\right]\bar{y}_1 = \bar{d}^F \tag{18}$$

By the adjustment and minimality properties, for any $\bar{d}^F \in R_+^f$ there is a corresponding $\bar{y}_1 > 0$. This is only possible if the number of components of vector $\bar{y}_1$ is at least equal to *f*. If it were greater than *f*, however, it would be possible to satisfy demand with less than the full amount of processes, which would imply that *H* is not minimal, contradicting the assumption from which we started. The conclusion is that *H* consists of exactly $f + k_H^*$ processes. ■

We now establish the main results.

Theorem 1. A sectoral economy has the super-adjustment property.

Proof. Let *H* be a minimal set of processes. There exists an activity vector *y* such that $(B - A)y \in ri(D)$ and *H* = supp(*y*). Moreover, let $S_\alpha, S_\beta, \ldots, S_\sigma$ be a sector-classification such that condition (12) for sectoral economy holds. Then let us consider the $(f + k)\times\sigma$ matrix *E* defined as follows:

$$E = \begin{bmatrix} E^F \\ E^K \end{bmatrix} = \begin{bmatrix} E^F \\ E^G \\ E^P \end{bmatrix} \equiv \begin{bmatrix} (B_\alpha^F - A_\alpha^F)y_\alpha & (B_\beta^F - A_\beta^F)y_\beta & \cdots & (B_\sigma^F - A_\sigma^F)y_\sigma \\ (B_\alpha^G - A_\alpha^G)y_\alpha & (B_\beta^G - A_\beta^G)y_\beta & \cdots & (B_\sigma^G - A_\sigma^G)y_\sigma \\ (B_\alpha^P - A_\alpha^P)y_\alpha & (B_\beta^P - A_\beta^P)y_\beta & \cdots & (B_\sigma^P - A_\sigma^P)y_\sigma \end{bmatrix} \tag{19}$$

The σ columns of *E* represent the net outputs of the *f* final goods, the *g* general capital goods, and the *p* proper capital goods in the sectors $S_\alpha, S_\beta, \ldots, S_\sigma$. By definition of proper capital goods, we have $E^P = 0$. If the number of operated sectors *s*(*y*) = $\hat{\sigma}$ is smaller than the total number of sectors σ, let us rearrange the sectors such that the first $\hat{\sigma}$ sectors are the operated ones, and the last $(\sigma - \hat{\sigma})$ ones the inactive ones. In matrix *E* the columns corresponding to the inactive sectors are all equal to zero. After deleting the *p* zero rows (associated to the proper capital goods) and the $(\sigma - \hat{\sigma})$ zero columns (associated to the inactive sectors), we obtain the following reduced $(f + g_H) \times \hat{\sigma}$ matrix:

$$\hat{E} = \begin{bmatrix} \hat{E}^F \\ \hat{E}^G \end{bmatrix} \tag{20}$$

We will treat the integrated sectors represented in matrix $\hat{E}$ as separate processes of what we call the 'reduced economy'; the set of these processes will be denoted as $\Omega = \{\alpha, \beta, \ldots, \hat{\sigma}\}$.

There is a straightforward relationship between the reduced and the original economies. Let $\hat{v} \geqq 0$ be the activity vector of the reduced economy $\hat{E}$; it leads to a net product equal to $\begin{bmatrix} \hat{E}\hat{v} \\ 0 \end{bmatrix}$. The Hadamard product $z = \hat{v} * y \geqq 0$, where $z_\iota = \hat{v}_\iota y_\iota$ (all activity levels within sector S_ι are multiplied by the constant $\hat{v}_\iota$), is an activity vector which, if applied to the original economy, yields the same net product $(B - A)z$ as the one obtained by applying activity vector $\hat{v}$ to the reduced economy $\hat{E}$. In particular, the net product obtained by applying activity vector $\hat{u}$ (a vector composed of all ones) to the reduced economy $\hat{E}$ is equal to the net product $(B - A)y$.

If Ω were not minimal, there would exist a vector $\hat{v} \geq 0$ with at least one zero component and an associated net product vector which belongs to *ri*(*D*). Suppose, without loss of generality, that $\hat{v}_{\hat{\sigma}} = 0$; then the corresponding activity vector *z* has zeroes for the block $\hat{v}_{\hat{\sigma}} y_{\hat{\sigma}}$, which means that *H* is not minimal, contrary to the assumption. Therefore Ω is minimal.

Next we show that the number $\hat{\sigma}$ of processes of the reduced economy is equal to the sum of the number *f* of final goods and the number k_{Ω}^{*} of independent pure capital goods of the reduced economy. According to implication (14) the columns of $\hat{E}$ are independent, hence their number $\hat{\sigma}$ is equal to $rk(\hat{E})$. On the other hand, the rank of $\hat{E}$ cannot exceed $rk(\hat{E}^{F})+rk(\hat{E}^{G})$; since obviously $rk(\hat{E}^{F})\leq f$, we obtain:

$$\hat{\sigma}=rk(\hat{E})\leq rk(\hat{E}^{F})+rk(\hat{E}^{G})\leq f+rk(\hat{E}^{G}) \tag{21}$$

By definition, we have $k_{\Omega}^{*}=rk(\hat{E}^{G})$ and $g_{H}^{*}=rk(B_{H}^{G}-A_{H}^{G})$. Because $rk(\hat{E}^{G})=rk(E^{G})$ and the columns of E^{G} are linear combinations of those of $(B_{H}^{G}-A_{H}^{G})$, we can moreover derive:

$$f+rk(\hat{E}^{G})=f+k_{\Omega}^{*}\leq f+rk(B_{H}^{G}-A_{H}^{G})=f+g_{H}^{*} \tag{22}$$

Given that *H* is minimal and the economy sectoral, it follows from Definition 10 that *s*(*y*) = $\hat{\sigma}$ = $f+g_{H}^{*}$. Therefore, all inequalities in (21) and (22) are equalities, and in particular $\hat{\sigma}=f+k_{\Omega}^{*}$. Now Lemma 2 can be applied to the set of processes Ω: it is minimal and its number $\hat{\sigma}$ of processes is equal to that of final goods plus that of independent pure capital goods; therefore Ω has the adjustment property. But then so does *H*. Hence we have proved that every minimal set of processes has the adjustment property, which means that the economy has the super-adjustment property. ■

Conversely:

Theorem 2. A strictly viable economy which has the super-adjustment property is a sectoral economy.

Proof. Consider the classification which treats every process as a separate sector. Under this classification, all pure capital goods are general, which means that for any set of processes H $k_H^* = g_H^*$. Lemma 2 states that every minimal set H which has the adjustment property has $f + k_H^*$ processes. Hence $f + g_H^*$ sectors are operated, and the economy is sectoral. ■

7. NON-SUBSTITUTION RESULTS

Up to now, we have only considered whether demand can be met, not the mechanism by which a subset of methods is selected within the initial set. In a market economy, competition determines the choice of methods. In a long-run equilibrium, only those methods which yield the given profit rate r can be operated and no available method pays extra-profits. In formal terms, for a given demand vector d, the activity levels y and the prices p (prices are measured in terms of the wage, i.e. w = 1) satisfy relations:

$$\left.\begin{array}{ll} By = Ay + d & \\ pB \leqq (1+r)pA + l & [y] \\ y \geq 0, \quad p \geq 0 & \end{array}\right\} \qquad (23)$$

This is a Lippi (1979) model. Theorems 3 and 4 are non-substitution results, respectively with and without pure capital goods. Theorem 3 shows that a dominant technique exists and does not depend on final demand. Theorem 4 sharpens this result for the case in which there are no pure capital goods. (The theorem specifies ‘positive’ final demand in order to avoid the complications occurring with reducible systems; moreover, in case of switching, uniqueness holds for the price vector, not for the technique itself. These details are inessential.)

Theorem 3 generalizes the results obtained in fixed capital theory (Stiglitz, 1970; Salvadori, 1988).

Theorem 3. Let $(A, l) \rightarrow B$ be a sectoral economy with pure capital goods. Assume that:

- the profit rate r is given and nonnegative;
- the economy is strictly r-viable, i.e. there exists a semipositive vector y such that $\left[B-(1+r)A\right]y \in ri(D)$;
- labour is necessary for the production of a surplus, i.e.:

$$\{y \geqq 0, \quad (B-A)y \in D-\{0\}\} \quad \Rightarrow \quad ly > 0 \tag{24}$$

- the pure capital goods can be disposed of freely.

Then there exists at least one competitive technique and an associated price vector p_0 which are independent of final demand. The prices of the final goods are positive, those of the capital goods nonnegative.

Proof. Let us temporarily assume that *all* goods can be freely disposed of. Then for any $d_0 \in D$ the assumptions of Lippi's (1979) theorem concerning general joint production are met, and hence there exists a solution (y_0, p_0) to the system:

$$\left.\begin{array}{ll} By_0 \geqq Ay_0 + d_0 & [p_0] \\ p_0 B \leqq (1+r)p_0 A + l & [y_0] \\ y_0 \geq 0, \quad p_0 \geq 0 & \end{array}\right\} \tag{25}$$

Given d_0 and y_0, let e be the excess supply vector, i.e. $e = (B-A)y_0 - d_0$. Good j is overproduced if $e^j > 0$. For each pure capital good j that is overproduced, we increase the activity level of the free disposal process of good j so as to eliminate the excess production. Let the new activity vector be $\bar{y}_0$. For each final good j that is overproduced, we increase the corresponding component of the demand vector by the excess amount. Let the new demand vector be $\bar{d}_0$. It is clear that we have found a vector $\bar{d}_0 \in D$ such that $(\bar{y}_0, p_0)$ is a solution to the system:

$$\left.\begin{array}{l} B\bar{y}_0 = A\bar{y}_0 + \bar{d}_0 \\ p_0 B \leqq (1+r) p_0 A + l \quad [\bar{y}_0] \\ \bar{y}_0 \geq 0, \quad p_0 \geq 0 \end{array}\right\} \tag{26}$$

Since no final good is disposed of, the temporary assumption about their free diposal may now be dropped. According to Theorem 1, the same set of processes that can produce demand $\bar{d}_0$ can produce any semipositive demand $d \in D$ by changing its activity levels from $\bar{y}_0$ to y (supp(y) $\subseteq$ supp($\bar{y}_0$)). Therefore for any of these y the profitability condition:

$$p_0 B \leqq (1+r) p_0 A + l \quad [y] \tag{27}$$

continues to hold with the same price vector p_0. Hence for any $d \in D$ the same set of processes constitutes a competitive technique. Moreover, for any $d \in D$ we have:

$$p_0 d = p_0 (B - A) y \geqq p_0 [B - (1+r) A] y = ly > 0 \tag{28}$$

By taking d equal to one unit of final good j, it turns out that the price of final good j is positive. ■

A more precise result can be proved for sectoral economies without pure capital goods. For this we need the following lemma:

Lemma 3. Let (A, l) $\rightarrow$ B be a strictly viable economy without pure capital goods. Assume that H is a minimal set of methods which has the adjustment property, and that its labour input vector is positive. Suppose moreover that the profit rate is nonnegative and that there exists a price vector $p_H \geq 0$ such that:

$$p_H\left[B_H-(1+r)A_H\right]=l_H>0 \tag{29}$$

Then matrix $\left[B_H-(1+r)A_H\right]$ is square and regular, and admits a semipositive inverse.

Proof. Since H is minimal and has the adjustment property, and $k_H^*=0$, Lemma 2 implies that matrices B_H and A_H are square. According to Lemma 1, matrix (B_H-A_H) is regular (implication (14)), and its inverse semipositive (implication (13)). The same holds for the transposed[3] matrix $(\tilde{B}_H-\tilde{A}_H)$. Let us treat the columns of $\tilde{B}_H$ and $\tilde{A}_H$ as the output and input vectors of (obviously fictitious) processes. Because system $(\tilde{B}_H-\tilde{A}_H)y=d$ admits a positive solution y for any $d\in ri(D)$, this set of processes is minimal and has the adjustment property. Next consider another set of fictitious processes, this time described by the columns of the output matrix $\tilde{B}_H$ and the input matrix $(1+r)\tilde{A}_H$. According to assumption (29) this set of processes is strictly viable. Because r is nonnegative, any strictly feasible activity vector y for this set of processes is also strictly feasible for the first set of fictitious processes. Since this first set is minimal and has the adjustment property, activity vector y is positive. This means that the second set of fictitious processes is also minimal and has the adjustment property. Therefore, matrix $\left[\tilde{B}_H-(1+r)\tilde{A}_H\right]$ admits a semipositive inverse. The same goes for matrix $\left[B_H-(1+r)A_H\right]$. ■

Theorem 4. Let $(A, l)\rightarrow B$ be a sectoral economy without pure capital goods. Assume that:

- the profit rate r is given and nonnegative;
- the economy is strictly r-viable;

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[3] We use the symbol $\tilde{X}$ to denote the transposition of matrix X.

- labour is necessary for the production of a surplus;
- final demand is strictly positive.

Then there exists a competitive technique which is uniquely defined (except when the rate of profit corresponds to a switch point) and independent of final demand. The associated price vector is positive and such that the price of any commodity is minimal among all strictly viable techniques with semipositive prices, i.e. the real wage is maximal.

Proof. The existence of a competitive technique *T*, which is independent of final demand and has an associated positive price vector p_T, follows from Theorem 3, assuming the number of pure capital goods equal to zero. Consider any strictly viable set of processes *H* associated with semipositive prices, i.e. for which there exists a price vector $p_H \geq 0$ such that $p_H\left[B_H - (1+r)A_H\right] = l_H$. There is no loss of generality in assuming that *H* is minimal (otherwise, extract a minimal subset from *H*). Let us temporarily suppose that the labour vector l_H is positive. All the conditions of Lemma 3 are now satisfied: this means that $\left[B_H - (1+r)A_H\right]^{-1}$ exists and is semipositive. By definition, a competitive price vector p_T is such that no process yields extra-profits, which implies that:

$$p_T\left[B_H - (1+r)A_H\right] \leqq l_H \tag{30}$$

Multiplication of both sides of relation (30) by matrix $\left[B_H - (1+r)A_H\right]^{-1}$ shows that:

$$p_T \leqq l_H\left[B_H - (1+r)A_H\right]^{-1} = p_H \tag{31}$$

Relationship (31) establishes the price-minimality property, under the additional hypothesis that the labour vector is positive. A continuity argument allows us to maintain this property under the hypothesis that labour is directly or indirectly necessary to produce a net output.

Only at switch points will two different techniques have exactly the same price vector. Therefore the competitive technique is also uniquely defined, except at these points. ■

8. CONCLUDING REMARKS

In this note we have extended the concept of ‘industry’, linked to the single-product hypothesis, to the more general concept of ‘sector’. We have shown that ‘sectoral economies’ have the super-adjustment property (and vice versa), which means that a viable subset of methods can produce *any* positive final demand vector by means of an adequate choice of its activity levels. To characterize an economy as ‘sectoral’, it suffices to find one sector-classification which satisfies the condition mentioned in Definition 10. One of the limits of our paper is that we remain silent on how such a classification could be found in an efficient way. It would be useful if one disposed of a simple and fast algorithm which either constructs a classification with the desired property, or proves that no such classification exists. For the latter, it is important to note that if ever one stumbles upon a strictly viable set composed of fewer processes than the sum of the number of final goods and the number of independent pure capital goods associated to that set, it is certain that the economy cannot be sectoral.

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