On the Leontief Inverse of a Beta Distributed Input Matrix

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Abstract

For a stochastic input matrix with beta distributed input coefficients lower bounds of the Leontief inverse are investigated with respect to densities and first moments.

1 Introduction

Input-output models are used to be based on a deterministic input matrix A. However, the observed values of inputs should be seen as realisations of random variables since inputs are affected by random effects, e. g. weather, prices, factor substitution, technical progress, product and process mix. In the literature, only a small number of contributions start from a stochastic input matrix and draw some conclusions concerning the stochastic Leontief inverse $L(A)$, for instance, on bounds of its expected values based on Jensen's inequality. However, it is a hard problem to derive the distributions of the elements of $L(A)$, even if the input coefficients are supposed to be normally distributed which is mainly assumed.

In this paper, the input coefficients are assumed to be beta distributed. The standard beta distribution has the domain of the input

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coefficients, the interval $[0, 1]$. It depends on two parameters which allow for a high degree of flexibility. In particular, great skewness is admitted. These properties seem to be adequate for modelling the distribution of the large number of very small input coefficients.

To derive the distributions of $L(A)$ seems to be out of reach. However, for good lower bounds of $L(A)$ the densities will be given for its diagonal elements and the first two moments for the other elements. The latter result is achieved by applying an approximation of the density of a product of beta random variables proposed by Fan (1991).

A first proxy of the parameters of the beta distributions may be computed even from a single input-output table.

2 Properties of the Beta Distribution

Density of the standard beta distribution $Be(r, s)$:

$$f_X(x) = \frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1}\mathbf{1}_{[0,1]}(x) \quad \text{for } r, s > 0,$$

where $B(r, s)$ denotes the beta function ($B(r,s) = \frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r+s)}$).

$$X \sim Be(r, s) \Longrightarrow 1 - X \sim Be(s, r)$$

Moments:

$$E(X) = \mu_X = \frac{r}{r+s}, \quad E(X^2) = \frac{r(r+1)}{(r+s)(r+s+1)} = \frac{\mu_X}{1 + s/(r+1)},$$
$$Var(X) = \sigma_X^2 = \frac{rs}{(r+s)^2(r+s+1)} = \frac{\mu_X\,(1-\mu_X)}{r+s+1}.$$

If $r, s > 1$, then $Var(X) < 1/12$ and f_X is unimodal with mode

$$m = \frac{r-1}{r+s-2}.$$

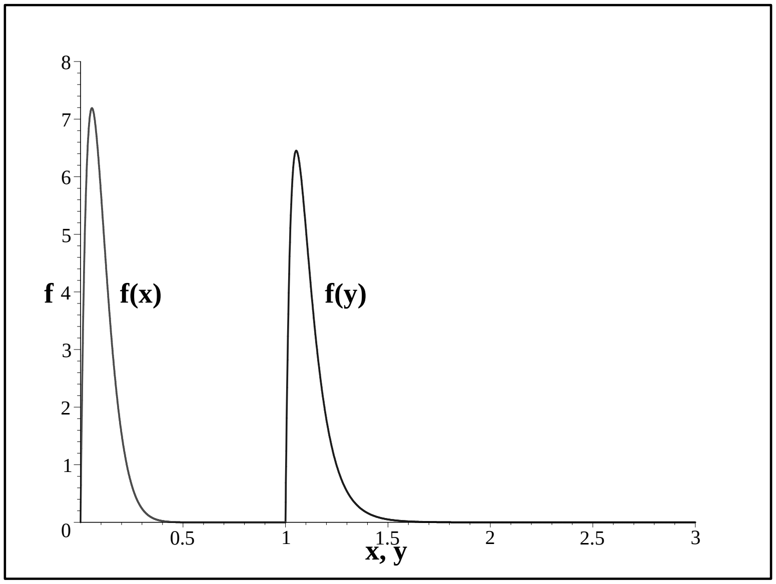

Figure 1: $r = 2, s = 18 \Rightarrow \mu_X = 0.1$

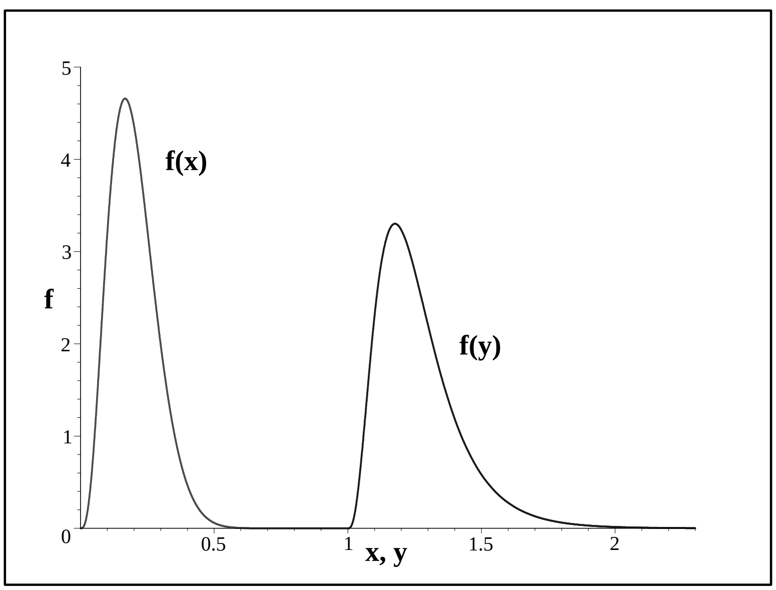

Figure 2: $r = 4, s = 16 \Rightarrow \mu_X = 0.2$

Distribution of $Y = (1-X)^{-1}$

$$\begin{aligned}
f_Y(y) &= \frac{1}{y^2}\frac{1}{B(r,s)}\left(1-\frac{1}{y}\right)^{r-1}\left(\frac{1}{y}\right)^{s-1}\mathbf{1}_{[1,\infty)}(y)\\
E(Y) &= \frac{r+s-1}{s-1} = 1+\frac{r}{s-1}\\
Var(Y) &= \frac{r\,(r+s-1)}{(s-1)^2(s-2)} = \mu_Y\,(\mu_Y-1)\frac{1}{s-2},\\
E(Y^2) &= \frac{(r+s-1)(r+s-2)}{(s-1)(s-2)} = \left(1+\frac{r}{s-1}\right)\left(1+\frac{r}{s-2}\right),
\end{aligned}$$

provided that $s > 2$, which implies $Var(X) < 1/16$. In the sequel, for the beta parameter s it will be assumed $s > 2$, whenever $Var(Y)$ or $E(Y^2)$ is mentioned.

Taylor-approximation of moments for $Y = (1-X)^{-1}$

(second order approximation at μ_X)

$$\begin{aligned}
E(Y) &\approx \frac{1}{1-\mu_X}\left(1+\frac{r}{s\,(r+s+1)}\right) \leq \frac{1}{1-\mu_X}\left(1+\frac{\mu_X}{s}\right)\\
Var(Y) &\approx \sigma_X^2\left(1+\frac{r}{s}\right)^4 \geq \frac{r}{s^2}\,(1+\mu_X)\,.
\end{aligned}$$

The figures show two beta densities X with corresponding transformed distributions of $Y = 1/(1-X)$ which will be relevant for the densities of the diagonal coefficients of the Leontief inverse. Remarkable is the skewness of f_X which seems adequate for the great number of very small input coefficients. A normal density does not seem to fit to this situation, even if truncated, because of its symmetry.

3 Approximation and Moments of the Leontief Inverse

If all the elements A_{ij} of the input matrix A have beta distributions, it seems impossible to determine the densities of the elements L_{ij} of the

Leontief inverse $L(A)$, since they depend on a ratio of two determinants which are sums of products of the A_{ij}.

Therefore, an approximation of $L(A)$ for a deterministic matrix A is regarded which will allow to approximate the distributions of its diagonal elements and the moments of the other elements after returning to random input coefficients.

The approximation by minors (cf. Kogelschatz, 1978), which gives good lower bounds for L, reads in its simplest form

$$\tilde{l}_{ii} = \frac{1}{1-a_{ii}} \ \forall\, i \quad \text{and} \quad \tilde{l}_{ij} = \frac{a_{ij}}{(1-a_{ii})(1-a_{jj})} \ \forall\, i,j \text{ with } j \neq i.$$

In the following, this approximation is studied for random variables $A_{ij} \sim Be(r_i, s_j)$.

For the diagonal elements of L the densities are given as for $Y = 1/(1-X)$ above:

$$f_{\tilde{L}_{ii}}(\tilde{l}_{ii}) = \frac{1}{\tilde{l}_{ii}^2} \frac{1}{B(r_{ii}, s_{ii})} \left(1 - \frac{1}{\tilde{l}_{ii}}\right)^{r_{ii}-1} \left(\frac{1}{\tilde{l}_{ii}}\right)^{s_{ii}-1} \mathbf{1}_{[1,\infty)}(\tilde{l}_{ii}) \quad \forall\, i.$$

$$\text{Furthermore,} \quad E(\tilde{L}_{ii}) = 1 + \frac{r_{ii}}{s_{ii}-1} > 1 + \frac{r_{ii}}{s_{ii}} = \frac{1}{1-E(A_{ii})}$$
$$\text{and} \quad Var(\tilde{L}_{ii}) = E(\tilde{L}_{ii}) \frac{E(\tilde{L}_{ii})-1}{s_{ii}-2} \geq Var(A_{ii}) \left(1 + \frac{r_{ii}}{s_{ii}}\right)^4.$$

For the off-diagonal elements of L the situation is complicated. The distribution of $\tilde{L}_{ij}$ depends on a ratio and a product of beta random variables. In the literature, several very complicated formulae can be found, mainly for special cases of products (Johnson, Kotz, Balakrishnan (1995, p. 256f)). A very good approximation of a product of beta random variables, which usually is not beta distributed, was suggested by Fan (1991, p. 4045). By construction, it ensures exact first two moments and, furthermore, it performs very well in approximating higher moments as his computations show.

Fan's approximation theorem

If $X_i \sim Be(r_i, s_i), X_i$ are independent random variables and $Z = \prod_{i=1}^{k} X_i$, then Z has an approximate $Be(R, S)$ distribution with true first two moments, where

$$R := \frac{U\,(U-T)}{T-U^2}, \qquad S := \frac{(1-U)(U-T)}{T-U^2}$$

$$\text{and} \quad U := \prod_{i=1}^{k} \frac{r_i}{r_i+s_i}, \qquad T := \prod_{i=1}^{k} \frac{r_i}{r_i+s_i} \cdot \frac{r_i+1}{r_i+s_i+1}$$

By the way, in the book of Johnson, Kotz, Balakrishnan (1995, p. 262), where Fan's result is reported, the formula for p has to be multiplied by $(S-T)$, where p and S correspond to R and U here. The following interpretations result from the independence assumption:

$$U = \prod_{i=1}^{k} E(X_i) = E(Z), \qquad T = \prod_{i=1}^{k} E(X_i^2) = E(Z^2),$$

$$\text{hence} \quad T-U^2 = Var(Z).$$

Obviously, $T = UV$ where $V = \prod_{i=1}^{k}(r_i+1)/(r_i+s_i+1)$.

Fan's method will now be applied to the approximation by minors with random input coefficients A_{ij} and $\tilde{L}_{ij}$, where $A_{ij} \sim Be(r_{ij}, s_{ij})$ with $r_{ij}, s_{ij} > 1$. In order to get some information about the distribution of the off-diagonal elements of $\tilde{L}_{ij}$, the denominator $(1-A_{ii})(1-A_{jj}) =: Z_{ij}$ is considered first. It seems realistic to assume the diagonal input coefficients to be independent random variables. As Z_{ij} is a product of $(1-A_{ii}) \sim Be(s_{ii}, r_{ii})$ and $(1-A_{jj}) \sim Be(s_{jj}, r_{jj})$ it can be approximated by $\tilde{Z}_{ij} \sim Be(S_{ij}, R_{ij})$ using the notation from above with additional indices ($i \neq j$ in the following)

$$S_{ij} = \frac{U_{ij}\,(U_{ij}-T_{ij})}{T_{ij}-U_{ij}^2}, \qquad R_{ij} = \frac{(1-U_{ij})\,(U_{ij}-T_{ij})}{T_{ij}-U_{ij}^2}$$

with

$$U_{ij} = \frac{s_{ii}}{r_{ii}+s_{ii}} \cdot \frac{s_{jj}}{r_{jj}+s_{jj}} \quad \text{and} \quad T_{ij} = U_{ij}\frac{s_{ii}+1}{r_{ii}+s_{ii}+1} \cdot \frac{s_{jj}+1}{r_{jj}+s_{jj}+1}$$

In the next step, the distribution of $Y_{ij} := 1/\tilde{Z}_{ij} = 1/(1-\tilde{Z}'_{ij}) = U_{ij}V_{ij} = E(Z_{ij}^2)$, where $\tilde{Z}'_{ij} := 1-\tilde{Z}_{ij}$ with $\tilde{Z}'_{ij} \sim Be(R_{ij}, S_{ij})$, is given as that of $Y := 1/(1-X)$ above

$$f_{Y_{ij}}(y_{ij}) = \frac{1}{y_{ij}^2}\frac{1}{B(R_{ij},S_{ij})}\left(1-\frac{1}{y_{ij}}\right)^{R_{ij}-1}\left(\frac{1}{y_{ij}}\right)^{S_{ij}-1}\mathbf{1}_{[1,\infty)}(y_{ij}).$$

The expected value of $1/\tilde{Z}_{ij}$ is obtained as

$$E\left(\frac{1}{\tilde{Z}_{ij}}\right) = 1 + \frac{R_{ij}}{S_{ij}-1} = \ldots = \frac{1+U_{ij}-2V_{ij}}{2U_{ij}-V_{ij}(1+U_{ij})} > \frac{1}{E(Z_{ij})},$$

where $U_{ij} = E(Z_{ij})$ and $V_{ij} \geq U_{ij}$ for r_{kk}, s_{kk} large. In this case, $E(1/\tilde{Z}_{ij}) \geq (1-U_{ij})/(U_{ij}-U_{ij}^2) = 1/U_{ij} = 1/E(Z_{ij})$.

In the last step, the nominator A_{ij} of $\tilde{L}_{ij} = A_{ij}/Z_{ij}$ has to be taken into account. The distribution of $\tilde{L}_{ij}$ which is a product of a beta and a transformed approximate beta random variable will not be investigated here. Evidently, no beta density can result since the domain of $\tilde{L}_{ij}$ is $\mathbb{R}_+$. Nevertheless, the density of $\tilde{L}_{ij}$, which is mainly concentrated on $[0,1]$, might be approximated by a suitable (standard) beta distribution. For empirical input matrices A the off-diagonal elements of $L(A)$ are smaller than 1.

In the sequel, expected value and variance of $\tilde{L}_{ij}$ will be derived. As before, Z_{ij} is substituted by $\tilde{Z}_{ij}$. Additionally, it is assumed that A_{ij} and A_{ii}, A_{jj} are independent which may be doubted. Then, the expected value of $\tilde{L}_{ij}$ turns out to be

$$\begin{aligned} E(\tilde{L}_{ij}) &= E(A_{ij})\, E\left(\frac{1}{\tilde{Z}_{ij}}\right) = E(A_{ij})\left(1+\frac{R_{ij}}{S_{ij}-1}\right) \\ &> E(A_{ij})\frac{1}{E(Z_{ij})} = E(A_{ij})\frac{(r_{ii}+s_{ii})(r_{jj}+s_{jj})}{s_{ii}s_{jj}} \\ &= E(A_{ij})\left(1+\frac{r_{ii}}{s_{ii}}\right)\left(1+\frac{r_{jj}}{s_{jj}}\right). \end{aligned}$$

Obviously, $\tilde{L}_{ij}$ it has not only a greater mean but also greater variance than A_{ij}. This is shown by the usual computation of a variance

$$\begin{aligned} Var(\tilde{L}_{ij}) &= E(\tilde{L}_{ij}^2) - \left(E(\tilde{L}_{ij})\right)^2. \\ E(\tilde{L}_{ij}^2) &= E\left(A_{ij}\cdot\frac{1}{\tilde{Z}_{ij}}\right)^2 = E(A_{ij}^2)\, E\left(\frac{1}{\tilde{Z}_{ij}^2}\right) \quad \text{by independence} \\ &= E(A_{ij}^2)\left(1+\frac{R_{ij}}{S_{ij}-1}\right)\left(1+\frac{R_{ij}}{S_{ij}-2}\right) \\ &\geq E(A_{ij}^2)\left(1+\frac{R_{ij}}{S_{ij}-1}\right)^2. \end{aligned}$$

$$\begin{aligned} Var(\tilde{L}_{ij}) &\geq E(A_{ij}^2)\left(1+\frac{R_{ij}}{S_{ij}-1}\right)^2 - (E(A_{ij}))^2\left(1+\frac{R_{ij}}{S_{ij}-1}\right)^2 \\ &= Var(A_{ij})\left(1+\frac{R_{ij}}{S_{ij}-1}\right)^2 \\ &> Var(A_{ij})\left[\left(1+\frac{r_{ii}}{s_{ii}}\right)\left(1+\frac{r_{jj}}{s_{jj}}\right)\right]^2 \end{aligned}$$

which follows from the lower bound of $E(\tilde{L}_{ij})$ mentioned above.

Both lower bounds for $E(\tilde{L}_{ij})$ and $Var(\tilde{L}_{ij})$, respectively, differ from the corresponding moments of A_{ij} by augmenting factors which depend only on the ratio of beta parameters r and s of the corresponding diagonal elements. The factor for $Var(\tilde{L}_{ij})$ is just obtained by squaring the factor for $E(\tilde{L}_{ij})$. Also the 3σ-region will be extended by the latter factor. For the values of the example in Figure 2, where $\mu = 0.2$ is realistic for average diagonal elements A_{ii} of input matrices with 10 to 15 sectors, one would obtain $E(\tilde{L}_{ij}) > 1.56 \cdot E(A_{ij})$ and $Var(\tilde{L}_{ij}) > 2.44 \cdot Var(A_{ij})$.

4 Proxies of Beta Parameters

After these theoretical considerations the question arises how to estimate the parameters r and s of the beta distributions within this model for the input coefficients. Estimation from a time series is doubtful since coefficients are changing over time for several reasons. Only input-output tables based on fixed prices should be used. For estimation procedures of the parameters r, s see Johnson/Kotz/Balakrishnan (1995, p. 221–238) and with special regard of skewness Moitra (1990).

Even from a single input matrix a first proxy for r, s may be given. A practical proposal made by Bamberg (1976, p. 16) for the moments of an apriori distribution in Bayesian estimation can be applied here. He suggests to ask for the mode m as a proxy of μ and for the greatest possible deviation d from μ. According to the 3σ-rule, which says that 99 % of the probability of a normal density lies in the 3σ-region and 89 % according to Chebychev's inequality for the least favorable distribution he suggests to take $d/3$ as a proxy of σ. For unimodal beta densities this probability will be close to that of the normal distribution.

For stochastic input coefficients one may modify this proposal as follows. Take the observed value a_{ij} a) as mode m_{ij} or b) as expected value of the distribution and also as deviation d in case b). It is assumed that the probability that A_{ij} exceeds $2a_{ij}$ may be neglected. With respect to the skewness (to the right) of adequate beta densities it might be preferred to take a larger region, say, up to $3a_{ij}$ in order to capture about 95 % of the probability. Thus, two equations are obtained for case b)

$$E(A_{ij}) = \frac{r_{ij}}{r_{ij} + s_{ij}} = a_{ij}$$
$$Var(A_{ij}) = \frac{a_{ij}\,(1 - a_{ij})}{r_{ij} + s_{ij} + 1} = \left(\frac{a_{ij}}{3}\right)^2$$

which may be solved for r_{ij} and s_{ij}.

In case a), which corresponds to the idea of maximum likelihood estimation, $a_{ij} = m_{ij}$ has to be transformed to $E(X)$ by

$$E(A_{ij}) = a_{ij} \frac{r_{ij}}{r_{ij} - 1} \frac{r_{ij} + s_{ij} - 2}{r_{ij} + s_{ij}} = \mu_{ij}$$

and in the equation for the variance a_{ij} has to be substituted by μ_{ij}, a third unknown. For input coefficients $Be(r, s)$ usually is skewed to the right so that $E(X) > m$. Usually, the smaller the input coefficients the greater the skewness and the greater μ_{ij}/m_{ij}. Since $\mu_{ij}/m_{ij} \leq 2$ for $r_{ij} \geq 2$ it is proposed to take $2(1 - a_{ij})$ as a proxy of this ratio in order to eliminate the third unknown.

5 Further Research

As a next step, the densities for the lower bounds of the off-diagonal elements of $L(A)$ may be approximated by suitable beta distributions the moments of which are given above.
An appealing extension of the framework would be to start with a multi-dimensional standard beta distribution (Dirichlet-distribution) for each sector since the input coefficients and the value added coefficient add up to one (columnwise). Thus, this restriction may be taken into account. In particular, this would be important for estimation procedures.

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