THE USE OF INTERVAL ARITHMETIC AS AN ALTERNATIVE METHOD TO EVALUATE UNCERTAINTY IN INPUT-OUTPUT MODELS

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ABSTRACT

Economic input-output (I-O) models are empirical realizations of general equilibrium economic model, which are based on the linear structure of inter-industry production linkages. In compact form, an I-O model can be written as $\mathbf{x} = \mathbf{A}\,\mathbf{x} + \mathbf{b}$, where $\mathbf{x}$ is a vector of sectorial outputs, $\mathbf{A}$ is the input-output technical coefficient matrix and $\mathbf{b}$ is the vector of final demands. From here: $(\mathbf{I}-\mathbf{A})\,\mathbf{x} = \mathbf{L}\mathbf{x} = \mathbf{b}$, where $\mathbf{L}$ is called the Leontief matrix. If final demand is known, then the amount of the goods needed to satisfy this demand can be found by solving the linear system $\mathbf{L}\mathbf{x} = \mathbf{b}$.

However, the technical coefficients of the Leontief matrix are not known but must be estimated and therefore are subject to some level of uncertainty. Some sources of uncertainties in I-O models are: Source data, assumptions inherent in I-O analysis such as linearity or proportionality, allocation and aggregation.

We can evaluate the effects of variation in both $\mathbf{L}$ and $\mathbf{b}$ on the solution $\mathbf{x}$, that is, if $\Delta\mathbf{L}$ and $\Delta\mathbf{b}$ are perturbation on $\mathbf{L}$ and $\mathbf{b}$, we can evaluate what are the effects on $\mathbf{x}$.

This paper shows the use of Interval Arithmetic as an alternative method to calculate how technical coefficients uncertainties are propagated, that is how system outputs vary as input parameters vary.
If we define an interval matrix $\mathbf{L}^I$ bounding $\mathbf{L}$ and an interval vector $\mathbf{b}^I$ bounding $\mathbf{b}$, then we need to solve the interval system: $\mathbf{L}^I\,\mathbf{x}^I = \mathbf{b}^I$. However, the solution of interval linear equations is a very different proposition from the solution of ordinary linear equations. In general, the solution set is complicated in shape and requires solving 2^n linear systems.

Interval Arithmetic can consider simultaneously variation of all the technical coefficients. We propose the use of efficient algorithms, which are able to provide very good *outer* and *inner* approximations to the solution set. An example related to an economic input-output model is presented. Strict bounds are obtained with only one linear system evaluation.

KEYWORDS: Input-Output Models, Sensitivity, Uncertainty and Interval Arithmetic

1. Introduction

Economic input-output (I-O) models are empirical realizations of general equilibrium economic models. An I-O is based on the linear structure of inter-industry production linkages and has been successfully applied in many countries for policy making (Beletskyy et al, 2001).

In compact form, an I-O model can be written as **x** = **A x** + **b**, where **x** is a vector of sectorial outputs, **A** is the input-output technical coefficient matrix and **b** is the vector of final demands. From here: (**I**-**A**) **x** = **Lx** = **b**, where **L** is called the Leontief matrix. If final demand is known, then the amount of the goods needed to satisfy this demand can be found by solving the linear system **Lx** = **b**.

However, the technical coefficients of the Leontief matrix are not known but must be estimated and therefore are subject to some level of uncertainty. Thus we need to evaluate how these uncertainties are propagated.

Recently Lenzen (2001) list some sources of uncertainties in I-O models: Source data, assumptions inherent in I-O analysis such as linearity or proportionality, allocation and aggregation. As an example, the author mentions that "the Australian Bureau of Statistics does not estimate uncertainties for I-O data, but keeps information on the standard error of source data items, which ranges mostly from 15 % to 30 %, with many being as low as 1 % to 2 % and the highest being 58 %".

Modeling uncertainty in input-output models can be based on two general approaches. The first is a probabilistic approach where probability distributions for all of the uncertainties are assumed. The second approach is called "Unknown but bounded" in which upper and lower limits on the uncertainties are assumed without a probability or possibility structure (Merryl et al, 1982).

For examining the effects of uncertain inputs, there are various analytic and computational techniques. These include (Granger et al, 1990):

- Methods for computing the effect of changes in inputs on model predictions, i.e. *sensitivity analysis*
- Methods for calculating the uncertainty in the model outputs induced by the uncertainties in its inputs, i.e., *uncertainty propagation*, and
- Methods for comparing the importance of the input uncertainties in terms of their relative contributions to uncertainty in the outputs, i.e., *uncertainty analysis*.

Techniques used in sensitivity and uncertainty analysis may include (Granger et al, 1990):

- Deterministic, one-at-time analysis of each factor holding all others constant at nominal value;
- Deterministic joint analysis, changing the value of more than one factor at a time;
- Parametric analysis, moving one or a few inputs across reasonably selected ranges such as from low to high values in order to examine the shape of the response;
- Probabilistic analysis, using probability density functions, Monte Carlo simulation, or other means to examine how much uncertainty in conclusions is attributable to which inputs.

A tentative condensed list of reasons why and instances where sensitivity analysis should be considered can be found in Saltelli (2000). Traditionally, all parameters are set to nominal values and each parameter is varied independently to determine its effect on the outcome. Those parameters that have significant effects are viewed as sensitive. In most cases sensitivity analysis does not deal with the possibility that several parameters varying simultaneously can cause significant variations in the output. Simultaneous variations of parameters model more accurately the real world situations.

To perform a sensitivity analysis on $\mathbf{Lx} = \mathbf{b}$, we can study the effects of variation in both $\mathbf{L}$ and $\mathbf{b}$ on the solution $\mathbf{x}$, that is, if $\Delta\mathbf{L}$ and $\Delta\mathbf{b}$ are perturbation on $\mathbf{L}$ and $\mathbf{b}$, we want to evaluate what are the effects on $\mathbf{x}$.

Using standard error analysis, Deif (1986) shows that $\Delta\mathbf{x} \approx \mathbf{L}^{-1}\Delta\mathbf{b} - \mathbf{L}^{-1}\Delta\mathbf{A}\mathbf{x}$. This expression is valid if perturbations are small enough, so that the first order perturbation is considered.

Another basic technique for carrying out sensitivity analysis of the I-O model is to evaluate multipliers. A multiplier for an output variable shows how this variable changes as a result of changes in input variable. For example (Jerrel, 1996), if:

$$h_i = \frac{\text{household income generated by industry i}}{\text{total outlays in industry i}}$$

then the wage income multiplier for the i-th industry is calculated by:

$$I_i = \frac{\left\lfloor (L^{-1})'h \right\rfloor i}{h_i}$$

Thus, the multipliers depend on the Leontief matrix. As the coefficients of the Leontief matrix have uncertainties, we need to evaluate how these uncertainties are propagated into the multipliers.

West (1986) has derived a formal expression for the probability of the multipliers assuming that the technical coefficients are independent and that they can be characterized as probability distributions with small variances. If probability distributions are assumed, then the West’s method will generate confidence intervals.

Other authors have used Monte Carlo method to perform this type of analysis (Lansen, 2001; West 1986; Raa et al, 1994)

Recently Beletskyy et al (2001) present a Leontief model and a tool for its sensitivity analysis, applied to the Ukrainian economy. The tool for the investigation of the model sensitivity analysis is a special program that generates changes in input parameter values and solves linear equation systems. Basically an interval coefficient of the technology matrix [parmin, parmax] is selected. Next, the algorithm forms (n+1) values of the coefficient: parmin, parmin+h, parmin+2h, ..., parmax, where h = (parmax-parmin)/n, and solves n linear system.

Instead of probabilities, we usually know only estimates and accuracy for the coefficients of the Leontief matrix, that is we only know intervals $[L_{ij} - \Delta L_{ij}, L_{ij} + \Delta L_{ij}]$.

This paper shows the use of Interval Arithmetic as an alternative method to calculate how system outputs vary as input parameters vary. It is shown that it is possible to perform sensitivity and uncertainty analysis by using Interval Arithmetic, assigning bounds to some or all the input parameters and observing the effect on the final interval outcome, that will contain all possible solutions due to the variations in input parameters. Strict bounds are obtained with only one linear system evaluation.

2. INTERVAL ARITHMETIC

2.1 Introduction

Interval arithmetic originates from the recognition that frequently there is uncertainty associated with the parameters used in a computation (Moore, 1979; Neumaier, 1990). This form of mathematics uses interval "numbers", which are actually an ordered pair of real numbers representing the lower and upper bound of the parameter range (Hansen, 1992). For example, if we know that a technological coefficient a_{ij} is between 12 and 15 %, the corresponding interval number would be written as follows: a_{ij} = [12,15] %.

Interval arithmetic is built upon a basic set of axioms. If we have two interval numbers T=[a,b] and W=[c,d] with a ≤ b and c ≤ d then (Hansen ,1992; Moore, 1979; Neumaier, 1990):

- $T + W = [a,b] + [c,d] = [a+c, b+d]$
- $T - W = [a,b] +(- [c,d]) = [a-d,b-c]$
- $T*W = [\min\{ac,ad,bc,bd\}, \max\{ac,ad,bc,bd\}]$
- $1/T = [1/b , 1/a], 0 \notin [a,b]$
- $T/W = [a,b] / [c,d] = [a,b]*[1/d,1/c], 0 \notin [c,d]$
- $kT = k*[a,b] = [a,b]*k = [k*a,k*b]$, k a real constant

Only some of the algebraic laws valid for real numbers remain valid for intervals. It is easy to show that interval addition and multiplication are associative as well commutative. However, the distributive law does not always hold for interval arithmetic (Moore, 1979). As an example: [0,1] (1-1) = [0,0], while [0,1] - [0,1] = [-1,1].

An important property referred to as subdistributivity does hold. It is given mathematically by the set inclusion relationship: $T(W + Z) \subseteq TW + TZ$. The failure of the distributive law often causes overestimation. It is also interesting to note that $T-T \neq 0$, and $T/T \neq 1$. In general, when a given variable occurs more than once in an interval computation, it is treated as a different variable in each occurrence. Thus T-T is the same as T-W with W equal to but independent of T and causes the widening of computed interval. This effect is called the "dependency problem" (Moore, 1979). However, there are expressions where dependence does not lead to overestimation. For example, T + T (Neumaier, 1990).

2.2 LINEAR SYSTEMS

Interval Arithmetic can also be applied to evaluate the effect of uncertain parameters in Linear Systems (Deif, 1986; Neumaier, 1990).
Let **L** be a real matrix with elements l_{ij} and let $\mathbf{L}^I$ be an interval matrix with interval elements $L_{i,j}$. We say "**L** is contained in $\mathbf{L}^I$ ($\mathbf{L} \subseteq \mathbf{L}^I$)" if and only if $l_{ij} \subseteq L_{ij}$, (Neumaier, 1990). Consider the real equation:

$$\mathbf{Lx} = \mathbf{b} \tag{1}$$

If we know an interval matrix $\mathbf{L}^I$ bounding **L** and an interval vector $\mathbf{b}^I$ bounding **b**, we can replace (1) by

$$\mathbf{L}^I\, \mathbf{x}^I = \mathbf{b}^I \tag{2}$$

The interval solution to (2) will contain the solution to (1).

However, the solution of interval linear equations is a very different proposition from the solution of ordinary linear equations. The solution set is not generally an interval vector.

Figure 1 illustrates the solutions set for the system (Hansen, 1992):

$$\begin{aligned} [2,3]\, x + [0,1]\, y &= [\ 0,120] \\ [1,2]\, x + [2,3]\, y &= [60,240] \end{aligned} \tag{3}$$

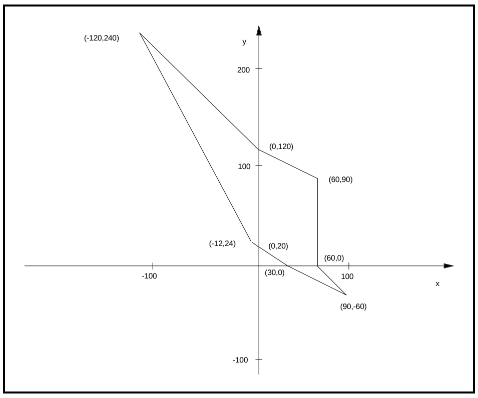

Figure 1: The solution set for equation (3) (Hansen, 1992)

The inner region is the exact solution set. In general, the solution set is complicated in shape and requires solving 2^n linear systems (Rossier, 1982). Kreinovich et al (1991) prove that the solution of the interval system is NP-hard.

For this reason, it is common practice to seek an interval vector $\mathbf{x}^I$ containing the solution set: the hull of the solution set (Alvarado et al, 1992). The hull of the solution set associated to (3) is the interval vector:

$$\mathbf{x}^I = [[-120,90], [-60,240]]^t$$

The hull contains, in addition to the entire solution set, many non-solutions.

Solving interval linear equations means obtaining the hull of the solution set. By analogy with the real cases, we try to solve (2) by Gaussian elimination or by the Gauss-Seidel iterative method (Burden et al, 1985). That means that we have only to change every real operation by an interval operation, using for example object-oriented techniques (Hyvönen et al, 1994) or calling special interval routines (Kearfott et al, 1994).

But ordinary forward and back substitutions result in considerable overestimation of the solution, due to the dependency problem and failure of the distributive law. Some ordering scheme may improve the solution (Alvarado et al, 1993). Simply replacing a real Gaussian elimination algorithm by an interval one cannot be recommended in practice (Hansen, 1992). However, an algorithm, which obtains excellent results, can be obtained by doing some extra work (Neumaier, 1990; Hansen, 1992).

In order to obtain the hull of the solution set, interval methods requires that the matrix $\mathbf{L}^I$ be an M-matrix. M-matrices are defined as:

> An n x n interval matrix $\mathbf{L}^I$ is an M-matrix if $L_{ij} \leq 0$ for all $i \leq j$ and $\mathbf{L}^I\mathbf{u} > 0$ for some positive vector $\mathbf{u} \in R^n$ (Ning et al, 1997).

If $\mathbf{L}^I$ is an M-Matrix and $\mathbf{b}^I \geq 0$, interval Gaussian elimination computes the hull. For example, let the interval equation system $\mathbf{L}^I\,\mathbf{x}^I = \mathbf{b}^I$ be (Ning et al, 1997):

$$\begin{pmatrix} [3.7,4.3] & [-1.5,-0.5] & [0,0] \\ [-1.5,-0.5] & [3.7,4.3] & [-1.5,-0.5] \\ [0,0] & [-1.5,-0.5] & [3.7,4.3] \end{pmatrix} \mathbf{x}^I = \begin{pmatrix} [0,14] \\ [0,9] \\ [0,3] \end{pmatrix} \tag{4}$$

Using interval Gaussian elimination, we obtain the box:

$$\mathbf{x}^I = \begin{pmatrix} [0,6.38] \\ [0,6.40] \\ [0,3.40] \end{pmatrix}$$

As the matrix is an interval M-matrix and $\mathbf{b}^I \geq 0$, this box is the smallest one containing the solution set.

Let $\mathbf{L}^I = [\underline{L}, \overline{L}]$. Ning et al (1997) show that:

1. If $\underline{L}$ and $\overline{L}$ are M-matrices then: $(\mathbf{L}^{-1})^I = [\overline{L}^{-1}, \underline{L}^{-1}]$
2. If $\underline{L}$ and $\overline{L}$ are M-matrices and $b^I \geq 0$ then $\mathbf{x}^I = [\overline{L}^{-1}\underline{b}, \underline{L}^{-1}\overline{b}]$

In the previous system (4):

$$\mathbf{L}^I = \begin{pmatrix} [3.7,4.3] & [-1.5,-0.5] & [0,0] \\ [-1.5,-0.5] & [3.7,4.3] & [-1.5,-0.5] \\ [0,0] & [-1.5,-0.5] & [3.7,4.3] \end{pmatrix}$$

$\underline{L}$ and $\overline{L}$ matrices are:

$$\underline{L} = \begin{pmatrix} 3.7 & -1.5 & 0 \\ -1.5 & 3.7 & -1.5 \\ 0 & -1.5 & 3.7 \end{pmatrix} \text{ and } \overline{L} = \begin{pmatrix} 4.3 & -0.5 & 0 \\ -0.5 & 4.3 & -0.5 \\ 0 & -0.5 & 4.3 \end{pmatrix}$$

with the following inverses:

$$\underline{L}^{-1} = \begin{pmatrix} 0.33644 & 0.16322 & 0.06617 \\ 0.16322 & 0.40261 & 0.16322 \\ 0.06617 & 0.16322 & 0.3364 \end{pmatrix} \text{ and } \overline{L}^{-1} = \begin{pmatrix} 0.23578 & 0.02779 & 0.00323 \\ 0.02779 & 0.23902 & 0.02779 \\ 0.00323 & 0.02779 & 0.235789 \end{pmatrix}$$

Then, $(\mathbf{L}^{-1})^I$ is given by:

$$(\mathbf{L}^{-1})^I = \begin{pmatrix} [0.23578,0.33644] & [0.02779,0.16322] & [0.00323,0.06617] \\ [0.02779,0.16322] & [0.23902,0.40261] & [0.02779,0.16322] \\ [0.00323,0.06617] & [0.02779,0.16322] & [0.235789,0.3364] \end{pmatrix}$$

Jerrel (1996) shows that Leontief matrix is an M-matrix and proposes an algorithm to obtain the inverse of $\mathbf{L}^I$ based on the use of an interval version of Gaussian elimination method.

Shary (1996) presents a different approach to find the hull of a linear interval system, based on an algorithm that combines high computational efficiency, good adaptability to various specific interval linear systems and high quality enclosures of the solution set.

The approach is based on the fact that the solution set of the interval system $\mathbf{L}^I\ \mathbf{x}^I = \mathbf{b}^I$ coincides with the solution set of the interval system $\mathbf{x}^I = \mathbf{C}^I\ \mathbf{x}^I + \mathbf{b}^I$, if $\mathbf{C}^I = \mathbf{I}^I - \mathbf{L}^I$. In order to avoid the dependence problem, the proposed approach is based on an iterative algorithm (sub-differential Newton method) that solves the original problem using one non-interval equation in the Euclidean space of double dimension R^{2n}. Details of the algorithm can be found in (Shary, 1996; Shary, 1997).

The idea in our paper is to use the approach proposed by Jerrel (1996), changing the interval Gaussian elimination with the Shary's method. For the interval linear system (4), the Shary's method also finds the hull of the solution but in one iteration.

2.3 Inner box

The hull of the solution set contains points that have nothing to do with the solution of the system $\mathbf{L}^I\ \mathbf{x}^I = \mathbf{b}^I$ for some $L \in \mathbf{L}^I$ and $b \in \mathbf{b}^I$, and due to this, such a problem statement may turn out unacceptable in many practical situations. However, this "box" gives information of minimum and maximum values independently.

We can avoid this problem by definining an internal box. In this case we can guarantee that any point selected within the box complies with the problem.

Therefore the problem we shall address is: Find an interval vector that is included in the solution set of the interval linear system. This problem is the identification problem for the interval algebraic system (Shary, 1997).

The algorithm proposed by Shary (1996) to obtain the hull of the solution can also be applied to the problem of inner interval estimation, using some concepts related to the identification problem for the interval algebraic system and an extension of the interval arithmetic.

An interval vector $\mathbf{x_a}$ is said to be an algebraic solution to the interval system of equations if substituting it into the system and execution of all interval arithmetic operations result in a valid equality.

For example, consider the following interval system of equation:

$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \tag{5}$$

The algebraic solution is $\mathbf{x_a} = \begin{pmatrix} \left[-\frac{1}{3}, \frac{1}{3}\right] \\ \left[-\frac{1}{3}, \frac{1}{3}\right] \end{pmatrix}$. Figure 2 shows the algebraic solution

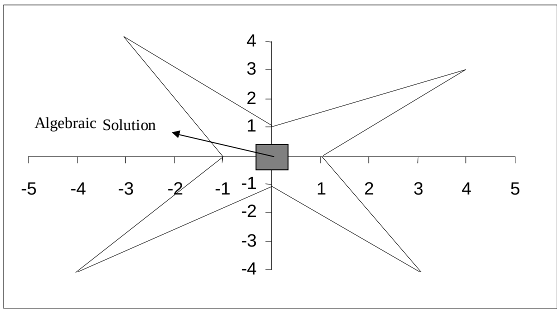

Figure 2: Algebraic solution for (5)

2.4 Extended Interval Arithmetic

Extended Interval Arithmetic (Kaucher, 1980; Gardenes et al, 1980)) is made up by adding improper interval $[a^-,a^+]$, $a^- \geq a^+$, to the set of proper interval $[a^-,a^+]$, $a^- \leq a^+$. Additionally the Dual operator is defined as: *Dual* [a,b] = [b,a].

Using the *Dual* operator, Shary (1997) proves that if $\mathbf{x_a}$ is an algebraic solution to the system $\mathbf{L}^I\ \mathbf{x}^I = Dual\ \mathbf{b}^I$, and all its components are improper, then dual $\mathbf{x_a}$ is a solution to the interval linear identification problem.

As an example, let us consider the interval linear system (5). The algebraic solution of the system:

$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} \mathbf{x}^I = Dual \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} = \begin{pmatrix} [2,-2] \\ [2,-2] \end{pmatrix}$$

is

$$\mathbf{x_a} = \begin{pmatrix} [1,-1] \\ [1,-1] \end{pmatrix}$$

From here:

$$Dual\ \mathbf{x_a} = \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}$$

provides a good inner approximation for the united solution set. Figure 3 shows the inner and outer boxes, along with the solution set.

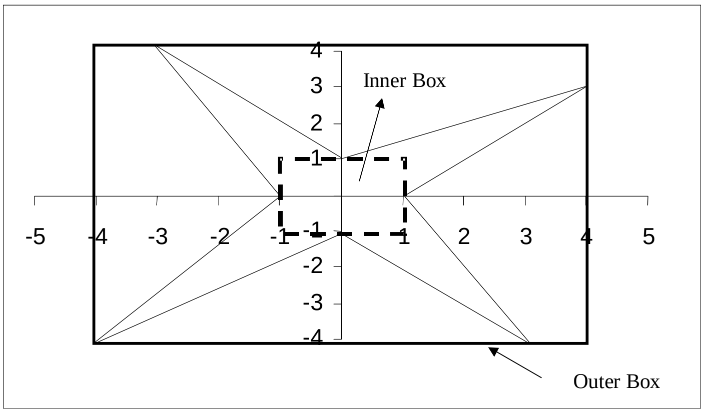

Figure 3: Solution set and its inner and outer approximation for the interval system (5)

The algebraic approach to the inner estimation has been advanced also by Kupriyanova (1995). Indeed she shows that the algebraic solution of interval linear algebraic systems $Dual\ \{\mathbf{A}^I\}\ \mathbf{x}^I = \mathbf{b}^I$ is the maximum inner box. However, in this case, the dual operator is applied to each element of the matrix while in the Shary's approach the dual operator is applied to the righ-hand side, therefore requiring less operations.

As an example, let us consider the following interval linear system (Kupriyanova, 1995):

$$\begin{pmatrix} [2,4] & [-1,1] \\ [-1,1] & [2,4] \end{pmatrix} \mathbf{x}^I = \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix} \qquad (6)$$

The algebraic solution to the system:

$$Dual\left\{\begin{pmatrix} [2,4] & [-1,1] \\ [-1,1] & [[2,4] \end{pmatrix}\right\}\mathbf{x}^{\mathrm{I}} = \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix}$$

provides the inner box. Using the *Dual* operator, we obtain:

$$\begin{pmatrix} [4,2] & [1,-1] \\ [1,-1] & [4,2] \end{pmatrix}\mathbf{x}^{\mathrm{I}} = \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix}$$

The algebraic solution is:

$$\mathbf{x_a} = \begin{pmatrix} [0,1] \\ [0,1] \end{pmatrix}$$

Using the Shary's approach, first we have to obtain the algebraic solution of:

$$\begin{pmatrix} [2,4] & [-1,1] \\ [-1,1] & [2,4] \end{pmatrix}\mathbf{x}^{\mathrm{I}} = Dual\left\{\begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix}\right\}$$

That is:

$$\mathbf{x_a} = \begin{pmatrix} [1,0] \\ [1,0] \end{pmatrix}$$

The inner box is then:

$$Dual\left\{\begin{pmatrix} [1,0] \\ [1,0] \end{pmatrix}\right\} = \begin{pmatrix} [0,1] \\ [0,1] \end{pmatrix}$$

Figure 4 shows the inner box.

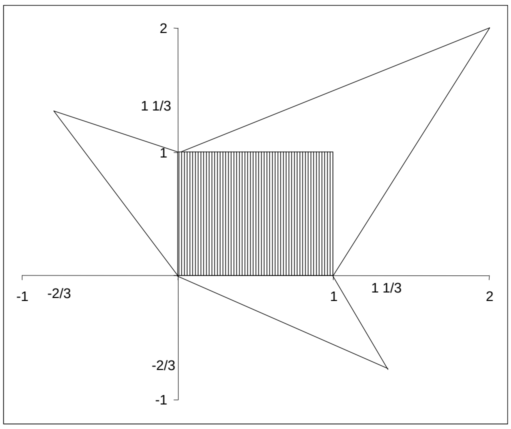

Figure 4: Inner Box for system (6)

3. EXAMPLE

Table 1 presents the 1987 Washington State direct purchase coefficient table estimated from an aggregated model (Chase et al, 1993). To evaluate the total economic impact of a $50 million increase in manufacturing exports, the author solved the system **Lx** = **b**, where **L** is the

Leontief matrix and **b** = $(0,50,0,0)^t$ MM\$. The vector **x** obtained is: $(3.0,60.0,27.0,42.0)^t$ MM\$ (Base Case). That means, for example, that a \$50 million increase in manufacturing exports is expected to result in a \$3 million production increase in natural resource industries (Chase et al, 1993).

Let us consider that the coefficient matrix has an uncertainty of ± 10 %. Table 2 shows the interval technology matrix.

Table 1: 1987 Washington State Direct Purchase Coefficient Table
(Chase et al, 1993)

	Natural Resource	**Manu-Facturing**	**Trade and Services**	**Personal Consumption**
Natural Resource	0.10453	0.04279	0.00287	0.00305
Manufacturing	0.08263	0.10870	0.05835	0.03212
Trade and Services	0.08667	0.10188	0.20319	0.35550
Personal Consumption	0.62531	0.34483	0.61063	0.07981

Table 2: 1987 Washington State Input-Output Study:
Direct Purchase Coefficient Table with ± 10 % uncertainty

	Natural Resource	**Manu-Facturing**	**Trade and Services**	**Personal Consumption**
Natural Resource	[0.0947, 0.115]	[0.0385, 0.04707]	[0.00258, 0.00316]	[0.00274, 0.00336]
Manufacturing	[0.0743,0.0910]	[0.097,0.1196]	[0.0525,0.0642]	[0.0289,0.03534]
Trade and Services	[0.0780,0.0953]	[0.091692,0.1121]	[0.1828,0.2235]	[0.3199,0.3911]
Personal Consumption	[0.5627,0.6879]	[0.3103,0.3794]	[0.5495,0.6717]	[0.0718,0.0878]

Table 3 shows the Leontief interval matrix, obtained using the approach proposed by Shary (1996). This matrix represents a standardized solution for impact analysis (Chase et al, 1993). For example, the table shows that for every dollar of natural resource industry, the output required from manufacturing is between \$0.1564 and \$0.2686. Without considering uncertainty, the amount required is \$0.202 (Chase et al, 1993).

Table 3: 1987 Washington State Input-Output study:
Inverse Coefficient Matrix with ± 10 % uncertainty

[1.1152,1.1548]	[0.0522,0.0731]	[0.0133,0.0268]	[0.0095,0.0186]
[0.1564,0.2686]	[1.1561,1.2440]	[0.1293,0.2328]	[0.0810,0.1490]
[0.5327,1.0733]	[0.3886,0.7578]	[1.6398,2.1992]	[0.5789,0.9760]
[1.0439,1.7727]	[0.6483.1.1304]	[1.022,1.7363]	[1.4530,1.8909]

Let us evaluate the economic impact of a \$50 million increase in manufacturing exports taking into account the uncertainty on the coefficient matrix. Using [$\overline{L}^{-1}$b, $\underline{L}^{-1}$b], and the vector **b**: $(0,50,0,0)^t$ MM\$, we obtain the following interval production vector:

	Production in MM$
Natural Resource	[2.61,3.66]
Manufacturing	[57.81,62.20]
Trade and Services	[19.43,37.89]
Personal Consumption	[32.41,56.52]

The effects on the production due to changes in the model input are known as swing weights (Shooman, 1990). A practical form of interpreting swing weights is by means of a "Tornado" graph, as shown in Figure 5.

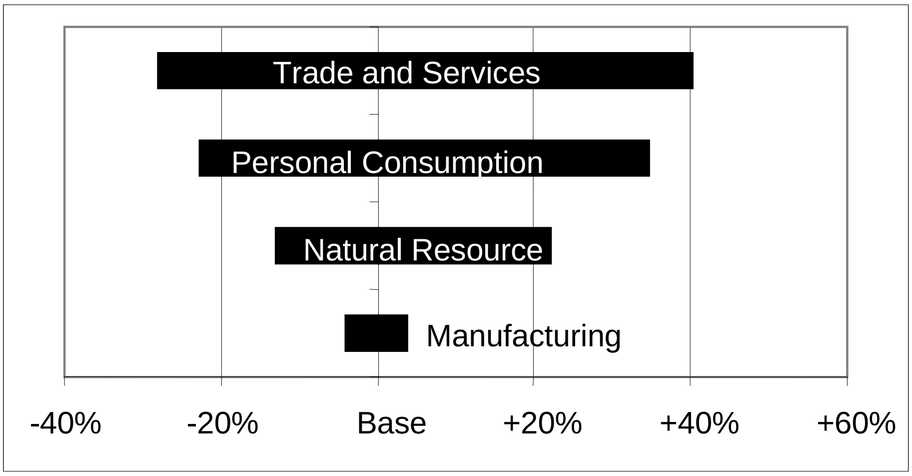

Figure 5: Tornado diagram: Effects on the production due to ± 10 % uncertainty on the coefficient matrix for the 1987 Washington State Input-Output study

This figure shows relative swing weights in percent ordered in ascending magnitude. For example, uncertainty on the coefficient matrix causes more impact on Trade and Services production than on Manufacturing production.

Let us evaluate the economic impact of a \$50 ± 10 % million increase in manufacturing exports. Using [$\overline{L}^{-1}\underline{b}$, $\underline{L}^{-1}\overline{b}$], we obtain the following interval production vector:

	Production in MM$
Natural Resource	[2.35,4.02]
Manufacturing	[52.03,68.42]
Trade and Services	[17.49,41.68]
Personal Consumption	[29.17,62.17]

Thus, according to the model, a \$50 ± 10 % million increase in manufacturing exports along with a ± 10 % uncertainty on the coefficient matrix, is expected to result in a

$[2.35,4.02] million production increase in natural resource industries. Figure 6 shows the Tornado graph related to uncertainty in both coefficient matrix and **b** vector.

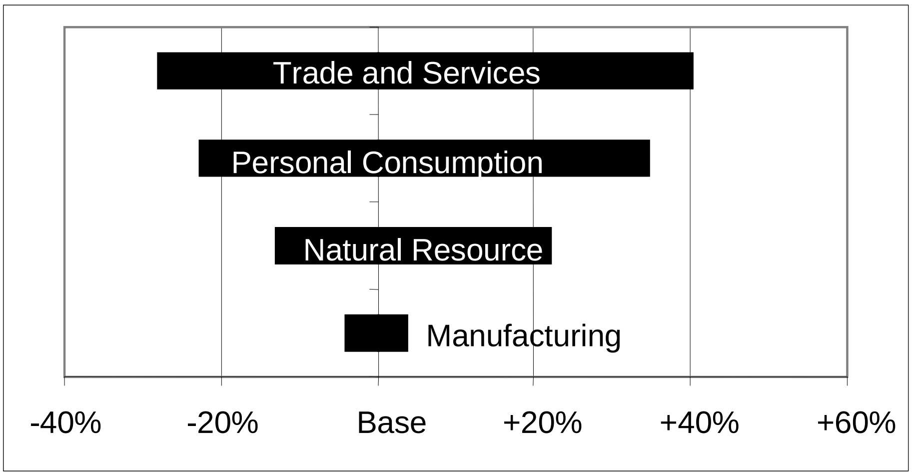

Figure 6: Tornado diagram: Effects on the production due to ± 10 % uncertainty on the coefficient matrix and demand vector for the 1987 Washington State Input-Output study

The interval vector previously obtained corresponds to the outer box and there are points in that box that do not belong to the solution set. Using the approach proposed by Shary (1997), we obtain the following inner box (only 2 iterations):

	Production In MM$
Natural Resource	[3.02,3.06]
Manufacturing	[53.15,66.19]
Trade and Services	[24.26.,28.67]
Personal Consumption	[40.97,42.24]

Figure 7 compares the swing weights for the outer and inner boxes.

5. Conclusions

There is a general agreement to perform systematic sensitivity and uncertainty analysis in order to structure incomplete knowledge. Decision-makers need a full display of the sources and magnitudes of the uncertainties before making an informed judgement.

Among the techniques, we presented Interval Arithmetic as an alternative to performing sensitivity/uncertainty analysis in Input-Output models. Interval Arithmetic can be used when uncertainty can not be expressed as a probabilistic or possibilistic function.

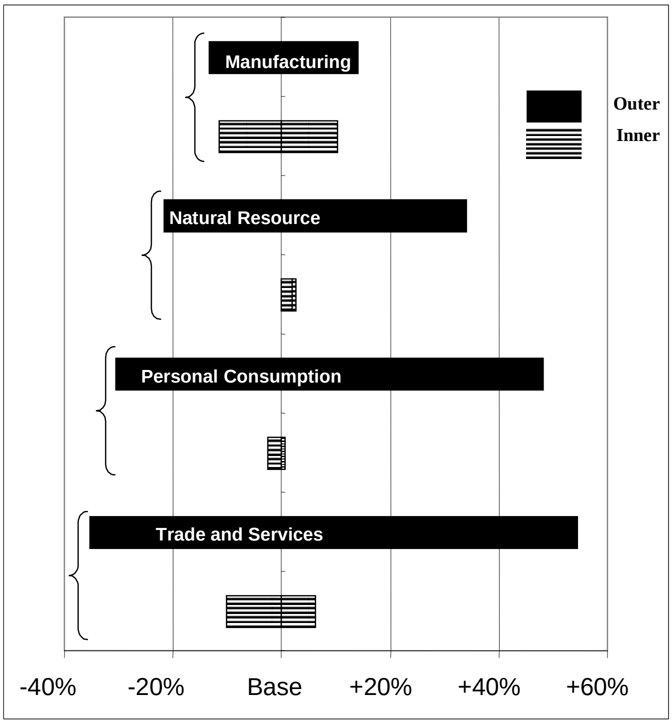

Figure 7: Swing weights for outer and inner production boxes, due to ± 10 % uncertainty on the coefficient matrix and demand vector for the 1987 Washington State Input-Output study

We have shown that it is possible to perform sensitivity/uncertainty analysis by using Interval Arithmetic, assigning bounds to some or all of the input parameters and observing the effects on the final interval outcome, that will contain all possible solutions due to the variations in input parameters. Among the techniques to solve linear interval system, we selected the approach presented by Shary, as it has high computational efficiency and can be used to obtain both outer and inner boxes to the solution set.

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