Construction Modelling of Input-Output Coefficients Matrices in an Axiomatic Context: Some Further Considerations.

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Abstract

Kop Jansen and ten Raa (1990) established a pure theoretical solution to the model selection problem of constructing input-output technical coefficients matrices using make and use tables. In an axiomatic context, these authors provide a characterization result pertaining to the construction of input-output coefficients, which lead to single out one of the seven models considered: the so-called commodity technology model as the best one according to some desirable properties.

The aim of this paper consists of giving answer to what restrictions must be imposed on the relevant data sets for each model to ensure fulfillment of most desirable properties. We delineate regions in data space where axioms are fulfilled, for each input-output construct.

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1 Introduction

Input-output analysis concentrates its attention on a so-called input-output matrix of technical coefficients, $A = \{a_{ij}\}, \forall i, j = 1, \ldots, n$ (with n different commodities), representing the direct requirements of commodity i needed per physical unit of commodity j's production. For instance, if industry 1 corresponds to agriculture and industry 2 corresponds to chemicals, then a_{21} will be the amount of chemical products consumed by agriculture per physical unit of peach, apple and so on. In more general terms, the standard reference for a more detailed explanation is Leontief (1966).

The matrix of technical coefficients A has been used as the point of departure for economic analysis by means of the so-called quantity equation or material balance (supply and demand) and the value equation or financial balance (costs and revenues). That is the following:

$$\begin{aligned} x &= Ax + y \\ p &= pA + v \end{aligned}$$

where x is a column vector of gross outputs, y is another column vector of final demand, p is a row vector of prices, and lastly, v is a another row vector of value-added coefficients.

Some of the most common applications of the quantity equation are derived from the need to develop a national or regional economic planning, which involves more detailed studies where, for instance, the output requirements to satisfy a certain final demand level could be analysed.Furthermore, final demand could also be influenced by an exports or investments policy. Thus, there will be a direct effect over the output levels which will depend on the final demand variations (Δy) and additional indirect effects that will be valued through the A technical coefficients matrix, as it is shown in the material balance equation. With respect to the value equation, it can be used to assess the price effects resulting from an energy shock, which surely will bring about variations in the value-added shares of a product.

There is little doubt about the usefulness of the W. Leontief methodological framework due to the huge quantity of books and papers related to it and its variants. Undoubtedly, the Leontief inverse $(I - A)^{-1}$, which gives a solution to both equations, is one of the most important points of reference in input-output analysis.

However, the main effort of the National and Regional Statistical Offices has been concentrated almost exclusively on industry input-output tables instead of commodity ones. Obviously, from a practical point of view, the varied complexity in data compilation and its reliability make advisable to set up a so called transactions table (ten Raa, 1994) $T = \{t_{ij}\}, \forall i, j = 1, \ldots, n+1$ (used for sectors or industries).[1] In such a table, each element displays the inputs requirements of sector i per unit of sector j's production, as well as the final demand compartments (household and government consumption, investment and net exports).

According to ten Raa (1994), if reality were to present itself through a simple

[1]Note that, for simplicity, we assume the same number of industries as of commodities (n).

input-output transactions table, T, the construction of a matrix of technical coefficients, A, would be a straightforward matter of divisions:

$$a_{ij} = \frac{t_{ij}}{\sum_{k=1}^{n+1} t_{jk}}$$

Nevertheless, if we assume this to be true, there are some further implications that we have to consider. They are described as follows:

1. The very existence of a transactions table presumes that commodities and sectors can be classified in the same way.
2. It is also suggested that sectors have a multitude of inputs, but only single outputs.

In 1967, Professor Richard Stone suggested accounting for inputs and outputs separately in order to give solutions to both implications. Thus, let us define an USE TABLE, $U = \{u_{ij}\}, \forall i, j = 1, \ldots, n$ of commodities i consumed by sector j, and a MAKE TABLE $V = \{v_{ij}\}, \forall i, j = 1, \ldots, n$ where sector i will produce commodity j (U.N., 1967; ten Raa, Chakraborty and Small, 1984; Kop Jansen and ten Raa, 1990). This kind of systems with input and output matrices has been generalized in input-output analysis since then until nowadays thanks to the contributions of authors like Professor Thijs ten Raa, among others. Note that, according to Kop Jansen and ten Raa (1990) industry tables and mixed tables are not considered here. Moreover, we will assume the same number of industries as of commodities.

Hence, the aim of this paper for the present is how to construct a technical coefficients matrix $A = \{a_{ij}\}, \forall i, j = 1, \ldots, n$ of commodities i needed for the production of one physical unit of commodity j.

2 The Standard Case for the Construction of Technical Coefficients Matrices

The standard case for the construction of technical coefficients matrices, A, could be expressed as a particular case in terms of U and V tables. In the absence of secondary outputs and by-products, V turns into a diagonal matrix and one simply puts:

$$a_{ij}(U, V) = \frac{u_{ij}}{v_{jj}}, \forall i, j = 1, \ldots, n$$

By this way, U will be equal to the transactions table T with only their first n rows and columns. In conclusion, the standard case formula for the construction of technical coefficients matrices would be as follows:

$$A(U, V) = UV^{-1}$$

3 Models Description

Nevertheless, V is not necessarily diagonal because economic reality shows that sectors have multiple outputs and productive processes which automatically generate by-products .In this sense, Kop Jansen and ten Raa (1990) deal with seven different methods to construct an input-output coefficients matrix. They divide them in two different groups:

1. *Statistical Models*; they consist basically of statistical methods which remove secondary products from the make table.
2. *Economic Models*; their foundations belong to economic reasons and their sources for motivation and derivations could be obtained by consulting ten Raa, Chakraborty and Small (1984).

3.1 Statistical Models

1. The *Lump-Sum Method* (used in the japanese Input-Output Tables, 1974); it presents the following analytical expressions[2]:

$$A_L(U,V) = U(\widehat{Ve})^{-1}$$

$$a_{ij}^L(U,V) = \left\{ \frac{u_{ij}}{v_j} \quad \forall i,j = 1,\ldots,n \right.$$

By the *Lump-Sum method*, technical coefficients could be obtained by dividing all the entries of each of the columns from the use table by the total output of sector j, specified in row j of the V table. Also, this total output includes not only primary products but secondary products and by-products (n different commodities). That is,

$$v_j = \sum_{k=1}^{n} v_{jk}$$

2. The *European System Accounting* method (1979) specifies the following analytical expressions:

$$A_E(U,V) = U(\widehat{V^T e})^{-1}$$

$$a_{ij}^E(U,V) = \left\{ \frac{u_{ij}}{v_j^{'}} \quad \forall i,j = 1,\ldots,n \right.$$

Unlike the *Lump-Sum* method, here we construct the technical coefficients by dividing all the entries of each of the columns from the use table by the total output of commodity k, specified in the column k of the V table. Note that this total output is not exclusively obtained by one single sector.[3] That is,

[2] In what follows, e denotes a column vector with all entries equal to one. ^{T} denotes transposition and $^{-1}$ inversion of a matrix. Since the latter two operations commute, their composition may be denoted $^{-T}$. Also,$\widehat{}$ denotes a diagonal matrix either by placement of null entries instead of the off-diagonal elements or by placement of the entries of a vector. $\widetilde{}$ denotes, finally, a matrix with all its diagonal elements null.

[3] As we are assuming that commodities and sectors can be classified in the same way, in what follows, we will use v_j to denote the total output of sector j and $v_j^{'}$ $(= v_k^{'})$ to denote the total output of commodity j.

$$v_k^{'} = \sum_{j=1}^{n} v_{jk}$$

3. The *Transfer* method yields :

$$A_T(U,V) = (U + \widetilde{V})(\widehat{Ve} + \widehat{V^T e} - \widehat{V})^{-1}$$

$$a_{ij}^T(U,V) = \begin{cases} \dfrac{u_{ij}}{v_j + v_j^{'} - v_{jj}} & \text{if } i = j \\ \dfrac{u_{ij} + v_{ij}}{v_j + v_j^{'} - v_{jj}} & \text{if } i \neq j \end{cases}$$

This method was first established by Professor Stone (1961) for the O.E.C.D.

3.2 Economic Models

1. The *Commodity Technology Model* (used in Germany and proposed in 1967 by the United Nations); it presents the following expression:

$$A_C(U,V) = UV^{-T}$$

The economic foundations of such a expression is easy to understand (ten Raa, Chakraborty and Small, 1984). The *Commodity Technology Model* rests on the assumption that each commodity has its own input structure independently of what sector could produce it. Hence, if a_{ik} represents the direct requirements of commodity i needed by sector j for the production of one physical unit of commodity k and also, v_{jk} stands for the total secondary output of commodity k produced by sector j, it can be derived that the amount $a_{ik}v_{jk}$ is the total inputs requirements of commodity i needed for the production of v_{jk} units of output k. Then, if we also assume that sector j has multiple outputs and all different from commodity k, we finally could sum over outputs k ($m < n$ different kind of commodities produced by sector j) to obtain the sector j's total demand for input i. Thus, u_{ij} is written as:

$$u_{ij} = \sum_{k=1}^{m} a_{ik}v_{jk}, \quad \forall i,j = 1,\ldots,n, \quad \forall k = 1,\ldots,m$$

and, therefore,

$$A_C(U,V)V^T = U$$

where the first expression mentioned above is easily derived.

2. The *By-Product Technology Model* (used in Japan and proposed by Professor Stone in 1961) yields the following analytical expression:

$$A_B(U,V) = (U - \widetilde{V}^T)\widehat{V}^{-1}$$

$$a_{ij}^B(U,V) = \begin{cases} \dfrac{u_{jj}}{v_{jj}} & \text{if } i = j \\ \dfrac{u_{ij} - v_{ji}}{v_{jj}} & \text{if } i \neq j \end{cases}$$

All secondary products are considered by-products in this model. Therefore, they can be treated as negative inputs, yielding a result that is, in a sense, similar to those derived from the *Lump-Sum* method except for the net input structure for the primary outputs and the divisor, which refers to the total primary outputs of sector j instead of the total economy output. Let us examine this implications in more detail. Sector j would need a net amount $u_{ij} - v_{ji}$ of commodity i, which is actually a secondary product of sector j, for the production of v_{jj} units of its primary output. Note that, necessarily $i \neq j$. One of the main problems to put this model in practice is that, in some cases, v_{ij} could be larger than u_{ij} and consequently, negative values of a_{ij}^{B} will be obtained.

3. The *Industry Technology Model* (used in the United States and proposed in 1967 by the United Nations) specifies the following analytical expression :

$$A_I(U,V) = U\widehat{Ve}^{-1}V\widehat{V^Te}^{-1}$$

$$a_{ij}^{I}(U,V) = \sum_{k=1}^{n}\left(\frac{u_{ik}}{v_k}\right)\left(\frac{v_{kj}}{v_j'}\right)$$

This model is characterized by the following two different aspects:

(a) (*Industry technology assumption*). Each industry or sector j has the same inputs requirements for any unit of output (this time, measured in value).

(b) The commodity market shares of industries (or sectors) are fixed.

Let us examine in more detail the last expression written above in order to cast light on the economic foundations of the *industry technology model.* $\frac{u_{ik}}{v_k}$ represents the direct requirements of commodity i needed for the production of one physical unit of commodity k. On the contrary, $\frac{v_{kj}}{v_j'}$ denotes the proportion of the commodity j output produced by sector k to the total output of such commodity. In short, this result is the so-called market share. Hence, according to this model, the technical coefficient a_{ij}, i.e. the amount of input i required for one unit of output j results from a (market share) weighted average over industries k.

4. The *CB-Mixed Technology Model* (proposed in 1984 by ten Raa, Chakraborty and Small and which is based on Gigantes mixed model (1970)) yields :

$$A_{CB}(U,V) = \left(U - V_2^T\right)V_1^{-T}$$

where the authors split the make table V into a table V_1 of primary products and "ordinary secondary products", i.e. those products which involve an alternative activity and which are not being generated automatically by the primary productive process, and a table V_2 of by-products. For a more detailed explanation, it is recommended ten Raa, Chakraborty and Small (1984).

4 The Choice of Model

These seven input-output constructs have not been subjected to an axiomatic analysis until Kop Jansen and ten Raa (1990). Thus far, the choice of model has been made on the basis of the reasonableness of the assumptions from which they are derived. In this sense, four desirable properties of input-output coefficients, $A(U,V)$, are defined by these authors in an axiomatic context. Namely,

1. Axiom M; it is referred to the material balance or the quantity equation which, with the same notation as above, could be denoted as (Kop Jansen and ten Raa, 1990):

$$A(U,V)V^T e = Ue$$

   In other words, the total input requirements must be equal to the observed total input.

2. Axiom F; it is referred to the financial balance or the value equation which, according to the same notation as above, is represented as follows (Kop Jansen and ten Raa, 1990):

$$e^T A(U,V)V^T = e^T U$$

   In words, the input cost of output must match the observed value of input.

3. Axiom P; it is referred to the invariance of the resulting A-matrix with respect to units of measurement or, in other terms, to prices. It is called the price invariance axiom (Kop Jansen and ten Raa, 1990):

$$A(\hat{p}U, V\hat{p}) = \hat{p}A(U,V)\hat{p}^{-1} \quad \forall p > 0$$

   Evidently, this property tries to avoid that a change in the base year prices could affect the technical coefficients. Variations in the internal structure of $A(U,V)$ should be caused by real economic phenomena and not by those kind of methodological changes.

4. Axiom S; or the so called scale invariance axiom (Kop Jansen and ten Raa, 1990):

$$A(U\hat{s}, \hat{s}V) = A(U,V) \quad \forall s > 0$$

   In this sense, if every commodity inputs requirements is augmented by a certain constant percentage, $(U\hat{s})$, the total output actually produced must match the previous total output augmented by the same constant percentage $(\hat{s}V)$. Thereby, this basically involves the absence of technological change and that technical coefficients will never change as a result of constant variations in inputs requirements.

Kop Jansen and ten Raa (1990) develop and examine how well the seven different models presented above fulfill axioms M, F, P and S. More precisely, the authors proved that the just described structure of input-output analysis, involving the four axioms, not only imposes restrictions on the choice of model of construction, but determines it uniquely, namely the *commodity technology model*. That is, either by the real sphere theorem or the nominal sphere theorem, if axioms M and S fulfill in the former case or if axioms F and P fulfill in the latter, the A-matrix must be constructed only by the so called *commodity technology model*.

It is also illustrative to show the main results from Kop Jansen and ten Raa (1990) for the rest of the construct models and which are represented in Table 1.

After all, the characterization theorems do not necessarily favor the *commodity technology model* over other alternative constructs.If, however, an alternative method of constructing input-output matrices is used, then one must be prepared to revise the basic structure of input-output analysis, since at least one of the properties must be violated (Kop Jansen and ten Raa, 1990). Therefore, as we will examine thereinafter, the *commodity technology model* has its own limitations which are so important that the very ten Raa (1988) confessed to be frustrated when he proved that the negative values yielded in the construction of the A-matrix could not be ascribed to errors of measurement but to the initial methodological assumptions. A few years later, ten Raa (1994) concludes that a more reasonable approach of the problem would be to accept the possibility of coexisting technologies for the production of one single commodity in two or more sectors.

TABLE 1: AXIOMS FULFILLMENT OF INPUT-OUTPUT COEFFICIENTS CONSTRUCTS

Model	Axiom M	Axiom F	Axiom P	Axiom S
Lump-Sum				$\surd$
European System	$\surd$		$\surd$	
Transfer				
Commodity Technology	$\surd$	$\surd$	$\surd$	$\surd$
By-product Technology			$\surd$	$\surd$
Industry Technology	$\surd$			
CB-Mixed Technology			$\surd$	$\surd$

Source: Kop Jansen and ten Raa (1990)

The aim of this paper consists of examining more thoroughly the Table 1 presented in Kop Jansen and ten Raa (1990). We will try not only answer to *what* model and *which* axiom is fulfilled but to give an answer to what restrictions must be imposed on the relevant data sets for each model to ensure fulfillment of most desirable properties. Here, we delineate regions in data space where axioms are fulfilled, for each input-output construct. Note that, in the absence of any additional assumption, the fulfillment of both M and S, or F and P implies that the best derived construct is the *commodity technology model* as it has been proved in Kop Jansen and ten Raa (1990). However, we will widen the conclusions obtained by these authors.

5 Equivalent conditions for axioms M and F

As a result of the theorems developed in the this section, two important issues can be deduced for axioms M and F with respect to the *Commodity Technology* model.

Theorem 5.1 *Let $A(U,V)$ be a technical coefficients matrix, axiom M fulfills for all U and non-singular V if and only if,*

$$\sum_{j=1}^{n} a_{ij} v'_j = \sum_{j=1}^{n} a^C_{ij} v'_j \quad \forall i = 1, \ldots, n$$

which in matrix terms is,

$$A(U,V)V^T e = A_C(U,V)V^T e$$

Proof. Sufficiency is proved as follows. Bearing in mind that the *commodity technology* model construct is defined by,

$$A_C(U,V) = UV^{-T}.$$

by substitution,

$$A(U,V)V^T e = UV^{-T}V^T e.$$

Hence, it can be derived that,

$$A(U,V)V^T e = Ue.$$

which is the axiom M indeed.

For the necessary proof, since $V^{-T}V^T = I$ it can be easily derived that,

$$A(U,V)V^T e = UV^{-T}V^T e = A_C(U,V)V^T e = Ue\ .$$

■

Theorem 5.2 *Let $A(U,V)$ be a technical coefficients matrix, axiom F fulfills if and only if the sum of each column of $A(U,V)$ match the sum of each column of $A_C(U,V)$ for all U and non-singular V.*

$$\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ij}^C \quad \forall j = 1,\ldots,n$$

which in matrix terms is,

$$e^T A(U,V) = e^T A_C(U,V)$$

Proof. Sufficiency is proved as follows. Let us suppose that the sum of each column of any $A(U,V)$ and $A_C(U,V)$ match. That is,

$$e^T A(U,V) = e^T A_C(U,V).$$

if we know that,

$$A_C(U,V) = UV^{-T}.$$

by substitution, the former expression yields,

$$e^T A(U,V) = e^T UV^{-T}.$$

Therefore, it can be derived that,

$$e^T A(U,V)V^T = e^T UV^{-T}V^T.$$

or

$$e^T A(U,V)V^T = e^T U.$$

which is the axiom F indeed.

Necessity is proved as follows. Let us assume that axiom F is fulfilled. This implies that,

$$e^T A(U,V)V^T = e^T U.$$

which is the same as,

$$e^T A(U,V)V^T V^{-T} = e^T U V^{-T}.$$

and consequently,

$$e^T A(U,V) = e^T A_C(U,V).$$

which, strictly speaking, is,

$$\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ij}^C \quad \forall j = 1, \ldots, n$$

■

Indeed, all the statistical and economic methods considered so far match the standard case when V is a diagonal matrix. In fact, when there are no secondary products and by-products the problem of the off-diagonal elements of V really does not exist and consequently, the standard case would be the most adequate method for constructing the A-matrix. It is also straightforward that all axioms fulfill if $A(U,V) = UV^{-1}$.

6 Some other additional assumptions

6.1 The *Lump-Sum* Method

Corollary 6.1 *The* Lump-Sum *method fulfills the material balance axiom if the total output of sector j matches the total output of commodity j, for all U and V such that $v_j = v_j^{'}, \forall j$.*

Proof. Under the *Lump-Sum* method the A-matrix is defined as:

$$A_L(U,V) = U(\widehat{Ve})^{-1}$$

and, therefore, if we assume that $Ve = V^T e$, namely $v_j = v_j^{'}, \forall j$ we can obtain:

$$A_L(U,V)V^T e = U(\widehat{Ve})^{-1} V^T e = U(\widehat{V^T e})^{-1} V^T e = Ue$$

since $V^T e = \left(\widehat{V^T e}\right) e$.

So, we can conclude that axiom M will be fulfilled in the *Lump-Sum* model if the total output of e.g. sector j is equal to the total output of commodity j ($v_j = v_j^{'}$). On the contrary, in case axiom M is fulfilled, this does not imply that always $v_j = v_j^{'}$.

The *Lump-sum* model fulfills the invariance scale axiom S as it is demonstrated in Kop Jansen and ten Raa (1990). Indeed, it is the only axiom which holds under the assumptions made by these authors.

6.2 The Commodity Technology Model

The *Commodity Technology* model fulfills all axioms or desirable properties established in Kop Jansen and ten Raa (1990), which is recommended for a more detailed analysis. In short, there is no need to assume any additional hypothesis in order to achieve the fulfillment of axioms M, P, F and S.

6.3 The By-Product Technology Model

Corollary 6.2 *The* By-Product Technology *model fulfills the financial balance axiom for all U and V when for any secondary production (or by-product), say v_{jk}, the sector which primary output corresponds to this secondary product (sector k) does not have positive value-added since the total inputs requirements of sector k would match the total sector k's outputs. That is,*

$$e^T V^T = e^T U$$

Proof. Under the financial balance axiom, the *By-Product Technology* model should verify,

$$e^T A_B(U,V)V^T = e^T U$$

with the left-hand side of this equality such as,

$$e^T(U - \tilde{V}^T)\hat{V}^{-1}V^T = (e^T U - e^T\tilde{V}^T)\hat{V}^{-1}V^T = e^T U\hat{V}^{-1}V^T - e^T\tilde{V}^T\hat{V}^{-1}V^T$$

Moreover, since $\tilde{V}^T = V^T - \hat{V}^T$ and $\hat{V}^T = \hat{V}$ it yields,

$$e^T U\hat{V}^{-1}V^T - e^T\tilde{V}^T\hat{V}^{-1}V^T = e^T U\hat{V}^{-1}V^T - e^T V^T\hat{V}^{-1}V^T + e^T\hat{V}^T\hat{V}^{-1}V^T$$

which is the same as,

$$e^T A_B(U,V)V^T = (e^T U - e^T V^T)\hat{V}^{-1}V^T + e^T V^T$$

So, let us assume now that $e^T U = e^T V^T$ then,

$$e^T A_B(U,V)V^T = e^T V^T = e^T U$$

■

With respect to price and scale invariance axioms it is easily verified that the *By-Product Technology* model fulfills both of them without any additional assumption. As a result, it yields

$$A_B(\hat{p}U, V\hat{p}) = \hat{p}A_B(U,V)\hat{p}^{-1} = \begin{cases} \dfrac{u_{ij}}{v_{jj}} & \text{if } i = j \\ \dfrac{p_i(u_{ij} - v_{ji})}{p_j v_{jj}} & \text{if } i \neq j \end{cases}$$

and,

$$A_B(U\hat{s}, \hat{s}V) = A_B(U,V) = \begin{cases} \dfrac{u_{ij}}{v_{jj}} & \text{if } i = j \\ \dfrac{u_{ij} - v_{ji}}{v_{jj}} & \text{if } i \neq j \end{cases}$$

6.4 The Mixed Technology Model

Kop Jansen and ten Raa (1990) demonstrate that each of axioms M and F holds if and only if the *CB-Mixed Technology* model reduces to the *Commodity Technology* model. In other words, both axioms hold only when the V_2 table is null, i.e. when there are no by-pruducts, although according to ten Raa, Chakraborty and Small (1984), the so-called ordinary secondary products are included in table V_1.

But, what happens when indeed there exist by-products? Under what restrictions on the data do axioms M and F still hold?. We will take some preliminary results from Kop Jansen and ten Raa (1990) as our point of departure in order to cast light on these issues.

Corollary 6.3 *The* CB-Mixed Technology *model fulfills the material balance axiom M for all U and non-singular V_1 when U and V^T match, which is the same that $A_C(U,V) = I$.*

Proof. Under the *CB-Mixed Technology* model construct, the axiom M should verify that,

$$A_{CB}(U,V)V^T e = \left(U - V_2^T\right) V_1^{-T} V^T e = Ue$$

where V_1 stands for the primary outputs and those secondary products considered as "ordinary" according to ten Raa, Chakraborty and Small (1984) definition, and V_2, for the by-products.

Since we are assuming that $U = V^T = V_1^T + V_2^T$ it can be shown that,

$$(U - V_2^T)V_1^{-T}V^T e = V_1^T V_1^{-T} V^T e = V^T e = Ue$$

■

Corollary 6.4 *The* CB-Mixed Technology *model fulfills the financial balance axiom when for any secondary production (or by-product), say v_{jk}, the sector which primary output corresponds to this secondary product (sector k) must not have positive value-added since the total inputs requirements of sector k would match the total sector k's outputs. That is,*

$$e^T V^T = e^T U$$

Proof. As Kop Jansen and ten Raa (1990) demonstrate, under the *CB-Mixed Technology* model construct the axiom F should verify that,

$$e^T A_{CB}(U,V)V^T = e^T \left(U - V_2^T\right) V_1^{-T} V^T = e^T U$$

This can also be expressed as,

$$e^T A_{CB}(U,V)V^T = e^T \left(U - V_2^T\right) V_1^{-T} V^T = (e^T U - e^T V_2^T) V_1^{-T} V^T$$

and substituting $e^T U = e^T V^T$, it yields,

$$e^T A_{CB}(U,V)V^T = (e^T V^T - e^T V_2^T)V_1^{-T}V^T =$$

$$e^T(V^T - V_2^T)V_1^{-T}V^T = e^T V_1^T V_1^{-T} V^T = e^T V^T = e^T U$$

$\blacksquare$

With respect to the price and scale invariance axioms, it can be seen in Kop Jansen and ten Raa (1990) how the *CB-Mixed Technology* model fulfills both of them.

7 Summary and conclusions

In view of the theorems and corollaries enunciated thus far, we can group the most relevant additional assumptions needed to improve the level of fulfillment of all axioms in each one of the technical coefficients constructs together into:

1. General assumptions with respect to the Commodity Technology Model (Theorems 5.1 and 5.2)

   Perhaps, the most interesting conclusion is referred to the fact that axioms M and F will be always fulfilled under some certain restrictions which relate any technical coefficients matrix construct to the *Commodity Technology* model.

2. Equivalence of sectors and commodities productions (Corollary 6.1)

   This restriction involves somehow that commodities and sectors could be classified in the same way. Since this is not unusual to find in several input-output tables published to date, it seems reasonable assume this hypothesis in order to improve the fulfillment of axiom M.

3. Null value-added. (Corollaries 6.2 to 6.4)

   The assumption related to the absence of value added in certain sectors have little real and economic sense. Therefore, it should be certainly neglected.

4. Absence of secondary products and by-products.

   This assumption is certainly nonsense due to the fact that the motivation of the several input-output technical coefficients constructs developed in this paper relies on a make table V which is actually not diagonal.

Now, it is presented in the Appendix a brief summary of the main results (Table 2) classified according to models and axioms. It is shown, as well, that the *Transfer* and the *Industry Technology* model are the ones which need more restrictive conditions in order to fulfill all axioms. Moreover, since axioms P and S are defined for all p and s respectively, they will never hold under those input-output technical coefficients construct models which actually do not fulfill them.

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Appendix

TABLE 2: ADDITIONAL ASSUMPTIONS OVER AXIOMS ACCORDING TO MODELS.

Model	Axiom M	Axiom F	Axiom P	Axiom S
Lump-Sum	$A_L(U,V)V^Te =$ $A_C(U,V)V^Te$ or $Ve = V^Te$	$e^TA_L(U,V) =$ $e^TA_C(U,V)$	*Never*	√
European System	√	$e^TA_E(U,V) =$ $e^TA_C(U,V)$	√	*Never*
Transfer	$A_T(U,V)V^Te =$ $A_C(U,V)V^Te$	$e^TA_T(U,V) =$ $e^TA_C(U,V)$	*Never*	*Never*
Commodity Technology	√	√	√	√
By-Product Technology	$A_B(U,V)V^Te =$ $A_C(U,V)V^Te$	$e^TA_B(U,V) =$ $e^TA_C(U,V)$ or $e^TV^T = e^TU$	√	√
Industry Technology	√	$e^TA_I(U,V) =$ $e^TA_C(U,V)$	*Never*	*Never*
CB-Mixed Technology	$A_{CB}(U,V)V^Te =$ $A_C(U,V)V^Te$ or $U = V^T$	$e^TA_{CB}(U,V) =$ $e^TA_C(U,V)$ or $e^TV^T = e^TU$	√	√

Source: own elaboration