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A Global Robustness Measure for Input-Output Projections from ESA and SNA Tables

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Abstract

Input-output (interindustry) data are in wide use in empirical research and constitute an integral part of the European System of Accounts (ESA) and of the System of National Accounts (SNA). In a strict sense, however, these data are merely estimates of the true economic relationships. Therefore, we suggest a measure of robustness of input-output projections with respect to errors or changes in the underlying Leontief matrix. Our measure is based on the mathematical theory of norms and characterizes a complete Leontief matrix. Thereby, no assumptions are required on the distribution of the matrix elements. We discuss alternative numerical-computing algorithms and provide useful bounds and approximation formulas. The paper concludes with a large set of empirical sample applications.

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1 Introduction [*]

In late 1996, the new 1995 European System of National and Regional Accounts, referred to as 'ESA 95', entered into force. It had been adopted by the Council of the European Union (1996) in order to provide "... a methodology on common standards, definitions, classifications and accounting rules, intended to be used for compiling accounts and tables on comparable bases for the purposes of the Community" (Article 1). The 1995 ESA replaced the former European System of Integrated Economic Accounts which was introduced in 1970 and revised in 1978. A remarkable feature of the ESA 95 is its full consistency with the 1993 System of National Accounts, henceforth SNA 93, of the United Nations Statistical Commission (United Nations et al. (1993)). The SNA 93 evolved from its forerunners, the 1953 SNA and 1968 SNA, under the joint responsability of the United Nations, the Commission of the European Communities, the IMF, the OECD and the World Bank. Notably, with the advent of the ESA 95, the European member states have accepted the duty to supply for their countries symmetric input-output tables according to unified statistical guidelines and within specified time limits, i.e., in no case later than by January 1, 2005. These tables constitute a main part both of the SNA 93 and ESA 95 frameworks and are expected to be "... extensively used for purposes of economic analysis and projections" (United Nations et al. (1993, p. 4)).

Input-output theory has prepared the methodological ground for this task. In particular, consider the following n-sector economy ($n \geq 2$) where $\mathbf{A} = (a_{ij})$ stands for the $n \times n$ matrix of production coefficients for intermediate inputs and, respectively, $\mathbf{x}$ and $\mathbf{y}$ represent $n \times 1$ vectors of sectoral gross outputs and final demands. Output prices and the sectoral total cost shares of primary inputs shall be collected in $n \times 1$ vectors $\mathbf{p}$ and $\mathbf{w}$. We assume that all matrix elements and vector components are non-negative real numbers. Furthermore, denote as $\mathbf{I}$ the $n \times n$ identity matrix and let $\mathbf{B} := \mathbf{I} - \mathbf{A}$. The superscript T will be attached in order to indicate vector and matrix transposition, where necessary. Then $\mathbf{x}$ and $\mathbf{p}$ satisfy the fundamental equations of static input-output analysis (e.g., see Nikaido (1975, Chapters 1 & 3), Takayama (1997, Chapter 4)):

$$\text{Primal:} \quad \mathbf{B}\,\mathbf{x} = \mathbf{y}\,, \quad \text{Dual:} \quad \mathbf{B}^T\mathbf{p} = \mathbf{w}\,. \tag{1}$$

These equations always possess unique and non-negative solutions $\mathbf{x}$ and $\mathbf{p}$ for arbitrary non-negative right-hand sides $\mathbf{y}$ and $\mathbf{w}$ (component-wise), if and only if the so-called Leon-

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tief matrix $\mathbf{B} = (b_{ij})$ is regular and no element of the Leontief inverse $\mathbf{B}^{-1}$ comes out less than zero. Accordingly, every Leontief matrix which has been compiled from empirical interindustry data can be shown to inherit this property. Such a matrix may be used, in principle, to analyse and forecast the effects, e.g., of demand shocks, government spending, energy consumption, wage policy and tax reforms, on sector outputs (and thus employment) and on output prices. Embedded input-output systems could thereby become an integral part of advanced econometric or computable general equilibrium models. The international INFORUM project (Interindustry Forecasting University of Maryland) links 13 such models for major industrial countries (cf. Nyhus (1988)).[1]

Both the ESA 95 and SNA 93 recommend that symmetric input-output tables with product or industry classification be constructed from rectangular so-called supply and use tables which display the provision of products and the demand for inputs per industry. In practice, these tables are arranged from different and often incompatible statistical sources. The conversion of two related tables, e.g., into a symmetric product-by-product input-output table involves the non-trivial transfer of the by-products or secondary outputs of each industry to the industry to which these outputs principally belong. Such a transfer can be achieved by exploiting as much supplementary statistical information on the sectoral production processes as possible. Otherwise, the conversion must rely on a-priori assumptions like the so-called industry-technology and product-technology assumptions which may be more or less appropriate in individual cases (cf. United Nations et al. (1993, p. 307), Almon (2000)). Also, deterministic scaling algorithms like the RAS technique (Bacharach (1970), Gilchrist/St Louis (1999)) frequently have to be imposed on the generated data in order to guarantee that the column and row sums of an input-output table are balanced. This indicates that the respective table entries are essentially estimates of the underlying true economic relationships and that the tables are likely to be distorted. In passing, note that the aggregation problem is another serious cause of data biases as well as plain measurement errors and rounding. After all, even good estimates may soon become obsolete if technology changes. Ideally, all this has only a small impact on the Leontief inverse of an empirical input-output model, in which case the model's projective power can be maintained. However, consider the following counter-example of a presumed hierarchical or recursive 5×5 economy where a single output unit of each sector j calls for one unit of intermediate inputs from every preceding sector i such that $a_{ij} = 1$ whenever $j > i$, whereas all other input coefficients are zero. Hence:

[1] The INFORUM URL is provided in the References.

$$\mathbf{B} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{B}^{-1} = \begin{pmatrix} 1 & 1 & 2 & 4 & 8 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Instead, if the input coefficient a_{51} was in fact equal to 0.1 we would obtain:

$$\mathbf{B} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ -0.1 & 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{B}^{-1} = \begin{pmatrix} 5 & 5 & 10 & 20 & 40 \\ 2 & 3 & 5 & 10 & 20 \\ 1 & 1 & 3 & 5 & 10 \\ 0.5 & 0.5 & 1 & 3 & 5 \\ 0.5 & 0.5 & 1 & 2 & 5 \end{pmatrix}. \tag{3}$$

We thus conclude that the Leontief matrix in (2) is ill-conditioned in the sense that small variations in its elements may induce large changes in the elements of the associated Leontief inverse. Consequently, input-output projections which are computed from such matrices may not be robust and, therefore, should be treated with caution.

This problem has already attracted a lot of attention in the literature and can be tackled in various ways. For instance, many input-output systems will incorporate several so-called key production sectors which have strong backward and forward linkages to other sectors in the economy (e.g. Rasmussen (1957), Hazari (1970), Hazari/Krishnamurty (1970), Cuello/Mansouri (1992)). We conclude that statistical input-output data for such leading sectors must be most accurate in order to prevent large projection errors. The same idea can also be applied to critical single input coefficients or groups of input coefficients in a Leontief matrix (see Jilek (1971), Schintke (1976), Maaß (1980), Schintke/Stäglin (1988), Aroche-Reyes (1996)). Changes in these coefficients would induce more than average changes in the endogenous variables. Hence, important input coefficients have to be measured with great care. Ideally, the user of an input-output table would be informed of key production sectors and would also know how various potential errors or changes in different selections of input-output coefficients will affect the table's associated Leontief inverse. However, this means to provide the user with a vast extra amount of statistical material. Therefore, we suggest in this paper a unified global robustness measure which characterizes a complete Leontief matrix. Our measure can detect condition problems without resorting to additional assumptions on the distribution of errors or changes in the matrix elements. The measure's numerical value could be reported along with the underlying input-output table and should serve as a supporting information for users. They may then wish to isolate critical input coefficients if the size of this value indicates the presence of an overall condition problem.

Formally speaking, our measure corresponds to the inverse of the so-called spectral condition number which can be attached to a Leontief matrix **B**. Section 2 provides the underlying mathematical concepts. The calculation of this measure from statistical input-output data involves extensive number crunching. Therefore, Section 3 will focus on selected computational issues. Section 4 introduces convenient bounds and approximation formulas which can be calculated directly from $\mathbf{B}$ without too much loss of knowledge. A large number of sample applications will finally be presented in Section 5. Section 6 concludes and provides a summary of results.

2 The Condition of a Leontief Matrix

Assume that the true Leontief matrix equals $\mathbf{B} + \epsilon\,\mathbf{F}$ rather than $\mathbf{B}$ where $\mathbf{F}$ is a real-valued $n \times n$ matrix and ϵ is a real-valued scalar parameter which reflects the 'size' of the deviation of the true matrix from $\mathbf{B}$. This deviation may be the result of errors in the compilation of the underlying matrix of input coefficients $\mathbf{A}$. In a broader sense, the term $\epsilon\,\mathbf{F}$ can also capture unforseen technology changes which might lie ahead.

Next, consider the following parameterized input-output system which replaces the primal model in (1) (see Golub/Van Loan (1996, pp. 80-82)):

$$(\mathbf{B} + \epsilon\,\mathbf{F})\,\mathbf{x}(\epsilon) = \mathbf{y}\,. \tag{4}$$

Note that the true Leontief matrix equals $\mathbf{B}$ provided that $\epsilon = 0$ in which case (4) has a unique solution $\mathbf{x} = \mathbf{x}(0)$ due to the assumed regularity of $\mathbf{B}$. Hence, $\mathbf{x}(\epsilon)$ will be differentiable for small ϵ. Therefore, if $\mathbf{x}'(\epsilon)$ indicates the respective derivative and $\mathbf{o}$ is the $n \times 1$ null vector, differentiation of both sides of (4) with respect to (small) ϵ yields $\mathbf{F}\mathbf{x}(\epsilon) + (\mathbf{B} + \epsilon\,\mathbf{F})\,\mathbf{x}'(\epsilon) = \mathbf{o}$ and hence $\mathbf{x}'(0) = -\mathbf{B}^{-1}\mathbf{F}\mathbf{x}$. For the same reason, we can expand $\mathbf{x}(\epsilon)$ into a Taylor series around $\mathbf{x}$. In particular, if R stands for second-order and higher-order terms: $\mathbf{x}(\epsilon) = \mathbf{x} + \epsilon\,\mathbf{x}'(0) + R$. After all, we conclude that

$$\mathbf{x}(\epsilon) = \mathbf{x} - \epsilon\,\mathbf{B}^{-1}\mathbf{F}\mathbf{x} + R \quad \Leftrightarrow \quad \mathbf{x}(\epsilon) - \mathbf{x} = -\epsilon\,\mathbf{B}^{-1}\mathbf{F}\mathbf{x} + R\,. \tag{5}$$

At this point, consider the Euclidean length or norm $\|\mathbf{x}\|_2 = (\mathbf{x}^T\mathbf{x})^{\frac{1}{2}}$ of $\mathbf{x}$. A consistent distance measure for $\mathbf{B}$ is given by the so-called spectral norm $\|\mathbf{B}\|_2$ of $\mathbf{B}$ which is defined as the positive square root of the maximum eigenvalue of $\mathbf{B}^T\mathbf{B}$. These norms possess the submultiplicative property and will thus satisfy the inequality $\|-\epsilon\,\mathbf{B}^{-1}\mathbf{F}\mathbf{x}\|_2 \leq |\epsilon|\,\|\mathbf{B}^{-1}\|_2\,\|\mathbf{F}\|_2\,\|\mathbf{x}\|_2$. We thereby obtain from (5) an upper bound for the relative error in an input-output projection of $\mathbf{x}$ from a false Leontief matrix $\mathbf{B}$:

$$\begin{aligned}\frac{\|\mathbf{x}(\epsilon) - \mathbf{x}\|_2}{\|\mathbf{x}\|_2} &\leq |\epsilon|\,\|\mathbf{B}^{-1}\|_2\,\|\mathbf{F}\|_2 + \frac{R}{\|\mathbf{x}\|_2} \\ &= \|\mathbf{B}\|_2\,\|\mathbf{B}^{-1}\|_2 \frac{|\epsilon|\,\|\mathbf{F}\|_2}{\|\mathbf{B}\|_2} + \frac{R}{\|\mathbf{x}\|_2} \end{aligned} \tag{6}$$

$$= \kappa(\mathbf{B})\,\rho_{\mathbf{B}} + r\,,$$

where $\kappa(\mathbf{B}) := \|\mathbf{B}\|_2\,\|\mathbf{B}^{-1}\|_2\,,\ \ \rho_{\mathbf{B}} := |\epsilon|\,\|\mathbf{F}\|_2\,/\,\|\mathbf{B}\|_2$ and $r := R\,/\,\|\mathbf{x}\|_2\,.$

We see from (6) that for each given relative error $\rho_{\mathbf{B}}$ in $\mathbf{B}$ the associated bound on the relative error in $\mathbf{x}$ increases with the size of the factor $\kappa(\mathbf{B})$. This factor is called the *spectral condition number* of $\mathbf{B}$ (cf. Wilkinson (1999, p. 87)). Now note that $\mathbf{B}^T\mathbf{B}$ is a symmetric and strictly positive definite matrix.[2] Therefore, its eigenvalues λ_i will always come out as positive real numbers. We may accordingly assume without loss of generality that these numbers satisfy $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. A corresponding assumption will then hold for their positive roots σ_i (the so-called *singular values* of $\mathbf{B}$). Thereby, as the maximum eigenvalue of $(\mathbf{B}^{-1})^T\mathbf{B}^{-1}$ always equals the reciprocal of the smallest eigenvalue of $\mathbf{B}^T\mathbf{B}$, we have $\kappa(\mathbf{B}) = \sigma_n(\mathbf{B})/\sigma_1(\mathbf{B}) \geq 1$. We then say that $\mathbf{B}$ is ill-conditioned if $\kappa(\mathbf{B})$ is large.

Consequently, in the presence of a condition problem with respect to $\mathbf{B}$, the inverse ratio $\tau(\mathbf{B}) := 1/\kappa(\mathbf{B}) = \sigma_1(\mathbf{B})/\sigma_n(\mathbf{B})$ will be close to zero. We may thus also consider $\tau(\mathbf{B})$ as a measure of the robustness of the input-output model with respect to errors in the Leontief matrix $\mathbf{B}$. Such a measure has several advantages:

1. As $0 < \sigma_1 \leq \sigma_n$, we conclude that always $\tau(\mathbf{B}) \in (0, 1]$ which facilitates the interpretation and comparison of different τ-figures.

2. The underlying spectral matrix norm is known to be subordinate to the Euclidean vector norm in the sense that the spectral norm is the only matrix norm $\|\cdot\|$ for which the expression $\|-\epsilon\,\mathbf{B}^{-1}\mathbf{F}\mathbf{x}\| \leq |\epsilon|\,\|\mathbf{B}^{-1}\|\,\|\mathbf{F}\|\,\|\mathbf{x}\|$ can hold as a strict equality (cf. Bauer/Fike (1960, p. 1)). This allows to have a sharp bound on the right-hand side of (6).

3. We emphasize that $\tau(\mathbf{B})$ is independent of both gross outputs $\mathbf{x}$ and the unknown error structure $\mathbf{F}$. In this sense, our notion of robustness refers to an invariant property of $\mathbf{B}$.

4. The measure $\tau(\mathbf{B})$ also does not depend on sector numbering. If $\mathbf{P}$ is a $n \times n$ permutation matrix, then $\tilde{\mathbf{B}} := \mathbf{P}^T\mathbf{B}\mathbf{P} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ because of the regularity and orthogonality of $\mathbf{P}$. For the same reason, $\tilde{\mathbf{B}}^T\tilde{\mathbf{B}} = \mathbf{P}^T\mathbf{B}^T(\mathbf{P}^{-1})^T\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \mathbf{P}^{-1}\mathbf{B}^T\mathbf{B}\mathbf{P}$. Hence, $\tilde{\mathbf{B}}^T\tilde{\mathbf{B}}$ and $\mathbf{B}^T\mathbf{B}$ are similar matrices and thereby share the same eigenvalues.

[2]The symmetry is due to the fact that $(\mathbf{B}^T\mathbf{B})^T = \mathbf{B}^T(\mathbf{B}^T)^T = \mathbf{B}^T\mathbf{B}$. Furthermore, introduce as $\mathbf{z}$ an arbitrary real-valued column vector of length n with $\mathbf{z} \neq \mathbf{o}$ in the sense that $\mathbf{z}$ has at least one non-zero component. Then, $\mathbf{w} := \mathbf{B}\mathbf{z} \neq \mathbf{o}$, for otherwise $\mathbf{z} = \mathbf{B}^{-1}\mathbf{o} = \mathbf{o}$ by the assumed regularity of $\mathbf{B}$. We thus obtain $\mathbf{z}^T\mathbf{B}^T\mathbf{B}\mathbf{z} = (\mathbf{B}\mathbf{z})^T\mathbf{B}\mathbf{z} = \mathbf{w}^T\mathbf{w} > 0$.

5. Likewise, note that our measure carries over to the dual input-output model since $\mathbf{B}\mathbf{B}^T$ is again a similarity transformation $\mathbf{S}^{-1}\mathbf{B}^T\mathbf{B}\mathbf{S}$ of $\mathbf{B}^T\mathbf{B}$ with $\mathbf{S} = \mathbf{B}^T$. Hence, the singular values of $\mathbf{B}^T$ and $\mathbf{B}$ coincide.

In particular, we find that $\tau(\mathbf{B}) = 0.03398$ in the case of (2). Note for this case that $\mathbf{B}$ is triangular and thus $\det(\mathbf{B}) = b_{11} \cdot \ldots \cdot b_{nn} = 1$. This shows that the poor condition of $\mathbf{B}$ need not be reflected in a low determinant value $\det(\mathbf{B})$. In order to illustrate this point even further, consider the example of $n = 20$ isolated production sectors with an associated Leontief matrix $\mathbf{B} = \mathrm{diag}(0.5, \ldots, 0.5)$. If in this example an arbitrary off-diagonal entry to $\mathbf{B}$ was in fact non-zero rather than zero, this would have no impact on all but the single corresponding entry to $\mathbf{B}^{-1}$. Accordingly, $\tau(\mathbf{B}) = 1$ (because of $\sigma_1 = \sigma_n = 0.5$), although $\det(\mathbf{B}) = 9.53674 \cdot 10^{-7}$.

Statistical offices and research institutes could henceforth publish along with their input-output tables and related Leontief matrices the respective associated τ-values. This supplementary information would provide the user with a measure of the overall condition of the supplied data. Such a measure could be an important general indicator of the robustness of projections which are made from the data set. However, the calculation of the τ-value of a given Leontief matrix $\mathbf{B}$ essentially involves the non-trivial task of finding reasonable approximations to the largest and smallest of the eigenvalues of the associated product matrix $\mathbf{B}^T\mathbf{B}$. This raises important numerical issues which are treated in the next section.

3 Computational Issues

Almost all advanced numerical software packages like, e.g., EISPACK, IMSL, LINPACK, MATLAB, or the NAG library, provide canned routines for the (approximate) computation of selected eigenvalues or of all eigenvalues of real (as well as complex) matrices. These routines may be firmly used in applied work. However, the user should then be able to make an appropriate choice of an algorithm. Moreover, empirical research, e.g., in the field of computational general equilibrium modeling, often involves the development of proprietary hand-coded software which may have a distinct interface and which also frequently requires thorough optimization for speed of program execution and memory allocation. Therefore, we now turn to the important question of how to exploit the particular structure of an input-output model for computational design. Fortunately, our interest is only in largest and smallest eigenvalues rather than in a complete matrix eigensystem. We can also benefit from the fact that our matrix in question $\mathbf{H} := \mathbf{B}^T\mathbf{B}$ is symmetric and even positive definite. Algebraic eigenvalue problems for such matrices are, in general, wellconditioned. There are essentially two straightforward methods which serve our purpose. These are the power method and the method of matrix squaring.

The classical numerical technique to compute selected eigenvalues of a matrix like $\mathbf{H} = (h_{ij})$ is called power method or vector iteration. The underlying rationale is as follows (see, e.g., Golub/Van Loan (1996, pp. 406-408) and Wilkinson (1999, pp. 570-572)). Let $\mathbf{z}_0 \neq \mathbf{o}$ stand for an arbitrary real-valued column vector of length n and consider the sequence of vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots$ as defined by $\mathbf{z}_{k+1} = \mathbf{H}\mathbf{z}_k$ starting from $k = 0$. Note that $\mathbf{H}$ is a symmetric matrix and will thus possess n independent (moreover: orthonormal) eigenvectors $\mathbf{x}_i$ which span the n-dimensional vector space. Hence, there always exist real-valued scalars $c_1, \ldots, c_n$ such that $\mathbf{z}_0 = c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n$. Consequently, as $\mathbf{H}\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for all eigenvalues λ_i of $\mathbf{H}$ and associated eigenvectors $\mathbf{x}_i$, we conclude that $\mathbf{H}\mathbf{z}_0 = \sum_i c_i\mathbf{H}\mathbf{x}_i = \sum_i c_i\lambda_i\mathbf{x}_i$. Finally:

$$\mathbf{z}_k = \sum_{i=1}^{n} c_i\lambda_i^k\mathbf{x}_i = \lambda_n^k[c_n\mathbf{x}_n + \sum_{i=1}^{n-1} c_i(\frac{\lambda_i}{\lambda_n})^k\mathbf{x}_i]\,. \tag{7}$$

Since $0 < \lambda_i/\lambda_n \leq 1$ for all $i < n$, the bracketed term $[\ldots]$ will converge to a vector of constants such that $\mathbf{z}_{k+1} = \mathbf{H}\mathbf{z}_k \rightarrow \lambda_n\mathbf{z}_k$. The ratios of each two corresponding (non-zero) components of $\mathbf{z}_{k+1}$ and $\mathbf{z}_k$ thereby tend towards the maximum eigenvalue λ_n of $\mathbf{H}$.[3] Next turn to the shifted matrix $\mathbf{H} - \lambda_n\mathbf{I}$ and its related eigenvalues $\mu_i = \lambda_i - \lambda_n \leq 0$ which satisfy $|\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_n| = 0$. The power method will thereupon provide us with μ_1 from which we also obtain the minimum eigenvalue λ_1 of $\mathbf{H}$ according to $\lambda_1 = \mu_1 + \lambda_n$. In all, the power method is computationally straightforward. A respective computer program would add a stopping rule and would also control for overflow and underflow errors by means of an appropriate normalization of the iterates $\mathbf{z}_k$.

Note that for a symmetric matrix like $\mathbf{H}$ the companion sequence of so-called Rayleigh quotients $R(\mathbf{z}_k) := \mathbf{z}_k^T\mathbf{H}\mathbf{z}_k/\mathbf{z}_k^T\mathbf{z}_k$ also converges to λ_n. This can be seen from the gradient $\nabla R(\mathbf{z})$ which comes out as $2[\mathbf{H}\mathbf{z} - R(\mathbf{z})\mathbf{z}]/\mathbf{z}^T\mathbf{z}$ and thus vanishes for an extremum of $R(\mathbf{z})$ if $R(\mathbf{z})$ and $\mathbf{z}$ are an eigenvalue and associated eigenvector, respectively, to $\mathbf{H}$.[4] As λ_n is by assumption the largest such eigenvalue, we conclude that $\lambda_n = \max_{\mathbf{z}\neq\mathbf{o}} R(\mathbf{z})$. The Rayleigh quotients provide, in general, more accurate measures of λ_n than the ratios of two related components of successive vector iterates (see Wilkinson (1999, pp. 172-176)). A smaller number of iterations may thus be needed to extract a tolerable approximation to λ_n (at the

[3] Note that the iterates $\mathbf{z}_k$ converge to an associated eigenvector. This eigenvector will come out as $\mathbf{x}_n$ if λ_n is unique, i.e.: $\lambda_n > \lambda_{n-1} \geq \ldots \geq \lambda_1 > 0$ such that $\lim_{k\rightarrow\infty}\left(\frac{\lambda_i}{\lambda_n}\right)^k = 0$ for all $i < n$. If $\mathbf{H}$ has a multiple maximum root $\lambda_n = \lambda_{n-1} = \ldots = \lambda_p$, then $\mathbf{z}_k$ approaches a point in the subspace spanned by $\mathbf{x}_n, \mathbf{x}_{n-1}, \ldots, \mathbf{x}_p$, depending on the choice of $\mathbf{z}_0$.

[4] Cf. Johnston (1972, pp. 114-116) for differential calculus in vector and matrix notation.

computational cost of calculating $R(\mathbf{z}_1)$, $R(\mathbf{z}_2)$, $\ldots$). We also conclude that the sequence of vectors generated by $\mathbf{z}_{k+1} = \mathbf{H}\mathbf{z}_k/R(\mathbf{z}_k)$ would progress towards $\mathbf{z}_k$ rather than $\lambda_n \mathbf{z}_k$. This demonstrates that the power method is essentially a fixed-point algorithm for the mapping $\mathbf{H}\mathbf{z}/R(\mathbf{z})$ over the set $I\!R^n \setminus \{\mathbf{o}\}$. A sample program in pseudo code can be found in the Appendix.

However, a specific problem may arise in empirical applications if the matrix of production coefficients for intermediate inputs $\mathbf{A}$ satisfies the so-called Brauer-Solow column sum and row sum conditions (see Nikaido (1975, p. 18) and Takayama (1997, pp. 363-364)). We thereby focus on the important class of applications where interindustry flows are measured in money value at current prices or at the prices of some base year:

$$\sum_{j=1}^{n} a_{ji} < 1\,, \sum_{j=1}^{n} a_{ij} < 1 \quad \text{for all } i. \tag{8}$$

The first group of inequalities then demands that the sectoral total cost shares of intermediate inputs must all be less than one as no sector can do without primary inputs. The inequalities in the second group hold for many production sectors as an empirical matter of fact and require that all sectors could make a positive money contribution to final demand from producing just one unit of gross output each. As a consequence of (8), the sums of the absolute column entries, respectively, of the Leontief matrix $\mathbf{B} = \mathbf{I} - \mathbf{A}$ will all be less than 2. Now consider the so-called column sum norm $\|\mathbf{H}\|_1 := \max_i \sum_j |h_{ji}|$ of $\mathbf{H}$ and observe that this norm also has the submultiplicative property $\|\mathbf{H}\|_1 = \|\mathbf{B}^T\mathbf{B}\|_1 \leq \|\mathbf{B}^T\|_1 \|\mathbf{B}\|_1$. Hence, because of (8), $\|\mathbf{H}\|_1 < 4$ regardless of the number of industries n. At this point, we can draw upon the fact that $\|\mathbf{H}\|_1$ is an upper bound for the eigenvalues of $\mathbf{H}$ (cf. Wilkinson (1999, p. 58)). Consequently, if n increases, more eigenvalues will fall into the interval $(0, 4)$ and the distances between the maximum root λ_n and the succeeding eigenvalues of $\mathbf{H}$ are likely to become small. This, in turn, would reduce the speed of convergence of the ratios $(\lambda_i/\lambda_n)^k$ on the right-hand side of (7).

From this viewpoint, note for the power method that $\mathbf{z}_{k+1} = \mathbf{H}^{k+1}\mathbf{z}_0$ and consider the following alternative technique which operates directly on the powers of $\mathbf{H}$ (cf. Wilkinson (1999, pp. 615-617)). This technique is based on the theorem that a symmetric matrix like $\mathbf{H}$ with eigensystem $(\lambda_1, \mathbf{x}_1)$, $\ldots$, $(\lambda_n, \mathbf{x}_n)$ will always possess a representation of the form $\mathbf{H} = \mathbf{X}\,\mathrm{diag}(\lambda_i)\,\mathbf{X}^T$ where $\mathbf{X} := (\mathbf{x}_1 \ \ldots \ \mathbf{x}_n)$ is an orthonormal matrix such that $\mathbf{X}^T\mathbf{X} = \mathbf{I}$. Hence, $\mathbf{H}^k = \mathbf{X}\,\mathrm{diag}(\lambda_i^k)\,\mathbf{X}^T$, i.e.:

$$\mathbf{H}^k = \sum_{i=1}^{n} \lambda_i^k \mathbf{x}_i \mathbf{x}_i^T = \lambda_n^k [\mathbf{x}_n \mathbf{x}_n^T + \sum_{i=1}^{n-1} (\frac{\lambda_i}{\lambda_n})^k \mathbf{x}_i \mathbf{x}_i^T]\,. \tag{9}$$

Again, as k increases starting from $k = 1$, $\mathbf{H}^{k+1} \to \lambda_n \mathbf{H}^k$ such that the ratios of two arbitrary non-zero elements of $\mathbf{H}^{k+1}$ and $\mathbf{H}^k$ like, e.g., of any diagonal elements, approach

λ_n.[5] We then receive $\lambda_1 = \mu_1 + \lambda_n$ in a subsequent step from applying the same algorithm to $\mathbf{H} - \lambda_n \mathbf{I}$.

The successive product matrices $\mathbf{H}^{k+1} = \mathbf{H}^k\mathbf{H}$ $(k = 1, 2, \ldots)$ can be evaluated in a straightforward way by accumulating n^2 dot products (inner products) of corresponding columns and rows, respectively, of $\mathbf{H}^k$ and $\mathbf{H}$ which involves n multiplications and $n-1$ additions each. This amounts to a total of $2n^3 - n^2$ arithmetic operations. A single matrix iterate $\mathbf{H}^{k+1}$ will thereby have an $O(n^3)$ operation count which means that the associated workload is essentially cubic in the number of industries n.[6] At first glance, this compares unfavorably to the fact that the computation of all n components of a single vector iterate $\mathbf{z}_k$ is an $O(n^2)$ operation and thus requires an amount of work that is quadratic in n. However, it only takes s iterations to compute $\mathbf{H}^{2^s}$ by means of repeated matrix squaring! An arbitrary diagonal element of $\mathbf{H}^{2^s+1}$ can then be obtained from a single dot product of the respective row of $\mathbf{H}^{2^s}$ and column of $\mathbf{H}$. Apart from this, matrix squaring greatly benefits from symmetry. This method can thus be effective even in the case of a large Leontief matrix with singular values which are poorly separated. A sample computer program which terminates according to an appropriate stopping rule and which also controls for arithmetic overflow and underflow is in the Appendix.

4 Bounds and Estimates

As we have seen in Section 3, the computation of the τ-value of an empirical Leontief matrix $\mathbf{B}$ will, in general, involve extensive number crunching over a sequence of successive iterations. Yet, quick and useful approximations of this value can already be derived from straight formulas. We suggest three such formulas in this section. They provide an upper bound for τ as well as two τ-estimates. Each formula operates directly on $\mathbf{B}$ and exploits several eigenvalue bounds which are known from the literature.

To begin with, we refer to the fact that the moduli of the roots of a square matrix are bounded by the smallest and largest of the singular values of this matrix (cf. Zurmühl (1964, p. 208)). In particular, if $\mu_1, \ldots, \mu_n$ are the roots of a Leontief matrix $\mathbf{B}$, then $\sigma_1 \leq |\mu_i| \leq \sigma_n$ for all i. We may assume without loss of generality that $|\mu_1| \leq |\mu_2| \leq \ldots \leq |\mu_n|$. Thereby:

$$\tau := \sigma_1/\sigma_n \leq |\mu_1|/|\mu_n| \,. \tag{10}$$

[5] All rows and all columns of $\mathbf{H}^k$ will become parallel either to $\mathbf{x}_n$ or to a point in the subspace spanned by some eigenvectors $\mathbf{x}_n, \mathbf{x}_{n-1}, \ldots, \mathbf{x}_p$, depending on whether there is a distinct root λ_n or a root λ_n of multiplicity $n - p + 1$.

[6] There are more efficient, however also much more complicated, algorithms with an associated minimum $O(n^{2.496})$ burden. The interested reader is referred to Pan (1984).

Now observe that μ_1 is always real as it equals $1-\chi$ where χ is the Frobenius root of $\mathbf{A}$. As $0<\chi<1$ (see Nikaido (1975, pp. 127-131)), μ_1 will be strictly positive. Furthermore, since $\chi \geq \max_i \{a_{ii}\}$ (cf. Taussky (1951)), we conclude that $|\mu_1| \leq \min_i \{b_{ii}\}$. At this point, denote by $P := \min_i \{1-\sum_j a_{ij}\}$ and $Q := \min_i \{1-\sum_j a_{ji}\}$ the minimum row and column sums of $\mathbf{B}$. Then, $|\mu_1| \geq \max\{P, Q\}$ (cp. Brauer (1946, p. 388)). In all:

$$\max\{0, P, Q\} \leq |\mu_1| \leq \min_i \{b_{ii}\}\,. \tag{11}$$

Next recall that the trace of a quadratic real-valued matrix always equals the sum of the matrix eigenvalues. Hence, $\mathrm{tr}(\mathbf{B}) := \sum_i b_{ii} = \sum_i \mu_i \leq |\mu_1| + (n-1)|\mu_n|$. Again, due to $|\mu_1| \leq \min_i \{b_{ii}\}$ because of (11), it follows that $|\mu_n| \geq \frac{1}{n-1}(\mathrm{tr}(\mathbf{B}) - \min_i \{b_{ii}\})$. Finally, let $R := \max_i \{1 - a_{ii} + \sum_{j\neq i} a_{ij}\}$ and $S := \max_i \{1 - a_{ii} + \sum_{j\neq i} a_{ji}\}$ stand for the maxima of the sums of the absolute entries to the n rows and columns, respectively, of $\mathbf{B}$. Then, $|\mu_n| \leq \min\{R, T\}$ (cf. Brauer (1946, p 388)). After all:

$$\frac{1}{n-1}(\mathrm{tr}(\mathbf{B}) - \min_i \{b_{ii}\}) \leq |\mu_n| \leq \min\{R, T\}\,, \tag{12}$$

which completes our preparatory remarks on suitable bounds for the eigenvalues of $\mathbf{B}$.

We are now in the position to establish an upper bound $\bar{\tau}$ for our robustness measure τ and to suggest two deterministic τ-estimates $\hat{\tau}_1$ and $\hat{\tau}_2$. Both the bound and the estimates are build from the right-hand and left-hand sides in (11)-(12). The respective expressions have been collected in Table 1 for notational convenience:

$a := \max\{0, P, Q\}$	$b := \min_i\{b_{ii}\}$
$c := \frac{1}{n-1}(tr(\mathbf{B}) - \min_i\{b_{ii}\})$	$d := \min\{R, T\}$

Table 1: Shortcuts

According to (11)-(12), we verify from (10) that the subsequent ratio $\bar{\tau}$ constitutes an upper bound for τ:

$$\tau \leq \frac{b}{c} =: \bar{\tau}\,. \tag{13}$$

We stress that this bound depends exclusively on the diagonal entries to $\mathbf{B}$ and that it can be evaluated with very little computational effort. A small further effort is required if we replace $|\mu_1|$ and $|\mu_n|$ by the arithmetic means $\frac{1}{2}(a+b)$ and $\frac{1}{2}(c+d)$ of their respective bounds in (11)-(12). The resulting ratio $\hat{\tau}_1$ can be considered an estimate of τ which also takes advantage of off-diagonal information on $\mathbf{B}$:

$$\hat{\tau}_1 := \frac{a+b}{c+d}\,. \tag{14}$$

Our second estimate $\hat{\tau}_2$ is a variation on $\hat{\tau}_1$. We resort to weighted arithmetic means of the bounds a and b and, respectively, c and d:

$$\hat{\tau}_2 := \frac{Aa + (1-A)b}{Cc + (1-C)d}\,, \tag{15}$$

where $A := a/(a+b)$ and $C := c/(c+d)$. All of the above three formulas (13)-(15) can provide valuable approximations to the τ-values of empirical Leontief matrices as will now be illustrated in our final section.

5 Empirical Results

We conclude with a selection of empirical studies. They are meant to illustrate the range of possible and reasonable sizes of our robustness measure τ in applied cases. Another objective of these studies is to examine the usefulness of the concepts which we introduced in the preceding section. Our sample set consists of a total of 26 Leontief matrices for domestic production (one matrix) or for domestic production and competitive imports (all other matrices) of either 12, 58, or 59 industries. We compiled these matrices from input-output tables of the German Federal Statistical Office over the period 1978-1997. Each such table has been constructed by the Office from underlying supply and use accounts as recommended in the ESA and SNA (see Stahmer (1979) for the conversion procedure). We emphasize that we use these tables for the purpose of demonstration only. Therefore, we do not go into further statistical details. The interested reader is referred to the References at the end of this paper.

The τ-measures which can be associated with the Leontief matrices in the data set are provided in Table 2. Each measure has been computed both from the power method (PM) and the method of matrix squaring (MSQ) as indicated in the Appendix. Note that the threshold ϵ was set to 10^{-8}. The resulting total number of iterations k needed to calculate approximations to $\sigma_1(\mathbf{B})$ and $\sigma_n(\mathbf{B})$ is reported in the last two columns of Table 2. For each sample matrix $\mathbf{B}$, we also simulated the effect of a 1% increase in the underlying non-zero input coefficients upon the corresponding matrix inverse. Let ε_{ij} stand for the arithmetic mean of the absolute changes (in %) in all non-zero elements of $\mathbf{B}^{-1}$ which result from such an increase in a single coefficient a_{ij}. The largest mean $\varepsilon := \max\{\varepsilon_{ij}\}$ is communicated in the fourth column of Table 1. The next column holds the respective variance Σ in the elements of $|\Delta\mathbf{B}^{-1}|$.

Table 2 shows that the τ-measures fall into the interval [0.3528, 0.4583] for the smaller 12-sector input-output tables and into the interval [0.0717, 0.1435] for the medium-size 58-sector and 59-sector tables. This indicates that a Leontief matrix tends to be more

Year	n	τ	ε	Σ	k PM	k MSQ
1978	12	0.3984	0.2364%	0.0016%	333	19
1978	58	0.0991	0.3027%	0.0168%	213	19
1980	12	0.3528	0.2559%	0.0020%	174	18
1980	58	0.0842	0.3187%	0.0195%	229	18
1982	12	0.3577	0.2241%	0.0015%	184	18
1982	58	0.0721	0.3843%	0.0275%	210	17
1984	12	0.3620	0.2196%	0.0015%	187	18
1984	58	0.0717	0.3870%	0.0276%	204	17
1985	12	0.3651	0.2184%	0.0015%	224	18
1985	58	0.0765	0.3513%	0.0229%	212	17
1986	12	0.3946	0.1903%	0.0011%	268	19
1986	58	0.0975	0.2965%	0.0157%	203	18
1987	12	0.4098	0.1727%	0.0010%	367	19
1987	58	0.0922	0.3231%	0.0191%	214	18
1988	12	0.4042	0.1797%	0.0011%	368	18
1988	58	0.1111	0.2669%	0.0129%	210	19
1990	12	0.4011	0.1582%	0.0008%	135	19
1990	58	0.1204	0.2263%	0.0098%	216	18
1991	12	0.3962	0.1566%	0.0008%	187	20
1991	58	0.1110	0.2580%	0.0121%	235	19
1993	12	0.3811	0.1427%	0.0006%	253	19
1993	58	0.1186	0.2295%	0.0096%	214	19
1995	12	0.4583	0.1207%	0.0002%	85	17
1995	59	0.1380	0.3169%	0.0099%	154	18
1997	12	0.3909	0.1003%	0.0002%	203	18
1997	59	0.1435	0.2999%	0.0089%	147	18

Table 2: Empirical τ-Measures

robust for less disaggregated tables. We consider this a plausible and intuitive outcome as the condition of a limit 1×1 Leontief 'matrix' is always equal to 1. The maximum average changes ε which are associated with the lowest τ-figures turn out to be as high as 0.2559% (1980) and 0.3870% (1984). We conclude from this that the respective Leontief inverse can be strongly affected even by small changes in the involved input coefficients. The underlying Leontief matrix should thus be taken with caution as potential errors in the matrix elements or unforseen technology changes may have a significant impact on projections which are made from this matrix, on the one hand. On the other hand, the German Federal Statistical Office has gained much experience in the making of input-output tables in the last more than 20 years. This suggests that τ-values in the range

Year	n	τ	$\bar{\tau}$	$\hat{\tau}_1$	$\hat{\tau}_2$
1978	12	0.3984	0.5861	0.3270	0.3461
1978	58	0.0991	0.1460	0.0475	0.0937
1980	12	0.3528	0.5544	0.2939	0.3107
1980	58	0.0842	0.1385	0.0445	0.0873
1982	12	0.3577	0.5928	0.3066	0.3295
1982	58	0.0721	0.1202	0.0387	0.0761
1984	12	0.3620	0.5979	0.3041	0.3296
1984	58	0.0717	0.1199	0.0386	0.0759
1985	12	0.3651	0.6004	0.3082	0.3298
1985	58	0.0765	0.1298	0.0417	0.0820
1986	12	0.3946	0.6387	0.3448	0.3542
1986	58	0.0975	0.1511	0.0486	0.0955
1987	12	0.4098	0.6511	0.3497	0.3622
1987	58	0.0922	0.1390	0.0448	0.0882
1988	12	0.4042	0.6382	0.3401	0.3564
1988	58	0.1111	0.1632	0.0525	0.1031
1990	12	0.4011	0.6859	0.3662	0.3783
1990	58	0.1204	0.1823	0.0587	0.1154
1991	12	0.3962	0.6850	0.3631	0.3782
1991	58	0.1110	0.1673	0.0539	0.1061
1993	12	0.3811	0.7158	0.3817	0.3939
1993	58	0.1186	0.1820	0.0591	0.1167
1995	12	0.4583	0.7661	0.4545	0.4620
1995	59	0.1380	0.2016	0.0596	0.1119
1997	12	0.3909	0.8118	0.4031	0.4640
1997	59	0.1435	0.2108	0.0704	0.1408

Table 3: Empirical Bounds and Estimates

of 0.45 for small tables and 0.15 for medium-size and bigger tables can be considered a benchmark result for robust Leontief matrices.

The τ-measures in our sample are compared in Table 3 to their upper bounds $\bar{\tau}$ and to the estimates $\hat{\tau}_1$ and $\hat{\tau}_2$ which we introduced in Section 4. The content of Table 3 is further illustrated in Figures 1-3. In view of Figure 1, it appears that τ and its bound $\bar{\tau}$ are highly correlated, although $\bar{\tau}$ does not contain any information on the off-diagonal elements of the respective Leontief matrices. Hence, the main diagonal entries to a Leontief matrix already seem to account for a large share in the variation of τ. More formally, the OLS estimation of the model of simple regression $\tau = \beta\bar{\tau} + u$ yields the following results to be

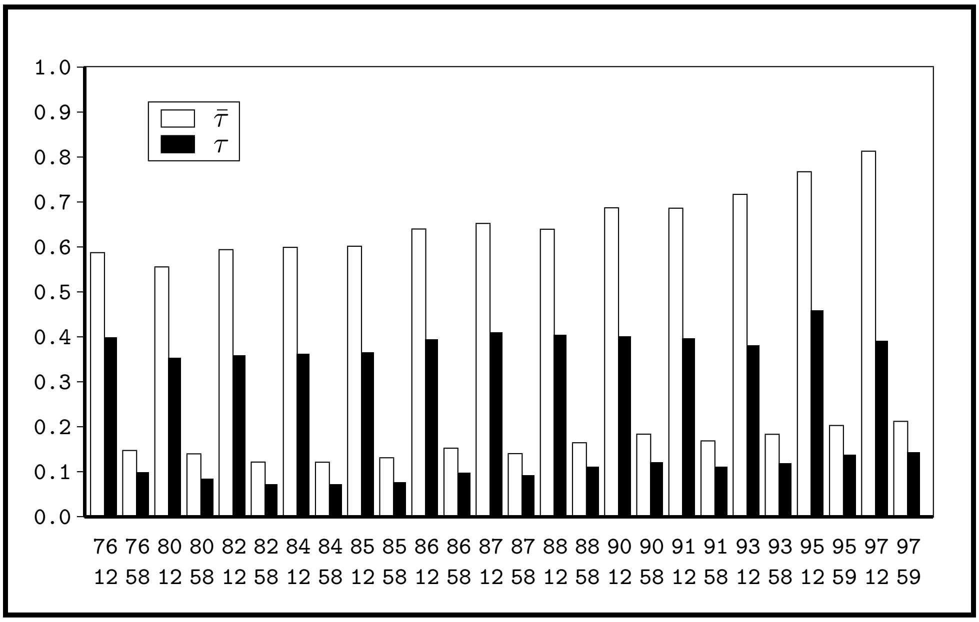

Figure 1: τ-Value and Upper Bound $\bar{\tau}$

explained below:

$$\tau = \underset{(57.41)}{0.5942}\,\bar{\tau}\,, \qquad DW = 2.139\,, \quad R^2 = 0.9868\,. \tag{16}$$

The t-statistic given in parentheses has 25 degrees of freedom. We infer from a single-tail t-test that the estimate $\hat{\beta} = 0.5942$ is statistically significant at the 1% level of significance. We also cannot from the Durbin-Watson statistic DW reject the null hypothesis that there is no first-oder autocorrelation in the disturbances u. Finally, we find that the coefficient of determination R^2 is close to 1. In passing, note for the above model of simple regression with zero intercept that R^2 coincides with the Bravais-Pearson correlation coefficient for a linear relationship between the dependent and explanatory variables.

An inspection of Figures 2 and 3 reveals that $\hat{\tau}_1$ and, in particular, $\hat{\tau}_2$ appear to be useful approximations to τ. For example, the OLS estimation of $\tau = \beta\hat{\tau}_2 + u$ gives:

$$\tau = \underset{(56.26)}{1.0473}\,\hat{\tau}_2\,, \qquad DW = 2.292\,, \quad R^2 = 0.9849\,. \tag{17}$$

The interpretation of this equation is much like the interpretation of equation (16). However, now we should also test the null hypothesis $H_0 : \beta = 1$ against the alternative hypothesis $H_1 : \beta \neq 1$. The corresponding t-statistic comes out as 2.5419 and indicates that we cannot reject the null hypothesis at the 1% significance level.

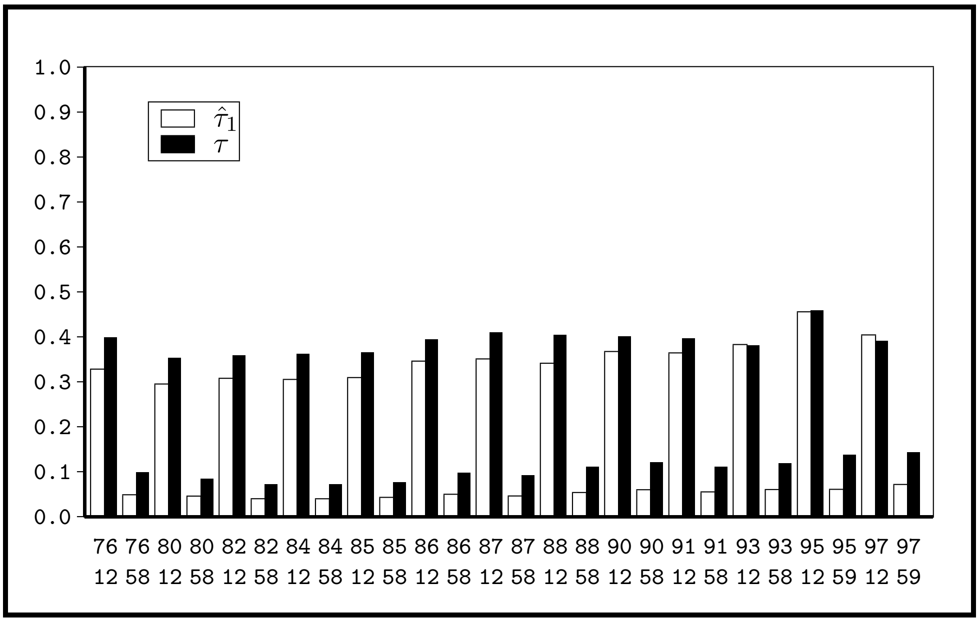

Figure 2: τ-Value and Arithmetic-Mean Ratio $\hat{\tau}_1$

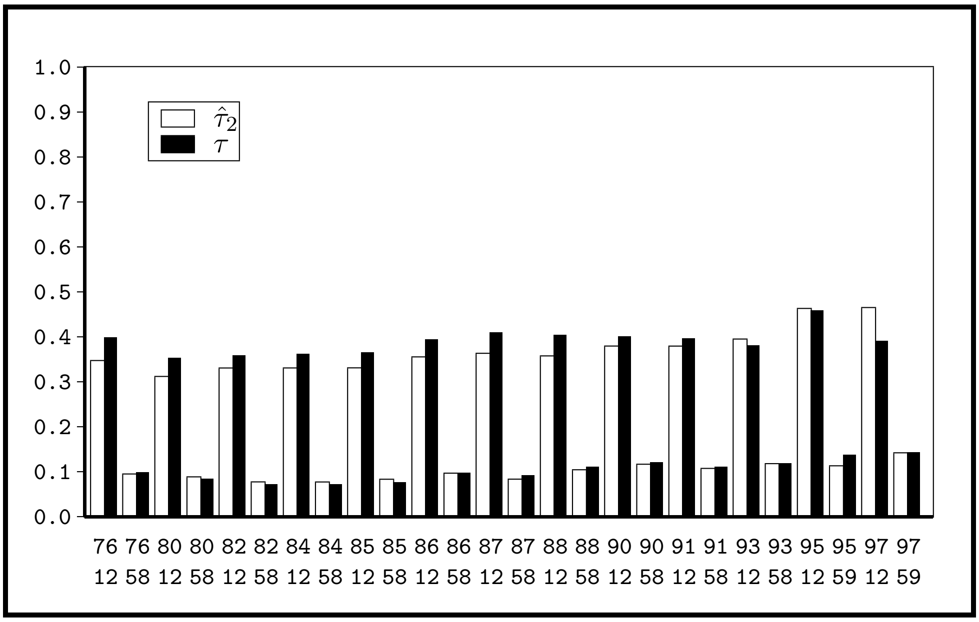

Figure 3: τ-Value and Weighted-Arithmetic-Mean Ratio $\hat{\tau}_2$

We end this section with a short remark on computational evidence. In almost all applications, we found that the respective norm $\|\mathbf{H}\|_1$ was less (often: much less) than 4. Only three applications had an associated norm $\|\mathbf{H}\|_1$ which was slightly above 4. However, even in the cases of up to 59 production sectors, both the power method and the method of matrix squaring worked fine and consumed at most a few seconds of computing time on a modern personal computer in order to find tolerable approximations to the respective τ-values. It turned out that the power method operated noticeably faster on bigger matrices from which we conclude that the problem of poorly separated eigenvalues of $\mathbf{H}$ does not seem to have been an important issue for the matrices in the sample set.

6 Conclusion

This study has investigated the problem of robustness of input-output projections in the context of the new European System of National and Regional Accounts (ESA) and of the System of National Accounts (SNA). We argued that the entries to a symmetric input-output table should be considered as estimates of the underlying true economic relationships and that they may be distorted for various reasons. In particular, both the ESA and SNA recommend that symmetric input-output tables be constructed from separate make and use matrices. These matrices may come from very different and incompatible statistical sources. The conversion of two such matrices into a symmetric input-output table involves the non-trivial transfer of the by-products of each industry to the industry to which these goods belong. Deterministic scaling algorithms which are frequently used in the making of input-output tables are a further cause of data biases, as well as aggregation errors, measurement errors and rounding. Apart from all this, even good estimates may soon be rendered obsolete if technology changes.

These problems may seriously affect the projective power of an empirical input-output model. Therefore, we proposed a measure τ of robustness of input-output projections with respect to errors or changes in the underlying input coefficients for intermediate flows. In contrast to the literature, τ is a global measure which characterizes a complete Leontief matrix. We suggested that statistical offices and research institutes report this measure along with their input-output tables and that it may thus serve as a helpful supplementary information on the overall condition of the supplied data. Formally speaking, our measure τ corresponds to the inverse of the spectral condition number which can be associated with every empirical Leontief matrix. We argued that the numerical computation of this number can be performed with the help of standard routines. Furthermore, we showed that convenient approximations of our measure can already be derived from straight formulas without too much loss of knowledge.

We concluded with a large set of empirical sample applications which were taken from publications of the German Federal Statistical Office. These applications revealed that τ-figures in the range of 0.45 for small tables and 0.15 for medium-size and bigger tables may be considered a benchmark result for robust Leontief matrices. Our approximation formulas proved to be very useful.

Appendix

Table A1 provides a pseudo-code implementation of the power method. The user has to supply the matrix $\mathbf{H} = \mathbf{B}^T\mathbf{B}$ while the routine computes numerical approximations to the maximum root and an associated eigenvector, respectively, of $\mathbf{H}$. Output is returned in λ and $\mathbf{z}$. The routine will terminate if λ does not change by more than ϵ or otherwise after $k_{\max}$ iterations. Both ϵ and $k_{\max}$ are also user input:

Input: $\mathbf{H}, \epsilon, k_{\max}$ Output: $\lambda, \mathbf{z}$

$\lambda := 1,\ \mathbf{z} := \iota,\ k := 0$
`DO`
    $k := k+1,\ \theta := \lambda$
    $\mathbf{y} := \mathbf{Hz}$
    $\mathbf{z} := \mathbf{y}/\lVert\mathbf{y}\rVert_2$
    $\lambda := \mathbf{z}^T\mathbf{Hz}$
`LOOP WHILE` $(\lvert\lambda - \theta\rvert > \epsilon$ `AND` $k < k_{\max})$

Table A1: The Power Method

Note that $\lambda = 1$ and $\mathbf{z}^T = \iota^T := (1 \ldots 1)$ upon initialization. This may be modified, e.g., if helpful a-priori information is available to the user. Also note that the Euclidean norm $\lVert\mathbf{y}\rVert_2 := \sqrt{\mathbf{y}^T\mathbf{y}}$ of $\mathbf{y}$ has been utilized for scaling in order to prevent arithmetic overflow or underflow. The eigenvalue λ will thus be approximated by the Rayleigh quotient associated with a normalized $\mathbf{z}$ which satisfies $\mathbf{z}^T\mathbf{z} = 1$. In general, all vectors and matrices and also λ, θ and ϵ should be stored in a high-precision format, whereas k and $k_{\max}$ will be short (in exceptional cases long) integers.

The technique of matrix squaring is coded in Table A2. The resulting eigenvector to λ is returned in $\mathbf{h}_1$ which is the first column of $\mathbf{H}$. Observe that the user-supplied $\mathbf{H}$ will be destroyed. All elements of $\mathbf{H}$ are scaled by its entry $|h_{ij}|_{\max}$ of maximum modulus. The user may also consider to scale $\mathbf{H}$ with a power of 2 close to $|h_{ij}|_{\max}$ in order to avoid roundoff:

Input:	$\mathbf{H}, \epsilon, k_{\max}$
Output:	$\lambda, \mathbf{h}_1$
$\lambda := 1,\ \mathbf{y} := \mathbf{h}_1,\ k := 0$	
DO	
$\quad k := k+1,\ \theta := \lambda$	
$\quad \mathbf{H} := \mathbf{HH}$	
$\quad \mathbf{H} := \mathbf{H}/\lvert h_{ij}\rvert_{\max}$	
$\quad \lambda := \mathbf{h}_1^T\mathbf{y}/h_{11}$	
LOOP WHILE $(\lvert\lambda - \theta\rvert > \epsilon$ AND $k < k_{\max})$	

Table A2: Method of Matrix Squaring

The symmetry of **H** should be exploited when coding the expression $\mathbf{H} := \mathbf{HH}$. Table A3 gives an example. We assume that the user has allocated memory for a matrix $\mathbf{Q} = (q_{ij})_{n\times n}$ which can hold **H**. The columns of **Q** are labelled $\mathbf{q}_i$ $(i = 1, \ldots, n)$. We economize on the computation of $(n^2 - n)/2$ dot products as we can do without separate calculation of the sup-diagonal entries to **H**:

Input:	$\mathbf{H}, n$
Output:	$\mathbf{H} := \mathbf{HH}$
$\mathbf{Q} := \mathbf{H}$	
FOR $i = 1$ TO n	
$\quad$ FOR $j = 1$ TO i	
$\quad h_{ij} = \mathbf{q}_i^T\mathbf{q}_j$: IF $j < i$ THEN $h_{ji} = h_{ij}$	
$\quad$ NEXT j	
NEXT i	

Table A3: Exploiting Symmetry

References

Almon, C. (2000), "Product-to-Product Tables via Product-Technology with No Negative Flows," *Economic Systems Research*, **12**: 27-43.

Aroche-Reyes, F. (1996), "Important Coefficients and Structural Change: A Multi-Layer Approach," *Economic Systems Research*, **8**: 235-246.

Bacharach, M. (1970), *Biproportional Matrices and Input-Output Change*, Cambridge: Cambridge University Press.

Bauer, F.L., and C.T. Fike (1960), "Norms and Exclusion Theorems," *Numerische Mathematik*, **2**: 137-141.

Brauer, A. (1946), "Limits for the Characteristic Roots of a Matrix," *Duke Mathematical*

Journal, **13**: 387-395.

Council of the European Union (1996), "Council Regulation (EC) No 223/96 of 25 June 1996 on the European System of National and Regional Accounts in the Community," *Official Journal of the European Communities* L 310: 1-515.

Cuello, F.A., and F. Mansouri (1992), "The Identification of Structure at the Sectoral Level: A Reformulation of the Hirschman-Rasmussen Key Sector Indices," *Economic Systems Research*, **4**: 285-396.

German Federal Statistical Office / Statistisches Bundesamt (several years), *Fachserie 18: Volkswirtschaftliche Gesamtrechnungen, Reihe 2: Input-Output-Tabellen.*

Gilchrist, D.A., and L.V. St Louis (1999), "Completing Input-Output Tables using Partial Information with an Application to Canadian Data," *Economic Systems Research*, **11**: 185-193.

Golub, G.H., and C.F. Van Loan (1996), *Matrix Computations*, 3rd ed., Baltimore: Johns Hopkins University Press.

Hazari, B.R. (1970), "Empirical Identification of Key Sectors in the Indian Economy," *The Review of Economics and Statistics*, **52**: 301-305.

Hazari, B.R., and J. Krishnamurty (1970), "Employment Implications of India's Industrialization: Analysis in an Input Output Framework," *Review of Economics and Statistics*, **52**: 181-186.

INFORUM, http://www.inforum.umd.edu.

Jilek, J. (1971), "The Selection of Most Important Input Coefficients," *Economic Bulletin for Europe*, **23**: 86-105.

Johnston, J. (1972), *Econometric Methods*, 2nd ed., New York: McGraw-Hill.

Maaß, S. (1980), *Die Reagibilität von Prognosen mittels Input-Output-Modellen auf Fehler im Datenmaterial: Gezeigt am Beispiel des statischen offenen Leontief-Modells*, Berlin: Duncker & Humblot.

Nikaido, H. (1975), *Introduction to Sets and Mappings in Modern Economics*, 3rd ptg., Amsterdam: North-Holland.

Nyhus, D. (1988), "The INFORUM-ERI International System of Macroeconomic Input-Output Models", in: M. Ciaschini, ed., *Input-Output Analysis: Current Developments*, New York: Chapman and Hall, 391-410.

Pan, V. (1984), *How to Multiply Matrices Faster*, Berlin: Springer.

Rasmussen, P.N. (1957), *Studies in Inter-Sectoral Relations*, Amsterdam: North-Holland.

Schintke, J. (1976), *Sensitivitätsanalysen im statischen offenen Leontief-Modell*, Berlin: Duncker & Humblot.

Schintke, J., and R. Stäglin (1988), "Important Input Coefficients in Market Transactions Tables and Production Flow Tables", in: M. Ciaschini, ed., *Input-Output Analysis: Current Developments*, New York: Chapman and Hall, 43-60.

Stahmer, C. (1979), "Verbindung von Ergebnissen der herkömmlichen Sozialproduktsberechnung und der Input-Output-Rechnung: Überleitungsmodell des Statistischen Bundesamtes," *Allgemeines Statistisches Archiv*, Heft 4: 340-385.

Takayama, A. (1997), *Mathematical Economics*, 2nd ed., 10th ptg., Cambridge: Cambridge University Press.

Taussky, O. (1951), "Bounds for the Characteristic Roots of Matrices II," *Journal of Research of the National Bureau of Standards*, **46**: 124-125.

United Nations et al. (1993), *System of National Accounts 1993*, New York: United Nations Publication.

Wilkinson, J.H. (1999), *The Algebraic Eigenvalue Problem*, 10th ptg., Oxford: Oxford University Press.

Zurmühl, R. (1964), *Matrizen und ihre technischen Anwendungen*, 4th ed., Berlin: Springer.