Measuring the Dependence of the Weighted RAS Method Objective Function through the Analysis of the Dual Problem

F. Escobedo[1 2], R. Mínguez[3] and J. M. Urena[4]

Abstract

Traditionally, input-output tables are updated or regionalized using the well-known RAS approach, which allows an easy implementation by means of an iterative method. This paper uses a variant in which weights are attached to the percentage changes in the input-output cell values. Existing methods primarily focus on the solution of the primal problem, but it is known in mathematical programming that every primal optimization problem has an associated dual one which has important properties that provide extra information about the solution of the problem. This paper is devoted to the solution and interpretation of the weighted RAS dual problem from a sensitivity point of view. As it is well known in the restrictions of the RAS method one of them is not necessary as the condition for compatibility is that the sum of columns has to be equal to the sum of rows. It is found that the optimum values of the dual variables are dependent on the restriction we take out from the initial ones. For this reason it has been developed a method to attain sensitivities that are not affected by which restriction is taken out. In this way a new dual sensitivity matrix is created whose elements represent the sensitivity of the RAS method objective function to the economic interchange between two sectors. On the other hand the highest values of this dual sensitivity matrix can show us which economic transactions are most influential in the change of a certain regional economy.

Key Words: RAS Method, Sensitivity Analysis, Duality, Weighting Techniques.

1 Introduction

Sensitivity analysis consists of determining "how" and "how much" specific changes in the parameters of an optimization problem influence the optimal objective function value and the point (or points) where the optimum is attained.

[1]Current address: Faculty of Economics, University of Groningen, PO Box 800, 9700 AV Groningen, The Netherlands. F.Escobedo@rug.nl. Tel: + 31 50 363 3720. Permanent address: School of Civil Engineering, University of Castilla-La Mancha, Avenida de Camilo Jose Cela 2, 13071 Ciudad Real, Spain. Fernando.Escobedo@uclm.es

[2]The author wants to emphasize their thankfulness to Dr. Jan Oosterhaven for his comments and suggestions to improve the quality and accuracy of the research undertaken.

[3]School of Civil Engineering, University of Castilla-La Mancha, Avenida de Camilo Jose Cela 2, 13071 Ciudad Real, Spain. Roberto.Minguez@uclm.es

[4]School of Civil Engineering, University of Cantabria, Avenida de Los Castros s/n, 39005 Santander, Spain. jose.urena@unican.es

The problem of sensitivity analysis in nonlinear programming has been discussed by several authors, as, for example, Vanderplaats [19], Sobiesky et al. [17], Enevoldsen, [10], Roos, Terlaky and Vial [16], Bjerager and Krend [3], etc. There are at least three ways of deriving equations for the unknown sensitivities: (a) the Lagrange multiplier equations of the constrained optimum (see Sobiesky et al. [17]), (b) differentiation of the Karush–Kuhn–Tucker conditions to obtain the sensitivities of the objective function with respect to changes in the parameters (see Vanderplaats [19], Sorensen and Enevoldsen [18] or Enevoldsen, [10]), and (c) the extreme conditions of a penalty function (see Sobiesky et al. [17]).

The existing methods for sensitivity analysis may present four main limitations:

1. They provide the sensitivities of the objective function value and the primal variables values with respect to parameters, but not the sensitivities of the dual variables with respect to parameters.

2. For different cases there are diverse methods for obtaining each of the sensitivities (optimal objective function value or primal variable values with respect to parameters), but there is no integrated approach providing all the sensitivities at once.

3. They assume the existence of partial derivatives of the objective function or the optimal solutions with respect to the parameters; however, this is not always the case. In fact, there are cases in which partial or directional derivatives fail to exist. In addition, most techniques reported in the literature do not distinguish between right and left derivatives. Ross, Terlaky and Vial [16] state:

   > "It is surprising that in the literature on sensitivity analysis it is far from common to distinguish between left- and right-shadow prices".

   By left- and right-shadow prices they mean left- and right-derivatives of the objective function with respect to parameters at the current optimal value.

4. They assume that the active constraints remain active, which implies that there is no need to distinguish between equality or inequality constraints, because all the active constraints can be considered as equality constraints, and inactive constraints will remain inactive for small changes in the parameters. As a consequence, the set of possible changes (perturbations) has (locally) the structure of a linear space.

2 Methodology

In this section we analyze the sensitivity of the optimal solution of a nonlinear programming problem to changes in the data values. Many authors, as those already mentioned, have studied different versions of this problem. Some of them have dealt with the linear programming problem and discussed the effect of changes of (i) the cost coefficients, (ii) the right hand sides of the constraints or (iii) the constraint coefficients on either (a) the optimal value of the objective function or (b) the optimal solution. A similar analysis has been done for nonlinear problems. However, these authors have dealt only with changes that keep invariant the set of active constraints.

The goal consists in getting a interindustry transactions matxix $\boldsymbol{z}$ as close as posible to the original one $\boldsymbol{z}^o$, using a special measurement between matrixes. It is rather logical, if we do not have more detailed information, to modify as less as possible the economic structure [12].

Consider the following Nonlinear Programming Problem (NLPP) [1] and [13]:

$$\underset{\boldsymbol{z}}{\text{Minimize}} \quad f(\boldsymbol{z}, \boldsymbol{z}^o, \boldsymbol{w}) = \sum_{i=1}^{m} \sum_{\forall j;\ (i,j)\notin\Omega_0} z_{ij} \ln\left(w_{ij}\frac{z_{ij}}{z^o_{ij}}\right), \tag{1}$$

subject to

$$\sum_{j=1}^{m} z_{ij} \quad = \quad u_i;\ i = 1, \ldots, m, : \lambda_i \tag{2}$$

$$\sum_{i=1}^{m} z_{ij} \quad = \quad v_j;\ i = 2, \ldots, m : \mu_j, \tag{3}$$

where $\boldsymbol{w}$ is the vector which contains the associated weights to each transfer, $\boldsymbol{u}$ and $\boldsymbol{v}$ are the sum by rows and columns of the final interindustry transactions matxix, respectively, and the set Ω_0 contains the indexes of the zero-elements of the matrix $\boldsymbol{z}^o$; the use of this set is due to the fact that the objective function is not defined for those specific values. This involves that once the problem is solved (1)-(3) the elements of the new matrix whose indexes agreed with those of Ω_0 set are zero $z_{ij} = 0$; $\forall(i,j) \in \Omega_0$. In this way it could be added an aditional restriction to the original problem that would not change the solution:

$$z_{i,j} = z^o_{ij};\ \forall(i,j) \in \Omega_0 : \kappa_{ij}. \tag{4}$$

All the depicted elements so far belong to the primal problem. On the other hand it is known that any optimization problem has another problem very linked to it named *dual* *problem*, where $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\kappa}$ are their variables (*dual variables*), which are associated to the restrictions of primal problem (2)-(4), respectively.

We should note that the solution of the problem (1)-(4) has to fulfil the following condition:

$$\sum_{i=1}^{m} u_i = \sum_{j=1}^{m} v_j, \tag{5}$$

For that reason the number of restrictions is $2m-1$, that is, one of the restrictions has been eliminated to keep always the condition of compatibility. Later it is explained why it has been chosen to eliminate the sum of the first column and what is the effect if another restriction is eliminated.

The *dual problem* of primal problem (1)-(4) in this situation is defined as:

$$\underset{\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\kappa}}{\text{Maximize}} \quad \phi(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\kappa}) \ , \tag{6}$$

where the function ϕ is the *dual function*.

Using the Lagrangian function:

$$\mathcal{L}(\boldsymbol{z}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\boldsymbol{z}, \boldsymbol{z}^o, \boldsymbol{w}) + \boldsymbol{\lambda}^T \mathbf{h}(\boldsymbol{z}, \boldsymbol{u}) + \boldsymbol{\mu}^T \boldsymbol{g}(\boldsymbol{z}, \boldsymbol{v}) + \boldsymbol{\kappa}^T \boldsymbol{t}(\boldsymbol{z}, \boldsymbol{z}^o), \tag{7}$$

where $\boldsymbol{z}$ is the vector of the *decision variables*, $f : \mathbb{R}^{m^2-n_z} \to \mathbb{R}$ is the objective function, and $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^m$, $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^{m-1}$ and $\mathbf{t} : \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$, where n_z is the number of elements of set Ω_0, and $\mathbf{h}(\boldsymbol{z}, \boldsymbol{u})$, $\boldsymbol{g}(\boldsymbol{z}, \boldsymbol{v})$ and $\boldsymbol{t}(\boldsymbol{z}, \boldsymbol{z}^o)$ are the equality restraints linked to the sum by rows (2), to the sum by columns (3) and the null elements (4), respectively.

The dual problem (6) can be rewritten as:

$$\underset{\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\kappa}}{\text{maximize}} \left[\underset{\boldsymbol{z}}{\text{Infimum}} \quad \mathcal{L}(\boldsymbol{z}, \boldsymbol{z}^o, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\kappa}) \right] \tag{8}$$

Remark 1 *It is supposed that $f, \boldsymbol{h}$, $\boldsymbol{g}$, and $\boldsymbol{t}$ are in such way that the infimum of lagrangian function is reached in some point $\boldsymbol{z}$, in such way that the infimum operator in (8) can be replaced with the minimum operator. For this reason the problem (8) is known as dual problem max–min.* ■

The lagrangian function particularized for the problem (2)-(4) is:

$$\begin{aligned} \mathcal{L}(\boldsymbol{z}, \boldsymbol{z}^o, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\kappa}) \ = \ & \sum_{i=1}^{m} \sum_{\forall j;\ (i,j) \notin \Omega_0} z_{ij} \ln \left(w_{ij} \frac{z_{ij}}{z_{ij}^o} \right) + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{m} z_{ij} - u_i \right) + \\ & \sum_{j=2}^{m} \mu_j \left(\sum_{i=1}^{m} z_{ij} - v_j \right) + \sum_{\forall (i,j) \in \Omega_0} \kappa_{ij} \left(z_{ij} - z_{ij}^o \right). \end{aligned} \tag{9}$$

Definition 1 (Regular point). *The solution to an optimization problem $\boldsymbol{z}^*$, $\boldsymbol{\lambda}^*$, $\boldsymbol{\mu}^*$, $\boldsymbol{\kappa}^*$ and f^* is considered a regular point if the active restrictions gradients are linearly independient.* ■

Theorem 1 (Sensitivities with respect to the objetive function). *If we have the optimization problem (1)-(4) whose solution is a regular point, then it is carried out that:*

$$\begin{aligned}\frac{\partial f(\boldsymbol{z}^*, \boldsymbol{z}^o, \boldsymbol{w})}{\partial u_i} &= -\lambda_i;\ i = 1, \ldots, m, &(10)\\ \frac{\partial f(\boldsymbol{z}^*, \boldsymbol{z}^o, \boldsymbol{w})}{\partial v_j} &= -\mu_j;\ i = 2, \ldots, m, &(11)\\ \frac{\partial f(\boldsymbol{z}^*, \boldsymbol{z}^o, \boldsymbol{w})}{\partial a_{ij}} &= -\kappa_{ij};\ \forall (i,j) \in \Omega_0, &(12)\end{aligned}$$

that is, the sensitivities of the optimum value of the problem objective function (1)-(4) with respect to the changes in the parameters $\boldsymbol{u}$, $\boldsymbol{v}$ and the null terms of $\boldsymbol{z}^o$, that are in the right terms of the restrictions (2)-(4), respectively, agreed with the optimum values of the dual variables associated to each restriction with the opposite sign. ■

The demonstration of this theorem can be found, for example, in Luenberger [11] or Conejo et al. [9].

It is important to insist that the parameters whose sensitivities we want to know have to appear in the right terms of the restrictions for this theorem to be applied. For that reason, it is not possible with this method to know the sensitivity of the objective function with respect to the weights $\boldsymbol{w}$ and the non null terms of the initial matrix $\boldsymbol{z}^o$.

Now the starting point is the non-linear equations system but including the dual variables associated to all the sums by rows and by columns, that is, we take out the condition $\forall j/\ j \geq 2$. If the terms are reorganized we have the following linear equations system:

$$\begin{aligned}\lambda_i + \mu_j &= -1 - \ln\left(w_{ij}\frac{z_{ij}}{z^o_{ij}}\right);\ i = 1;\ \ldots, m;\ \forall j/\ (i,j) \notin \Omega_0 &(13)\\ \lambda_i + \mu_j + \kappa_{ij} &= 0;\ \forall (i,j) \in \Omega_0. &(14)\end{aligned}$$

First of all we focus on (13) which allow us to calculate the multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, and later we will get the values of $\boldsymbol{\kappa}$ using (14).

We have in (13) $m^2 - n_z$ equations while the number of unknowns is $2m$ if we take into account that we include the associated multipliers to every sum by rows and by columns and that the condition of compatibility (5) is kept.

As it can be seen in normal conditions, that is, if the number of null elements in the original matriz $\boldsymbol{z}^o$ fulfils that $n_z \leq m^2 - 2m$, then we have enough number of equations to solve the system (13). In fact, in the majority of the cases we have more equations than the necessary ones; in that case we have to select the system coefficients matrix (13), that is going to be called $\boldsymbol{c}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$, sub-matrix whose rank is equal to $2m$, and proceed to solve.

The starting point for the suitable selection of that coefficients sub-matrix is the original coefficients matrix.

In principle it is considered that there are not null elements in the original matrix $\boldsymbol{z}^o$, that is, $n_z = 0$ to simplify the analysis. In this case the structure of coefficients matrix $\boldsymbol{c}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ and the independent terms vector $\boldsymbol{b}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is:

$$
\boldsymbol{c}_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \left[\begin{array}{ccccccccccc}
\lambda_1 & 0 & 0 & \cdots & 0 & \mu_1 & 0 & 0 & \cdots & 0 \\
\lambda_1 & 0 & 0 & \cdots & 0 & 0 & \mu_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \mu_m \\
\hline
0 & \lambda_2 & 0 & \cdots & 0 & \mu_1 & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots & 0 & 0 & \mu_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \lambda_2 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \mu_m \\
\hline
0 & 0 & \lambda_3 & \cdots & 0 & \mu_1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \lambda_3 & \cdots & 0 & 0 & \mu_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \lambda_3 & \cdots & 0 & 0 & 0 & 0 & \cdots & \mu_m \\
\hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\hline
0 & 0 & 0 & \cdots & \lambda_m & \mu_1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & \lambda_m & 0 & \mu_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_m & 0 & 0 & 0 & \cdots & \mu_m
\end{array}\right]
\text{ and } \boldsymbol{b}_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \left[\begin{array}{c}
-1 - \ln\left(w_{11} z_{11} / z^o_{11}\right) \\
-1 - \ln\left(w_{12} z_{12} / z^o_{12}\right) \\
\vdots \\
-1 - \ln\left(w_{1m} z_{1m} / z^o_{1m}\right) \\
\hline
-1 - \ln\left(w_{21} z_{21} / z^o_{21}\right) \\
-1 - \ln\left(w_{22} z_{22} / z^o_{22}\right) \\
\vdots \\
-1 - \ln\left(w_{2m} z_{2m} / z^o_{2m}\right) \\
\hline
-1 - \ln\left(w_{31} z_{31} / z^o_{31}\right) \\
-1 - \ln\left(w_{32} z_{32} / z^o_{32}\right) \\
\vdots \\
-1 - \ln\left(w_{3m} z_{3m} / z^o_{3m}\right) \\
\hline
\vdots \\
\hline
-1 - \ln\left(w_{m1} z_{m1} / z^o_{m1}\right) \\
-1 - \ln\left(w_{m2} z_{m2} / z^o_{m2}\right) \\
\vdots \\
-1 - \ln\left(w_{mm} z_{mm} / z^o_{mm}\right)
\end{array}\right]. \tag{15}
$$

Using the ortogonalization method proposed by Castillo et al. [4, 5, 8], it can be demonstrated that the coefficients matrix rank $\boldsymbol{c}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is $2m - 1$; therefore the solution of the system (13) is a vectorial space (see Padberg [14] and Castillo et al. [6, 7]) in the following way:

$$
\left[\begin{array}{c} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{array}\right] = \left[\begin{array}{c} \boldsymbol{\lambda}_0 \\ \boldsymbol{\mu}_0 \end{array}\right] + \rho \left[\begin{array}{c} \boldsymbol{\lambda}_1 \\ \boldsymbol{\mu}_1 \end{array}\right], \tag{16}
$$

composed by a particular solution $(\boldsymbol{\lambda}_0, \boldsymbol{\mu}_0)^T$ plus a vectorial space where $\rho \in \mathbb{R}$, and

$(\boldsymbol{\lambda}_1, \boldsymbol{\mu}_1)^T$ are the solution of the homogeneous space.

As ρ can take infinite values, the system, and therefore the dual problem, has infinite solutions. For that reason in the primal problem (1)-(4) it is eliminated the restriction associated to the first row; in that way the linear system needed to solve the dual problem has $2m-1$ rank, so the number of equations $2m-1$ is equal to the number of unknowns $2m-1$ and the system has an unique solution.

Remark 2 *It is necessary to eliminate one of the equations sum by rows or sum by columns for the system (13) and therefore the dual problem has a unique solution. This is due to one of the equations means redundant information for the compatibility condition (5).* ■

In the Figure 1 it is shown a graphic interpretation of the effect of a redundant restraint in the solution of an optimization problem. It has to be underlined that if we take out any of the restrictions $h(\boldsymbol{z}) = 0$, $g_1(\boldsymbol{z}) = 0$ or $g_2(\boldsymbol{z}) = 0$ the solution of the primal problem is the same; nevertheless in the dual problem there are infinite combinations of the dual variables values λ, μ_1, and μ_2 so if we multiply the dual variables by the restraints gradients we get the objective function gradient with the opposite sign. For this reason it is said that there is a redundant restraint.

The next step consists in determining:

1. Which submatrix with dimensions $((2m-1) \times (2m-1))$ is selected.

2. If the selection of the submatrix is decisive in the solution.

3. Choose which of the dual variables associated to row and column restrictions is removed and, again, know which are the consequences if one specific dual variable is chosen to be removed.

To give an answer to these questions initially we are going to select the following $2m \times 2m$ dimension block from the matrix $\boldsymbol{c}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ and its corresponding independent terms vector:

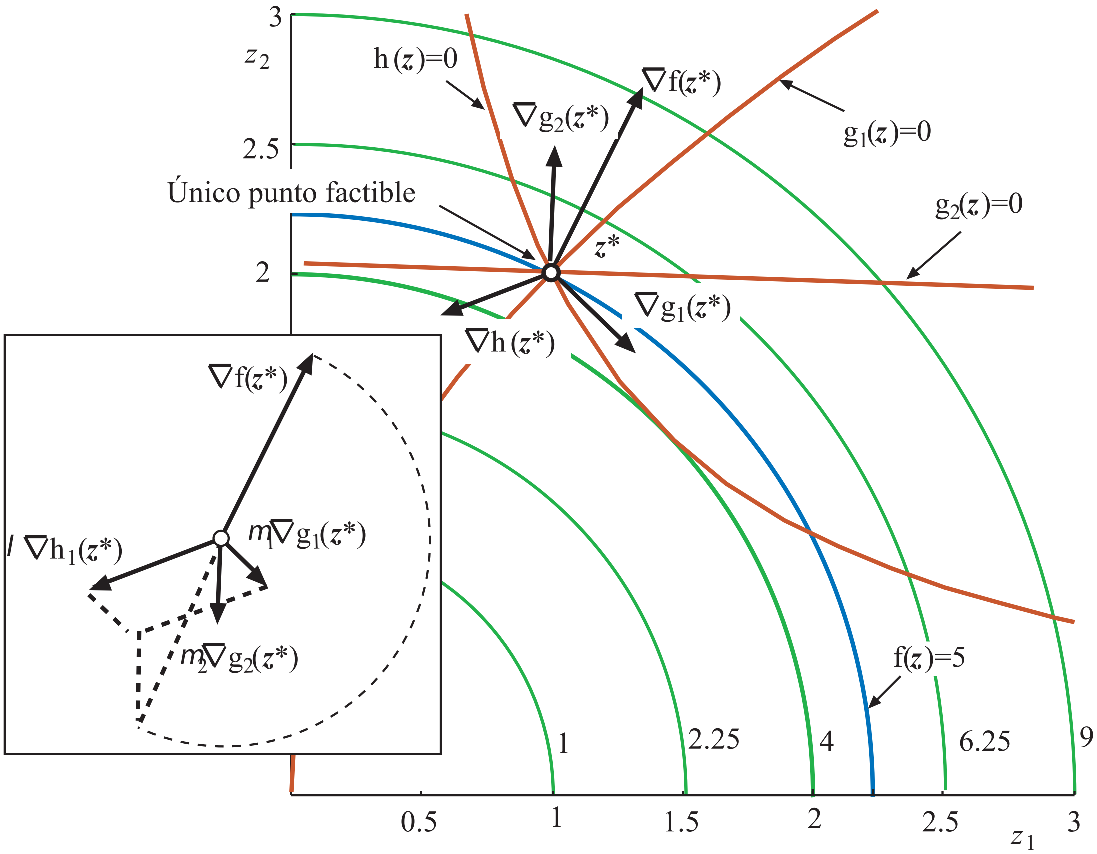

Figure 1: Graphic interpretation of a redundant restraint in the bidimensional case.

$$
\boldsymbol{c}_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \left[\begin{array}{ccccccc|c|cccc}
\lambda_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \mu_1 & 0 & 0 & \cdots & 0 \\ \hline
\lambda_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \mu_2 & 0 & \cdots & 0 \\ \hline
0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & 0 & \mu_1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \lambda_3 & \cdots & \cdots & \cdots & 0 & \mu_1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & \lambda_i & \ddots & 0 & \mu_i & 0 & 0 & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \lambda_m & \mu_1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \lambda_m & 0 & \mu_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \lambda_m & 0 & 0 & 0 & \cdots & \mu_m
\end{array}\right] \text{ and } \boldsymbol{b}_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \left[\begin{array}{c}
-1-\ln\left(w_{11}z_{11}/z^o_{11}\right) \\ \hline
-1-\ln\left(w_{12}z_{12}/z^o_{12}\right) \\ \hline
-1-\ln\left(w_{21}z_{21}/z^o_{21}\right) \\
-1-\ln\left(w_{31}z_{31}/z^o_{31}\right) \\
\vdots \\
-1-\ln\left(w_{i1}z_{i1}/z^o_{i1}\right) \\
\vdots \\
-1-\ln\left(w_{m1}z_{m1}/z^o_{m1}\right) \\
-1-\ln\left(w_{m2}z_{m2}/z^o_{m2}\right) \\
\vdots \\
-1-\ln\left(w_{mm}z_{mm}/z^o_{mm}\right)
\end{array}\right]. \tag{17}
$$

It can be demonstrated that the former matrix has the rank $2m-1$, so if the ortogonalization method is used the infinite solutions of the system (13) may be obtained. This system is finally in the following way:

$$
\left[\begin{array}{c}
\lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_i \\ \vdots \\ \lambda_m \\ \hline \mu_1 \\ \hline \mu_2 \\ \mu_3 \\ \mu_4 \\ \vdots \\ \mu_m
\end{array}\right] = \left[\begin{array}{c}
-1-\ln\left(w_{11}z_{11}/z^o_{11}\right) \\
-1-\ln\left(w_{21}z_{21}/z^o_{21}\right) \\
-1-\ln\left(w_{31}z_{31}/z^o_{31}\right) \\
\vdots \\
-1-\ln\left(w_{i1}z_{i1}/z^o_{i1}\right) \\
\vdots \\
-1-\ln\left(w_{m1}z_{m1}/z^o_{m1}\right) \\ \hline
0 \\ \hline
-\ln\left(w_{12}z_{12}/z^o_{12}\right)+\ln\left(w_{11}z_{11}/z^o_{11}\right) \\
-\ln\left(w_{m3}z_{m3}/z^o_{m3}\right)+\ln\left(w_{m1}z_{m1}/z^o_{m1}\right) \\
-\ln\left(w_{m4}z_{m4}/z^o_{m4}\right)+\ln\left(w_{m1}z_{m1}/z^o_{m1}\right) \\
\vdots \\
-\ln\left(w_{mm}z_{mm}/z^o_{mm}\right)+\ln\left(w_{m1}z_{m1}/z^o_{m1}\right)
\end{array}\right] + \rho \left[\begin{array}{c}
-1 \\ -1 \\ -1 \\ \vdots \\ -1 \\ \vdots \\ -1 \\ \hline 1 \\ \hline 1 \\ 1 \\ 1 \\ \vdots \\ 1
\end{array}\right], \tag{18}
$$

where only elements from the first row and column from the matrixes $\boldsymbol{z}$, $\boldsymbol{z}^o$ y $\boldsymbol{w}$ take part.

It has to be noticed, for example, that if we want the dual variable μ_1 value to be always zero, the only value of ρ that would fulfil that condition would be $\rho = 0$; in that case there would be one an only solution to the dual problem, that would be the same solution obtained when the system (17) is resolved if the second row from the coeficients matrix and from the independent terms vector and the $(m+1)$th column from the coeficients matrix are removed.

Precisely this solution corresponds with the system, in which the first restraint linked to the columns sum is taken out, that is, the multiplier μ_1.

The spatial or economic meaning of this solution is an easy one, as it was told in the Theorem 1; the dual variables, in particular $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, are the sensitivities of the objective function with respect to the changes in the parameters $\boldsymbol{u}$ y $\boldsymbol{v}$, respectively. As the partial derivative involves the change in the objective function when **only** one parameter changes, obviously if the rest of the parameters $\boldsymbol{u}$ and $\boldsymbol{v}$ do not change when one of them does, the contition of compatibility (5) is not fulfilled and the problem has no solution. For this reason it is necessary not to include one of the dual variables to allow the fulfillment of the contition of compatibility for every occasion, and in the particular case of the solution (18), the v_1 value is used to fulfil the contition of compatibility.

After this discussion the following question which may be raised is if it is possible to take out another dual variable or restraint and, in this case, which would be the solution of the dual problem. The answer is affirmative: if we start from the infinite solutions let's suppose that we want to take out the i th restraint in the row sums which corresponds to dual variable λ_i. For that it is enough to select the suitable value of parameter ρ to make zero λ_i in the final solution. In this case $\rho = -(1 + \ln(w_{i1}z_{i1}/z^o_{i1}))$ so the final solution is:

$$\left[\begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_i \\ \vdots \\ \lambda_m \\ \hline \mu_1 \\ \hline \mu_2 \\ \mu_3 \\ \mu_4 \\ \vdots \\ \mu_m \end{array}\right] = \left[\begin{array}{c} -1 - \ln(w_{11}z_{11}/z^o_{11}) + (1 + \ln(w_{i1}z_{i1}/z^o_{i1})) \\ -1 - \ln(w_{21}z_{21}/z^o_{21}) + (1 + \ln(w_{i1}z_{i1}/z^o_{i1})) \\ -1 - \ln(w_{31}z_{31}/z^o_{31}) + (1 + \ln(w_{i1}z_{i1}/z^o_{i1})) \\ \vdots \\ 0 \\ \vdots \\ -1 - \ln(w_{m1}z_{m1}/z^o_{m1}) + (1 + \ln(w_{i1}z_{i1}/z^o_{i1})) \\ \hline -(1 + \ln(w_{i1}z_{i1}/z^o_{i1})) \\ \hline -\ln(w_{12}z_{12}/z^o_{12}) + \ln(w_{11}z_{11}/z^o_{11}) - (1 + \ln(w_{i1}z^o_{i1}/a_{i1})) \\ -\ln(w_{m3}z_{m3}/z^o_{m3}) + \ln(w_{m1}z_{m1}/z^o_{m1}) - (1 + \ln(w_{i1}z^o_{i1}/a_{i1})) \\ -\ln(w_{m4}z_{m4}/z^o_{m4}) + \ln(w_{m1}z_{m1}/z^o_{m1}) - (1 + \ln(w_{i1}z^o_{i1}/a_{i1})) \\ \vdots \\ -\ln(w_{mm}z_{mm}/z^o_{mm}) + \ln(w_{m1}z_{m1}/z^o_{m1}) - (1 + \ln(w_{i1}z^o_{i1}/a_{i1})) \end{array}\right]. \tag{19}$$

In the same way we may proceed with the rest of the restraints or dual variables; therefore it is clear that regardless the restraint we remove the solution of the dual problem could be obtained for any removed restraint.

On the other hand it is interesting to examine what happens if in the obtained solution some of the terms (i, j) corresponds to a zero element in the matrix $\boldsymbol{z}^o$; in that case it could not be obtained the dual variable value as we would have an indetermined value $\frac{0}{0}$, and we would have to select another solution which allow us calculate the multiplier. As that process may be hardworking and many times the number of non-zero elements is not too high it is developed in this paper an alternative method which allows the calculation of the dual variables that cannot be calculated with the particular solution in (18).

After some previous tasks the following equations:

$$\lambda_i = -1 - \log \frac{u_i}{\sum_{j=1}^{m} \frac{z_{ij}^o}{w_{ij}} \exp(-\mu_j)}; \; i = 1, \ldots, m \tag{20}$$

$$\mu_j = -1 - \log \frac{v_j}{\sum_{i=1}^{m} \exp(-\lambda_i) \frac{z_{ij}^o}{w_{ij}}}; \; j = 2, \ldots, m. \tag{21}$$

allow us to calculate in an iterative way the rest of the dual variables.

With respect to the dual variables $\boldsymbol{\kappa}$, once we have the variables $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, they may be obtained by means of the equation:

$$\kappa_{ij} = -\lambda_i - \mu_j; \forall (i,j) \in \Omega_0, \tag{22}$$

that is, the sensitivity of the objective function with respect to a fixed parameter $z_{ij}^o = 0$ is the sum of the dual variables associated to its row λ_i and to its column μ_j.

The weighted RAS procedure for the dual problem solution may be resolved by means of the following algorithm:

Algorithm 1 (Dual solution of the RAS weighted problem).

Input: *Data including the initial transactions matrix $\boldsymbol{z}^o$, the weights associated to each element in the former matrix $\boldsymbol{w}$, and the rows and columns sums in the final transactions matrix $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively, and the solution of the primal problem $\boldsymbol{z}$. The maximum number of interactions $i_{\max}$, and a tolerance ϵ to control the process convergence.*

Output: *Final values of the dual variables $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\kappa}$ with a tolerance ϵ.*

Step 1: *Set creation Ω_0, that is, for all elements of the original matrix $\boldsymbol{z}^o$ check which ones are zero and, in that case, store the corresponding row and column indexes.*

Step 2: *Calculation of dual variables values* $\boldsymbol{\lambda}$*,* $\boldsymbol{\mu}$ *for the non-zero terms in the first row and column of the initial transactions matrix* $\boldsymbol{z}^o$ *by means of the solution provided by (18) for* $\rho = 0$*.*

Step 3: *Iterative procedure for the calculation of the multipliers that could not be obtained in an analitic way. While the error is bigger than the tolerance* ϵ *and the number of iteractions* n_{iter} *is lower than the allowed limit* $i_{\max}$*, proceed with the following stages:*

1. *Obtain the new values of* $\lambda_i; \forall i/z^o_{i1} = 0 \ \& \ u_i \neq 0$ *associated to the rows through (20).*

2. *Obtain the new values of* $\mu_j; \forall j/z^o_{j1} = 0 \ \& \ u_j \neq 0$ *associated to the columns through (21).*

3. *Calculate the error value as the relative maximum diference of the current value of all the elements of the vector* $[\boldsymbol{\lambda} \ \boldsymbol{\mu}]^T$ *with respect to the values in the former iteraction, and come back to the stage 1 of the* **step 3***.*

In the contrary case the procedure convergence has been achieved with the required tolerance adding the following step.

Step 4: *Calculate the dual variables associated to the zero terms in the initial transactions matrix and give back the solution of* $\boldsymbol{\lambda}$*,* $\boldsymbol{\mu}$ *and* $\boldsymbol{\kappa}$*.* ■

Once the multipliers have been obtained, the expression (18) allows calculate any combination of dual variables depending on the restraint which is removed in (2)-(3), and therefore the main issue is to know if the removed restraint has an influence in the dual variables values. In this point some questions are raised:

1. Does it mean that different sensitivities are obtained according to the removed restraint?

2. Would not it be better to obtain sensitivities which do not rely on the removed restraint? In this way the dual problem would correspond with the uniqueness of the primal problem.

The answer to these questions is affirmative so we have to rearrange the assesment of the dual problem in the way of obtaining sensitivities which are independent from the removed restraint. If the solution structure (18) is carefully analyzed, specially the associated term to the vectorial space, it can be observed that the associated terms to $\boldsymbol{\lambda}$

and $\boldsymbol{\mu}$ are equal but with the opposite sign. For this reason it can be demonstrated that the following condition is fulfilled:

$$\begin{array}{lcl} \lambda_i + \mu_j + \kappa_{ij} & = & \text{cte.};\ \forall i = 1, \ldots, m;\ j = 1, \ldots, m;\ (i,j) \in \Omega_0 \\ \lambda_i + \mu_j & = & \text{cte.};\ \forall i = 1, \ldots, m;\ j = 1, \ldots, m;\ (i,j) \notin \Omega_0. \end{array} \tag{23}$$

This means that the sum of the associated multipliers to the rows sums and to the columns sums, respectively, is constant independently of the restriction removed to fulfil the compatibility condition. In this way the **dual sensitivity $\boldsymbol{\Phi}$** matrix may be obtained in the following way:

$$\boldsymbol{\Phi} = \begin{bmatrix} \lambda_1 + \mu_1 + \kappa_{11} & \lambda_1 + \mu_2 + \kappa_{12} & \lambda_1 + \mu_3 + \kappa_{13} & \cdots & \lambda_1 + \mu_m + \kappa_{1m} \\ \lambda_2 + \mu_1 + \kappa_{21} & \lambda_2 + \mu_2 + \kappa_{22} & \lambda_2 + \mu_3 + \kappa_{23} & \cdots & \lambda_2 + \mu_m + \kappa_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_m + \mu_1 + \kappa_{m1} & \lambda_m + \mu_2 + \kappa_{m2} & \lambda_m + \mu_3 + \kappa_{m3} & \cdots & \lambda_m + \mu_m + \kappa_{mm} \end{bmatrix}, \tag{24}$$

where each term Φ_{ij} with the opposite sign represents the sensitivity of the objective function when the output economic sector ith increases one unit and that growth means economic sector jth purchases.

Remark 3 *It has to be underlined that in the matrix (24) the elements* $\kappa_{ij} = 0;\ (i,j) \notin \Omega_0$ *are zero.* ∎

3 Conclusions

This paper provides a procedure for solving the primal and dual weighted RAS method, obtaining not only the optimal solution of the primal problem but information of the sensitivities of the solution with respect the parameters. The main conclusions derived from the work reported can be summarized as follows:

1. Due to the excess of constraints in the mathematical definition of the RAS problem the dual problem has infinite solutions; for this, a new procedure to obtain a unique solution of the dual problem holding the compatibility condition is provided.

2. It turns out that this unique solution of the dual problem allows obtaining the sensitivity matrix, invariant sensitivities (measurements) of the primal RAS objective function with respect constraint parameters.

3. The meaning of the sensitivity matrix obtained is very useful for the economic analysis. That matrix shows us which is the influence of the interchanges between economic sectors in the development of the economic structure of a certain region, in the case of updating I-O tables, and the influence of those interchanges in the regional economic singularity in a national context in the case of regionalizing I-O tables.

4. The solution of the dual problem can be obtained efficiently once the solution of the primal RAS problem is known using an iterative method.

References

[1] Bazaraa, M. S., Jarvis, J. J. and Sherali, H. D. (1990). *Linear Programming and Network Flows* second ed. John Wiley & Sons, New York.

[2] Bazaraa, M. S., Sherali, H. D. and Shetty, C. M. (1993). *Nonlinear Programming. Theory and Algorithms* second ed. John Wiley & Sons, New York.

[3] Bjerager, P. and Krenk, S. (1989). Parametric sensitivity in first order reliability theory. *J. Engineering Mechanics, ASCE* **115,** 1577–1582.

[4] Castillo, E., Cobo, A., Jubete, F. and Pruneda, R. E. (1999). *Orthogonal Sets and Polar Methods in Linear Algebra: Applications to Matrix Calculations, Systems of Equations and Inequalities, and Linear Programming.* John Wiley & Sons, New York.

[5] Castillo, E., Cobo, A., Jubete, F., Pruneda, R. E. and C., C. (2000). An orthogonally based pivoting transformation of matrices and some applications. *SIAM Journal on Matrix Analysis and Applications* **22,** 666–681.

[6] Castillo, E., Conejo, A., Pedregal, P., García, R. and Alguacil, N. (2001). *Building and Solving Mathematical Programming Models in Engineering and Science.* John Wiley & Sons Inc., New York. Pure and Applied Mathematics: A Wiley-Interscience Series of Texts, Monographs and Tracts.

[7] Castillo, E. and Jubete, F. (2004). The γ-algorithm and some applications. *International Journal of Mathematical Education in Science and Technology* 369–389.

[8] Castillo, E., Jubete, F., Pruneda, E. and Solares, C. (2002). Obtaining simultaneous solutions of linear subsystems of equations and inequalities. *Linear Algebra and its Applications* 131–154.

[9] Conejo, A., Castillo, E., Mínguez, R. and García-Bertrand, R. (2006). *Decomposition techniques in mathematical programming. Engineering and science applications*. Springer Berlin Heidelberg, New York.

[10] Enevoldsen, I. (1994). Sensitivity analysis of reliability-based optimal solution. *Journal of Engineering Mechanics* **120,** 198–205.

[11] Luenberger, D. G. (1989). *Linear and Nonlinear Programming* second ed. Addison-Wesley, Reading, Massachusetts.

[12] Miller, R. E. (1998). *Methods of Interregional and Regional Analysis*. Ashgate, Brookfield, VE, USA. ch. Regional and interregional input-output analysis, pp. 41–133.

[13] Miller, R. E. and Blair, P. D., Eds. (1985). *Input-output analysis : Foundations and extensions*. Prentice-Hall, Englewood Cliffs, NJ, USA.

[14] Padberg, M. (1995). *Linear Optimization and Extensions*. Springer, Berlin, Germany.

[15] Press, W. H., Teukolsky, S. A., Vetterling, W. T. and Flannery, B. P. (1992). *Numerical Recipes in C*. Cambridge University Press, New York.

[16] Roos, C., Terlaky, T. and Vial, J. P. (1997). *Theory and Algorithms for Linear Optimization: An Interior Point Approach*. John Wiley & Sons, Chichester, UK. Wiley-Interscience Series in Discrete Mathematics and Optimization.

[17] Sobieski, J. S., Barthelemy, J. F. and Riley, K. M. (1982). Sensitivity of optimal solutions of problems parameters. *AIAA J.* **20,** 1291–1299.

[18] Sorensen, J. D. and Enevoldsen, I. (1992). Sensitivity analysis in reliability-based shape optimization. In *NATO ASI*. ed. B. H. V. Topping. vol. 2 21 of *E*. Kluwer Academic Publishers, Dordrecht, The Nethelands. pp. 617–638.

[19] Vanderplaats, G. N. (1984). *Numerical Optimization Techniques for Engineering Design*. McGraw-Hill, New York.