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WHY AND WHEN ARE THERE NEGATIVE COEFFICIENTS IN JOINT PRODUCTION SYSTEMS WITH "COMMODITY TECHNOLOGY"

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Modern input output statistics as well as some theoretical models of growth and distribution distinguish between input (use) and output (make) matrices, both of dimension commodity by activity (industry). Input output analysts, on the contrary, aim at the construction of commodity by commodity input-output tables. Amongst the several methods which have been developed the "commodity technology model" appear as the only one which fulfils important axioms. It is shown, that the "commodity technology assumption" is based on the proposition that all processes are single production processes and, hence, defines joint production away. A joint production approach is proposed, which formally yields the same results as the "commodity technology model" but does not rely on its restrictive assumptions. Furthermore, it is a well known but nevertheless cumbersome feature of the "commodity-technology" assumption that the matrix of input coefficients as well as the Leontief-inverse may contain negative elements. The main purpose of this paper is to debunk the economic reason for the occurrence of these negatives and to provide meaningful conditions for the existence of nonnegative solutions in joint production systems. Theoretical concepts such as "separately producible commodities" or "all-productive systems of production" developed by Sraffian scholars appear as useful tools and devices.

Joint production is – in actual fact – an important issue (Steedman, 1984). Most statistical offices publish frequently input output statistics which are based on make and use tables. There exist also some theoretical models of growth and distribution, in particular models built on the works of John von Neumann and Piero Sraffa, which are based on input and output tables and therefore are able to account for joint production. Nevertheless joint production is widely neglected. One will hardly find a textbook, which do not treat joint production as an unimportant complication rather than an interesting issue. The basic concepts of IO-analysis are also limited to single production. Input output coefficients are defined as inputs of commodity i per unit of output of a single product j. in the case of joint production inputs refer to bundles of outputs and can, by no means, be allocated to a single product. But exactly this is the aim of so called technology assumptions and other methods.

There are two basic technology assumptions, the commodity technology model (CTM) and the industry technology model, several mixed or hybrid assumptions and other methods. Kop Jansen and ten Raa (1990) and ten Raa and Rueda-Cantuche (2003) postulate some desirable and reasonable properties and test how well some methods to construct commodity by commodity coefficients perform. It appears that only the CTM exhibit all the desired properties.

In this paper it is argued (on pure theoretical grounds), that “technology assumptions” define joint production away and are, therefore, not suitable for that case. If there are some activities which produce jointly more than one homogenous product it is – in general – not possible to derive IO-coefficients which refer to an output of one unit of a homogenous product. The validity of the concepts of IO-coefficients and of a Leontief-Inverse is limited to single production systems and cannot easily be carried over to general joint production.

The characteristics of joint production are studied and the very reason of negative intensities and negative quantities is detected.

It is demonstrated that a joint production system may not always adjust completely to given final demand and, therefore, excess production is possible. Following the track of von Neumann and applying the rule of free goods (free disposal) the solutions for quantities and for prices can be determined by a linear Programme.

Finally, a class of joint production systems is defined which have all important characteristics of single production systems. Useful concepts developed by Neo-Ricardian scholars such as “adjustability” or “all-productiveness” are presented and are utilised.

Let $\mathbf{A},\mathbf{B}\in R_+^{n\times n}$ be semi-positive square matrices of inputs and outputs of dimension commodity by activity. Note that it is assumed that the number of commodities equals the number of activities.

Assume ***constant returns to scale***, i.e. if $\left(\mathbf{a}^k,\ \mathbf{b}^k\right)$ is a feasible process characterised by the k-th column of the input and of the output matrix, then, for all intensities $x_k > 0$, $\left(\mathbf{a}^k x_k,\ \mathbf{b}^k x_k\right)$ is also a feasible process. In other words, a proportional change in the inputs of an activity results into a proportional change in the outputs of that activity without changing the proportions of the outputs.

It is assumed, that the composition of outputs (product mix) is constant for each activity. However, this assumption does not postulate commodity technology.

The CTM assumes a single production process for each commodity *j*, characterised by a vector $\mathbf{c}^j$ whose elements are the quantities of inputs of commodity *i* required to produce one unit of product *j*. The inputs used by an activity *k* depend on the output structure of that activity. Hence each elements a_{ik} of the input matrix **A**, which refer to inputs of an activity *k*, reflects a weighted average of the underlying commodity by commodity coefficients c_{ij} where the weights are determined by the outputs of process *k*.

Hence the CTM is defined by

$$(1) \qquad \sum_{j=1}^{n} c_{ij} b_{jk} = a_{ik} \quad \Leftrightarrow \quad \mathbf{CB} = \mathbf{A}\,.$$

If **B** is regular the coefficients of the unobservable single production processes can be determined by

$$(2) \qquad \mathbf{C} = \mathbf{AB}^{-1}$$

The CTM assumes that there is – in principle – a single production system which is not directly observable. Observable are “industries” which produce some quantities of more than one commodity by combining more than one single production process. Joint production is thus considered as a statistical problem caused by aggregation. This assumption may have some justification if aggregate industries are concerned, which produce one characteristic primary product and, in addition, one or more secondary products which themselves are characteristic products of other industries. In this case, it is straightforward to calculate non-negative matrices via a systematic search of possible sources of inaccuracy in the basic make and use tables[1]. If, on the other hand, joint production proper is concerned, the CTA misses the point, is unacceptable and, after all, is not needed.

A straightforward approach which can be applied to joint production without any “technology assumption” is described above.

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[1] This route has been proposed by Steenge, 1990.

Given input and output matrices, **A** and **B**, and a vector of required net products (final demand), denoted by **d**. The problem is to find a vector of intensities **x** which support the bundle **d** such that

(3) $$(\mathbf{B}-\mathbf{A})\mathbf{x}=\mathbf{d}$$

The straightforward solution of (3) is

(4) $$\mathbf{x}=(\mathbf{B}-\mathbf{A})^{-1}\mathbf{d}\,.$$

Furthermore, we may calculate the vector of gross products by

(5) $$\mathbf{q}=\mathbf{B}\mathbf{x}=\mathbf{B}(\mathbf{B}-\mathbf{A})^{-1}\mathbf{d}\,.$$

Note that the inverse matrices in (4) and (5) may be written as

(6) $$(\mathbf{B}-\mathbf{A})^{-1}=\mathbf{B}^{-1}\left(\mathbf{I}-\mathbf{A}\mathbf{B}^{-1}\right)^{-1}$$

and

(7) $$\mathbf{B}(\mathbf{B}-\mathbf{A})^{-1}=\left(\mathbf{I}-\mathbf{A}\mathbf{B}^{-1}\right)^{-1}$$

Note that the latter inverse may be considered as a generalisation of the Leontief-inverse.

It is clear that the results for these solutions are the same as one would obtain by using the CTM. However, no “technology assumptions” are required nor any matrix of commodity by commodity input coefficients have been calculated. Furthermore, meaningful solutions may be obtained even if the output matrix is singular or if $\mathbf{A}\mathbf{B}^{-1}$ contains negative elements.

Though we have avoided the problem of a singular output matrix and negative commodity by commodity input coefficients, there still remains the possibility of negative intensities and negative quantities.

The following examples may be useful to detect the very reason for this problem of negative results.

Assume that there are two processes by which two commodities (wool and mutton) are jointly produced by means of wool, mutton and some primary factors.

Let $\mathbf{A}=\begin{pmatrix}6 & 4\\ 4 & 2\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}8 & 3\\ 2 & 6\end{pmatrix}$ are input and output matrices of dimension product by process. Neglecting the primary factors and consider a stationary economy, we may characterise the two processes also by the "netput" matrix $(\mathbf{B}-\mathbf{A})=\begin{pmatrix}2 & -1\\ -2 & 4\end{pmatrix}$, where the first row refers to process 1 and the second row refers to process 2.

The set of producible net outputs is given by the set $\forall\mathbf{x}\geqq 0;\quad (\mathbf{B}-\mathbf{A})\mathbf{x}=\begin{pmatrix}2\\ -2\end{pmatrix}x_1+\begin{pmatrix}-1\\ 4\end{pmatrix}x_2$ and is depicted in figure 1

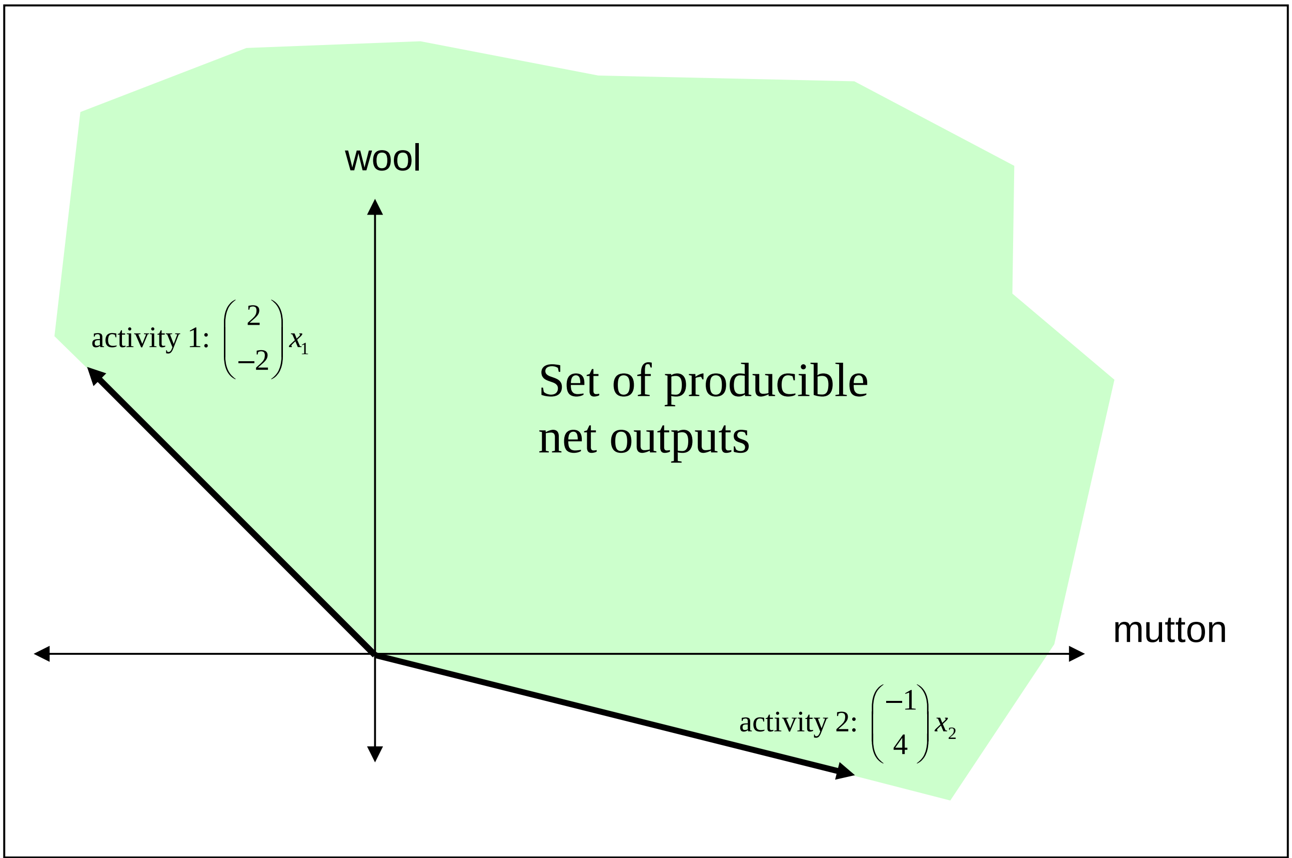

figure 1

Note that any nonnegative net product bundle of wool and mutton can be produced because the feasible set of net bundles contains the whole positive orthant.[2]

This observation is crucial for the non negativity of the solutions:

(i) For all non negative bundles of net products **d** there exist a non negative vector of intensities **x** such that $(\mathbf{B}-\mathbf{A})\mathbf{x}=\mathbf{d}$.

(ii) The inverse $(\mathbf{B}-\mathbf{A})^{-1}=\begin{pmatrix} 2/3 & 1/6 \\ 1/3 & 1/3 \end{pmatrix}>0$.

(iii) The generalized Leontief-inverse $(\mathbf{I}-\mathbf{AB}^{-1})^{-1}=\begin{pmatrix} 19/3 & 7/3 \\ 10/3 & 7/3 \end{pmatrix}>0$

(iv) Therefore, the bundle of gross outputs $\mathbf{q}=(\mathbf{I}-\mathbf{AB}^{-1})^{-1}\mathbf{d}$ is non negative for all $\mathbf{d}\geq 0$.

These nice features disappear, if we change the input matrix slightly such that instead of 4 units of wool just 2 units of wool are required for the second process. While the output matrix remains unchanged, the input matrix is now given by $\mathbf{A}=\begin{pmatrix} 6 & 2 \\ 4 & 2 \end{pmatrix}$. The new "netput" matrix is $\mathbf{B}-\mathbf{A}=\begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix}$ and the cone of producible net outputs is depicted in figure 2.

In contrast to example 1 that set does not contain the whole positive orthant. Hence some proportions of net outputs are not producible. If more than four times more quantities of mutton than quantities of wool are demanded, than there exist no positive intensities by which the two processes can produce the required bundle.

The inverses $(\mathbf{B}-\mathbf{A})^{-1}=\begin{pmatrix} 2/5 & -1/10 \\ 1/5 & 1/5 \end{pmatrix}$ exist but contain negative elements. Hence solutions for any demanded bundle of net products can be calculated, but negative intensities for the first activity will result if the bundle of net products demanded is not an element of the producible set.

[2] Note that also some negative amounts of wool or mutton are "producible". That means that the system is also capable to dispose off some existing quantities of wool or mutton.

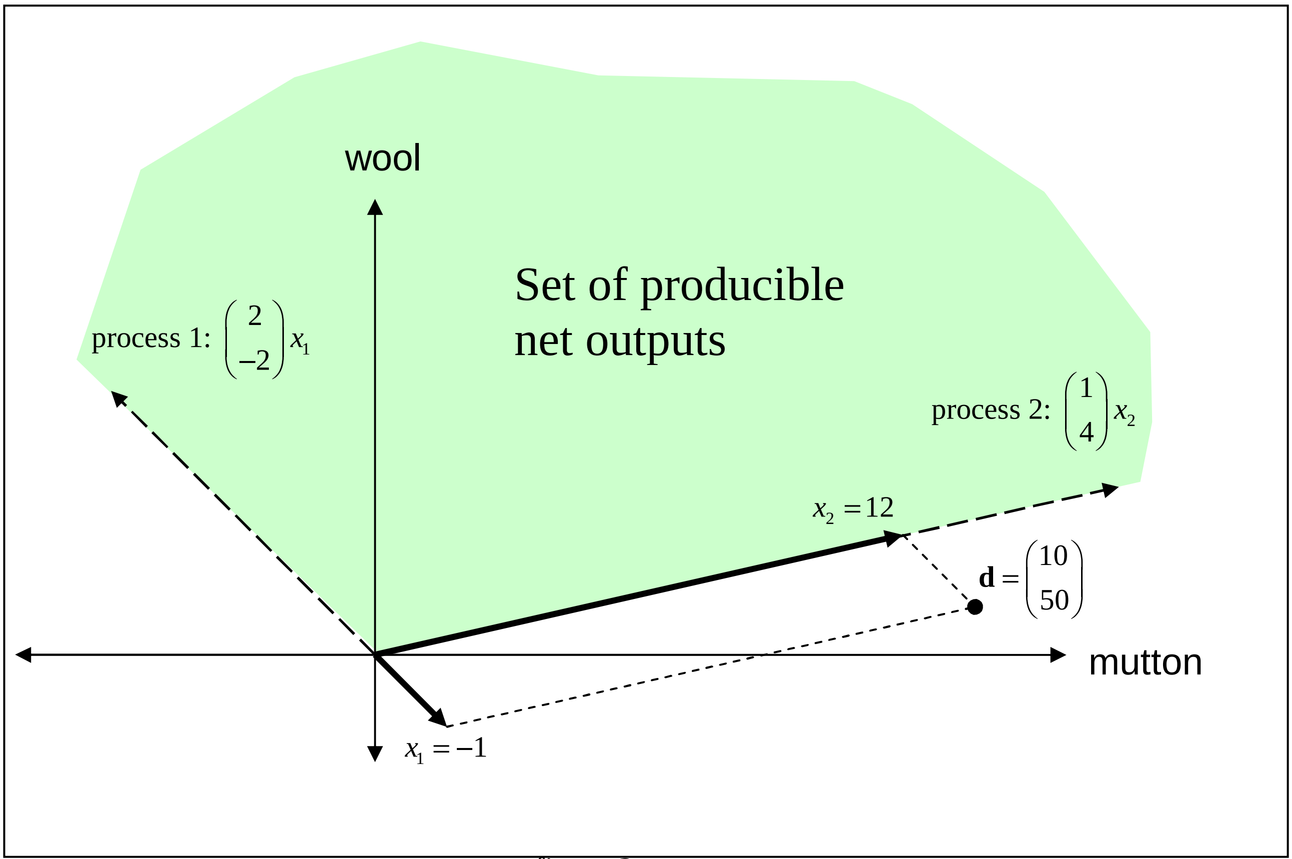

figure 2

Assume, for instance, that 10 units of wool and 50 quantities of mutton are demanded. These proportions are not supported by the system. Even if only process 2, which specialises in the production of mutton, will be activated there is too much wool or to less mutton. A solution for (4) can be obtained but will result in a negative intensity for process 1 such that some wool will be “absorbed” and additional mutton will be “produced” by that process. Hence the linear combination $\begin{pmatrix} 2 \\ -2 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 4 \end{pmatrix} x_2 = \begin{pmatrix} 10 \\ 50 \end{pmatrix}$ has the solution $x_1 = -1$ and $x_2 = 12$.

Negative intensities are not acceptable. Hence the method proposed by (4) will work for the case that the demanded bundle is producible but cannot be applied in general.

Another route is suggested by von Neumann: Allow for excess production, i.e. substitute the equation (3) by the inequality $(\mathbf{B}-\mathbf{A})\mathbf{x} \geq \mathbf{d}$, and, if necessary, invoke the rule of free goods (free disposal).

In example 2 this approach will result in $x_1 = 0$ and $x_2 = 12,5$. The demand for mutton is exactly met but wool is produced in excess and has to be disposed off by an (free) disposal process. For a more careful and general discussion von Neumann systems with free or costly disposal see Lager (2001).

In order to obtain some more rigorous and more general findings we may utilize the following definitions and concepts developed by scholars working in the Sraffian tradition:

Definition 1: A system of production is **strictly viable** if it is possible to produce a positive net output.

$$\exists \mathbf{x} \geq, \quad (\mathbf{B} - \mathbf{A})\mathbf{x} > 0$$

Note, that in this case it is possible to produce **any** net output or more than that. Hence it is possible to match any vector of finished products if there is free disposal.

Definition 2: A product i is said to be **separately producible** if it is possible to produce a net output consisting of one unit of that product but nothing else, i.e.

$\exists \mathbf{x}^i \geq 0, \quad (\mathbf{B} - \mathbf{A})\mathbf{x}^i = \mathbf{e}_i$, where $\mathbf{e}_i$ is a vector whose i-th element is equal to 1 and all other elements are equal to zero.

Definition 3: A system of production is **all-productive** if all products are separately producible.[3]

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[3] The concept of an all-productive system has been introduced by Schefold (1971, 1989). A similar concept, “the adjustment property”, has been presented by Bidard and Erreygers (1998).

All-productive systems have the following nice properties:

- All-productive systems can produce **any** semi-positive net output, i.e. $\forall \mathbf{d} \geq 0, \quad \exists \mathbf{x} \geq 0, \quad (\mathbf{B} - \mathbf{A})\mathbf{x} = \mathbf{d}$.

- If and only if the system is all-productive, then the inverse $(\mathbf{B} - \mathbf{A})^{-1}$ is semi-positive. Note that that the vector $\mathbf{x}^i$ in definition 2 is the *i*-th column of the inverse $(\mathbf{B} - \mathbf{A})^{-1}$. Therefore, an all-productive system has a semi-positive net-output inverse by definition and if $(\mathbf{B} - \mathbf{A})^{-1} \geq 0$, then all products are separately producible

- If – but not only if – the system is all productive, the generalised Leontief inverse is non-negative, i.e. $(\mathbf{B} - \mathbf{A})^{-1} \geq 0 \quad \Rightarrow \quad \mathbf{B}(\mathbf{B} - \mathbf{A})^{-1} = (\mathbf{I} - \mathbf{A}\mathbf{B}^{-1})^{-1} \geq 0$. The proof for the "if statement" follows from the fact that the product of two semi-positive matrices is non-negative. The evidence for the "not only if" statement is provided by the example in the annex.

- All-productive systems have always (semi-)positive solutions for intensities. $(\mathbf{B} - \mathbf{A})^{-1}\mathbf{d} = \mathbf{x} \geq 0$

- All-productive systems have always (semi-)positive solutions for gross outputs. $\mathbf{B}\mathbf{x} = \mathbf{B}(\mathbf{B} - \mathbf{A})^{-1}\mathbf{d} = \mathbf{q} \geq 0$

- All-productive systems have always (semi-)positive solutions for total capital requirements. $\mathbf{A}\mathbf{x} = \mathbf{A}(\mathbf{B} - \mathbf{A})^{-1}\mathbf{d} = \mathbf{k} \geq 0$

- All-productive systems have always (semi-)positive solutions for labour values. $\mathbf{l}'(\mathbf{B} - \mathbf{A})^{-1} = \mathbf{v} \geq 0$

We may also explore the relation between semi-positive commodity by commodity input coefficients and all-productive systems

- If – but not only if – the commodity by commodity coefficients are semi-positive it follows that the generalised Leontief inverse is semi-positive, i.e.

$\mathbf{AB}^{-1} \geq 0 \quad \Rightarrow \quad \mathbf{B}\left(\mathbf{B}-\mathbf{A}\right)^{-1} = \left(\mathbf{I}-\mathbf{AB}^{-1}\right)^{-1}$. The proof for the "if statement" follows from the fact that the generalised Leontief inverse is equal to the sum of a Neumann series of convergent semi-positive matrices, i.e. $\left(\mathbf{I}-\mathbf{AB}^{-1}\right)^{-1} = \sum_{t=0}^{\infty}\left(\mathbf{AB}^{-1}\right)^{t}$. The evidence for the "not only if" statement is provided by the example in the annex.

- It follows that semi-positive commodity by commodity coefficients guarantee semi-positive gross outputs, i.e. $\mathbf{AB}^{-1} \geq 0 \quad \Rightarrow \quad \forall \mathbf{d} \geq 0; \quad \exists \mathbf{q} \geq 0: \quad \mathbf{q} = \left(\mathbf{I}-\mathbf{AB}^{-1}\right)^{-1}\mathbf{d}$

- But semi-positive commodity by commodity coefficients **do not** guarantee semi-positive intensity vectors

Summary and conclusion:

In this paper it is argued on pure theoretical grounds that a "technology assumption" or any other method to estimate commodity by commodity (or industry by industry) input output matrices neglects the very nature of joint production. In particular, the commodity technology model, defines joint production away by assuming that there are only single production processes and observed joint production is a result of aggregation of single production processes to industries. A straightforward joint production model is proposed which avoids calculating commodity by commodity input-output matrices. The very reason of the still remaining problem of negative solutions is detected. The common feature of systems which may have negative solutions is that those systems do not support any bundle of net products. A class of joint production systems is identified which rules negative solutions out and exhibit some other nice features of single production systems.

Annex

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 9 & 3 \\ 3 & 4 \end{pmatrix}$.

The netput matrix $(\mathbf{B}-\mathbf{A}) = \begin{pmatrix} 8 & 1 \\ -2 & 3 \end{pmatrix}$ shows clearly, that not all net bundles in the positive orthant are producible, i.e. the system is not all productive.

Consequently, $(\mathbf{B}-\mathbf{A})^{-1} \approx \begin{pmatrix} 0{,}12 & -0{,}04 \\ 0{,}08 & 0{,}31 \end{pmatrix} \not\geq 0$.

Nevertheless the generalised Leontief-inverse, $\mathbf{B}(\mathbf{B}-\mathbf{A})^{-1} = \left(\mathbf{I}-\mathbf{AB}^{-1}\right)^{-1} \approx \begin{pmatrix} 1{,}27 & 0{,}58 \\ 0{,}65 & 1{,}12 \end{pmatrix} \geq 0$, is positive.

Hence it is possible that there is a solution with negative intensities but gross outputs will be positive.

Though there is a positive generalised Leontief inverse, the commodity by commodity input matrix calculated by the CTA contains negative coefficients: $\mathbf{C} = \mathbf{AB}^{-1} \approx \begin{pmatrix} -0{,}07 & 0{,}56 \\ 0{,}63 & -0{,}22 \end{pmatrix}$

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