RAS-ing the Transactions or the Coefficients: It Makes No Difference

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Abstract

The biproportional RAS technique has become one of the most important tools to update, regionalize or balance input-output tables. In this note we rigorously prove that the estimation of the intermediate transactions matrix yields the same results as the estimation of the input coefficients matrix or the output coefficients matrix. We also show that this does not hold for any of the other updating procedures that have been commonly proposed as an alternative to RAS.

Keywords

Input-output analysis; Updating; RAS technique

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1. INTRODUCTION

The RAS procedure is widely used for adjusting and estimating matrices. In its simplest form, the true "new" matrix $\mathbf{Z}(1)$ is estimated by $\tilde{\mathbf{Z}}(1)$ on the basis of the given "old" matrix $\mathbf{Z}(0)$, taking into account that the new row and column sums are given. The technique has become an indispensable tool in research that involves using data from input-output tables or social accounting matrices. It has been applied for a wide variety of purposes, such as updating, regionalizing or reconciling. Updating is often necessary, because input-output tables are generally only published once every five years, say. The matrix of intermediate transactions $\mathbf{Z}(0)$ in year $t = 0$ is then updated so as to match the margins for year $t = 1$. Regionalization involves the estimation of a regional matrix $\tilde{\mathbf{Z}}(1)$ on the basis of the national matrix (or the matrix for another region) $\mathbf{Z}(0)$, given the object region's margins. Reconciliation is applied when an initial estimate $\mathbf{Z}(0)$ is obtained (for example, from a partial survey or from industry experts) that does not satisfy the given margins. RAS is then adopted to reconcile or balance the initial estimate, by deriving a new matrix $\tilde{\mathbf{Z}}(1)$ that does satisfy the margins.

The RAS estimate is obtained by multiplying each row i of $\mathbf{Z}(0)$ by r_i and simultaneously each row j by s_j. In matrix notation this yields $\tilde{\mathbf{Z}}(1) = \hat{\mathbf{r}}\mathbf{Z}(0)\hat{\mathbf{s}}$, where a "hat" is used to indicate a diagonal matrix. It appears that Deming and Stephan (1940) first used this biproportional technique that later became known as RAS. Leontief (1941) suggested a similar pair of influences (on rows and columns) to account jointly for coefficient change. Stone and his colleagues at Cambridge (Stone and Brown, 1962) apparently were unaware of this work when they proposed the same approach in 1962 (Bacharach, 1970, p. 4; see also Lahr and de Mesnard, 2004). The Cambridge work seems to have concentrated on operating on a base year matrix $\mathbf{A}(0)$ of input *coefficients*, even though Bacharach (1970, p. 20) writes that the ultimate interest was in a target year *transactions* matrix.

In this note we address the question whether updating a transactions matrix yields a different result than updating the corresponding coefficients matrix. Many publications include language such as "... RAS adjustment of input-output coefficients" in the title, perhaps implying (incorrectly) that the procedure is appropriate exclusively for coefficients matrices. (For example, Hewings, 1985, and Miller and Blair, 1985, discuss the RAS procedure exclusively in terms of coefficients.) The question also arises frequently at conferences and workshops, either explicitly or implicitly (i.e., researchers reporting the numerical equivalence in empirical examples). We prove that updating (or regionalizing) a transactions matrix yields the same result as updating (or regionalizing) the corresponding coefficients matrix.

It should be mentioned that in some cases it might be argued that it seems more natural to update the coefficients rather than the transactions, while the opposite might apply to other cases. For example, taking the foundations of production theory as a starting-point for the input-output model, it seems natural to focus on the coefficients. From a pure accounting point of view, however, it seems more natural to update the transactions. RAS procedures have also been applied widely to demographic issues. For example, if the focus is on transition probabilities one would choose to update the coefficients, whereas a focus on transportation flows between origin and destination calls for updating the "transactions". We would like to emphasize that the purpose of this note is *not* to advocate the use of one over the other, just that one can safely use either because the results are the same.

2. FORMAL STATEMENT OF THE PROBLEM

We address the issue in the updating context. Consider an input-output table in year $t = 0$. The intermediate deliveries or transactions between the industries are given by the matrix $\mathbf{Z}(0)$, the final demands by the (column) vector $\mathbf{f}(0)$ and the gross outputs by the vector $\mathbf{x}(0)$. Using $\mathbf{e}$ for the summation vector consisting of ones, the first set of accounting equations yield $\mathbf{Z}(0)\mathbf{e} + \mathbf{f}(0) = \mathbf{x}(0)$. The row sums of the transactions matrix are denoted by $\mathbf{u}(0) = \mathbf{Z}(0)\mathbf{e} = \mathbf{x}(0) - \mathbf{f}(0)$. Let the row vector of primary inputs (value added and imports) be denoted by $\mathbf{m}(0)'$, where a prime is used to indicate transposition. The second set of accounting equations then yields $\mathbf{e}'\mathbf{Z}(0) + \mathbf{m}(0)' = \mathbf{x}(0)'$. The column sums of the transactions matrix are denoted by $\mathbf{v}(0)' = \mathbf{e}'\mathbf{Z}(0) = \mathbf{x}(0)' - \mathbf{m}(0)'$. Similar expressions hold for year t = 1.

RAS-updating the transactions matrix $\mathbf{Z}(0)$ yields $\tilde{\mathbf{Z}}^Z$, which satisfies the following three conditions.

$$\tilde{\mathbf{Z}}^Z = \hat{\mathbf{r}}^Z \mathbf{Z}(0) \hat{\mathbf{s}}^Z \tag{1a}$$

$$\tilde{\mathbf{Z}}^Z \mathbf{e} = \mathbf{u}(1) \tag{1b}$$

$$\mathbf{e}'\tilde{\mathbf{Z}}^Z = \mathbf{v}(1)' \tag{1c}$$

with $u_i(1) > 0$ and $v_i(1) > 0$ for all i. Next we will introduce the approach for the input coefficients, which for t = 0 are defined as $a_{ij}(0) = z_{ij}(0) / x_j(0)$ or $\mathbf{A}(0) = \mathbf{Z}(0)\hat{\mathbf{x}}(0)^{-1}$. Substituting $\mathbf{Z}(0) = \mathbf{A}(0)\hat{\mathbf{x}}(0)$ into the two sets of accounting equations yields $\mathbf{A}(0)\mathbf{x}(0) + \mathbf{f}(0) = \mathbf{x}(0)$ and $\mathbf{e}'\mathbf{A}(0)\hat{\mathbf{x}}(0) + \mathbf{m}(0)' = \mathbf{x}(0)'$. Equivalently, $\mathbf{u}(0) = \mathbf{x}(0) - \mathbf{f}(0) = \mathbf{A}(0)\mathbf{x}(0)$ and $\mathbf{v}(0)' = \mathbf{x}(0)' - \mathbf{m}(0)' = \mathbf{e}'\mathbf{A}(0)\hat{\mathbf{x}}(0)$ or $\mathbf{v}(0)'\hat{\mathbf{x}}(0)^{-1} = \mathbf{e}'\mathbf{A}(0)$. Again, similar expressions hold for year t = 1. RAS-updating the coefficients matrix $\mathbf{A}(0)$ yields $\tilde{\mathbf{A}}^A$, which satisfies the following three conditions.

$$\tilde{\mathbf{A}}^A = \hat{\mathbf{r}}^A \mathbf{A}(0) \hat{\mathbf{s}}^A \tag{2a}$$

$$\tilde{\mathbf{A}}^A \mathbf{x}(1) = \mathbf{u}(1) \tag{2b}$$

$$\mathbf{e}'\tilde{\mathbf{A}}^A = \mathbf{v}(1)'\hat{\mathbf{x}}(1)^{-1} \tag{2c}$$

In principle, there are now two ways of obtaining an estimate for the transactions matrix in year 1. First, $\tilde{\mathbf{Z}}^Z$ from (1) and, second, transforming the estimate $\tilde{\mathbf{A}}^A$ from (2) into a transactions matrix. That is, we define $\tilde{\mathbf{Z}}^A \equiv \tilde{\mathbf{A}}^A \hat{\mathbf{x}}(1)$. In the same fashion, if we are interested in the coefficients matrix, the two estimates are given by $\tilde{\mathbf{A}}^A$ directly from (2) and $\tilde{\mathbf{A}}^Z \equiv \tilde{\mathbf{Z}}^Z \hat{\mathbf{x}}(1)^{-1}$ by applying (1) first. The issue is whether $\tilde{\mathbf{Z}}^A = \tilde{\mathbf{Z}}^Z$ or, equivalently, whether $\tilde{\mathbf{A}}^A = \tilde{\mathbf{A}}^Z$?

Several recent articles and interpretations of those articles appears to have raised the question of whether or not one obtains different results by using RAS on coefficients or transactions in the sense of suggesting that $\tilde{\mathbf{A}}^A \neq \tilde{\mathbf{A}}^Z$ or $\tilde{\mathbf{Z}}^A \neq \tilde{\mathbf{Z}}^Z$. Okuyama et al. (2002) wrote: "Because the adjustment process ... operates on **A** matrices, the adjustment process is conservative, making only the minimally necessary adjustments to ensure agreement with the vectors **u**(1) and **v**(1)." (p. 364, changed notation). (Virtually identical language can be found in Hewings, 1986, p. 52, describing RAS in the context of generating a regional from a national input-output coefficient matrix.) Citing the quotation from Okuyama et al. (2002), Jackson and Murray (2004) wrote: "One might reasonably ask why, given the available data, one does not update the matrix of intermediate transactions directly, and then derive the associated new coefficients matrix." (p. 137). Finally, Oosterhaven (2005) observed: "JM [Jackson and Murray], following Okuyama et al. (2002), suggest that RAS defined on coefficients produces a different outcome than RAS defined on transactions. This suggestion is incorrect. A careful inspection of JM's own description of RAS in terms of coefficients ... simply reveals its mathematical equivalence with RAS in terms of transactions." (p. 300, footnote 1).

Whether in fact Okuyama et al. (2002) intended this interpretation, it is not correct, as Oosterhaven suggests. And whether or not simple inspection of the procedure as described in Jackson and Murray is sufficient to make the case convincingly is debatable. In the next section, we give a rigorous proof for $\tilde{\mathbf{Z}}^A = \tilde{\mathbf{Z}}^Z$ and, hence, $\tilde{\mathbf{A}}^A = \tilde{\mathbf{A}}^Z$.

3. THE PROOF

The proof is based on the fact that in procedure (1), the outcome $\tilde{\mathbf{Z}}^Z$ is unique (see Bacharach, 1970, pp. 47-9). Now consider equations (2). In (2a), post-multiply both sides by $\hat{\mathbf{x}}(1)$ and write $\mathbf{A}(0) = \mathbf{Z}(0)\hat{\mathbf{x}}(0)^{-1}$. Also in (2c), post-multiply both sides by $\hat{\mathbf{x}}(1)$. Then equations (2) change into

$$\tilde{\mathbf{A}}^{A}\hat{\mathbf{x}}(1) = \hat{\mathbf{r}}^{A}\mathbf{Z}(0)\hat{\mathbf{x}}(0)^{-1}\hat{\mathbf{s}}^{A}\hat{\mathbf{x}}(1) \tag{3a}$$

$$\tilde{\mathbf{A}}^{A}\hat{\mathbf{x}}(1)\mathbf{e} = \mathbf{u}(1) \tag{3b}$$

$$\mathbf{e}'\tilde{\mathbf{A}}^{A}\hat{\mathbf{x}}(1) = \mathbf{v}(1)' \tag{3c}$$

Using the definition $\tilde{\mathbf{Z}}^{A} \equiv \tilde{\mathbf{A}}^{A}\hat{\mathbf{x}}(1)$, equations (3) then become

$$\tilde{\mathbf{Z}}^{A} = \hat{\mathbf{r}}^{A}\mathbf{Z}(0)\hat{\hat{\mathbf{s}}} \tag{4a}$$

$$\tilde{\mathbf{Z}}^{A}\mathbf{e} = \mathbf{u}(1) \tag{4b}$$

$$\mathbf{e}'\tilde{\mathbf{Z}}^{A} = \mathbf{v}(1)' \tag{4c}$$

with the diagonal matrix $\hat{\hat{\mathbf{s}}} \equiv \hat{\mathbf{x}}(1)\hat{\mathbf{x}}(0)^{-1}\hat{\mathbf{s}}^{A}$.

Note that the sets of equations in (1) and (4) are equivalent. That is, both sets express that we are searching for a matrix $\tilde{\mathbf{Z}}$ that satisfies certain requirements (i.e., a biproportional relationship to $\mathbf{Z}(0)$ and given row and column sums). Note also that there is only one matrix that satisfies these requirements, because the updated matrix as the solution to (1) is unique. Hence, the outcomes of (1) and (4) must be the same, i.e., $\tilde{\mathbf{Z}}^{A} = \tilde{\mathbf{Z}}^{Z}$. Postmultiplying both sides by $\hat{\mathbf{x}}(1)^{-1}$ yields $\tilde{\mathbf{Z}}^{A}\hat{\mathbf{x}}(1)^{-1} = \tilde{\mathbf{Z}}^{Z}\hat{\mathbf{x}}(1)^{-1}$. Using the definitions $\tilde{\mathbf{Z}}^{A} \equiv \tilde{\mathbf{A}}^{A}\hat{\mathbf{x}}(1)$ and $\tilde{\mathbf{A}}^{Z} \equiv \tilde{\mathbf{Z}}^{Z}\hat{\mathbf{x}}(1)^{-1}$ immediately implies $\tilde{\mathbf{A}}^{A} = \tilde{\mathbf{A}}^{Z}$. In conclusion, it does not matter whether procedure (1) or procedure (2) is applied, they yield the same transactions matrix ($\tilde{\mathbf{Z}}^{A} = \tilde{\mathbf{Z}}^{Z}$) and the same coefficients matrix ($\tilde{\mathbf{A}}^{A} = \tilde{\mathbf{A}}^{Z}$).

It should be stressed that the equivalence of the two updating procedures has a much wider validity than just for the forms in (1) and (2). Adding restrictions to (1) and (2) does not alter our finding, although it may alter the outcome.

For example in (1), such restrictions may have the form of certain elements being known a priori (e.g., to be zero), or the sum of a set of elements may be known a priori, or inequalities for sets of elements may be imposed. The basis of our proof was the uniqueness of the outcome to procedure (1). Adding a restriction implies that the outcome will either be unique or will not exist. The same applies to procedure (2). Hence, *if* both procedures have an outcome, each will be unique and the two will be equivalent (i.e., exhibit a one-to-one correspondence), simply because the solution also must satisfy (1) and (2). This means that our main conclusion is not affected.

The outcome, however, might be affected. It may happen that (1), (2) or both have no solution. The possibility where only one of the two procedures has no solution can be prevented by appropriately "translating" the restrictions. That is, any restriction in (sets of) elements of $\tilde{\mathbf{Z}}^Z$ in (1) should be translated according to $\tilde{\mathbf{A}}^Z \equiv \tilde{\mathbf{Z}}^Z \hat{\mathbf{x}}(1)^{-1}$ so as to yield the restriction in (2). Vice versa, any restriction in (sets of) elements of $\tilde{\mathbf{A}}^A$ in (2) should use $\tilde{\mathbf{Z}}^A \equiv \tilde{\mathbf{A}}^A \hat{\mathbf{x}}(1)$ to yield the restriction in (1). In conclusion, if the restrictions have been appropriately translated there are two possibilities. First, (1) and (2) have a unique solution and the solutions correspond to each other. Second, both (1) and (2) have no solution.

4. MORE COEFFICIENTS

The two sets of accounting equations, i.e., $\mathbf{Z}(0)\mathbf{e} + \mathbf{f}(0) = \mathbf{x}(0)$ and $\mathbf{e}'\mathbf{Z}(0) + \mathbf{m}(0)' = \mathbf{x}(0)'$, are also the basis of the Ghosh (or supply-driven) input-output model, which can be interpreted as a price model (see Dietzenbacher, 1997). The output (or allocation) coefficients are defined as $b_{ij}(0) = z_{ij}(0)/x_i(0)$, or $\mathbf{B}(0) = \hat{\mathbf{x}}(0)^{-1}\mathbf{Z}(0)$, and they indicate the fraction of industry *i*'s output that is delivered to industry *j*. Substituting $\mathbf{Z}(0) = \hat{\mathbf{x}}(0)\mathbf{B}(0)$ into the accounting equations yields $\hat{\mathbf{x}}(0)\mathbf{B}(0)\mathbf{e} + \mathbf{f}(0) = \mathbf{x}(0)$ and $\mathbf{x}(0)'\mathbf{B}(0) + \mathbf{m}(0)' = \mathbf{x}(0)'$. Equivalently, $\mathbf{u}(0) = \mathbf{x}(0) - \mathbf{f}(0) = \hat{\mathbf{x}}(0)\mathbf{B}(0)\mathbf{e}$ or $\mathbf{B}(0)\mathbf{e} = \hat{\mathbf{x}}(0)^{-1}\mathbf{u}(0)$ and $\mathbf{v}(0)' = \mathbf{x}(0)' - \mathbf{m}(0)' = \mathbf{x}(0)'\mathbf{B}(0)$. Again, similar expressions hold for year $t = 1$. RAS-updating the coefficients matrix $\mathbf{B}(0)$ yields $\tilde{\mathbf{B}}^B$, which satisfies the following three conditions.

$$\tilde{\mathbf{B}}^B = \hat{\mathbf{r}}^B \mathbf{B}(0)\hat{\mathbf{s}}^B \quad (5a)$$

$$\tilde{\mathbf{B}}^B \mathbf{e} = \hat{\mathbf{x}}(1)^{-1}\mathbf{u}(1) \quad (5b)$$

$$\mathbf{x}(1)'\tilde{\mathbf{B}}^B = \mathbf{v}(1)' \quad (5c)$$

The procedure in (5) yields an update for the matrix with output coefficients from which the transactions matrix may be obtained as $\tilde{\mathbf{Z}}^B = \hat{\mathbf{x}}(1)\tilde{\mathbf{B}}^B$. In the same way, from the updated transactions matrix from (1), the estimate for the output coefficients matrix yields $\tilde{\mathbf{B}}^Z = \hat{\mathbf{x}}(1)^{-1}\tilde{\mathbf{Z}}^Z$. Following the same steps as in the previous section it follows that $\tilde{\mathbf{Z}}^B = \tilde{\mathbf{Z}}^Z$ and $\tilde{\mathbf{B}}^Z = \tilde{\mathbf{B}}^B$.

5. How Non-unique is the Set of r and s Vectors?

It is well known that the vectors $\mathbf{r}^Z$ and $\mathbf{s}^Z$ in (1) are not unique. It is easily seen that also $\lambda\mathbf{r}^Z$ and $\mathbf{s}^Z/\lambda$ for any $\lambda \neq 0$ satisfy (1), if $\mathbf{r}^Z$ and $\mathbf{s}^Z$ do (see Bacharach, 1970, p. 22; Lahr and de Mesnard, 2004, use the term hyperbolically homogeneous for this). However, this does not imply "the outcomes being unique only up to a scalar" as one of us stated (see van der Linden and Dietzenbacher, 2000, p. 2209). The following simple example suffices to show that this claim does not hold in general.

$$\mathbf{Z}(0) = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}, \ \mathbf{u}(1) = \begin{pmatrix} 15 \\ 8 \end{pmatrix}, \text{ and } \mathbf{v}(1)' = \begin{pmatrix} 8 & 15 \end{pmatrix}$$

The solution is then given by

$$\tilde{\mathbf{Z}}^Z = \begin{bmatrix} 0 & 15 \\ 8 & 0 \end{bmatrix}, \ \mathbf{r}^Z = \begin{pmatrix} \lambda \\ 4\mu \end{pmatrix}, \text{ and } \mathbf{s}^Z = \begin{pmatrix} 1/\mu \\ 5/\lambda \end{pmatrix} \text{ with } \lambda, \mu \neq 0.$$

Lahr and de Mesnard (2004, p. 119) write that "the elements within each of these two vectors [i.e., $\mathbf{r}^Z$ and $\mathbf{s}^Z$] have constant relative values" and normalization is the solution to arrive at a unique result. Further they state: "… due to issues of degrees of freedom, normalization can be affected on either $\mathbf{r}^Z$ and $\mathbf{s}^Z$, not both!" (notation adapted). Their statements are made against the background of an example for which $\mathbf{Z}(0)$ has only positive elements. In our example above, however, we have two degrees of freedom, and normalization of either $\mathbf{r}^Z$ or $\mathbf{s}^Z$ would not suffice.

The claim that the vectors $\mathbf{r}^Z$ and $\mathbf{s}^Z$ are unique, apart from a single scalar $\lambda \neq 0$, *does* hold if the matrix $\mathbf{Z}(0)$ satisfies certain conditions that are usually met when dealing with input-output tables. The requirements are that: (*i*) the matrix $\mathbf{Z}(0)$ is square; (*ii*) all elements on the man diagonal of $\mathbf{Z}(0)$ are positive, i.e., $z_{ii}(0) > 0$ for all i; and (*iii*) the matrix $\mathbf{Z}(0)$ is not block diagonal (Bacharach, 1970, p 44, uses the terminology "disconnected" in this respect). The matrix $\mathbf{Z}(0)$ is block diagonal if – after a suitable permutation of its rows and columns – it can be written as follows.

$$\begin{bmatrix} \mathbf{Z}_1(0) & 0 \\ 0 & \mathbf{Z}_2(0) \end{bmatrix} \tag{6}$$

Next we prove that under these three conditions, $\mathbf{r}^Z$ and $\mathbf{s}^Z$ are unique, only up to a single scalar $\lambda \neq 0$. First, note that $\mathbf{r}^Z$ and $\mathbf{s}^Z$ are both strictly positive or strictly negative (see Bacharach, 1970, p. 45). Without loss of generality we take $r_i^Z > 0$ and $s_i^Z > 0$ for all i. Since there is at least one degree of freedom, normalize $(\mathbf{r}^Z, \mathbf{s}^Z)$ such that the first element of the **r**-vector is one. That is, define $\bar{\mathbf{r}}^Z = \mathbf{r}^Z / r_1^Z$ and $\bar{\mathbf{s}}^Z = r_1^Z \cdot \mathbf{s}^Z$. Clearly, $(\bar{\mathbf{r}}^Z, \bar{\mathbf{s}}^Z)$ satisfies (1) and suppose that also $(\bar{\mathbf{p}}^Z, \bar{\mathbf{q}}^Z)$ satisfies (1). We will show that $\bar{\mathbf{r}}^Z = \bar{\mathbf{p}}^Z$ and $\bar{\mathbf{s}}^Z = \bar{\mathbf{q}}^Z$ when conditions (*i*) – (*iii*) hold. From $\tilde{z}_{ii}^Z = \bar{r}_i^Z z_{ii}(0)\bar{s}_i^Z = \bar{p}_i^Z z_{ii}(0)\bar{q}_i^Z$ with $z_{ii}(0) > 0$ it follows that either we have that both $\bar{r}_i^Z = \bar{p}_i^Z$ and $\bar{s}_i^Z = \bar{q}_i^Z$, or we have that both $\bar{r}_i^Z \neq \bar{p}_i^Z$ and $\bar{s}_i^Z \neq \bar{q}_i^Z$. Suppose that (after a suitable permutation of industry indexes) $\bar{r}_i^Z = \bar{p}_i^Z$ and $\bar{s}_i^Z = \bar{q}_i^Z$ for $i = 1, ..., k$, and that

$\bar{r}_i^Z \neq \bar{p}_i^Z$ and $\bar{s}_i^Z \neq \bar{q}_i^Z$ for i = k+1, ..., n. Then for all i = 1, ..., k and j = k+1, ..., n we have $\tilde{z}_{ij}^Z = \bar{r}_i^Z z_{ij}(0)\bar{s}_j^Z = \bar{p}_i^Z z_{ij}(0)\bar{q}_j^Z$ or $z_{ij}(0)\bar{s}_j^Z = z_{ij}(0)\bar{q}_j^Z$ because $\bar{r}_i^Z = \bar{p}_i^Z$. Since the vectors are strictly positive and since $\bar{s}_j^Z \neq \bar{q}_j^Z$ for any j = k+1, ..., n, it must be true that $z_{ij}(0) = 0$. In the same fashion, for all i = k+1, ..., n and j = 1, ..., k, $\tilde{z}_{ij}^Z = \bar{r}_i^Z z_{ij}(0)\bar{s}_j^Z = \bar{p}_i^Z z_{ij}(0)\bar{q}_j^Z$ implies $\bar{r}_i^Z z_{ij}(0) = \bar{p}_i^Z z_{ij}(0)$ because $\bar{s}_j^Z = \bar{q}_j^Z$. Since the vectors are strictly positive and since $\bar{r}_i^Z \neq \bar{p}_i^Z$ for any i = k+1, ..., n, it must be true that $z_{ij}(0) = 0$. Summarizing we have that $z_{ij}(0) = 0$ for all i = 1, ..., k and j = k+1, ..., n and for i = k+1, ..., n and j = 1, ..., k. This implies that the matrix **Z**(0) is block diagonal which contradicts condition (*iii*). Consequently, either $\bar{r}_i^Z = \bar{p}_i^Z$ and $\bar{s}_i^Z = \bar{q}_i^Z$ holds for all i = 1, ..., n, or $\bar{r}_i^Z \neq \bar{p}_i^Z$ and $\bar{s}_i^Z \neq \bar{q}_i^Z$ holds for all i. For i = 1 we find $\tilde{z}_{11}^Z = \bar{r}_1^Z z_{11}(0)\bar{s}_1^Z = \bar{p}_1^Z z_{11}(0)\bar{q}_1^Z$, which implies $\bar{s}_1^Z = \bar{q}_1^Z$ because $\bar{r}_1^Z = \bar{p}_1^Z = 1$ by construction and $z_{11}(0) > 0$ due to condition (*ii*). This completes the proof that $\bar{r}_i^Z = \bar{p}_i^Z$ and $\bar{s}_i^Z = \bar{q}_i^Z$ holds for all i = 1, ..., n.

With respect to the plausibility of the conditions, input-output transactions matrices are always square, and the main diagonal is typically strictly positive. Only condition (*iii*) is sometimes violated in real world cases. It should be emphasized, however, that this causes no problem at all. Economies that exhibit a block diagonal structure can be separated into two sub-economies, each with its own transactions matrix. In this case the matrices $\mathbf{Z}_1(0)$ and $\mathbf{Z}_2(0)$ are updated separately.

6. THE RELATIONSHIP BETWEEN THE DIFFERENT SETS OF r AND s VECTORS

In this section we assume that the matrix **Z**(0) satisfies the conditions derived in the previous section. Because the vectors **r** and **s** in (1) and (4) are unique apart from a scalar $\lambda \neq 0$, we can derive a simple relationship between $\mathbf{r}^Z$ and $\mathbf{s}^Z$ on the one hand and $\mathbf{r}^A$ and $\mathbf{s}^A$ on the other hand. The only issue we have to deal with is the fact that uniqueness holds up to a scalar multiple. This implies $\mathbf{r}^Z = \lambda \mathbf{r}^A$ and $\mathbf{s}^Z = \tilde{\mathbf{s}} / \lambda = \hat{\mathbf{x}}(1)\hat{\mathbf{x}}(0)^{-1}\mathbf{s}^A / \lambda$. Next, we normalize the vectors $\mathbf{r}^Z$ and $\mathbf{r}^A$ such that their first element equals 1. That is, $\bar{\mathbf{r}}^Z = \mathbf{r}^Z / r_1^Z$. Similarly, we have $\bar{\mathbf{r}}^A = \mathbf{r}^A / r_1^A$.

This normalization has three consequences. First, $\mathbf{r}^Z = \lambda \mathbf{r}^A$, $\bar{\mathbf{r}}^Z = \mathbf{r}^Z / r_1^Z$ and $\bar{\mathbf{r}}^A = \mathbf{r}^A / r_1^A$ imply $\bar{\mathbf{r}}^Z = \lambda \bar{\mathbf{r}}^A (r_1^A / r_1^Z)$. The normalization yields that $\bar{r}_1^Z = \bar{r}_1^A = 1$, by definition. Then, it immediately follows from $\bar{\mathbf{r}}^Z = \lambda \bar{\mathbf{r}}^A (r_1^A / r_1^Z)$ that $\lambda = r_1^Z / r_1^A$, and that $\bar{\mathbf{r}}^Z = \bar{\mathbf{r}}^A$. Second, because we have normalized the **r**-vectors, the **s**-vectors in (1) and (4) have also changed. That is, $\bar{\mathbf{r}}^Z = \mathbf{r}^Z / r_1^Z$ implies $\bar{\mathbf{s}}^Z = r_1^Z \cdot \mathbf{s}^Z$, and $\bar{\mathbf{r}}^A = \mathbf{r}^A / r_1^A$ implies $\bar{\tilde{\mathbf{s}}} = r_1^A \cdot \tilde{\mathbf{s}}$. Using $\mathbf{s}^Z = \tilde{\mathbf{s}} / \lambda$ then yields $\bar{\mathbf{s}}^Z = (r_1^Z / r_1^A)\bar{\tilde{\mathbf{s}}} / \lambda$. Because $\lambda = r_1^Z / r_1^A$, we now have $\bar{\mathbf{s}}^Z = \bar{\tilde{\mathbf{s}}}$. Third, normalizing $\mathbf{r}^A$ also affects $\mathbf{s}^A$ in (2), i.e., $\bar{\mathbf{s}}^A = r_1^A \cdot \mathbf{s}^A$. From the definition $\tilde{\mathbf{s}} \equiv \hat{\mathbf{x}}(1)\hat{\mathbf{x}}(0)^{-1}\mathbf{s}^A$ we get $\bar{\tilde{\mathbf{s}}} = r_1^A \cdot \tilde{\mathbf{s}} \equiv \hat{\mathbf{x}}(1)\hat{\mathbf{x}}(0)^{-1}(r_1^A \cdot \mathbf{s}^A) = \hat{\mathbf{x}}(1)\hat{\mathbf{x}}(0)^{-1}\bar{\mathbf{s}}^A$.

In conclusion, once the **r**-vectors from procedures (1) and (2) are normalized such that their first element equals 1, we have that the normalized **r**-vectors are equal to each other (i.e. $\bar{\mathbf{r}}^Z = \bar{\mathbf{r}}^A$), while the corresponding **s**-vectors exhibit a simple relationship [i.e., $\bar{\mathbf{s}}^Z = \hat{\mathbf{x}}(1)\hat{\mathbf{x}}(0)^{-1}\bar{\mathbf{s}}^A$].

The relationship between $\mathbf{r}^Z$ and $\mathbf{s}^Z$ in (1) on the one hand and $\mathbf{r}^B$ and $\mathbf{s}^B$ in (5) on the other hand, can be obtained in the same fashion. We now normalize the **s**-vectors (instead of the **r**-vectors as we did above) such that their first element equals 1. In that case we have $\ddot{\mathbf{s}}^Z = \mathbf{s}^Z / s_1^Z$ and $\ddot{\mathbf{s}}^B = \mathbf{s}^B / s_1^B$. Then we find $\ddot{\mathbf{s}}^Z = \ddot{\mathbf{s}}^B$ and for the corresponding **r**-vectors it follows that $\ddot{\mathbf{r}}^Z = \hat{\mathbf{x}}(1)\hat{\mathbf{x}}(0)^{-1}\ddot{\mathbf{r}}^B$. This can also be rewritten such that the **r**-vectors are normalized (as we did above). In that case we have $\bar{\mathbf{r}}^Z = \mathbf{r}^Z / r_1^Z$ again and $\bar{\mathbf{r}}^B = \mathbf{r}^B / r_1^B$. Further $\bar{\mathbf{r}}^Z = [x_1(0)/x_1(1)]\hat{\mathbf{x}}(1)\hat{\mathbf{x}}(0)^{-1}\bar{\mathbf{r}}^B$ and for the corresponding **s**-vectors we then obtain $\bar{\mathbf{s}}^Z = [x_1(1)/x_1(0)]\bar{\mathbf{s}}^B$.

7. OTHER OBJECTIVE FUNCTIONS

It is well known that RAS yields the optimal solution for the following minimization problem in case of the transactions.

$$\text{Minimize } \Sigma_i \Sigma_j \tilde{z}_{ij}^{Z} \ln[\tilde{z}_{ij}^{Z} / z_{ij}(0)] \tag{7}$$

subject to (1b), (1c), and $\tilde{z}_{ij}^{Z} \geq 0$

In case of coefficients we have

$$\text{Minimize } \Sigma_i \Sigma_j \tilde{a}_{ij}^{A} \ln[\tilde{a}_{ij}^{A} / a_{ij}(0)] \tag{8}$$

subject to (2b), (2c), and $\tilde{a}_{ij}^{A} \geq 0$

In the literature, several other objective functions to minimize have been suggested (see, for example, Hewings and Janson, 1980; Lahr and de Mesnard, 2004; Jackson and Murray, 2004). In this section, we examine whether the property that it doesn't matter whether transactions or coefficients are updated also holds for updating procedures with a different objective function. We find that this property does not hold for the updating procedures that have been proposed most commonly as an alternative to RAS. It turns out that a single, simple counterexample suffices.

Our starting point is the input-output table in year t = 0 as given in Table 1. It immediately follows that

$$\mathbf{Z}(0) = \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix} \text{ and } \mathbf{A}(0) = \begin{bmatrix} 0.2 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

Insert Table 1

The information available for the new input-output table in year t = 1 is given in Table 2. Hence, we have

$$\mathbf{u}(1)=\begin{pmatrix}10\\110\end{pmatrix},\ \mathbf{v}(1)=\begin{pmatrix}25\\95\end{pmatrix},\text{ and }\mathbf{x}(1)=\begin{pmatrix}30\\150\end{pmatrix}.$$

Insert Table 2

Applying RAS yields the following results

$$\tilde{\mathbf{Z}}^Z=\tilde{\mathbf{Z}}^A=\begin{bmatrix}1.5312 & 8.4688\\23.4688 & 86.5312\end{bmatrix}\text{ and }\tilde{\mathbf{A}}^A=\tilde{\mathbf{A}}^Z=\begin{bmatrix}0.0510 & 0.0565\\0.7823 & 0.5769\end{bmatrix}$$

As alternatives for the RAS approach we have minimized the following six couples of objective functions (see Jackson and Murray, 2004; or Lahr and de Mesnard, 2004). The constraints are the same as given in (7) and (8) for the transactions and coefficients, respectively.

Function 1 (absolute differences): $\Sigma_i\Sigma_j\left|\tilde{z}_{ij}^Z-z_{ij}(0)\right|$ and $\Sigma_i\Sigma_j\left|\tilde{a}_{ij}^A-a_{ij}(0)\right|$

Function 2 (weighted absolute differences):

$$\Sigma_i\Sigma_j z_{ij}(0)\left|\tilde{z}_{ij}^Z-z_{ij}(0)\right| \text{ and } \Sigma_i\Sigma_j a_{ij}(0)\left|\tilde{a}_{ij}^A-a_{ij}(0)\right|$$

Function 3 (normalized absolute differences):

$$\Sigma_i\Sigma_j\left|\tilde{z}_{ij}^Z-z_{ij}(0)\right|/z_{ij}(0) \text{ and } \Sigma_i\Sigma_j\left|\tilde{a}_{ij}^A-a_{ij}(0)\right|/a_{ij}(0)$$

Function 4 (squared differences): $\Sigma_i\Sigma_j[\tilde{z}_{ij}^Z-z_{ij}(0)]^2$ and $\Sigma_i\Sigma_j[\tilde{a}_{ij}^A-a_{ij}(0)]^2$

Function 5 (weighted squared differences):

$$\Sigma_i\Sigma_j z_{ij}(0)[\tilde{z}_{ij}^Z-z_{ij}(0)]^2 \text{ and } \Sigma_i\Sigma_j a_{ij}(0)[\tilde{a}_{ij}^A-a_{ij}(0)]^2$$

Function 6 (normalized squared differences):

$$\Sigma_i\Sigma_j[\tilde{z}_{ij}^Z-z_{ij}(0)]^2/z_{ij}(0) \text{ and } \Sigma_i\Sigma_j[\tilde{a}_{ij}^A-a_{ij}(0)]^2/a_{ij}(0).$$

Jackson and Murray (2004) provide linearized versions of functions 1-3 that are easy to solve with standard software for linear programming problems. The functions 4-6, however, cannot be linearized and involve constrained nonlinear optimization. Although software is available, it should be emphasized that one may possibly find a local optimum instead of a global optimum. In the present, simple 2×2 example we have four unknown variables with three independent equality constraints, which leaves one degree of freedom. We can calculate the global optimum and do not need to use nonlinear optimization software. One possibility would be to write $\tilde{\mathbf{Z}}^Z$ and $\tilde{\mathbf{A}}^A$ as follows

$$\tilde{\mathbf{Z}}^Z = \begin{bmatrix} \varepsilon & 10-\varepsilon \\ 25-\varepsilon & 85+\varepsilon \end{bmatrix} \text{ and } \tilde{\mathbf{A}}^A = \begin{bmatrix} 5\delta & \frac{10}{150}-\delta \\ \frac{25}{30}-5\delta & \frac{85}{150}+\delta \end{bmatrix}$$

with $0 \le \varepsilon \le 10$ and with $0 \le \delta \le 10/150$. Each of the functions 1-6 can then be expressed as a function of ε (or δ, respectively) and the minimum can be found algebraically.

When updating the transactions, we find for *all* six functions:

$$\tilde{\mathbf{Z}}^Z = \begin{bmatrix} 0 & 10 \\ 25 & 85 \end{bmatrix}, \text{ so that } \tilde{\mathbf{A}}^Z = \begin{bmatrix} 0 & 0.0667 \\ 0.8333 & 0.5667 \end{bmatrix}.$$

When updating the coefficients, we find the results in Table 3. For each of the functions 1-6 we have $\tilde{\mathbf{Z}}^Z \neq \tilde{\mathbf{Z}}^A$ and $\tilde{\mathbf{A}}^A \neq \tilde{\mathbf{A}}^Z$. This example shows that for the six most commonly proposed alternative updating procedures proposed in the literature it does indeed make a difference whether transactions or coefficients are updated. Although in some specific cases one might have a preference for transactions or for coefficients, generally one is indifferent between the two. This implies, however, that one is confronted with two different answers to the same question, and there are usually no arguments on which a choice can be based. It thus seems that generating the same answer whether updating the transactions or the coefficients is a very attractive property that holds exclusively for RAS, at least within the set of commonly applied updating procedures.

Insert Table 3

8. CONCLUDING REMARKS

In this note we have shown that the biproportional RAS technique yields the same results, regardless of whether we update the transactions, the input coefficients, or the output coefficients. That is, updating the transactions directly yields $\tilde{\mathbf{Z}}^Z$ from equations (1). Updating the input coefficients yields $\tilde{\mathbf{A}}^A$ from equations (2) and the transactions are then obtained as $\tilde{\mathbf{Z}}^A \equiv \tilde{\mathbf{A}}^A\hat{\mathbf{x}}(1)$. Similarly, updating the output coefficients gives $\tilde{\mathbf{B}}^B$ from equations (5) after which the transactions are obtained from $\tilde{\mathbf{Z}}^B \equiv \hat{\mathbf{x}}(1)\tilde{\mathbf{B}}^B$. In Sections 3 and 4, these three estimates for the transactions matrix are shown to be exactly the same, i.e., $\tilde{\mathbf{Z}}^Z = \tilde{\mathbf{Z}}^A = \tilde{\mathbf{Z}}^B$. Also we have found $\tilde{\mathbf{A}}^Z = \tilde{\mathbf{A}}^A$ (in Section 3) and $\tilde{\mathbf{B}}^Z = \tilde{\mathbf{B}}^B$ (in Section 4). Therefore, $\tilde{\mathbf{A}}^B \equiv \tilde{\mathbf{Z}}^B\hat{\mathbf{x}}(1)^{-1} = \tilde{\mathbf{Z}}^Z\hat{\mathbf{x}}(1)^{-1} = \tilde{\mathbf{A}}^Z$, so $\tilde{\mathbf{A}}^Z = \tilde{\mathbf{A}}^A = \tilde{\mathbf{A}}^B$. Similarly, $\tilde{\mathbf{B}}^A \equiv \hat{\mathbf{x}}(1)^{-1}\tilde{\mathbf{Z}}^A = \hat{\mathbf{x}}(1)^{-1}\tilde{\mathbf{Z}}^Z = \tilde{\mathbf{B}}^Z$, so $\tilde{\mathbf{B}}^Z = \tilde{\mathbf{B}}^A = \tilde{\mathbf{B}}^B$. Table 4 summarizes our results.

Insert Table 4

Although the proof was given for the standard case, we have also indicated that the result is also valid when additional restrictions are included. Further, we have compared the RAS technique with six other updating procedures that have been frequently proposed in the literature. It turns out that RAS is the only approach that exhibits the property that updating the transactions yields the same answer as updating the coefficients. This property is very attractive in practical work, because usually there are no reasons to favor one over the other. Within the set of commonly used updating procedures this property is thus a distinctive feature of RAS.

As a final remark, it should—for the sake of completeness—be mentioned that the direct estimation of the transactions matrix (i.e., by means of $\tilde{\mathbf{Z}}^Z$) requires less exogenous information than its indirect estimation (i.e., through $\tilde{\mathbf{Z}}^A$ or $\tilde{\mathbf{Z}}^B$). The direct estimation of $\mathbf{Z}(1)$ only requires the row and column sums $\mathbf{u}(1)$ and $\mathbf{v}(1)$. The indirect estimation additionally requires the vector $\mathbf{x}(1)$, or equivalently $\mathbf{f}(1)$ or $\mathbf{m}(1)$, because $\mathbf{x}(1)$

= $\mathbf{u}(1) + \mathbf{f}(1)$ and $\mathbf{x}(1) = \mathbf{v}(1) + \mathbf{m}(1)$. The estimation of the coefficients matrices requires the same information in each of the three procedures, i.e., $\mathbf{u}(1)$, $\mathbf{v}(1)$ and $\mathbf{x}(1)$.

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TABLE 1: Transactions table in year $t = 0$

	Industries 1	Industries 2	Subtotal	Final demand	Total
Industry 1	10	20	30	20	50
Industry 2	30	40	70	30	100
Subtotal	40	60	100	50	150
Primary inputs	10	40	50	--	50
Total	50	100	150	50	

TABLE 2: Transactions table in year $t = 1$

	Industries 1	Industries 2	Subtotal	Final demand	Total
Industry 1			10	20	30
Industry 2			110	40	150
Subtotal	25	95	120	60	180
Primary inputs	5	55	60	--	60
Total	30	150	180	60	

TABLE 3: Results from updating the coefficients

Function	$\tilde{\mathbf{A}}^A$	$\tilde{\mathbf{Z}}^A$
1	$\begin{bmatrix} 0.2000 & 0.0267 \\ 0.6333 & 0.6067 \end{bmatrix}$	$\begin{bmatrix} 6 & 4 \\ 19 & 91 \end{bmatrix}$
2	$\begin{bmatrix} 0.2333 & 0.2000 \\ 0.6000 & 0.6133 \end{bmatrix}$	$\begin{bmatrix} 7 & 3 \\ 18 & 92 \end{bmatrix}$
3	$\begin{bmatrix} 0.2000 & 0.0267 \\ 0.6333 & 0.6067 \end{bmatrix}$	$\begin{bmatrix} 6 & 4 \\ 19 & 91 \end{bmatrix}$
4	$\begin{bmatrix} 0.1795 & 0.0308 \\ 0.6538 & 0.6026 \end{bmatrix}$	$\begin{bmatrix} 5.3846 & 4.6154 \\ 19.6154 & 90.3846 \end{bmatrix}$
5	$\begin{bmatrix} 0.1958 & 0.0275 \\ 0.6375 & 0.6058 \end{bmatrix}$	$\begin{bmatrix} 5.8738 & 4.1262 \\ 19.1262 & 90.8738 \end{bmatrix}$
6	$\begin{bmatrix} 0.1683 & 0.0330 \\ 0.6651 & 0.6003 \end{bmatrix}$	$\begin{bmatrix} 5.0478 & 4.9522 \\ 19.9522 & 90.0478 \end{bmatrix}$

TABLE 4: Summary of findings

Start	$\mathbf{Z}(0)$	$\mathbf{A}(0)$	$\mathbf{B}(0)$
Biproportional adjustment	*RZS*	*RAS*	*RBS*
Initial result	$\tilde{\mathbf{Z}}^{Z}$	$\tilde{\mathbf{A}}^{A}$	$\tilde{\mathbf{B}}^{B}$
Initial result + derived results	$\tilde{\mathbf{Z}}^{Z}$ $\tilde{\mathbf{A}}^{Z} \equiv \tilde{\mathbf{Z}}^{Z}\hat{\mathbf{x}}(1)^{-1}$ $\tilde{\mathbf{B}}^{Z} \equiv \hat{\mathbf{x}}(1)^{-1}\tilde{\mathbf{Z}}^{Z}$	$\tilde{\mathbf{A}}^{A}$ $\tilde{\mathbf{Z}}^{A} \equiv \tilde{\mathbf{A}}^{A}\hat{\mathbf{x}}(1)$ $\tilde{\mathbf{B}}^{A} \equiv \hat{\mathbf{x}}(1)^{-1}\tilde{\mathbf{Z}}^{A} = \hat{\mathbf{x}}(1)^{-1}\tilde{\mathbf{A}}^{A}\hat{\mathbf{x}}(1)$	$\tilde{\mathbf{B}}^{B}$ $\tilde{\mathbf{Z}}^{B} \equiv \hat{\mathbf{x}}(1)\tilde{\mathbf{B}}^{B}$ $\tilde{\mathbf{A}}^{B} \equiv \tilde{\mathbf{Z}}^{B}\hat{\mathbf{x}}(1)^{-1} = \hat{\mathbf{x}}(1)\tilde{\mathbf{B}}^{B}\hat{\mathbf{x}}(1)^{-1}$