A Structural Growth Model and its Applications to Sraffa's System

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ABSTRACT *This paper presents a discrete-time growth model based on the classical growth framework to describe the disequilibrium dynamics of an m-agent, n-good economy. And an exchange function is formulated to describe the exchange process among agents, which serves as the exchange part of the growth model. For concreteness a system of Sraffa (1960) is utilized to exemplify the growth model and simulations are performed. First, business cycles in the growth model are discussed, which are found to be limit cycles in some sense. Then a method is presented to compute the equilibrium land rent in a Sraffian system including homogeneous land, and the fluctuation of land rent is also simulated. Finally, the system of Sraffa is extended to a two-country economy, and the dynamic economic effects of free trade and trade protectionism are investigated.*

KEY WORDS: Growth, business cycle, land rent, international trade

1. Introduction

As Kurz and Salvadori (2000, 2001) pointed out, the input-output models of Leontief (1936, 1941) and the growth model of von Neumann (1945) have a classical root, and should be viewed as essential components of the classical growth framework which consists of Quesnay's (1972) *Tableau Economique* (*Economic Table*), Marx's (1956) reproduction models, Sraffa's (1960) *Production of Commodities by Means of Commodities* etc.

The classical growth framework has two major characteristics, i.e. its dynamic perspective and

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structural perspective. That is, the framework regards the economy as a circular flow containing interdependent sectors (or agents), and provides deep insights into the structure and interplay of all parts of the economy. Specifically, the models under the classical growth framework usually are characterized by following features:

(i) The models contain multiple sectors (or agents) and goods, and each sector (or agent) needs products of other sectors (or agents) as inputs of production;

(ii) The models usually assume constant returns to scale (e.g., see Samuelson and Etula, 2006);

(iii) Equilibrium paths usually are balanced growth paths with a uniform growth rate (i.e profit rate);

(iv) Equilibrium growth rate, equilibrium prices and equilibrium output structure are determined by technologies of sectors (or agents);

(v) Fixed capital usually is dealt with in a joint production framework, as a result joint production plays an important role (e.g., see Salvadori and Steedman, 1990);

(vi) Matrices and Perron-Frobenius theorem are (or can be) used widely as mathematical tools;

(vii) Dynamic models usually are discrete-time systems.

Up to now issues related to equilibrium have been analyzed thoroughly under the classical growth framework, such as the existence of equilibrium (e.g. Kemeny, Morgenstern and Thompson, 1956), exchange equilibrium (e.g. Gale, 1960), optimality of equilibrium (e.g. Dorfman, Samuelson and Solow, 1958; McKenzie, 1963, 1976), perturbations of equilibrium (e.g. Dietzenbacher, 1988), stability of equilibrium (e.g. Morishima, 1964), equilibrium models of Marx (e.g. Morishima, 1973). However, it seems that some disequilibrium issues such as the fluctuation of prices and land rent, business cycles, international trade under disequilibrium circumstances etc., haven't been investigated sufficiently, and this paper is a tentative attempt to explore the method for the analysis of these disequilibrium issues under the classical growth framework.

The main idea in this paper is to develop an exchange function describing the (disequilibrium) exchange process among agents (or sectors), and then combine it with the input-output production processes to obtain a growth model capable of describing the disequilibrium dynamics of an m-agent,

n-good economy. Then by simulations the model will provide some insights to those disequilibrium issues aforementioned.

The paper is organized as follows. Section 2 introduces concepts about technology and production, and an economy system given by Sraffa (1960) is also introduced. Section 3 presents the exchange function, and the exchange process is illustrated with the example of the Sraffa's system. Section 4 introduces the structural growth model, and a computable specific form of the model is also given to serve as the simulation platform in following sections. Section 5 analyzes the business cycles and the land rent in a one-country economy. Section 6 analyzes the international trade in a two-country economy. The final section contains some concluding remarks.

In the sequel the following notations and terms will be used. $\mathbf{e}$ denotes the vector $(1,1,\cdots,1)'$. A vector **x** is called positive (or nonnegative) and we write $\mathbf{x} \gg \mathbf{0}$ (or $\mathbf{x} \geq \mathbf{0}$) if all its components are positive (or nonnegative). **x** is called semipositive and we write $\mathbf{x} > \mathbf{0}$ if $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$. For vectors **x** and **y**, we write $\mathbf{x} \gg \mathbf{y}$, $\mathbf{x} > \mathbf{y}$ and $\mathbf{x} \geq \mathbf{y}$ analogously. Such notations and terms are also used for matrices. A semipositive column (or row) vector **x** is said to be normalized if $\mathbf{e}'\mathbf{x} = 1$ (or $\mathbf{xe} = 1$) holds. $\hat{\mathbf{x}}$ denote diag(**x**), i.e. the diagonal matrix with the vector **x** as the main diagonal.

2. Technology and Production

Suppose there are m agents and n goods in an economy, which are indexed by $1,2,\cdots,m$ and $1,2,\cdots,n$ respectively. An agent may stand for a firm or a sector. If we regard a household as a producer of labor power (or human capital, service, etc.), which absorbs consumer goods, education, trainings and medical treatment etc, and regard its consumption process as an investment and production process, then such an agent can also stand for a household roughly. And such treatment of the consumption process is generally used (e.g., see Solow and Samuelson, 1953).

2.1 Input Coefficient Matrix and Output Coefficient Matrix

When each agent has only one technology and joint production is allowed for, as in the growth model of von Neumann (1945) all technologies can be represented by an ($n \times m$) input coefficient matrix **A** and an ($n \times m$) output coefficient matrix **B**, and in either matrix the ith row and ith column correspond to good *i* and agent *i* respectively.

Below is an example of (4 × 6) input and output coefficient matrices.

$$\mathbf{A}=\begin{bmatrix}0.28 & 0.50 & 0.53 & 0 & 0 & 0\\ 0.84 & 0 & 0 & 0 & 0 & 0.77\\ 0 & 0.49 & 0.45 & 0.50 & 0.48 & 0\\ 0 & 0 & 0 & 0.51 & 0.57 & 0.29\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0.25 & 1 & 1 & 0.25 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \tag{1}$$

Let $\mathbf{a}^{(i)}$ and $\mathbf{b}^{(i)}$ denote the *i*th columns of **A** and **B** respectively, which are supposed to be semipositive, then the vector pair $\left(\mathbf{a}^{(i)},\mathbf{b}^{(i)}\right)$ stands for the technology of agent *i*, and $\mathbf{a}^{(i)}$ is called the *standard input bundle* of agent *i*. Given a positive price vector **p**, whose *i*th component p_i denotes the price of good *i*, $\left(\mathbf{p}'\mathbf{b}^{(i)}\right)\big/\left(\mathbf{p}'\mathbf{a}^{(i)}\right)-1$ is the *profit rate* of agent *i* under **p**.

Let's assume constant returns to scale, thus each feasible production process of agent *i* can be represented by $\left(\xi\mathbf{a}^{(i)},\ \xi\mathbf{b}^{(i)}\right)$, where ξ is a nonnegative real number, $\xi\mathbf{a}^{(i)}$ is the input bundle and $\xi\mathbf{b}^{(i)}$ is the output bundle; moreover, ξ is called the *production intensity* of the production process. In the special case $\mathbf{B}=\mathbf{I}$ the production intensity of one agent is also the output amount of its sole product.

When each agent has multiple technologies and will adjust the technology in use for maximizing profit when market prices changes, the input and output coefficient matrices may be treated as variables with respect to prices.

2.2 An Economic System of Sraffa

Let's write an economic system given by Sraffa (1960) here, which contains two agents (or sectors) and two goods, and the system will be utilized to exemplify the growth model presented in this paper. In the initial period (or year) the system runs as follows.

$$280 \text{ quarters wheat} \quad +12 \text{ tons iron} \quad \to 575 \text{ quarters wheat} \tag{2a}$$

$$120 \text{ quarters wheat} \quad +8 \text{ tons iron} \quad \to 20 \text{ tons iron} \tag{2b}$$

Formula (2a) represents the production process of agent 1 (i.e. the wheat producer) in the initial period, and Formula (2b) represents the production process of agent 2 (i.e. the iron producer) in the initial period. The input coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} \frac{56}{115} & 6 \\ \frac{12}{575} & \frac{2}{5} \end{bmatrix} \tag{3}$$

and the output coefficient matrix is $\mathbf{B} = \mathbf{I}$. The first column of **A**, i.e. $\left(\frac{56}{115}, \frac{12}{575}\right)'$, is the standard input bundle of agent 1, and the second column, i.e. $\left(6, \frac{2}{5}\right)'$, is the standard input bundle of agent 2.

It's well known that the equilibrium price vectors and equilibrium output vectors in the system (2a)-(2b) are the left and right P-F (i.e. Perron-Frobenius) eigenvectors of **A** respectively. Here a left and right P-F eigenvector of **A** are $\mathbf{p}^* = \left(\frac{1}{15}, 1\right)'$ and $\mathbf{z}^* = (575, 30)'$ respectively. That is, with iron as the numeraire, the equilibrium price of wheat is $\frac{1}{15}$ or 0.0667 approximately. And the equilibrium output amount of iron should be 30 tons when the output amount of wheat is 575 quarters, and the system will be in equilibrium if Formula (2b) is substituted by

$$\text{180 quarters wheat} \quad +\text{12 tons iron} \quad \rightarrow \text{30 tons iron} \tag{2b'}$$

The P-F eigenvalue of **A** is $\lambda = 0.8$, which implies the equilibrium growth rate of the system (2a)-(2b) is $1/\lambda - 1 = 0.25$.

The outputs of two agents in the initial period are represented by an output matrix

$$\mathbf{Y} = \begin{bmatrix} 575 & 0 \\ 0 & 20 \end{bmatrix} \tag{4}$$

The outputs of agents may also be represented by an output vector $(575, 20)'$ here. However, the output matrix will become necessary when there is joint production.

3. Exchange Process among Agents

A major weakness of some models under the classical growth framework is that the market mechanism is not reflected in them (Los, 2001). That is, the exchange process, price fluctuation, and their economic effects, are ignored. In this paper we try to integrate the market mechanism into the growth model by embedding an exchange process in it. And this section is devoted to developing an exchange function describing the exchange process among *m* agents.

3.1 The Exchange Function

Let's consider the exchange process among m agents under a given price vector $\mathbf{p}$, in which each agent sells its outputs and purchases an input bundle for its next production process.

Let $\mathbf{S}$ denote the ($n \times m$) supply matrix, whose (i, j) entry denotes agent j's supply amount of good i. Let $\mathbf{s} \equiv \mathbf{Se}$ denote the supply vector, which is supposed to be positive.

For example, in the system (2a)-(2b) of Sraffa, when two agents put their products into market the supply matrix is

$$\mathbf{S} = \begin{bmatrix} 575 & 0 \\ 0 & 20 \end{bmatrix} \tag{5}$$

and the supply vector is $\mathbf{s} = (575, 20)'$. Here the supply matrix $\mathbf{S}$ equals the output matrix $\mathbf{Y}$ since both agents have no inventory in the initial period.

Demand structures of agents are represented by the input coefficient matrix $\mathbf{A}$ and each agent intends to purchase some standard input bundles indicated by $\mathbf{A}$ for its production. That is, in the exchange process the bundle purchased by agent i must be $\xi \mathbf{a}^{(i)}$, where ξ is a nonnegative real number and $\mathbf{a}^{(i)}$ is the ith column of $\mathbf{A}$. ξ is called the *purchase amount* of agent i. Let $\mathbf{z}$ denote the vector consisting of purchase amounts of m agents, and $\mathbf{z}$ is called the *purchase vector* or *exchange vector* (of standard input bundles), and $\mathbf{Az}$ is called the *sales vector of goods*.

For example, in the system (2a)-(2b) the bundles purchased by agent 1 and agent 2 must be $\left(\frac{56}{115}\alpha, \frac{12}{575}\alpha\right)'$ and $\left(6\beta, \frac{2}{5}\beta\right)'$ respectively, where α and β are nonnegative real numbers. Then the exchange vector is $(\alpha, \beta)'$, which indicates the purchase amounts of standard input bundles, and the corresponding sales vector of goods is

$$\mathbf{Az} = \left(\frac{56}{115}\alpha + 6\beta, \frac{12}{575}\alpha + \frac{2}{5}\beta\right)' \tag{6}$$

which indicates the sales amount of two goods.

Let $\hat{\mathbf{x}}$ denote diag($\mathbf{x}$), i.e. the diagonal matrix with the vector $\mathbf{x}$ as the main diagonal. For example, for the supply vector $\mathbf{s} = (575, 20)'$ we have

$$\hat{\mathbf{s}} = \begin{bmatrix} 575 & 0 \\ 0 & 20 \end{bmatrix} \tag{7}$$

The sales rate of a good refers to the proportion of its sales amount to its supply amount. Suppose for one good all its suppliers share the same sales rate, and let $\mathbf{u}$ be the n-dimensional sales rate vector indicating the sales rates of n goods, that is,

$$\mathbf{u} \equiv \hat{\mathbf{s}}^{-1}\mathbf{Az} \tag{8}$$

For example, if $\mathbf{s} = (575, 20)'$ and $\mathbf{z} = \left(\frac{2875}{6}, 25\right)'$, then the sales rate vector is

$$\mathbf{u} \equiv \hat{\mathbf{s}}^{-1}\mathbf{Az} = \begin{bmatrix} 575 & 0 \\ 0 & 20 \end{bmatrix}^{-1} \begin{bmatrix} \frac{56}{115} & 6 \\ \frac{12}{575} & \frac{2}{5} \end{bmatrix} \left(\frac{2875}{6}, 25\right)' = \begin{bmatrix} 575 & 0 \\ 0 & 20 \end{bmatrix}^{-1} \left(\frac{1150}{3}, 20\right)' = \left(\frac{2}{3}, 1\right)' \tag{9}$$

Obviously, the matrices $\mathbf{A}\hat{\mathbf{z}}$ and $\hat{\mathbf{u}}\mathbf{S}$ indicate each agent’s purchase and sales amounts of goods respectively. For the example above we have

$$\mathbf{A}\hat{\mathbf{z}} = \begin{bmatrix} \frac{700}{3} & 150 \\ 10 & 10 \end{bmatrix}, \quad \hat{\mathbf{u}}\mathbf{S} = \begin{bmatrix} \frac{1150}{3} & 0 \\ 0 & 20 \end{bmatrix} \tag{10}$$

Under the given price vector $\mathbf{p}$, the purchase and sales values of m agents are $\mathbf{p}'\mathbf{A}\hat{\mathbf{z}}$ and $\mathbf{p}'\hat{\mathbf{u}}\mathbf{S}$ respectively. Suppose the value each agent purchases must equal the value it sells, that is,

$$\mathbf{p}'\mathbf{A}\hat{\mathbf{z}} = \mathbf{p}'\hat{\mathbf{u}}\mathbf{S} \equiv \mathbf{p}'\widehat{\hat{\mathbf{s}}^{-1}\mathbf{Az}}\mathbf{S} \tag{11}$$

Eq. (11) is the equivalent exchange condition. When Eq. (11) holds and $\mathbf{S}'\mathbf{A}$ is indecomposable the following proposition shows that there exists a unique normalized exchange vector.

Proposition 1. Let $\mathbf{A}$ and $\mathbf{S}$ be ($n \times m$) semipositive matrices such that $\mathbf{s} \equiv \mathbf{Se}$ is positive and $\mathbf{S}'\mathbf{A}$ is indecomposable. Let $\mathbf{p}$ be an n-dimensional positive vector and $\mathbf{z}$ be an m-dimensional semipositive vector. Then:

(i) $\mathbf{Z} \equiv \widehat{\mathbf{A}'\mathbf{p}}^{-1}\mathbf{S}'\hat{\mathbf{s}}^{-1}\hat{\mathbf{p}}\mathbf{A}$ is an indecomposable nonnegative matrix possessing the P-F eigenvalue 1;

(ii) $\mathbf{z}$ satisfies $\mathbf{p}'\mathbf{A}\hat{\mathbf{z}} = \mathbf{p}'\widehat{\hat{\mathbf{s}}^{-1}\mathbf{Az}}\mathbf{S}$ if and only if $\mathbf{z}$ is a right P-F eigenvector of $\mathbf{Z}$, i.e. $\mathbf{Zz}=\mathbf{z}$ holds; moreover, if $\mathbf{z}$ satisfies $\mathbf{p}'\mathbf{A}\hat{\mathbf{z}} = \mathbf{p}'\widehat{\hat{\mathbf{s}}^{-1}\mathbf{Az}}\mathbf{S}$ then $\mathbf{z}$ is positive.

The proof of Proposition 1 is in the Appendix.

Let $\mathbf{x}$ denote the normalized right P-F eigenvector of $\mathbf{Z}$. Then by Proposition 1(ii) we have $\mathbf{z} = \xi\mathbf{x}$, where ξ is a nonnegative real number. Since the sales amount of each good is no more than its supply amount, we find $\mathbf{Az} \leq \mathbf{s}$ holds, that is, $\xi\mathbf{Ax} \leq \mathbf{s}$. Hence ξ is no greater than the minimal component of $\widehat{\mathbf{Ax}}^{-1}\mathbf{s}$. Suppose all agents attempt to obtain maximal exchange amounts. The unique maximal exchange vector can be found by following steps, which stands for the outcome of the exchange process:

Step 1. Compute the matrix $\mathbf{Z} \equiv \widehat{\mathbf{A}'\mathbf{p}}^{-1}\mathbf{S}'\hat{\mathbf{s}}^{-1}\hat{\mathbf{p}}\mathbf{A}$;

Step 2. Find the normalized right P-F eigenvector of $\mathbf{Z}$, denoted by $\mathbf{x}$;

Step 3. Find the minimal component of $\widehat{\mathbf{Ax}}^{-1}\mathbf{s}$, denoted by ξ;

Step 4. Compute the exchange vector $\mathbf{z} \equiv \xi\mathbf{x}$.

Thus the exchange process can be represented by a function as follows:

$$(\mathbf{u}, \mathbf{z}) = \mathrm{Z}(\mathbf{A}, \mathbf{p}, \mathbf{S}) \tag{12}$$

where $\mathbf{A}$, $\mathbf{S}$ and $\mathbf{p}$ satisfy those assumptions in Proposition 1, and $\mathbf{z}$ is computed by steps above and $\mathbf{u}$ equals $\hat{\mathbf{s}}^{-1}\mathbf{Az}$. Here we write $\mathbf{u}$ explicitly on the left side of Eq. (12) only for the expression convenience of the growth model in Section 4.

Note that given $\mathbf{A}$ and $\mathbf{s}$ there may be no nonnegative vector $\mathbf{z}$ such that $\mathbf{Az} = \mathbf{s}$, hence in such a case the market cannot clear whatever the market prices are.

Finally, let’s explain how to view that in the exchange process represented by Eq. (12) each agent may buy and sell its product at the same time. Let’s take the example of Sraffa, in which wheat producer will not only sell but also buy wheat in the market. Note that the wheat is a representative of consumer goods, and its producer is a representative of producers of consumer goods, the wheat producer sells and buys wheat at the same time in the market should be viewed as that producers of food, clothes, furniture etc., exchange distinct consumer goods among themselves.

3.2 Some Special Cases of the Exchange Function

As in the system (2a)-(2b), sometimes $\mathbf{S}$ is an ($n \times n$) diagonal matrix. In such a case $\mathbf{S}'\hat{\mathbf{s}}^{-1} = \mathbf{I}$

holds and the matrix **Z** becomes

$$\mathbf{Z} = \widehat{\mathbf{A}'\mathbf{p}}^{-1}\hat{\mathbf{p}}\mathbf{A} \tag{13}$$

And in this case the following proposition holds for the exchange process.

Proposition 2. Let the supply matrix **S** be an ($n \times n$) diagonal matrix such that the supply vector $\mathbf{s} \equiv \mathbf{Se}$ is positive. Let **A** be an ($n \times n$) indecomposable semipositive matrix and **p** be a positive n-dimensional price vector. For the exchange function $(\mathbf{u}, \mathbf{z}) = \mathrm{Z}(\mathbf{A}, \mathbf{p}, \mathbf{S})$ we have:

(i) **z** is a right P-F eigenvector of **A** if and only if **p** is a left P-F eigenvector of **A**;

(ii) if $\mathbf{Az} = \mathbf{s}$ holds (i.e. the market clears) and **p** is a left P-F eigenvector of **A,** then **s** is a right P-F eigenvector of **A**;

(iii) if $\mathbf{Az} = \mathbf{s}$ holds, **A** is nonsingular, and **s** is a right P-F eigenvector of **A,** then **p** is a left P-F eigenvector of **A**;

(iv) for agent i, let $\bar{\mathbf{p}}$ be another positive price vector satisfying $\bar{p}_i > p_i$ and $\bar{p}_j = p_j$ (for all $j \neq i$), and let $\bar{\mathbf{z}}$ be the exchange vector under $\bar{\mathbf{p}}$, i.e. $(\bar{\mathbf{u}}, \bar{\mathbf{z}}) = \mathrm{Z}(\mathbf{A}, \bar{\mathbf{p}}, \mathbf{S})$, then $\bar{z}_i \geq z_i$ and $\bar{z}_i / z_i \geq \bar{z}_j / z_j$ (for all j) hold; moreover, $\bar{z}_i / z_i > \bar{z}_j / z_j$ holds if $j \neq i$ and $a_{ij} > 0$.

The proof of Proposition 2 is in the Appendix.

Next let's suppose further $\mathbf{A} = (a_{ij})$ is a positive (2×2) matrix, as in the system (2a)-(2b) of Sraffa. By computing the exchange vector **z** we find

$$\frac{z_1}{z_2} = \frac{a_{12}}{a_{21}} \frac{p_1}{p_2} \tag{14}$$

That is, the exchange ratio z_1/z_2 is proportional to the price ratio p_1/p_2.

Furthermore, the sales amounts of two goods are indicated by **Az** and the sales amount ratio between two goods is computed to be

$$\kappa = \frac{a_{11}a_{12}\, p_1/p_2 + a_{12}a_{21}}{a_{21}a_{12}\, p_1/p_2 + a_{22}a_{21}} \tag{15}$$

Let θ denote p_1/p_2, then we have

$$\frac{d\kappa}{d\theta} = \frac{a_{12}(a_{11}a_{22} - a_{12}a_{21})}{a_{21}(a_{12}\theta + a_{22})^2} \tag{16}$$

Hence κ is monotonic with respect to θ, and by Eq. (15) it's clear that the sales amount ratio κ must fall between a_{11}/a_{21} and a_{12}/a_{22}. In the system (2a)-(2b) κ must fall between 15 and $\frac{70}{3} \approx 23.3$. When the supply ratio of the two goods isn't in that interval, the market won't clear whatever the market prices are.

Moreover, Eq. (16) indicates that depending on the sign of $(a_{11}a_{22} - a_{12}a_{21})$ κ may rises, fall or keep constant when θ rises. For instance, by Eq. (3) the value of $(a_{11}a_{22} - a_{12}a_{21})$ in the system (2a)-(2b) is positive, that is so say, when the price of wheat rises it will sell more relative to iron in the market due to that the wheat producer will purchase more wheat. The cause is that the technology of the wheat producer is more wheat-intensive than that of the iron producer, and when the price of wheat rises the increase of sales revenue of wheat producer results in more demand for wheat relative to iron. And here that the wheat producer buys and sells more wheat should be viewed as that the producers of consumer goods exchange more consumer goods among themselves.

3.3 Exchange Process in Sraffa's System

Here let's illustrate the exchange process among agents with the example of Sraffa. Formula (2a) and (2b) may be viewed as the production processes in the initial period. After the production processes two agents will exchange their products in the market, and the supply matrix is indicated by Eq. (5). Suppose the current market price vector is an equilibrium price vector $\left(\frac{1}{15}, 1\right)'$, then the exchange vector is computed to be $\mathbf{z} = \left(\frac{2875}{6}, 25\right)' \approx (479.2, 25)'$, and the sales vector of goods is $\mathbf{Az} = \left(\frac{1150}{3}, 20\right)' \approx (383.3, 20)'$, that is, iron sells out and wheat not. And the sales rate vector is $\mathbf{u} = \left(\frac{2}{3}, 1\right)'$. Accordingly, the inventory of wheat is about 191.7 quarters and there is no inventory of iron.

If the market prices change on the basis of supply and demand, the price of wheat will fall relatively and the price of iron will rise relatively after the first exchange process.

After the exchange process the two agents will perform production processes again, and note that the exchange vector $\mathbf{z}$ also indicates the ensuing production intensities (and output amounts) of agents. Wheat producer has purchased 479.2 standard input bundles in the market, and will yield 479.2 quarters of wheat in the next production process. Iron producer has purchased 25 standard input bundles, and will yield 25 tons of iron in the next production process. That is to say, the output of wheat will fall and the output of iron will rise. And then the exchange process and production process will repeat again and again.

In the first exchange process the market doesn't clear. In fact, recall that the ratio of sales amounts of two goods must fall between 15 and 23.33, and note that the supply ratio of wheat to iron now is $\frac{575}{20} = 28.75$, the market cannot clear whatever the market prices are.

The sales rate vector in the first exchange process is $\mathbf{u} = \left(\frac{2}{3}, 1\right)'$, thus the inventory rate vector is $\mathbf{e} - \mathbf{u} = \left(\frac{1}{3}, 0\right)$. Furthermore, the inventory matrix is defined as $\widehat{\mathbf{e}-\mathbf{u}}\mathbf{S}$, whose $(i,\ j)$ entry indicates the inventory amount of good *i* of agent *j*. After the first exchange process the inventory matrix is

$$\widehat{\mathbf{e}-\mathbf{u}}\mathbf{S} = \widehat{\left(\frac{1}{3}, 0\right)}\begin{bmatrix} 575 & 0 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 575 & 0 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} \frac{575}{3} & 0 \\ 0 & 0 \end{bmatrix} \approx \begin{bmatrix} 191.7 & 0 \\ 0 & 0 \end{bmatrix} \qquad (17)$$

After the first exchange process the inventory of the wheat producer is about 191.7 quarters, which may undergo depreciation before the next exchange process comes, say, it has a depreciation rate 0.2. Thus 153.3 quarters of wheat will be left when the next exchange process comes. Then in the next exchange process the supply of wheat will be $632.5(=153.3+479.2)$ quarters. Recall that the supply of iron will be 25 tons, hence the supply ratio will be 25.3 and the market cannot clear once again.

4. The Structural Growth Model

Let's regard the economy as a discrete-time dynamic system and suppose economic activities such as price adjustment, exchange and production etc. occur in turn in each period. And the state of the economy in period *t* is represented by following variables:

$\mathbf{p}^{(t)}$ Price vector, which is positive and consists of prices of *n* goods in period *t*;

$\mathbf{S}^{(t)}$ Supply matrix, whose (i, j) entry stands for the agent *j*'s supply amount of good *i* in period *t*;

$\mathbf{u}^{(t)}$ Sales rate vector, which consists of sales rates of *n* goods in period *t*;

$\mathbf{z}^{(t)}$ Exchange vector and production intensity vector, which represents the amounts of standard input bundles that are purchased and put into production by agents in period *t*; and in the special case $\mathbf{B} = \mathbf{I}$, $\mathbf{z}^{(t)}$ is also the output vector of goods which indicates the output amounts of all agents;

$\mathbf{Y}^{(t)}$ Output matrix, whose (i, j) entry stands for the output amount of good *i* by agent *j* in period *t*.

4.1 The Model

Suppose in period *t*+1 the economy runs as follows.

Firstly, the new price vector emerges on the basis of the price vector and sales rates of period *t*, which indicates the market prices of *n* goods in period *t*+1.

Secondly, outputs and depreciated inventories of period *t* constitute the supplies of period *t*+1.

Thirdly, supplies are exchanged under market prices, and the exchange vector and sales rate vector of period *t*+1 are obtained. Unsold goods constitute the inventories of period *t*+1, which will undergo depreciation and become a portion of the supplies of the next period.

Finally, each agent puts into production its input bundle purchased in the market, and outputs of period *t*+1 are obtained.

The structural growth model is as follows:

$$\mathbf{p}^{(t+1)} = \mathrm{P}\left(\mathbf{p}^{(t)}, \mathbf{u}^{(t)}\right) \tag{18a}$$

$$\mathbf{S}^{(t+1)} = \mathbf{Y}^{(t)} + \mathrm{Q}\left(\widehat{\mathbf{e} - \mathbf{u}^{(t)}}\mathbf{S}^{(t)}\right) \tag{18b}$$

$$\left(\mathbf{u}^{(t+1)}, \mathbf{z}^{(t+1)}\right) = \mathrm{Z}\left(\mathbf{A}, \mathbf{p}^{(t+1)}, \mathbf{S}^{(t+1)}\right) \tag{18c}$$

$$\mathbf{Y}^{(t+1)} = \mathbf{B}\widehat{\mathbf{z}^{(t+1)}} \tag{18d}$$

Let's explain equations above in turn.

Eq. (18a) stands for the adjustment process of market prices, and P is the price adjustment function. In this paper prices are assumed to be adjusted on the basis of supply and demand, and P may assume other forms to allow some prices are exogenous or controlled by agents.

Eq. (18b) stands for the formation of supplies. If $\mathbf{u}^{(t)} \neq \mathbf{e}$, then there are some unsold goods in period *t*. The inventory amounts of agents in period *t* are indicated by the inventory matrix $\widehat{\mathbf{e}-\mathbf{u}^{(t)}}\mathbf{S}^{(t)}$. Q is the inventory depreciation function, which stands for the depreciation process of inventories. The outputs of period *t*, which is denoted by $\mathbf{Y}^{(t)}$, plus the depreciated inventories of period *t*, which is denoted by $\mathrm{Q}\left(\widehat{\mathbf{e}-\mathbf{u}^{(t)}}\mathbf{S}^{(t)}\right)$, forms the supplies of period $t+1$, which is denoted by $\mathbf{S}^{(t+1)}$.

Eq. (18c) stands for the exchange process, and Z is the exchange function in Eq. (12).

Eq. (18d) stands for the production process. Since $z_i^{(t+1)}$ indicates the amount of the standard input bundle purchased by agent *i* in period $t+1$ and a standard input bundle corresponds to a unit of production intensity, $z_i^{(t+1)}$ also indicates the production intensity of agent *i* in period $t+1$.

We write Eq. (18d) explicitly in the model only for clarity. Eq. (18d) can be omitted if Eq. (18b) is written as

$$\mathbf{S}^{(t+1)} = \mathbf{B}\widehat{\mathbf{z}^{(t)}} + \mathrm{Q}\left(\widehat{\mathbf{e}-\mathbf{u}^{(t)}}\mathbf{S}^{(t)}\right) \qquad (18\mathrm{b}')$$

When input and output coefficient matrices are variables, an equation may be inserted between Eq. (18a) and (18b) to reflect the adjustment of input and output coefficient matrices by agents due to price change or technology progress etc.

4.2 A Specific Form of the Structural Growth Model

The following is a computable specific form of the model (18a)-(18d), which will be used in Section 5 and 6 for simulations.

$$p_i^{(t+1)} = \begin{cases} p_i^{(t)} & u_i^{(t)} > 0.99 \\ 0.98 p_i^{(t)} & u_i^{(t)} \le 0.99 \end{cases}, \text{ for } i = 1, 2, \cdots, n \qquad (19\mathrm{a})$$

$$\mathbf{S}^{(t+1)} = \mathbf{B}\widehat{\mathbf{z}^{(t)}} + 0.8\widehat{\mathbf{e}-\mathbf{u}^{(t)}}\mathbf{S}^{(t)} \qquad (19\mathrm{b})$$

$$\left(\mathbf{u}^{(t+1)}, \mathbf{z}^{(t+1)}\right) = \mathrm{Z}\left(\mathbf{A}, \mathbf{p}^{(t+1)}, \mathbf{S}^{(t+1)}\right) \qquad (19\mathrm{c})$$

Eq. (19a) stands for the price adjustment process, which means when a good nearly sells out its price won't change; otherwise its price will fall by 2 percent. Hence all prices won't change if and only if all goods nearly sell out. If there are goods far from clearing, the prices of nearly sold-out goods will rise relatively. Note that only relative prices matters in the model, such adjustment method is reasonable.

Eq. (19b) stands for the formation of supplies. Here we assume a simple inventory depreciation function $\mathrm{Q}(\mathbf{M}) \equiv 0.8\mathbf{M}$. Let's assume $\mathbf{z}^{(0)} \gg \mathbf{0}$ and $\mathbf{B}'\mathbf{A}$ is indecomposable to guarantee that $\mathbf{S}'^{(t)}\mathbf{A}$ is indecomposable for all $t = 1, 2, \cdots, \infty$.

Eq. (19c) stands for the exchange process.

In the model (19a)-(19c) **A** and **B** are exogenous, and let's always set $\mathbf{u}^{(0)} = \mathbf{e}$ so that $\mathbf{S}^{(1)}$ will equal $\mathbf{B}\widehat{\mathbf{z}^{(0)}}$ (i.e. $\mathbf{Y}^{(0)}$, the outputs in the initial period). Then when we set the values of $\mathbf{p}^{(0)}$ and $\mathbf{z}^{(0)}$ the model can run by itself.

A path of the model (19a)-(19c) is called an *equilibrium path* if $\mathbf{u}^{(t)} = \mathbf{e}$ holds in it for all $t = 0, 1, 2, \cdots, \infty$. That is, the model runs in an equilibrium path if all goods clear all the time. By Eq. (19a) the price vector will keep constant in an equilibrium path, which is called an *equilibrium price vector*. By Proposition 2 it can be readily verified that the model will run in an equilibrium path when $\mathbf{B} = \mathbf{I}$, $\mathbf{u}^{(0)} = \mathbf{e}$, $\mathbf{p}^{(0)}$ and $\mathbf{z}^{(0)}$ are a left and right P-F eigenvector of **A** respectively. And in such a case the output vector $\mathbf{z}^{(t)}$ in each period is a right P-F eigenvector of **A**.

5. One-country Economy

In this section business cycles in the model (19a)-(19c) will be illustrated with the example of the system (2a)-(2b). The system (2a)-(2b) will also be extended to include land, then the equilibrium land rent and the dynamics of land rent will be investigated.

5.1 Business Cycles

For the system (2a)-(2b) of Sraffa, the input coefficient matrix **A** is indicated by Eq. (3) and the input coefficient matrix $\mathbf{B}$ equals $\mathbf{I}$. Recall that an equilibrium price vector is $\left(\frac{1}{15}, 1\right)'$ and the output vector in the initial period is $(575, 20)'$, let's set them as $\mathbf{p}^{(0)}$ and $\mathbf{z}^{(0)}$ of the model (19a)-(19c) respectively, and let $\mathbf{u}^{(0)} = \mathbf{e}$. The simulation results are depicted in Figure 1.

Figure 1 shows that the price ratio, output ratio, profit rates and output growth rates fluctuate periodically, that is, there are business cycles in the system.

Recall that the equilibrium price ratio of wheat to iron is $\frac{1}{15} \approx 0.0667$, the equilibrium output ratio of wheat to iron is $\frac{575}{30} \approx 19.17$, and the equilibrium profit rates of both agents are 0.25, now it's clear that the price ratios, output ratios, profit rates all fluctuate around their equilibrium values, which exemplifies the argument of Kurz and Salvadori:

> 'The classical as well as the early neoclassical economists did not consider these [equilibrium] prices are purely ideal or theoretical; they saw them rather as "centers of gravitation," or "attractors," of actual or market prices.' (Kurz and Salvadori, 1995, p. 1)

The argument of Kurz and Salvadori also applies to the output ratio and profit rates here. Why these variables fluctuate around their equilibrium values rather than other values? A short answer is that the market mechanism will keep the supply ratio of two goods between 15 and 23.3 in the long run, consequently the two agents must have almost equal average output growth rates in the long run, and the equalization tendency of output growth rates forces those variables to fluctuate around their

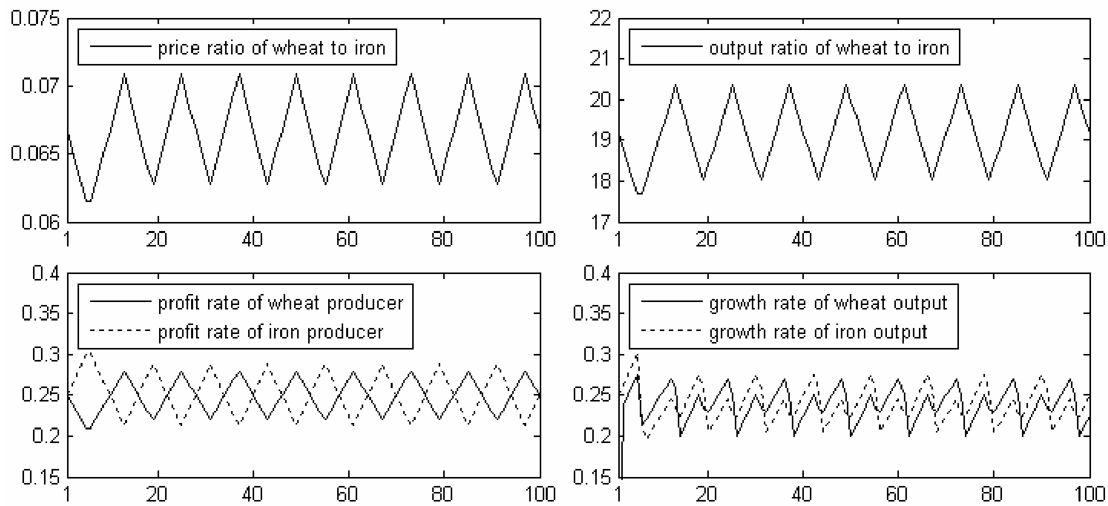

Figure 1. Price ratio, output ratio, profit rates and output growth rates in period 1 to 100

equilibrium values.

However, the "centers of gravitation" of output growth rates aren't the equilibrium growth rate 0.25, instead it's lower than 0.25 because the average output growth rates of both agents suffer a loss due to business cycles. In fact, the average output growth rates of both agents in a regular business cycle are 0.2381 approximately. And due to the utilization of inventory sometimes the output growth rates of both agents may be higher than the equilibrium growth rate 0.25, e.g. the output growth rates of two agents in period 5 are about 0.2745 and 0.3005 respectively.

When the price ratio and supply ratio of wheat to iron in period 1 to 1000 are depicted in one panel, as in Figure 2, business cycles show themselves from another angle and turn out be discrete-time limit cycles. In Figure 2 the left panel depicts the path with $\mathbf{p}^{(0)} = \left(\frac{1}{15}, 1\right)'$ and $\mathbf{z}^{(0)} = (575, 20)'$. And the right panel depicts the path with $\mathbf{p}^{(0)} = (0.0660, 1)'$, which is slightly apart from the equilibrium price vector, and $\mathbf{z}^{(0)} = (575, 30)'$, which is an equilibrium output vector. The asterisks at (19.17, 0.0667) stand for equilibrium paths.

Figure 2 shows that the economy runs into a limit cycle before long in both cases, and limit cycles correspond to business cycles. As the limit cycles in Figure 2 indicate, in both cases a regular business cycle contains 12 periods.

Hence, here the market mechanism leads the economy into limit cycles (i.e. business cycles) rather than the fixed point (i.e. equilibrium paths). As the left panel shows, when the economy starts running at a point far from the equilibrium, the market mechanism indeed pulls it towards the

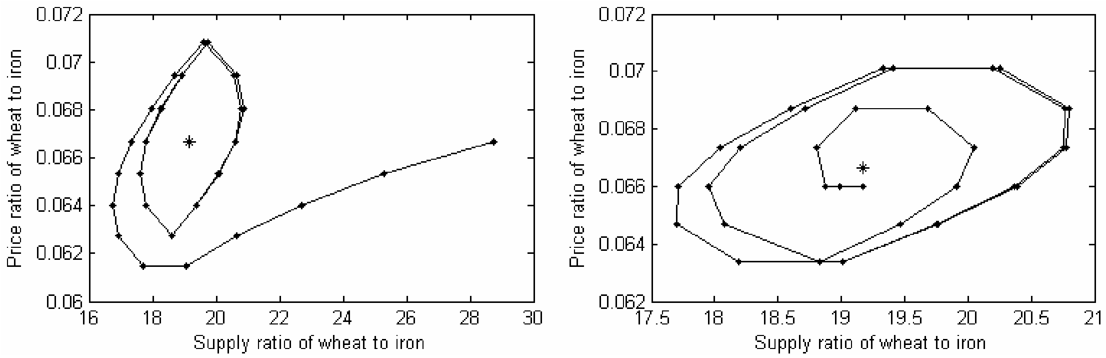

Figure 2. Fixed point (i.e. equilibrium paths) and limit cycles (i.e. business cycles)

equilibrium at first. However, the economy falls eventually into a limit cycle and will keep revolving around the fixed point rather than approach it. And the right panel shows that even the economy starts running at a point quite near to equilibrium paths, the invisible hand may also pull it away and finally leads it into business cycles.

Next let's focus on the path with $\mathbf{p}^{(0)} = \left(\frac{1}{15}, 1\right)'$ and $\mathbf{z}^{(0)} = (575, 20)'$, and investigate the change of output growth rates in a regular business cycle. For instance, the output growth rates and sales rates in a regular business cycle from period 86 to period 97 are depicted in Figure 3.

If in one period the output growth rate of an agent is higher than that in the preceding period, here let's call that period a rising period of the agent; and in the opposite case, that period is called a declining period of the agent. The left panel of Figure 3 shows that there are 8 rising periods and 4 declining periods in a regular business cycle for both agents. That is, the number of rising periods is much more than the number of declining periods, which implies that when the growth rate declines, it declines suddenly and fiercely, and when it rises, it rises slowly and moderately.

Furthermore, when output growth rates of both agents are compared with sales rates, it's clear that they rise and decline quite synchronously, that is, the rise of the growth rate is usually accompanied with the reduction in inventory.

5.2 Land Rent

Supplies of some factors of production may be fixed or grow at an exogenous rate $\gamma \in (-1, +\infty)$, e.g. land, mineral deposits, labor force etc., consequently their rents (or wage) have more complex

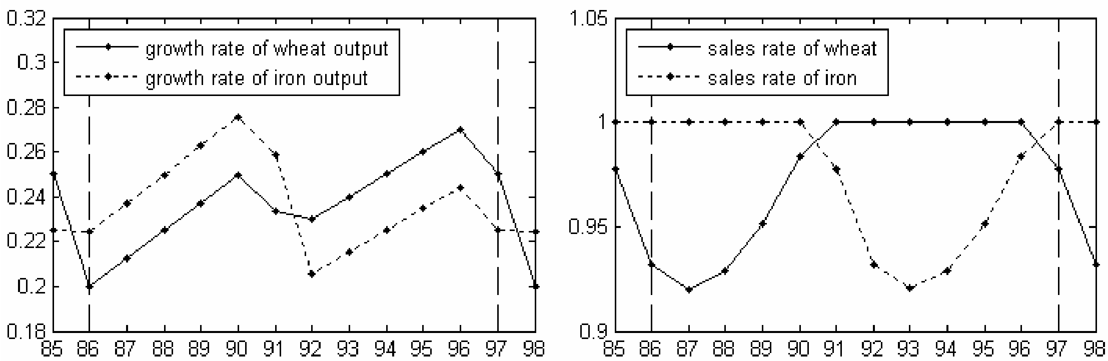

Figure 3. Output growth rates and sales rates in a regular business cycle

dynamics than common goods, and land rent may be taken as a representative. The theory of land rent of Sraffa (1960) has been discussed by a number of authors, e.g. Kurz (1978), Salvadori (1986), Woods (1987), Bidard (2004), and these discussions ignore the consumption structure of the landowner. Assuming simply that land is uniform in quality, here let's compute the equilibrium land rent with a method taking account of the supply growth rate of land and the consumption structure of the landowner, and then simulate the dynamics of land rent. First let's extend the system (2a)-(2b) to include land as follows, where the supply of land is assumed to be 1140 units all the time.

$$280 \text{ quarters wheat} \quad +12 \text{ tons iron} \quad +18 \text{ units land} \rightarrow 575 \text{ quarters wheat} \tag{20a}$$

$$120 \text{ quarters wheat} \quad +8 \text{ tons iron} \quad +5 \text{ units land} \rightarrow 20 \text{ tons iron} \tag{20b}$$

$$115 \text{ quarters wheat} \quad +6 \text{ tons iron} \quad +3 \text{ units land} \rightarrow 1140 \text{ units land} \tag{20c}$$

The first two formulas are self-evident. The third formula means that in the initial period the landowner (i.e. agent 3) have 1140 units of land for rent, and the landowner consumes 115 quarters of wheat and 6 tons of iron, and uses 3 units of land for his own living. Of course, the landowner has to rent out his land to exchange his consumption bundle.

For simplicity, let's suppose the consumption structure of the landowner will keep unchanged all the time, that is, his consumption bundle muse be $\left(115\xi, 6\xi, 3\xi\right)'$, where ξ is a nonnegative real number indicating his consumption intensity and is determined by the magnitude of land rent and the prices of goods etc. in the exchange process. Now the variable input coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} \frac{56}{115} & 6 & \frac{23}{228}\xi \\ \frac{12}{575} & \frac{2}{5} & \frac{1}{190}\xi \\ \frac{18}{575} & \frac{1}{4} & \frac{1}{380}\xi \end{bmatrix} \tag{21}$$

and the output coefficient matrix is $\mathbf{B} = \mathbf{I}$. The last column of $\mathbf{A}$ stands for the standard input bundle (i.e. standard consumption bundle) of the landowner, which is his consumption bundle divided by 1140. That is to say, here land is treated as the product of the landowner, and the production intensity equals 1140 all the time and the standard input bundle is variable.

Since land is indispensable for production in the economy (20a)-(20c) and technological change is excluded here, under the fixed supply of land the growth rate of the economy in an equilibrium path must be zero, that is, the P-F eigenvalue of $\mathbf{A}$ must be 1. Hence ξ are computed to be 40.

That is, in an equilibrium path the input coefficient matrix must be

$$\mathbf{A}^* = \begin{bmatrix} \frac{56}{115} & 6 & \frac{230}{57} \\ \frac{12}{575} & \frac{2}{5} & \frac{4}{19} \\ \frac{18}{575} & \frac{1}{4} & \frac{2}{19} \end{bmatrix} \tag{22}$$

Furthermore, a left P-F eigenvector of $\mathbf{A}^*$ is found to be $(0.0759, 1, 0.5777)'$ approximately, that is, the equilibrium land rent is about 0.5777 per unit per period with iron as the numeraire. By computing the right P-F eigenvector of $\mathbf{A}^*$ the production processes in the equilibrium path are found be:

11200 quarters wheat +480 tons iron +720 units land $\rightarrow$ 23000 quarters wheat (23a)

7200 quarters wheat +480 tons iron +300 units land $\rightarrow$ 1200 tons iron (23b)

4600 quarters wheat +240 tons iron +120 units land $\rightarrow$ 1140 units land (23c)

Now let's run the model (19a)-(19c) with $\mathbf{p}^{(0)} = (0.0759, 1, 0.5777)'$ and $\mathbf{z}^{(0)} = (575, 20, 1140)'$ to simulate the dynamics of prices, land rent, and outputs. In each period the supply of land in the model is set to be 1140 units. Figure 4 depicts the simulation results.

Figure 4 shows that the land rent decreases at first due to its oversupply. However, the demand for land increases as the economy grows, and eventually the demand exceeds the fixed supply. Then the land rent begins rising until the average growth rate of the economy becomes zero. Hence only after a long time the land rent begins fluctuating around its equilibrium value. And eventually the outputs of wheat and iron also fluctuate around their equilibrium values.

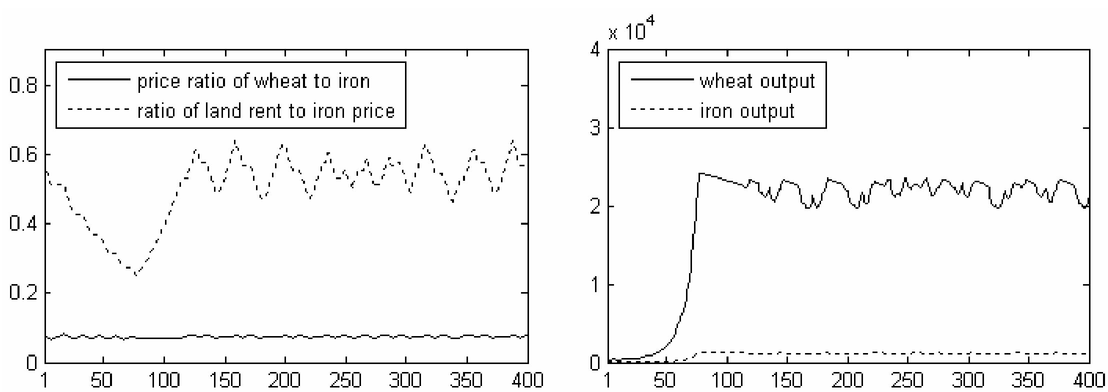

Figure 4. Price, land rent and outputs under fixed land supply in period 1 to 400

When the supply of land in the system (20a)-(20c) grows exogenously at a fixed rate γ, the equilibrium land rent can be computed analogously.

Note that the equilibrium growth rate of the original economy (2a)-(2b) without land is 0.25 and the supply of land is exogenous, it's clear that when γ is no less than 0.25 agent 1 and 2 can obtain land as much as needed and need pay nothing in an equilibrium path, that is, in such a case land becomes a free good and the equilibrium land rent is zero.

When γ falls between -1 and 0.25, say $\gamma = 0.2$, the equilibrium growth rate of the economy (20a)-(20c) must be $\gamma = 0.2$, which implies that the P-F eigenvalue of **A** in Eq. (21) must be $\frac{1}{1+\gamma} = \frac{5}{6}$. Hence we find $\xi \approx 6.1$, and a left P-F eigenvector is found to be $(0.0684, 1, 0.0907)'$ approximately, that is, the equilibrium land rent is about 0.0907 per unit per period with iron as the numeraire.

Furthermore, note that when the system (2a)-(2b) runs in a disequilibrium path the long-run average growth rate is lower than 0.25, hence in a disequilibrium path the supply of land may exceed the demand all the time even though γ is less than 0.25. And generally speaking, the exogenous supply disturbs the market mechanism and consequently the land rent does not necessarily fluctuate around its equilibrium value in a disequilibrium path. For instance, Figure 5 depicts the dynamics of the land rent and the wheat price with iron as the numeraire when $\gamma = 0.2$, wherein $\mathbf{p}^{(0)} = (0.0684, 1, 0.0907)'$ and $\mathbf{z}^{(0)} = (575, 20, 1140)'$. The figure shows that the wheat price fluctuates around 0.0684, however, the land rent fluctuates below its equilibrium value 0.0907.

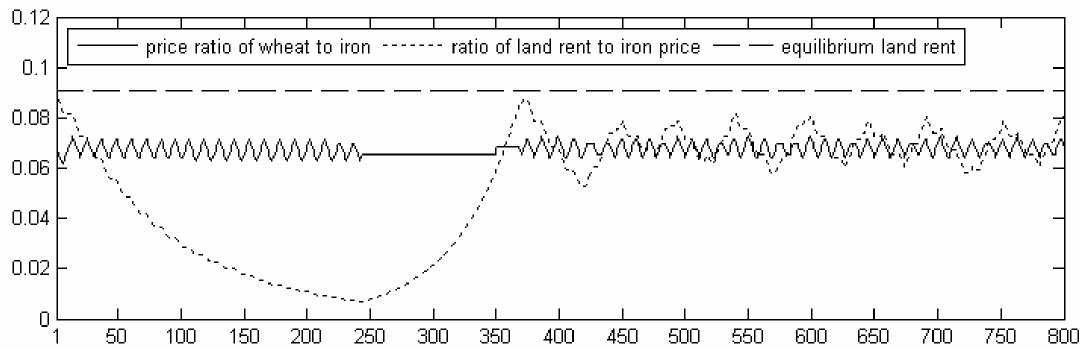

Figure 5. Wheat price and land rent under growing land supply ($\gamma = 0.2$) in period 1 to 800

6. Two-Country Economy

In this section let's extend the system of Sraffa to a two-country economy and analyze its dynamics under free trade and trade protectionism. Let's regard the system (2a)-(2b) as country 1, and suppose there is another country, namely country 2, which consists of agent 3 and 4, and runs in the initial period as follows:

$$215 \text{ quarters wheat} \quad +14 \text{ tons iron} \quad \rightarrow 575 \text{ quarters wheat} \tag{24a}$$

$$105 \text{ quarters wheat} \quad +10 \text{ tons iron} \quad \rightarrow 20 \text{ tons iron} \tag{24b}$$

The equilibrium growth rate of country 2 is also 0.25. An equilibrium price vector is $\mathbf{p}^* = \left(\frac{2}{35}, 1\right)' \approx (0.0571, 1)'$, and an equilibrium output vector is $\mathbf{z}^* = (345, 28)'$. Similar to country 1, with $\mathbf{p}^{(0)} = \left(\frac{2}{35}, 1\right)'$ and $\mathbf{z}^{(0)} = (575, 20)'$ the economy of country 2 will also exhibit regular business cycles, and the average output growth rates of both agents in a regular business cycle are about 0.2379.

Next let's investigate the economy consisting of the two countries.

6.1 Two-country Economy under Free Trade

When all goods are internationally tradable, the two-country economy contains 2 goods and 4 agents, and the input and output coefficient matrices are

$$\mathbf{A} = \begin{bmatrix} \frac{56}{115} & 6 & \frac{43}{115} & \frac{21}{4} \\ \frac{12}{575} & \frac{2}{5} & \frac{14}{575} & \frac{1}{2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{25}$$

Let's run the model (19a)-(19c) with $\mathbf{p}^{(0)} = \left(\frac{1}{15}, 1\right)$, i.e. an equilibrium price vector for country 1, and $\mathbf{z}^{(0)} = (575, 20, 575, 20)'$. The output growth rates of agents are depicted in Figure 6, and the price ratio of wheat to iron is depicted in the left panel of Figure 7.

Figure 6 shows that the output growth rate of wheat of country 1 is much lower than that of country 2, and the output growth rate of iron of country 1 is much higher than that of country 2. That is, the outputs of agent 1 and agent 4 grow at much smaller rates than that of agent 3 and 2. As a result, the share of agent 1 in the wheat market and the share of agent 4 in the iron market keep

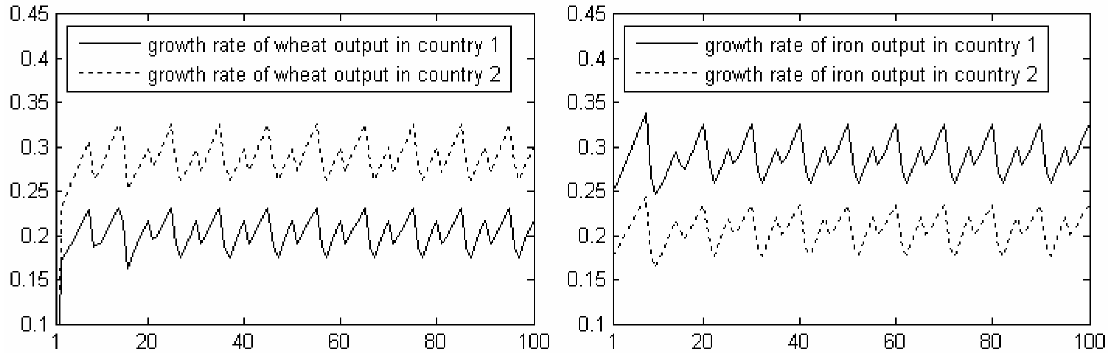

Figure 6. Growth rates of outputs under free trade in period 1 to 100

decreasing, thus they will be washed out from the market in the long run. As a matter of fact, both the wheat output ratio of country 1 to country 2 and the iron output ratio of country 2 to country 1 in period 100 are about 0.1%. Consequently, the pattern of international trade in the long run must be that country 1 exports iron and imports wheat and country 2 exports wheat and imports iron.

Hence in the long run the two-country economy is dominated by agent 2 and 3, and in fact the two agents can compose an autarkic sub-economy. The equilibrium price ratio of wheat to iron in the sub-economy is computed likewise to be 0.0616 approximately, and the equilibrium growth rate is about 0.2997. The left panel of Figure 7 shows that the prices ratio in the two-country economy fluctuates around 0.0616 due to the domination of the sub-economy.

Finally, let’s discuss the intensities of inputs in technologies briefly. By Eq. (25) it’s clear that the technologies producing iron are more iron-intensive than the technologies producing wheat for both countries. In other words, iron is the iron-intensive product and wheat isn’t. Furthermore, either technology of country 2 is more iron-intensive than the corresponding technology of country 1. So we may say that technologies of country 2 are more iron-intensive than country 1. Since country 2 exports wheat and imports iron, we see that a country with relatively iron-intensive technologies may import an iron-intensive product and export a non-iron-intensive product.

6.2 Two-country Economy under Trade Protectionism

Now let’s suppose wheat is internationally non-tradable due to trade protectionism. Since the wheat of country 1 and country 2 isn’t substitutable for each other and consequently may have different prices, they need to be treated as two distinct goods. Therefore the two-country economy contains 3 goods (i.e. wheat of country 1, iron, wheat of country 2) and 4 agents now, and the input and output coefficient matrices are

$$\mathbf{A}=\begin{bmatrix} \frac{56}{115} & 6 & 0 & 0 \\ \frac{12}{575} & \frac{2}{5} & \frac{14}{575} & \frac{1}{2} \\ 0 & 0 & \frac{43}{115} & \frac{21}{4} \end{bmatrix},\quad \mathbf{B}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{26}$$

Let $\mathbf{p}^{(0)}=(1,15,1)'$, that is, the initial prices of wheat of both countries are equal, and let $\mathbf{z}^{(0)}=(575,20,575,20)'$. The price ratios of wheat to iron are depicted in the right panel of Figure 7, which shows that the wheat price of country 1 now is higher than that of country 2.

Since either wheat producer cannot trade with the foreign iron producer, iron also becomes internationally non-tradable in effect because two iron producers needn't trade with each other. Thus there is virtually no international trade now, and two countries are linked only through the unified iron market and the same iron price. Now an equilibrium path in the two-country economy is merely a simple combination of the equilibrium paths in two one-country economies. So the equilibrium price ratios of wheat to iron of two countries are still 0.0667 and 0.0571 respectively now. The right panel in Figure 7 shows that the price ratios indeed fluctuate around them. And now the equilibrium growth rate of the two-country economy is still 0.25.

The output growth rates of agents are depicted in Figure 8, and the growth rates of output values of two countries are depicted in Figure 9. Here the output value of one country in one period is computed on the basis of outputs of its agents and the normalized market price vector in that period.

Now the average growth rates of agents 2 and 3 are lower than that under free trade, and the

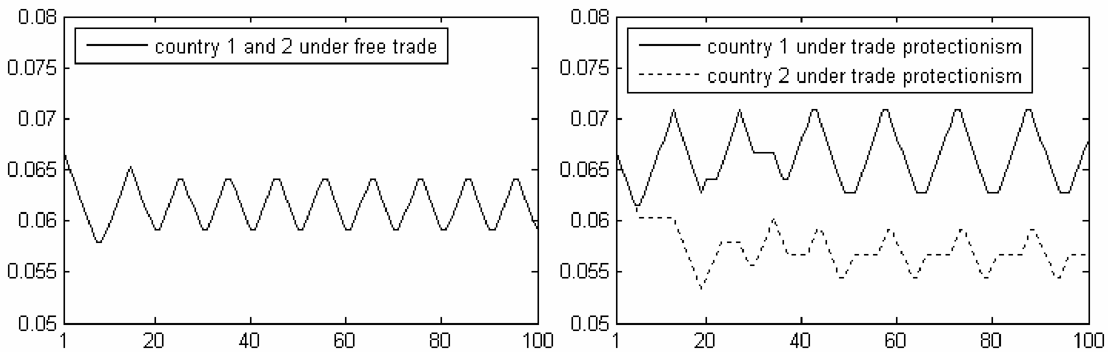

Figure 7. Price ratio of wheat to iron under free trade and trade protectionism

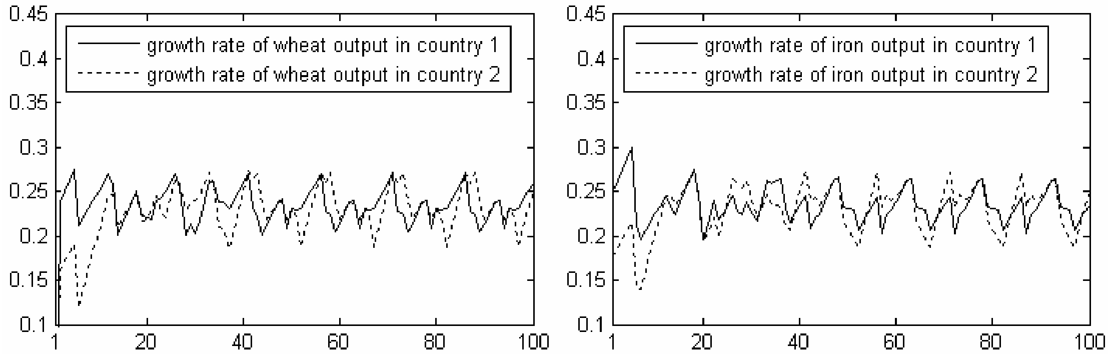

Figure 8. Growth rates of outputs under trade protectionism

average growth rates of agents 1 and 4 are higher. The average growth rates of agent 1 and 2 in a regular business cycle are computed to be about 0.2324 and that of agent 3 and 4 are about 0.2338.

Agent 1 and 4 survive due to trade protectionism, however, both countries suffer a growth rate loss in output values. In a regular business cycle, the average growth rates of output values of both countries under free trade are computed to be 0.2896, which are higher than 0.25 (i.e. the equilibrium growth rates of both countries when they are separated). Under trade protectionism the average growth rates of output values of two countries are computed to be 0.2324 and 0.2338 respectively.

Moreover, note that for the two countries the current average growth rates (i.e. 0.2324 and 0.2338) are lower than 0.2381 and 0.2379 respectively, that is, lower than the average growth rates of output values of two countries when two countries are separated, hence in this example the best arrangement for the two countries is free trade, the next is to separate two countries, and the worst is the trade protectionism with non-tradable wheat and tradable iron.

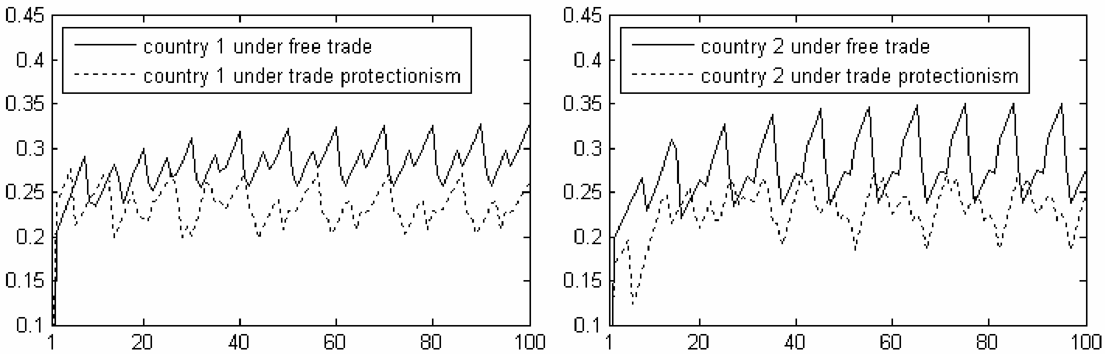

Figure 9. Growth rates of output values of two countries under free trade and trade protectionism

7. Concluding Remarks

The economy in the real world runs in disequilibrium, and in this sense the disequilibrium analysis is as crucial as the equilibrium analysis. The structural growth model presented in this paper provides a dynamic analytical tool based on the classical growth framework for the disequilibrium analysis. In the model a country is treated as a collection of agents, therefore the disequilibrium dynamics of a one-country economy and a multi-country economy can be analyzed in a unified way. By aggregating the individual variables of agents, the macroeconomic variables of countries such as output values and their growth rates, amounts of exports and imports, etc., can be analyzed readily.

For concreteness a system of Sraffa (1960) is utilized to exemplify the structural growth model and simulations are performed, and we arrive at some conclusions as follows.

First, simulations shows that in some sense the equilibrium paths in the growth model correspond to a fixed point and business cycles corresponds to limit cycles, and the market mechanism usually will pull the economy into a limit cycle rather than the fixed point. And in business cycles variables such as prices, profit rates, output ratios fluctuate around their equilibrium values. However, the "centers of gravitation" of output growth rates are lower than the equilibrium growth rate due to the growth rate loss resulting from business cycles.

As for the land rent under an exogenous supply of homogeneous land, its equilibrium value is determined by the consumption structure of landowner, the exogenous growth rate of land supply and technologies of other agents. Since wage is the rent of labor force, the equilibrium wage rate can be computed likewise based on the consumption structure of labor force, the exogenous growth rate of labor force supply and technologies of other agents. And due to the exogenous supply of land the land rent doesn't necessarily fluctuate around its equilibrium value in disequilibrium paths.

For the two-country economy extended from the system of Sraffa in this paper, the free trade boosts the growth rates of output values of both countries in comparison with trade protectionism. And under free trade those agents with lower growth rates will inevitably be washed out by international competitors in the long run, as a result the pattern of international trade emerges naturally. Hence, in a disequilibrium multi-country economy the pattern of international trade can be identified in the light of output growth rates of agents, and both the growth rates and the trade pattern can be investigated easily by simulations.

References

Bidard, C. (2004) *Prices, Reproduction, Scarcity*, Cambridge: Cambridge University Press.

Dietzenbacher, E. (1988) Perturbations of Matrices: A Theorem on the Perron Vector and its Applications to Input-output Models, *Journal of Economics*, 48(4), pp. 389-412.

Dorfman, R., P. A. Samuelson and R. M. Solow. (1958) *Linear Programming and Economic Analysis*. New York: McGraw-Hill.

Gale, D. (1960) *The Theory of Linear Economic Models.* New York: McGraw-Hill.

Kemeny, J. G., O. Morgenstern and G. L. Thompson. (1956) A Generalization of the von Neumann Model of an Expanding Economy, *Econometrica*, 24, pp. 115-135.

Kurz, H. (1978) Rent Theory in a Multisectoral Model, *Oxford Economic Papers*, 30(1), pp. 16-37.

Kurz, H., and N. Salvadori. (1995) *Theory of Production: A Long-Period Analysis*, Cambridge: Cambridge University Press.

Kurz, H. and N. Salvadori. (2000) ‘Classical’ Roots of Input-Output Analysis: A Short Account of its Long Prehistory, *Economic Systems Research*, 12(2), pp. 153-179.

Kurz, H. and N. Salvadori. (2001) Sraffa and von Neumann, *Review of Political Economy*, 13(2), pp. 161-180.

Leontief, W. (1936) Quantitative Input-Output Relations in the Economic System of the United States, *Review of Economics and Statistics*, 18, pp. 105-125.

Leontief, W. (1941) *Structure of the American Economy, 1919-1929*. Cambridge, Mass.: Harvard University Press.

Los, B. (2001) Endogenous Growth and Structural Change in a Dynamic Input–Output Model. *Economic Systems Research*, 13(1), pp. 3-34.

Marx, K. (1956) *Capital*, vol. II, (Moscow, Progress Publishers). English translation of Das Kapital, vol. II, edited by F. Engels, (Hamburg, Meissner, 1885).

McKenzie, L. W. (1963) Turnpike Theorems for a Generalized Leontief Model, *Econometrica*, 31, pp. 165-180.

McKenzie, L. W. (1976) Turnpike Theory, *Econometrica*, 44, pp. 841-865.

Morishima, M. (1964) *Equilibrium, Stability and Growth: A Multi-sectoral Analysis*. New York: Oxford University Press.

Morishima, M. (1973) *Marx's Economics: A Dual Theory of Value and Growth*. Cambridge: Cambridge University Press.

Quesnay, F. (1972) *Quesnay's Tableau Economique* [1759]. Edited by M. Kuczynski and R. L. Meek. London: Macmillan.

Salvadori, N. (1986) Land and Choice of Techniques within the Sraffa Framework, *Australian Economic Papers*, 25, pp. 94-105.

Salvadori, N. and I. Steedman. (1990) *Joint Production of Commodities*, Edward Elgar Publishing.

Samuelson, P. and E. Etula. (2006) Testing to Confirm that Leontief–Sraffa Matrix Equations for Input/Output Must Obey Constancy of Returns to Scale. *Economics Letters*, 90, pp. 183–188.

Solow, R and P. Samuelson. (1953) Balanced Growth under Constant Returns to Scale, *Econometrica*, 21, pp. 412-424.

Sraffa, P. (1960) *Production of Commodities by Means of Commodities*. Cambridge: Cambridge University Press.

von Neumann, J. (1945) A Model of General Economic Equilibrium, *Review of Economic Studies*, 13, pp. 1–9.

Woods, J. (1987) A Note on Rent, *Oxford Economic Papers*, 39, pp. 388-411.

Appendix. Mathematical Proofs

Proof of Proposition 1. (i) Because $\mathbf{S}'\mathbf{A}$ is indecomposable, each column of **A** must be semipositive. Then $\mathbf{A}'\mathbf{p}$ is a positive vector, and all entries on the main diagonals of $\widehat{\mathbf{A}'\mathbf{p}}^{-1}$, $\hat{\mathbf{s}}^{-1}$ and $\hat{\mathbf{p}}$ are positive. Hence if the (i, j) entry of $\mathbf{S}'\mathbf{A}$ is positive then the (i, j) entry of **Z** is also positive. Therefore **Z** is indecomposable.

And it can be readily verified that $\mathbf{p}'\mathbf{AZ} = \mathbf{p}'\mathbf{A}$ holds. By Perron-Frobenius theorem, the P-F eigenvalue of **Z** equals 1 and $\mathbf{p}'\mathbf{A}$ is a left P-F eigenvector of **Z**.

(ii) We have:

$$\mathbf{p}'\mathbf{A}\hat{\mathbf{z}} = \mathbf{p}'\hat{\mathbf{s}}^{-1}\widehat{\mathbf{Az}}\mathbf{S} \Leftrightarrow \mathbf{p}'\mathbf{A}\hat{\mathbf{z}} = \mathbf{p}'\widehat{\mathbf{Az}}\hat{\mathbf{s}}^{-1}\mathbf{S} \Leftrightarrow \widehat{\mathbf{A}'\mathbf{p}}\mathbf{z} = \mathbf{S}'\hat{\mathbf{s}}^{-1}\hat{\mathbf{p}}\mathbf{Az}$$

$$\Leftrightarrow \widehat{\mathbf{A}'\mathbf{p}}^{-1}\mathbf{S}'\hat{\mathbf{s}}^{-1}\hat{\mathbf{p}}\mathbf{Az} = \mathbf{z} \Leftrightarrow \mathbf{Zz} = \mathbf{z} \tag{A.1}$$

Hence by Perron-Frobenius theorem the statement holds. ■

Proof of Proposition 2. (i) Let **p** be a left P-F eigenvector of **A** and λ be the P-F eigenvalue, it's clear that $\mathbf{Z} = \widehat{\mathbf{A}'\mathbf{p}}^{-1}\hat{\mathbf{p}}\mathbf{A} = \mathbf{A}/\lambda$. Since **z** is a right P-F eigenvector of **Z**, **z** must be a right P-F eigenvector of **A**.

Given a positive exchange vector **z**, we have

$$\mathbf{Zz} = \mathbf{z} \Leftrightarrow \widehat{\mathbf{A}'\mathbf{p}}^{-1}\hat{\mathbf{p}}\mathbf{Az} = \mathbf{z} \Leftrightarrow \hat{\mathbf{p}}\mathbf{Az} = \widehat{\mathbf{A}'\mathbf{p}}\mathbf{z} \Leftrightarrow \widehat{\mathbf{Az}}\mathbf{p} = \hat{\mathbf{z}}\mathbf{A}'\mathbf{p} \Leftrightarrow \widehat{\mathbf{Az}}^{-1}\hat{\mathbf{z}}\mathbf{A}'\mathbf{p} = \mathbf{p} \tag{A.2}$$

Thus by Perron-Frobenius theorem a positive exchange vector **z** corresponds to a unique normalized positive price vector **p**; moreover, if **A** is nonsingular and $\mathbf{Az} = \mathbf{s} \equiv \mathbf{Se}$ holds then $\mathbf{s}$ and $\mathbf{S}$ correspond to a unique normalized positive market-clearing price vector. When **z** is a right P-F eigenvector of **A**, by $\widehat{\mathbf{Az}}^{-1}\hat{\mathbf{z}}\mathbf{A}'\mathbf{p} = \mathbf{p}$ it's clear that **p** must be a left P-F eigenvector of **A**.

(ii) And (iii) are immediate results of (i).

(iv) The proof is based on the work of Dietzenbacher (1988). Here let $\mathbf{m}_{j\bullet}$ and $\bar{\mathbf{m}}_{j\bullet}$ denote the *j*th rows of $\mathbf{Z} = \widehat{\mathbf{A}'\mathbf{p}}^{-1}\hat{\mathbf{p}}\mathbf{A}$ and $\bar{\mathbf{Z}} = \widehat{\mathbf{A}'\bar{\mathbf{p}}}^{-1}\hat{\bar{\mathbf{p}}}\mathbf{A}$ respectively. $\mathbf{a}_{j\bullet}$ and $\mathbf{a}_{\bullet j}$ denote the *j*th row and *j*th column of **A** respectively.

For a $j \neq i$, note that $\mathbf{m}_{j\bullet} = p_j\mathbf{a}_{j\bullet}/(\mathbf{p}'\mathbf{a}_{\bullet j})$ and $\bar{\mathbf{m}}_{j\bullet} = \bar{p}_j\mathbf{a}_{j\bullet}/(\bar{\mathbf{p}}'\mathbf{a}_{\bullet j}) = p_j\mathbf{a}_{j\bullet}/(\bar{\mathbf{p}}'\mathbf{a}_{\bullet j})$, thus $\bar{\mathbf{m}}_{j\bullet} \leq \mathbf{m}_{j\bullet}$ holds. And it can be verified readily that $\bar{\mathbf{m}}_{i\bullet} \geq \mathbf{m}_{i\bullet}$ holds. Hence $\bar{z}_i/z_i \geq \bar{z}_j/z_j$ (for all *j*) holds by Theorem 5.3 of Dietzenbacher (1988). The theorem says: if **M** and $\bar{\mathbf{M}}$ are two indecomposable nonnegative ($n \times n$) matrices possessing the same P-F eigenvalue and possessing the P-F eigenvector **x** and $\bar{\mathbf{x}}$ respectively, and each entry in the *i*th row of $\bar{\mathbf{M}}$ is no smaller than the corresponding entry of **M** and each other entry of $\bar{\mathbf{M}}$ is no greater than the corresponding entry of **M**, then $\bar{x}_i/x_i \geq \bar{x}_j/x_j$ holds for all j.

Note that if $a_{ij} > 0$ and $j \neq i$ then $\bar{\mathbf{m}}_{j\bullet} < \mathbf{m}_{j\bullet}$ holds. If there is a $j \neq i$ such that $a_{ij} > 0$ (thus $\bar{\mathbf{m}}_{j\bullet} < \mathbf{m}_{j\bullet}$ holds) and $\bar{z}_i/z_i = \bar{z}_j/z_j$ (thus $\bar{z}_j/z_j \geq \bar{z}_k/z_k$ holds for all *k*), then like the proof of Theorem 2.1 of Dietzenbacher (1988) a contradiction is found:

$$\bar{z}_j = \bar{\mathbf{m}}_{j\bullet}\bar{\mathbf{z}} \leq \bar{\mathbf{m}}_{j\bullet}\mathbf{z}\frac{\bar{z}_j}{z_j} < \mathbf{m}_{j\bullet}\mathbf{z}\frac{\bar{z}_j}{z_j} = z_j\frac{\bar{z}_j}{z_j} = \bar{z}_j \tag{A.3}$$

Furthermore, if $\bar{z}_i < z_i$ then by $\bar{z}_i/z_i \geq \bar{z}_j/z_j$ (for all *j*) we yield $\bar{\mathbf{z}} \ll \mathbf{z}$. Thus we find $\mathbf{A}\bar{\mathbf{z}} \ll \mathbf{Az} \leq \mathbf{s}$, i.e. $\mathbf{A}\bar{\mathbf{z}} \ll \mathbf{s}$, which implies each good doesn't sell out in the second exchange process, and this contradicts the definition of the exchange function. Hence $\bar{z}_i \geq z_i$ holds. ■