# A theory for measuring productivity change in the system with fixed capital

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## 1 Introduction

This paper explains a theory to measure social productivity in the system with fixed capital. Firstly, we will explain the labour productivity in our model by using my standard productivity index. Secondly we will explain the productivity of capital in our model. Finally, we will show the definitoion of total factor productivity as the geometric mean of the labour productivity and capital productivity of our model. It should be stressed that we connected the labour productivity, the capital productivity and the total factor productivity to the input output data. The significance of this paper lies in the new approach to define priductivity by the input output system and the Sraffa system.

## 2 Input Output System

Let us consider the Input Output system. We assume that an equal rate of prifit and an equal wage rate provails throughout industries. Let us consider $n$ industry system and let us denote the diagonal matrix of the real output by $\mathbf{Z}$, the matrix of real intermediate input by $\mathbf{M}$, the money wage vector by $\mathbf{E}$, the total profit vector ( the operating surplus vector) by $\mathbf{F}$, the depreciation vector by $\mathbf{d}$,and the price vector by $\mathbf{p}_{IO}$, then we have

$$\mathbf{Z}\mathbf{p}_{IO} = \mathbf{M}\mathbf{p}_{IO} + \mathbf{E} + \mathbf{F} + \mathbf{d} \tag{1}$$

In this equation, each term is measured in terms of money unit.

Let us denote the money wage rate by $w_M$, and the lebour input vector by $\mathbf{L}$. If the equal rate of money wage $w_M$ prevails throughout industries, then the money wage vector will berepresented as

$$\mathbf{E} = w_M \mathbf{L} \tag{2}$$

Let us denote the fixed capital vector measured in terms of money unit by $\mathbf{k}_F$.

If an equal rate of profit $r$ prevails throughout industries, we will have

$$\mathbf{F} = r\mathbf{M}\mathbf{p}_{IO} + r\mathbf{k}_F \qquad (3)$$

If we measure the fixed capital vector, and the depreciation vector in terms of commanded labour(wage unit), then

$$\mathbf{F} + \mathbf{d} = r\mathbf{M}\mathbf{p}_{IO} + w_M r\mathbf{k}_F/w_M + w_M\mathbf{d}/w_M \qquad (4)$$

If we use the notations of

$$\begin{aligned} \mathbf{k}_{Fw} &= \mathbf{k}_F/w_m \\ \mathbf{d}_w &= \mathbf{d}/w_m \end{aligned}$$

the equation (1) will become

$$\mathbf{Z}\mathbf{p}_{IO} = \mathbf{M}\mathbf{p}_m + r\mathbf{M}\mathbf{p}_{IO} + w_m\mathbf{L} + w_m r\mathbf{k}_{Fw} + w_m\mathbf{d}_w \qquad (5)$$

The transposed matrix of input coefficients and the labour coefficient vector will be represented as

$$\mathbf{A} = \mathbf{Z}^{-1}\mathbf{M} \qquad (6)$$
$$\mathbf{l}_A = \mathbf{Z}^{-1}\mathbf{L} \qquad (7)$$

Also let us define the following

$$\boldsymbol{\kappa} = \mathbf{Z}^{-1}\mathbf{k}_{Fw} \qquad (8)$$
$$\boldsymbol{\delta} = \mathbf{Z}^{-1}\mathbf{d}_w \qquad (9)$$

And we will denote the fixed capital and depreciation by

$$K_{Fw} = \mathbf{e}\mathbf{Z}\boldsymbol{\kappa} \qquad (10)$$
$$D_w = \mathbf{e}\mathbf{Z}\boldsymbol{\delta} \qquad (11)$$

These values are measured in terms of commanded labour. From (5), the price vector will be represented by

$$\mathbf{p}_{IO} = \mathbf{A}\mathbf{p}_{IO} + r\mathbf{A}\mathbf{p}_{IO} + w_M\mathbf{l}_A + w_M r\boldsymbol{\kappa} + w_M\boldsymbol{\delta} \qquad (12)$$

The output vector $\mathbf{x}$ will be represented as

$$\mathbf{x} = \mathbf{e}\mathbf{Z} \qquad (13)$$

The total amount of labour ($L_A$) will be represented by

$$L_A = \mathbf{x}\mathbf{l}_A = \mathbf{e}\mathbf{L} \tag{14}$$

And we can normalize $\mathbf{p}_{IO}, \mathbf{l}_A, \boldsymbol{\kappa}, \boldsymbol{\delta}$ by dividing them by $L_A$. The normalized vectors will be defined by

$$\hat{\mathbf{p}}_{IO} = \frac{1}{L_A}\mathbf{p}_{IO} \tag{15}$$

$$\mathbf{l}_S = \frac{1}{L_A}\mathbf{l}_A \tag{16}$$

$$\boldsymbol{\kappa}_{\mathbf{S}} = \frac{1}{L_A}\boldsymbol{\kappa} \tag{17}$$

$$\boldsymbol{\delta}_{\mathbf{S}} = \frac{1}{L_A}\boldsymbol{\delta} \tag{18}$$

Then we have

$$\mathbf{x}\mathbf{l}_S = 1 \tag{19}$$

By using the normalized vectors, we have the following price equation.

$$\hat{\mathbf{p}}_{IO} = \mathbf{A}\hat{\mathbf{p}}_{IO} + r\mathbf{A}\hat{\mathbf{p}}_{IO} + w_M\mathbf{l}_S + w_M r\boldsymbol{\kappa}_S + w_M\boldsymbol{\delta}_S \tag{20}$$

If we use the notations of Pasinetti's vertically integrated model, the notatoions will be

$$\mathbf{v}_S = [\mathbf{I} - \mathbf{A}]^{-1}\mathbf{l}_S \tag{21}$$

$$\mathbf{H} = [\mathbf{I} - \mathbf{A}]^{-1}\mathbf{A} \tag{22}$$

then the equation we have

$$\mathbf{p}_{IO} = w_m\mathbf{v}_A + r\mathbf{H}\mathbf{p}_{IO} + w_m\{r[\mathbf{I} - \mathbf{A}]^{-1}\boldsymbol{\kappa}_S + [\mathbf{I} - \mathbf{A}]^{-1}\boldsymbol{\delta}_{\mathbf{S}}\} \tag{23}$$

In this equation, the first term of the right member is the vertically integrated labour coefficient vector multiplied by the money wage rate $w_m$. The vertically integrated labour coefficient vector is measured in terms of labour. The equation (23) is popular in the economic analysis. However, we will reformulate the equation (20) by using the Sraffa system. In order to derive the equation (23), the (transposed) Leontief inverse matrix is used. We will derive the similar equation as (23), by using the Sraffian inverse matrix.

# 3 The Sraffa System

## 3.1 The Actual Quantity System

We will explain the two different notions of income and capital in the Sraffa system. We will concern only with the case of single product industries. There is no joint production, and land and fixed capital are excluded from the analysis. Each industry produces a single commodity by using a certain quantity of labour and certain quantities of commodities as a means of production. The number of industries and thus the number of products is equal to $n$. Let us assume that there is no heterogeneity in labour, and the amount of total labour is equal to $L_A$.

In the case of production of a surplus, the total output is divided into two components: the part which is required for the replacement, and the part of the surplus produced, which is over and above the replacement for reproduction. If we denote the actual net product vector by $\mathbf{y}$, then the quantity equation can be rewritten as

$$\mathbf{x} = \mathbf{y} + \mathbf{xA} \tag{24}$$

From (24), we also obtain

$$\mathbf{y} = \mathbf{x}(\mathbf{I} - \mathbf{A}) \tag{25}$$

where $\mathbf{I}$ is the identity matrix. The equation (24) gives the definition of the vector of the actual net product that makes up the actual national income. The $j$ th component of $\mathbf{y}$ represents the net product of industry $j$. If each industry produces a surplus, $\mathbf{y}$ will be strictly positive: $\mathbf{y}$ >0, and if there is no surplus in some industries, $\mathbf{y}$ will be semi-positive: $\mathbf{y}$ $\geq$0.

$$\mathbf{y} > 0 \tag{26}$$

Now let $\mathbf{k}_A$ be the vector of the produced means of production. Then it is defined by

$$\mathbf{k}_y = \mathbf{xA} = \mathbf{eZA} \tag{27}$$

The equation (24) can be rewritten as

$$\mathbf{x} = \mathbf{y} + \mathbf{k}_y \tag{28}$$

The equation (28) represents the physical relation between the total outputs, total net products and the products for replacement.

## 3.2 Sraffa's Standard System

The standard commodity is well known as an invariable standard. We will proceed to the discussions of the standard commodity introduced by Sraffa in Ch.IV of Sraffa[1960]. The standard system is defined as a virtual system whose quantity vector corresponds to the eigenvector of the input coefficient matrix $\mathbf{A}$ and which has a uniform rate of surplus in physical terms throughout industries.

If we denote the maxmum rate of physical surplus by $\Pi$. In the Sraffa system, the maximum rate of surplus is equal to the physical rate of surplus. Let us denote the row vector of output of the standard system by **h**, the standard system can be represented as

$$\mathbf{h} = (1 + \Pi)\mathbf{hA} \tag{29}$$

If we denote the standard commodity vector by **u**, it can be defined from (28) as

$$\mathbf{u} = \mathbf{h}(\mathbf{I} - \mathbf{A}) = \Pi\mathbf{hA} \tag{30}$$

The components of this vector correspond to the commodities constituting the standard commodity.

It is indeed true that Sraffa introduced the standard commodity for the analysis of price changes and referred to the term frequently, but what Sraffa adopted in his analysis was not the standard commodity in general but the standard national income. The standard national income was defined as the standard commodity with an additional assumption introduced in §26 of Sraffa[1960]. The standard national income is the standard commodity when the total labour of the standard system is equal to the total labour of the actual system. If we denote the row vector of the total output produced in the standard system when the total labour of the standard system is equal to the total labour of the actual system by **q** in place of **h**, the relationship between **q** and **h** will be represented by

$$\mathbf{q} = t\mathbf{h} \tag{31}$$

where $t$ is a positive scalar. Like (28), the vector **q** is defined as the vector that is given by

$$\mathbf{q} = (1 + \Pi)\mathbf{qA} \tag{32}$$

and, in addition, which satisfies the following assumption

$$\mathbf{ql}_S = \mathbf{xl}_S \tag{33}$$

From (32), we can obtain, like (30), the following equation

$$\mathbf{s} = \mathbf{q}(\mathbf{I} - \mathbf{A}) = \Pi\mathbf{qA} \tag{34}$$

This is called the standard net product or standard national income in §26 of Sraffa[1960]. The vector **s** is uniquely determined if the matrix **A** and the total labour $\mathbf{xl}_S$ are given exogenously. The standard net product vector **s** is a bundle of commodities potentially producible with the given technique $[\mathbf{A},\mathbf{l}_S]$ and the given total labour $\mathbf{xl}_S$.

The aggregate capital should be calculated in two ways: one is the aggregation of the produced means of production by the commodity prices and the other is the calculation of the discounted value of the rate of return. In the standard system, the aggregate capital is independent of the variations of prices and distribution. We will call the aggregate capital of the standard system the Standard capital. Let us first show the calculation of the standard capital by aggregating the produced means of production. Let us denote the vector of the produced means of production of the standard system by $\mathbf{k}_S$, then we can define it by

$$\mathbf{k}_S = \mathbf{qA} \tag{35}$$

The aggregate capital should be calculated in two ways: one is the aggregation of the produced means of production by the commodity prices and the other is the calculation of the discounted value of the rate of return. In the standard system, the aggregate capital is independent of the variations of prices and distribution. We will call the aggregate capital of the standard system the Standard capital. The quantity relation of the standard system will be represented as

$$\mathbf{q} = \mathbf{s} + \mathbf{k}_S \tag{36}$$

### 3.3 The Sraffian Evaluation System

We call the exchange-ratios, which enable the system to be economically viable, the prices of commodities or simply the prices. Let us denote the column vector of the commodity prices by $\mathbf{p}_S$. And let us denote the rate of profits by $r$, which is assumed to be uniform all over the economic system. Moreover let us denote the maximum rate of profits by $R$. Then the rate of profits will take real numbers ranging from 0 to $R$ $(0 \leq r \leq R)$. Similarly, a uniform rate of wage (post factum) is assumed to be prevailing in the economy. It is indicated by $w_A$. From above, the price system can be written as

$$\mathbf{p}_S = (1 + r)\mathbf{A}\mathbf{p}_S + w_S \mathbf{l}_S \tag{37}$$

When the wage is equal to zero, the price equation (37) will reduce to

$$\mathbf{p}_S = (1 + R)\mathbf{A}\mathbf{p}_S \tag{38}$$

This implies that the price vector $\mathbf{p}_S$ is the right-hand eigenvector of the matrix $\mathbf{A}$ and $1\ /(1+R)$ is its eigenvalue. On the other hand, in the equation (37), the vector $\mathbf{q}$ is the left-hand eigenvector of the matrix $\mathbf{A}$ and $1/(1+\mathit{\Pi})$ is its eigenvalue. From the Perron-Frobenius theorem, we have

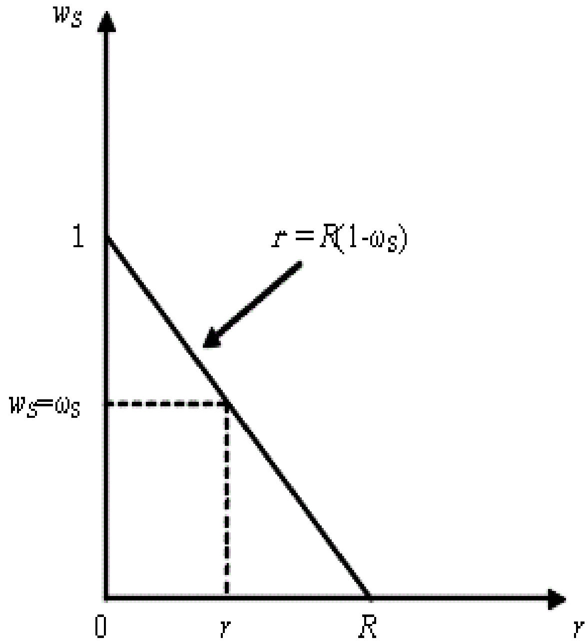
$$\mathit{\Pi} = R \tag{39}$$

Substituting (39) into (34), we obtain

$$\mathbf{s} = \mathbf{q}(\mathbf{I} - \mathbf{A}) = R\mathbf{qA} \tag{40}$$

From the price system (37), we can derive

*Figure 1*




$$\mathbf{sp}_S = \mathbf{xl}_S \iff r = R(1 - w_S) \tag{41}$$

Under the condition of

$$\mathbf{sp}_S = \mathbf{xl}_S \tag{42}$$

we can obtain the linear wage curve as

$$r = R(1 - w_S) \tag{43}$$

On the contrary, under the condition of linear wage curve, we can derive the equality between the standard income $\mathbf{sp}_S$ and the standard labour $\mathbf{xl}_S$.

In order to explain reasonably the equality between the standard national income and the standard total labour, which is measured in terms of unit of labour, we will introduce the value of labour $v_L$ into the equation $\mathbf{sp}_S = \mathbf{xl}_S$ as follows.

$$\mathbf{sp}_S = v_L \mathbf{xl}_S \tag{44}$$

From this equation we have

$$\mathbf{sp}_v = \mathbf{xl}_S \tag{45}$$

where $\mathbf{p}_v = \mathbf{p}_S \;/\; v_L$. The left member of this equation means that the aggregate value of the standard net product is obtained by post-multiplying the vector $\mathbf{s}$ by $\mathbf{p}_v$. The right member is the standard total labour. Then the standard national income in terms of the quantity of labour is equal to the standard total labour. The standard condition (normalization condition ) of is

$$v_L = 1 \tag{46}$$

In order to show the relationship between the standard condition $v_L = 1$ and the Sraffian price system, we will show the whole relation in the following Evaluation System. When a set of data ($\mathbf{x}$, $\mathbf{A}$, $\mathbf{l}_S$, $\mathbf{s}$, $R = \mathit{\Pi}$) is given, we have the following Evaluation System

$$\text{[}\textbf{Evaluation System 1}\text{]} \quad \begin{aligned} v_L &= 1 & (47)\\ \mathbf{s}\mathbf{p}_S &= v_L \mathbf{x}\mathbf{l}_S & (48)\\ w_v &= w_S \;/\; v_L & (49)\\ \mathbf{p}_v &= \mathbf{p}_S \;/\; v_L & (50)\\ \mathbf{p}_S &= (1+r)\mathbf{A}\mathbf{p}_S + w_S\mathbf{l}_S & (51) \end{aligned}$$

In this system, there are $(2n{+}3)$ independent equations and $(2n\ {+}4)$ unknowns(i.e. $v_L$, $\mathbf{p}_S$, $\mathbf{p}_v$, $w_S$, $w_v$, $r$). Then, if the rate of profits is given exogenously, the price system will become determinate.

When wage curve becomes linear, the value of labour embodied in the standard national income becomes equal to unity. It is true that, in the Sraffa system, the prices in terms of labour are not equal to the vertically integrated labour coefficients, i.e. $\mathbf{p}_v \neq \mathbf{v}_S$. However, the reduced form of $\mathbf{p}_v$ of the Sraffa system can be represented as

$$\mathbf{p}_v = (1 - r/R)[\mathbf{I} - (1-r)\mathbf{A}]^{-1}\mathbf{l}_A \tag{52}$$

The right-hand member of this equation is the Reduction equation of Sraffa [1960], which is measured in terms of quantities of labour. The prices of this equation are subject to the changes in the rate of profits and the changes in the technological condition of production. But, since the value of labour embodied in the standard national income is invariant, there is no effect of the changes in the value of the chosen standard on the prices of commodities. The Sraffa's theory can be considered not as a simple theory of price determination but as a theory of value.

In the Sraffa system, the profit share measured in terms of the standrad income $\pi_S$ is given as

$$\pi_S = r/R \tag{53}$$

The wage measured in terms of labour is $w_v = w_A \;/\; v_L$. This also means the wage share measured in terms of the standard income. If we denote the wage share measured in terms of the standard income by $\omega_S$,then we will have

$$\omega_S = w_v = 1 - r/R \tag{54}$$

## 3.4 Difference between the Actual Income and the Standard Income

Let us consider the difference between the notions of actual system ($\mathbf{x}$, $\mathbf{y}$, $\mathbf{k}_y$) and those of the standard system ($\mathbf{q}$, $\mathbf{s}$, $\mathbf{k}_S$). The quantity vectors of the standard system are different from those of the actual system. It is true that, in or multiple product model, we should confronted with the aggregation problem, but the standard system of Sraffa [1960] will make things very simple and transparent. The reason why we use the standard system is that the standard national income can be considered as a proxy for the actual national income and works as a helpful reference.

In order to explain this, let us take notice of the difference between the actual national income and the standard national. The social accounting corresponding to the actual national income can be obtained as

$$\mathbf{y}\mathbf{p}_v = r\mathbf{k}_y\mathbf{p}_v + w_v\mathbf{x}\mathbf{l}_S \tag{55}$$

On the other hand, we can obtain the social accounting corresponding to the standard national income as follows:

$$\mathbf{s}\mathbf{p}_v = r\mathbf{k}_S\mathbf{p}_v + w_v\mathbf{x}\mathbf{l}_S \tag{56}$$

Subtracting (56) from (55), we have

$$\mathbf{y}\mathbf{p}_v - \mathbf{s}\mathbf{p}_v = r(\mathbf{k}_y\mathbf{p}_v - \mathbf{k}_S\mathbf{p}_v) \tag{57}$$

This equation means that the difference between the actual national income and the standard national income is equal to the profit which can be obtained by the capitalist from the difference of two notions of capital, i.e. ($\mathbf{k}_y\mathbf{p}_v$ - $\mathbf{k}_S\mathbf{p}_v$ ).The meaning of this equation is the definition of profit.

$$\Delta\mathbf{\Pi}_\mathbf{y} = \mathbf{y}\mathbf{p}_v - \mathbf{s}\mathbf{p}_v \tag{58}$$

From (57), we can obtain

$$\mathbf{y}\mathbf{p}_v = \mathbf{s}\mathbf{p}_v + r(\mathbf{k}_y\mathbf{p}_v - \mathbf{k}_S\mathbf{p}_v) \tag{59}$$

This result should be important, because, though the standard system is a hypothetical construction, the wage curve obtained by using the standard system explains the same rate of profit as that of the actual system. And the value of

the standard income and the value of the standard capital is constant. Therefore we can use the standard system as a reference. It comes to that the linear wage curve of $r$ =$R$(1-$\omega_S$) can be considered as the relation of distribution of the actual economy.

# 4 Labour Productivity

## 4.1 The Standard Productivity Index

Let us consider *Period* 1 and *Period* 2 . Let us denote the actual total labour of *Period* 1 by $L_S^1$,and that of *Period* 2 by $L_S^2$.If we consider *Period* 1 as base,then the normarized total labour or the standard total labour will be represented as

$$L_S^2 = (1/\ L_S^1)\ L_S^2 \tag{60}$$

Let us denote the input coefficient matrix of *Period* 1 by $\mathbf{A}^1$, and that of *Period* 2 by $\mathbf{A}^2$. Let us denote the labour coefficient vector of of *Period* 1 by $\mathbf{l}_S^1$, and that of *Period* 2 by $\mathbf{l}_S^2$.The techniques are characterized by $[\mathbf{A}^1,\ \mathbf{l}_S^1,\ L_S^1]$ and $[\mathbf{A}^2,\ \mathbf{l}_S^2,\ L_S^2]$.

When the rate of profit is equal to zero, the Reduction equations, or the reduced forms of $\mathbf{p}_v^1$, $\mathbf{p}_v^2$, are represented as

$$\mathbf{v}_S^1 = [\mathbf{I} - \mathbf{A}^1]^{-1}\mathbf{l}_S^1 \tag{61}$$

$$\mathbf{v}_S^2 = [\mathbf{I} - \mathbf{A}^2]^{-1}\mathbf{l}_S^2 \tag{62}$$

The total labour of the economy can be represented by

$$L_S^1 = \mathbf{s}^1\mathbf{v}_S^1 \tag{63}$$

$$L_S^2 = \mathbf{s}^2\mathbf{v}_S^2 \tag{64}$$

Let us show the index numbers which correspond to the case when the rate of profit is equal to zero. We have the price indexes of Fisher type as follows

$$P_{sv} = \sqrt{\frac{\mathbf{s}^1\mathbf{v}_S^2}{\mathbf{s}^2\mathbf{v}_S^1} \cdot L_S^2} \qquad\qquad Q_{sv} = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_S^1}{\mathbf{s}^1\mathbf{v}_S^2} \cdot L_S^2} \tag{65}$$

Therefore, we can obtain the following relationship

$$1/P_{sv} = Q_{sv}/L_S^2 = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_S^1}{\mathbf{s}^1\mathbf{v}_S^2} \cdot \frac{1}{L_S^2}} \tag{66}$$

This is a quite interesting result because the inverse of the price index is equal to the output index divided by $L_S^2$. In (66), the productivity change is calculated both from the cost side as $1/P_{sv}$ and from the quantity side as $Q_{sv}/L_S^2$. We can define the productivity index by

$$\Lambda_{sv} = \sqrt{\frac{\mathbf{s}^2\mathbf{v}_S^1}{\mathbf{s}^1\mathbf{v}_S^2} \cdot \frac{1}{L_S^2}} \tag{67}$$

It should be stressed that $\Lambda_{sv}$ is obtained by the given production techniques $[\mathbf{A}^1,\ \mathbf{l}_S^1;\ \mathbf{A}^2,\ \mathbf{l}_S^2]$ and the total labour of each period, i.e. $L_S^1$ and $L_S^2$. $\Lambda_{sv}$ is independent of the changes in compositions of the net product (or demand and output). $\Lambda_{sv}$ is considered to reflect the changes in production techniques, the labour growth and the output growth.

## 4.2 The Effective Labour

From (66)(67), we have

$$Q_{sv} = \Lambda_{sv} L_S^2 \tag{68}$$

The left member is the product of the Standard productivity index and the standard labour. In other words, it is the total labour weighted by the Standard productivity index. It will be convenient to represent the right member of equation (68) as

$$L_E^2 = \Lambda_{sv} L_S^2 \tag{69}$$

Let us call this weighted total labour the *effective labour*. This is the same as the definition of effective labour which is considered in the case of Harrod-neutral technical progress or in the case of purely labour augmenting technical progress. The effective labour is determined by the Standard productivity index and the Standard labour. From above, we have

$$Q_{sv} = L_E^2 \tag{70}$$

The standard output index becomes equal to the effective labour. Corresponding to the effective labour, a new normalized labour coefficient vector can be defined as

$$\mathbf{l}_E^2 = \Lambda_{sv}\ \mathbf{l}_S^2 \tag{71}$$

We call $\mathbf{l}_E$ the effective labour coefficient vector. Then the effective labour is represented by

$$L_E^2 = \mathbf{x}^2\mathbf{l}_E^2 = \Lambda_{sv}\mathbf{x}^2\mathbf{l}_S^2 \tag{72}$$

From this, we obtain

$$\mathbf{x}^2\mathbf{l}_E^2 = \Lambda_{sv} L_S^2\ \mathbf{x}^1\mathbf{l}_S^1 \tag{73}$$

This means that the effective labour of *Period* 2 can be compared with that of *Period* 1. The equation (73) is an important basis for our income comparison.

The vertically integrated labour coefficient vector given by the effective labour coefficient vector $\mathbf{l}_E^b$ can be defined as

$$\mathbf{v}_E^2 = [\mathbf{I} - \mathbf{A}^2]^{-1}\mathbf{l}_E^2 \tag{74}$$

The relation between $\mathbf{v}_E^2$ and $\mathbf{v}_S^2$ is represented by

$$\mathbf{v}_E^2 = \Lambda_{sv}\mathbf{v}_S^2 \tag{75}$$

By $\mathbf{v}_E^2$, we have

$$\mathbf{q}^2\mathbf{l}_E^2 = \mathbf{q}^2[\mathbf{I}-\mathbf{A}^2][\mathbf{I}-\mathbf{A}^2]^{-1}\mathbf{l}_E^2 = \mathbf{s}^2\mathbf{v}_E^2 \tag{76}$$

Multiplying the both sides of the equation $\mathbf{q}^2\mathbf{l}_S^2 = \mathbf{x}^2\mathbf{l}_S^2$ by $\Lambda_{sv}L_S^2$, we have

$$\mathbf{q}^2\mathbf{l}_E^2 = \mathbf{x}^2\mathbf{l}_E^2 \tag{77}$$

Therefore, we have

$$\mathbf{s}^2\mathbf{v}_E^2 = \mathbf{x}^2\mathbf{l}_E^2 \tag{78}$$

We have obtained the equality between the standard national income and the effective labour.

From (78), we have

$$\mathbf{s}^2\mathbf{v}_E^2 = \Lambda_{sv}\mathbf{s}^2\mathbf{v}_S^2 \tag{79}$$

Therefore $\mathbf{s}^2\mathbf{v}_E^2$ means the income physically given by the effective labour.

The definition of the effective labour in our model will bring about a remarkable result into our analysis. It enables us to make an intertemporal comparison of income. The effective labour is measured in terms of labour unit. Therefore, the effective labour of Techinique $b$ can be compared with that of Technique $a$. The standard income of Technique $a$ is equal to the standard labour of *Period* 2. In *Period* 2, the equality between the standard income and the effective labour holds. Since $\mathbf{s}^1\mathbf{v}_S^1 = \mathbf{x}^1\mathbf{l}_S^1$, we can obtain

$$\mathbf{s}^2\mathbf{v}_E^2 = \Lambda_{sv}L_S^2\ \mathbf{s}^1\mathbf{v}_S^1 \tag{80}$$

This is quite an interesting result. Although they are composed of heterogeneous commodities, the standard incomes of both periods can be compared with each other, because the aggregated values of the standard incomes are measured in terms of labour.

## 4.3 Income Comparison

Now let $\mathbf{s}^1$ and $\mathbf{s}^2$ be the standard net product of each technique and let $\mathbf{p}_S^1$ and $\mathbf{p}_S^2$ be the price vector of each technique. $w_S^1$ and $w_S^2$ mean the wage rate of each *Period*. $v_L^1$ and $v_L^2$ mean the value of labour of each *Period*. Then we can construct our Model for price comparisons. A set of given data is ($\mathbf{x}^1$, $\mathbf{A}^1$, $\mathbf{l}_A^1$, $\mathbf{q}^1$, $\mathbf{s}^1$, $\mathbf{x}^2$, $\mathbf{A}^2$, $\mathbf{l}_E^2$, $\mathbf{q}^2$, $\mathbf{s}^2$). The superscript 1 indicates *Period* 1 and the superscript 2 indicates *Period 2*. We have the following system of evaluation.

[**Evaluation System 2**]

$$\begin{aligned}
\mathbf{s}^1\mathbf{p}_S^1 &= v_L^1\mathbf{x}^1\mathbf{l}_S^1 && (81)\\
\mathbf{s}^2\mathbf{p}_E^2 &= v_L^2\mathbf{x}^2\mathbf{l}_E^2 && (82)\\
\mathbf{p}_S^1 &= (1+r)\mathbf{A}^1\mathbf{p}_S^1 + w_S^1\mathbf{l}_S^1 && (83)\\
\mathbf{p}_E^2 &= (1+r)\mathbf{A}^2\mathbf{p}_E^2 + w_E^2\mathbf{l}_E^2 && (84)
\end{aligned}$$

In this system, there are $(2n{+}2)$ independent equations and $(2n{+}4)$ unknowns ($w_S^1$, $\mathbf{p}_S^1$, $v_L^1$, $w_S^2$, $\mathbf{p}_S^2$, $v_L^2$). If the condition of standard for each period is given, the above system will become determinate. In this system, the following proposition will hold.

**[Proposition]** In the above Evaluation System 2, the value of labour of *Period* 1 is equal to the value of labour of *Period* 2 if and only if

$$r^1 = R^1(1-\omega_S^1) \quad and \quad r^2 = R^2(1-\omega_S^2) \tag{85}$$

where $0 \leq r^1 < R^1, 0 \leq r^2 < R^2$.

**[Proof]** In Evaluation System 2, we have for all $r^1$ of $0 \leq r^1 < R^1$ and for all $r^2$ of $0 \leq r^2 < R^2$

$$v_L^1 = 1 \iff r^1 = R^1(1-\omega_S^1) \tag{86}$$

$$v_L^2 = 1 \iff r^2 = R^2(1-\omega_S^2) \tag{87}$$

Then, for all $r$ of $0 \leq r < R_{min}$, we have

$$v_L^1 = v_L^2 = 1 \iff r^1 = R^1(1-\omega_S^1) \quad and \quad r^2 = R^2(1-\omega_S^2) \tag{88}$$

From this, the proposition is verified.

Q.E.D.

This proposition states that the prices of different periods are measured in terms of a common unit, unit of labour. Therefore, under the condition of (85), the prices of different periods can be compared with each other. It also means that the equations of (85) can be replaced by the condition

$$v_L^1 = v_L^2 = 1 \tag{89}$$

Either (85) or (89) can make Evaluation System 1 determinate. It is more important to understand that, in Evaluation System 1, the labour can be considered as the intertemporal standard of value by the condition (85). Under the condition (85) or (89), the prices and wages of both periods are measured in terms of the same unit of labour.

The Reduction equations, or the reduced forms of $\mathbf{p}_v^1$, $\mathbf{p}_v^2$, are represented as

$$\mathbf{p}_v^1 = \mathbf{p}_A^1/v_L^1 = (1-r^1/R^1)[\mathbf{I}-(1+r)\mathbf{A}^1]^{-1}\mathbf{l}_S^1 \tag{90}$$

$$\mathbf{p}_E^2 = \mathbf{p}_E^2/v_L^2 = (1-r^2/R^2)[\mathbf{I}-(1+r)\mathbf{A}^2]^{-1}\mathbf{l}_E^2 \tag{91}$$

Let us introduce a new wage curve. If we define the following price vector,

$$\mathbf{p}_E^2 = \mathbf{p}_v^2 \Lambda_{sv} \tag{92}$$

We will have the price system of *Period* 2 as follows

$$\mathbf{p}_E^2 = (1+r)\mathbf{A}^2\mathbf{p}_E^2 + w_S^2\mathbf{l}_E^2 \tag{93}$$

Then if we define the following wage,

$$w_E^2 = w_S^2 \Lambda_{sv} \tag{94}$$

We will have the following wage curve

$$r^2 = R^2(1 - w_E^2/\Lambda_{sv}) \tag{95}$$

We will call this wage curve the Effective Wage Curve.

# 5 The I-O system and the Sraffa System

## 5.1 The Sraffian Inverse Matrix

Like the equation (23), we will rewrite the equation (20) as

$$\hat{\mathbf{p}}_{IO} = w_M\{[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{l}_S + r[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\boldsymbol{\kappa}_S + [\mathbf{I}-(1+r)\mathbf{A}]^{-1}\boldsymbol{\delta}_{\mathbf{S}}\} \tag{96}$$

We will call the term $[\mathbf{I}-(1+r)\mathbf{A}]^{-1}$ the Sraffian inverse matrix. Moreover, by using the wage share $\omega_S$, we can rewrite this equation as

$$\hat{\mathbf{p}}_{IO} = (1/\omega_S)w_M\{\omega_S[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\mathbf{l}_S + \omega_S\, r[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\boldsymbol{\kappa} + \omega_S[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\boldsymbol{\delta}_{\mathbf{S}}\} \tag{97}$$

If we use the followig notations,

$$\mathbf{p}_\kappa = \omega_S[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\boldsymbol{\kappa}_S \tag{98}$$

$$\mathbf{p}_\delta = \omega_S[\mathbf{I}-(1+r)\mathbf{A}]^{-1}\boldsymbol{\delta}_S \tag{99}$$

we will have

$$\hat{\mathbf{p}}_{IO} = (1/\omega_S)w_M\{\mathbf{p}_v + r\mathbf{p}_\kappa + \mathbf{p}_\delta\} \tag{100}$$

In order to link the analysis based on I-O system to the macroeconomic relation, let us enlarge the imaginary standard system to the proft for fixed capital and depricartion measured in commanded labour terms.

Let us normalize $\mathbf{s}_\kappa = t\mathbf{s}$ ($t$ is a positive scalar ) and $\mathbf{s}_\delta = t\mathbf{s}$ as

$$\mathbf{s}_\kappa \mathbf{p}_\kappa = \frac{K_{Fw}}{L_A} \tag{101}$$

$$\mathbf{s}_\delta \mathbf{p}_\delta = \frac{D_{Fw}}{L_A} \tag{102}$$

Then the per capita profit $\boldsymbol{\pi}_L$ and the per capita depriciation $\boldsymbol{\delta}_L$ will be represented by

$$\boldsymbol{\pi}_L = \frac{rK_{Fw}}{L_A} \tag{103}$$

$$\boldsymbol{\delta}_L = \frac{D_{Fw}}{L_A} \tag{104}$$

The value $(\boldsymbol{\pi}_L + \boldsymbol{\delta}_L)$ is the per capita gross profit. Let us define the following

$$\tilde{\mathbf{s}} = \mathbf{s} + \mathbf{s}_\kappa + \mathbf{s}_\delta \tag{105}$$

$$\tilde{\mathbf{p}} = \mathbf{p}_v + r\mathbf{p}_\kappa + \mathbf{p}_\delta \tag{106}$$

Then the aggregate value will become

$$\tilde{\mathbf{s}}\tilde{\mathbf{p}} = \mathbf{s}\mathbf{p}_\mathbf{v} + r\mathbf{s}_\kappa \mathbf{p}_\kappa + \mathbf{s}_\delta \mathbf{p}_\delta \tag{107}$$

If we denote $\tilde{\mathbf{s}}\tilde{\mathbf{p}}$ by $S$, we will have

$$S^1 = \mathbf{s}^1 \mathbf{p}_v^1 + \boldsymbol{\pi}_L^1 + \boldsymbol{\delta}_L^1 \tag{108}$$

$$= 1 + \boldsymbol{\pi}_L^1 + \boldsymbol{\delta}_L^1 \tag{109}$$

$$S^2 = \mathbf{s}^{\mathbf{2}} \mathbf{p}_E^2 + (\boldsymbol{\pi}_L^2 + \boldsymbol{\delta}_L^2)\ L_S^2 \tag{110}$$

$$= (\Lambda_{sv} + \boldsymbol{\pi}_L^2 + \boldsymbol{\delta}_L^2) L_S^2 \tag{111}$$

And the per capita standard income will become

$$S_L^1 = \mathbf{s}^1 \mathbf{p}_v^1 + \pi_L^1 + \delta_L^1 \tag{112}$$

$$= 1 + \boldsymbol{\pi}_L^1 + \boldsymbol{\delta}_L^1 \tag{113}$$

$$S_L^2 = \mathbf{s}^{\mathbf{2}} \mathbf{p}_E^2 / L_S^2 + \boldsymbol{\pi}_L^2 + \boldsymbol{\delta}_L^2 \tag{114}$$

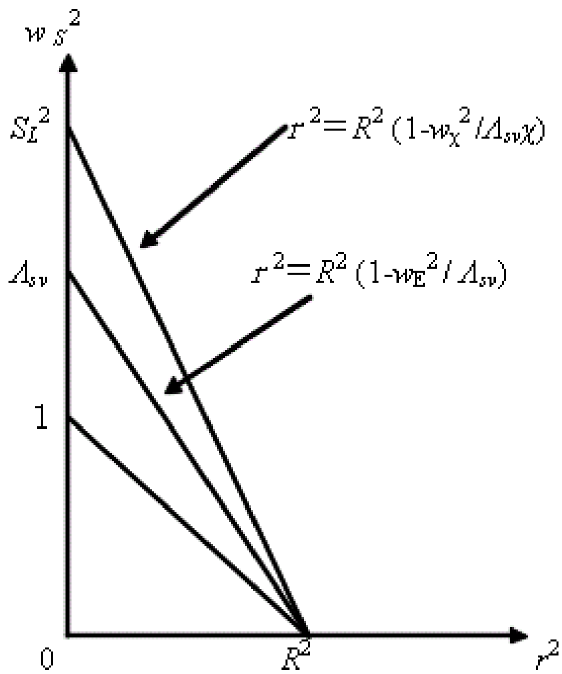
$$= \Lambda_{sv} + \boldsymbol{\pi}_L^2 + \boldsymbol{\delta}_L^2 \tag{115}$$

This is the definition of labour productivity derived from the Input Output system when the fixed capital exists. It will be convenient to define the ratio between the standard income $\mathbf{s}^1 \mathbf{p}_v^1$ and the gross profit $(\boldsymbol{\pi}_L + \boldsymbol{\delta}_L)$. They will be defined as

$$\chi^1 = 1 + \pi_L^1 + \boldsymbol{\delta}_L^1 \tag{116}$$

$$\chi^2 = \frac{\Lambda_{sv} + \boldsymbol{\pi}_L^2 + \boldsymbol{\delta}_L^2}{\Lambda_{sv}} \tag{117}$$

*Figure2*




The standard income is measured in terms of effective labour unit and it will be interpreted as the physically defined real standard income.

The wage curve will become

$$r = R(1 - \frac{w_\chi}{\Lambda_{sv}\chi}) \tag{118}$$

# 6 Productivity of Capital and Total Factor Productivity

## 6.1 Productivity of Capital

Let us turn to the difinition of productivity of capital $(S/K_{Fw})$ . If we consider the Sraffa System, the maximum rate of profit $R$ will be regarded as the productivity of (circulating) capital measured in terms of standard commodity.

This is a useful expression of the standard net product.Post-multiplying (39) by price vector $\mathbf{p}_A$, we have

$$\mathbf{s}\mathbf{p}_v = R\mathbf{q}\mathbf{A}\mathbf{p}_v \tag{119}$$

From (34)(40), we can obtain

$$\mathbf{s}\mathbf{p}_v = R\mathbf{k}_S\mathbf{p}_v \tag{120}$$

Then, we have

$$\mathbf{k}_S\mathbf{p}_v = \frac{1}{R} \tag{121}$$

This is the aggregate value of the standard capital. It is important to understand that the value of standard capital is independent of the variation of the prices and distribution. It is given simply as the inverse of $R$.

$$\frac{\mathbf{s}\mathbf{p}_v}{\mathbf{k}_S\mathbf{p}_v} = R \tag{122}$$

It will be convenient to define the ratio between the standard capital $\mathbf{k}_S\mathbf{p}_v$ and the actural fixed capital measured in terms of commanded labour. It will be represented as

$$\begin{aligned}
\mu^1 &= \frac{\mathbf{k}_S^1\mathbf{p}_v^1}{K_{Fw}^1} \\
\mu^2 &= \frac{\mathbf{k}_S^2\mathbf{p}_v^2}{K_{Fw}^2}
\end{aligned}$$

Then we can define the productivity of capital by

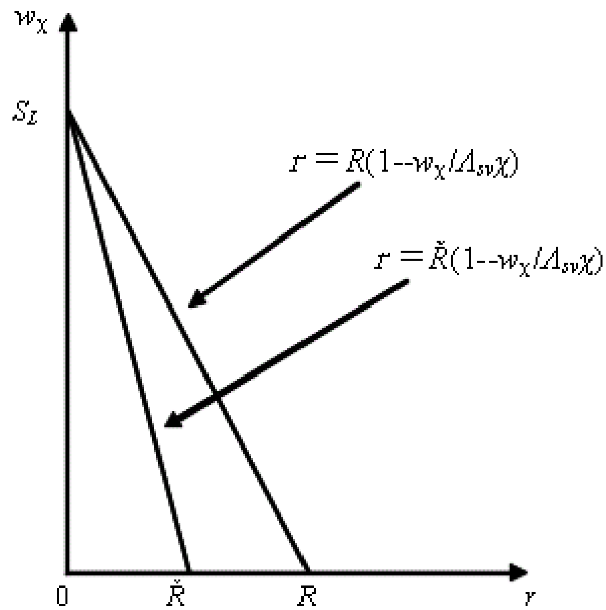
$$\begin{aligned}
\hat{R}^1 &= \frac{S^1}{K_{Fw}^1} && (123) \\
&= \frac{S^1}{\mathbf{s}^1\mathbf{p}_v^1} \times \frac{\mathbf{s}^1\mathbf{p}_v^1}{\mathbf{k}_S^1\mathbf{p}_v^1} \times \frac{\mathbf{k}_S^1\mathbf{p}_v^1}{K_{Fw}^1} && (124) \\
&= (1 + \boldsymbol{\pi}_L^1 + \boldsymbol{\delta}_L^1) \times R^1 \times \mu^1 && (125) \\
&= \chi^1 \times R^1 \times \mu^1 && (126)
\end{aligned}$$

And

$$\begin{aligned}
\hat{R}^2 &= \frac{S^2}{K_{Fw}^2} && (127) \\
&= \frac{S^2}{\mathbf{s}^2\mathbf{p}_v^2} \times \frac{\mathbf{s}^2\mathbf{p}_v^2}{\mathbf{k}_S^2\mathbf{p}_v^2} \times \frac{\mathbf{k}_S^2\mathbf{p}_v^2}{K_{Fw}^2} && (128) \\
&= \frac{\Lambda_{sv} + \boldsymbol{\pi}_L^2 + \boldsymbol{\delta}_L^2}{L_S^2} \times R^2 \times \mu^2 && (129) \\
&= \chi^2 \times R^2 \times \mu^2 && (130)
\end{aligned}$$

The wage curve will become

***Figure 3***



$$r = \hat{R}(1 - \frac{w_\chi}{\Lambda_{sv}\chi}) \quad (131)$$

$$r = \hat{R}(1 - \omega_S) \quad (132)$$

### 6.2 Total Factor Productivity

The total factor productivity is a geometric mean of labour productivity and capital productivity.

$$TFP = S_L{}^{\omega_S} \cdot \hat{R}^{1-\omega_S} \quad (133)$$

$$= S_L{}^{1-r/\hat{R}}\hat{R}^{r/\hat{R}} \quad (134)$$

It shoud be stressed that our definition of TFP is obtained in the I-O system by using the Sraffa system.

## 7 Conclusion

In this paper, we have given the definition of labour productivity, capital productivity and total factor productivity by using the I-O system and the Sraffa Systrem.

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