Joint Estimation of Supply and Use Tables*

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Abstract

We propose a new biproportional method specifically designed for joint projection of Supply and Use tables (SUTs). In contrast to standard input-output techniques, this method does not require the availability of total outputs by product for the projection year(s), a condition which is not often met in practice. The algorithm, called the SUT-RAS method, jointly estimates SUTs that are immediately consistent. It is applicable to different settings of SUTs, such as the frameworks with basic prices and purchasers' prices, and a setting in which Use tables are separated into domestic and imported uses. Our empirical evaluations show that the SUT-RAS method performs quite well compared to widely used short-cut methods.

Keywords: Supply, Use, biproportional adjustments, optimization

JEL Classification Codes: C61, C89, D57

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1 Introduction

Input-output (IO) tables provide a detailed picture of the interactions in an economy, summarizing the production and use of all goods and services, and of income generated in the production process. These tables are extensively used in a wide range of studies ranging from the analysis of international trade, productivity, efficiency, income inequality to ecological and environmental studies (see, for example, ten Raa (2005) and Miller and Blair (2009), and extensive references thereof). These analyses are relied on the use of symmetric input-output tables (SIOTs) either from the product-by-product or industry-by-industry type. An IO-table is often constructed from underlying Supply and Use tables (SUTs) by means of a particular technology assumption. In fact, SUTs provide more detailed and useful information, since they explicitly distinguish between commodities and industries that allows appropriately considering secondary products besides the main products of industries. However, many IO analyses require the linking of an IO table to additional data sets such as international trade and employment statistics. While the first dataset is organised by product, the second is typically collected at an industry basis. Linking these to a SIOT that is either from the product-by-product or industry-by-industry type is problematic. Instead, SUTs being industry-by-product provide a natural link to the additional data sources.

There exists, however, a problem of timeliness of SUTs (and SIOTs), which mainly has to do with the large financial costs and human efforts required to collect the appropriate data. Thus, the majority of countries provide benchmark tables based on the detailed surveys on mainly five years intervals. To fill the gap in-between the benchmark SUTs, it is necessary to use the so-called *non-survey* methods. Many different non-survey methods have been employed in updating the SIOTs. Jackson and Murray (2004) and Pavia et al. (2009) provide recent overviews and evaluations of the various methods.

The updating procedures for SIOTs could in principle also be used for projecting SUTs. In Temurshoev et al. (2010) eight different methods were studied and evaluated in the estimation of the Dutch and Spanish Use and Supply tables and it was concluded that the well-known (G)RAS algorithm was generally superior (for a recent reference on (G)RAS see e.g., Lahr and de Mesnard 2004).[1] There is, however, one important drawback of using SIOTs updating methods in order to estimate SUTs: one has to have both the row and column sums of the Use and Supply tables for the projection year(s). This is largely impractical for SUTs estimation because, although outputs by industry are available from other sources (such as national accounts), outputs by product are typically not available for the projection year. One, of course, can project SUTs on the base of only column totals information using, for example, a one-sided RAS method that has to satisfy only one constraint, namely, the column sums condition, rather than both the column sums and row sums conditions. This is the basis of one of the few existing SUT-updating procedures which was used in the construction of intermediate inputs in the EU KLEMS database and known as the EUKLEMS method (Timmer et al. 2005).[2] It is obvious that such estimation of SUTs is inefficient as it does not use the full potential of the original RAS algorithm, and requires *arbitrary* adjustments to make supply and use tables consistent with respect to the derived outputs by products.

In this paper, we propose a new method for *simultaneous* estimation of SUTs that does *not* require the availability of the use and supply totals by products, which are, instead, endogenously derived. We apply the traditional RAS procedure not separately to the Use and the Supply tables, but instead write the problem in terms of the *joint* estimation of SUTs such that the two requirements of the SUT framework are satisfied. These are the identities of total inputs and total outputs by industry, and total supply and total use by products. It is proved that, like the (G)RAS procedure, the estimates of SUTs are derived by biproportional

[1]They also found that two other, less commonly used, methods proposed, respectively, by Harthoorn and van Dalen (1987) and Kuroda (1988) performed as well as (G)RAS. See, e.g., Kalantari et al. (2008) who show the practicality of RAS in terms of its algorithmic complexity.

[2]See O'Mahony and Timmer (2009) for a description of this database.

adjustments of the original SUTs. However, unlike the (G)RAS algorithm, the process of updating supply and use tables are *not* independent: only three dependent (and not four pairwise independent) multipliers need to be computed to jointly estimate the Use and Supply tables. Because of its closeness to the original (G)RAS algorithm, we refer to this method as the *SUT-RAS approach*.

We show that the SUT-RAS method is rather flexible and can be used in a range of settings. In Section 2 we separately consider the situation in which Supply and Use tables are both given in basic prices, and when the Supply table is in basic prices and includes the transformation into purchasers' prices, while the Use table is given at purchasers' prices. The latter setting is most common in available datasets worldwide. Various interesting applications also require series in which the Use table is separated in domestic and imported uses and this is considered as well. Additionally, we also study the possibility of introducing additional projection data, besides the minimum information on the projection year(s) that are needed for the SUT-RAS implementation, including exogenous export and imports statistics, e.g., from international trade sources.

In Section 3 the performance of the SUT-RAS method is tested for a set of SUTs of Spain and Belgium. Comparisons are made with the results obtained from the EUKLEMS method. In addition, results are compared to another SUT-updating technique called the Euro method, widely used in Europe and advocated in the Eurostat handbook for IO table compilation (see Eurostat 2008, Chapter 14).[3] This method relies on two assumptions: first, the shares of industries in the production of commodities remain constant, and second, the fixed input coefficients determine the relations of all product inputs to production of industries (the so-called fixed product sales structure model). In contrast, the SUT-RAS method is a theory-based approach, which minimizes the deviations of the projected Use and Supply

[3]The Euro method was originally devised for updating symmetric input-output tables, but is also used in a SUT-setting, see a report prepared by Joerg Beutel to the European Commission (e.g., contract number 1508302007 FISC-D, April 2008). A detailed description is also provided by Temurshoev et al. (2010).

tables structure from that of some benchmark year. And while joint estimation in SUT-RAS immediately guarantees the consistency of the SUTs, the Euro needs an ad-hoc assumption of the fixed product sales structure to make SUTs consistent. It is found that the SUT-RAS method generally outperforms the EUKLEMS and Euro methods in estimation of Belgian and Spanish SUTs. Finally, Section 4 gives some concluding remarks.

2 The problem of joint estimation of Supply and Use tables

In this section we present the theoretical model of joint estimation of Supply and Use tables (SUTs) and provide its solution. Since Use tables in base year can be available both at basic and purchasers' prices, we consider both cases. Thus, first we give the details of the updating method when the original benchmark SUTs are at basic prices, while the second case is analyzed in one of the later sections.

2.1 Estimation of SUTs at basic prices

Let us first consider the case when the benchmark SUTs are both given at basic prices. For the projection year(s) the following data is available:[4] (i) $\mathbf{x}_b$ - output totals by industry, (ii) $\mathbf{v}_b$ - value-added totals by industry, (iii) $\mathbf{y}_b$ - totals of final demand categories, and (iv) M - overall sum of commodity imports. The subscript b indicates that the corresponding vector/matrix is expressed in basic prices. These data are typically available from the national accounts. The question is how the availability of this minimal information together with some benchmark SUT of the earlier period can be used to estimate the consistent SUTs for the projection year(s).

[4]Matrices are given in bold, capital letters; vectors in bold, lower case letters; and scalars in italicized capital case letters. Vectors are columns by definition, thus row vectors are obtained by transposition, indicated by a prime. $\hat{\mathbf{x}}$ denotes the $n \times n$ diagonal matrix with the elements of the vector $\mathbf{x}$ on its main diagonal.

Table 1: A framework of SUTs at basic prices

	p	s	f	$\sum$
p	$\mathbf{O}$	$\mathbf{U}_b$ - Intermediate Use	$\mathbf{Y}_b$ - Final demand	$\mathbf{q}_b$
s	$\mathbf{V}_b$ - Make matrix	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{x}_b$
m	$\mathbf{m}'$ - Imports vector	$\mathbf{0}'$	$\mathbf{0}'$	M
$\sum$	$\mathbf{q}_b'$	$\mathbf{u}_b' = \mathbf{x}_b' - \mathbf{v}_b'$	$\mathbf{y}_b'$	

To do so, let us first present a framework of SUTs at basic prices. Define the vector of commodity outputs at basic prices by $\mathbf{q}_b$ and the vector of sectoral intermediate use totals by $\mathbf{u}_b$. The last vector can be easily derived, since by definition it equals the difference between the vectors $\mathbf{x}_b$ and $\mathbf{v}_b$. Imports vector $\mathbf{m}$ is given at CIF prices. The null matrix and null vector of appropriate dimensions are denoted, respectively, by $\mathbf{O}$ and $\mathbf{0}$. Further, the summation vector of ones is denoted by $\boldsymbol{\imath}$. Table 1 gives SUTs' framework, where p, s, f and m determine a member of, respectively, products, industries (or sectors), final demand categories and total imports sets. This notation will be shown to be useful below. Thus, Table 1 shows that $\mathbf{U}_b\boldsymbol{\imath} + \mathbf{Y}_b\boldsymbol{\imath} = \mathbf{q}_b = \mathbf{V}_b'\boldsymbol{\imath} + \mathbf{m}$, i.e., total use by product is equal to total supply by product, and $\mathbf{U}_b'\boldsymbol{\imath} + \mathbf{v}_b = \mathbf{x}_b = \mathbf{V}_b\boldsymbol{\imath}$, i.e., total intermediate input and value added by industry is equal to sectoral total output.

Next we define the benchmark year (denoted by the subscript 0) SUTs matrix by

$$\mathbf{A} = \begin{pmatrix} \mathbf{O} & \overline{\mathbf{U}}_0 \\ \overline{\mathbf{V}}_0 & \mathbf{O} \end{pmatrix}, \tag{1}$$

where $\overline{\mathbf{U}}_0 = (\mathbf{U}_{b,0}, \mathbf{Y}_{b,0})$ and $\overline{\mathbf{V}}_0 = (\mathbf{V}_{b,0}', \mathbf{m}_0)'$ are, respectively, the Use table and Supply table at basic prices. The aim is to estimate the corresponding matrix for some projection year that is denoted by $\mathbf{X}$. Using the traditional RAS objective, we want $\mathbf{X}$ to be as close as possible to $\mathbf{A}$, but it should satisfy the two identities in the SUT framework. That is in the estimated matrix, total supply by product should be equal to total use by product, and total input by industry should be equal to total output by industry. Further, the overall sum of the estimated imports should match the given corresponding value $M > 0$. Note that in cases when $M = 0$, the entire import vector $\mathbf{m}$ should be harmlessly deleted from the Supply table $\overline{\mathbf{V}}_0$, hence will not be considered in the SUTs estimation procedure.

As in Junius and Oosterhaven (2003), we define $z_{ij} \equiv \frac{x_{ij}}{a_{ij}}$ whenever $a_{ij} \neq 0$, and set $z_{ij} = 1$ for $a_{ij} = 0$. This mathematical trick will be shown to be very useful, since it will allow to preserve signs of the original elements in the estimated matrix. Next, define sets $I = \{p\}$, $II = \{\{s\}, \{f\}\}$, and $III = \{\{s\}, m\}$, and the *expanded* vectors of total outputs and total uses, respectively, as $\overline{\mathbf{x}} = (\mathbf{x}_b', M)'$ and $\overline{\mathbf{u}} = (\mathbf{u}_b', \mathbf{y}_b')'$. We consider the following constrained optimization problem:

$$\min_{z_{ij} \geq 0} \sum_i \sum_j |a_{ij}| \left(z_{ij} \ln(z_{ij}/e) + 1 \right) \tag{2}$$

such that

$$\sum_{j \in II} a_{pj} z_{pj} - \sum_{k \in III} a_{kp} z_{kp} = 0 \quad \text{for all } p \in I, \tag{3}$$

$$\sum_{k \in I} a_{kj} z_{kj} = \overline{u}_j \quad \text{for all } j \in II, \tag{4}$$

$$\sum_{p \in I} a_{ip} z_{ip} = \overline{x}_i \quad \text{for all } i \in III, \tag{5}$$

where $|a_{ij}|$ is the absolute value of a_{ij} and e is the base of the natural logarithm.

Function (2) is the objective used in the Generalized RAS (GRAS) problem (see Lenzen et al. 2007, Huang et al. 2008). Minimizing this function implies that we want x_{ij} to be as close as possible to the original element a_{ij} for all i and all j. This is because for $z_{ij} = 1$ the value of (2) is zero, which is its minimum possible value. This objective function is the transformation of the well-known information-based entropy measure. Employing Table 1, let us find the meaning of the three constraints of our problem. Condition (3) by using the definition of z_{ij} boils down to $\sum_s x_{ps} + \sum_f x_{pf} - \sum_s x_{sp} - x_{mp} = 0$ (note that $x_{mp} = m_p$), which in matrix form is $\mathbf{U}_b \boldsymbol{\imath} + \mathbf{Y}_b \boldsymbol{\imath} = \mathbf{V}_b' \boldsymbol{\imath} + \mathbf{m}$. Thus, this constraint ensures the identity of supply and use by products, and as a result the commodity output vector is endogenously determined. Constraint (4) guarantees that the column totals of the estimated intermediate use matrix and final demand matrix are equal to $\overline{\mathbf{u}}'$, while (5) requires that the row totals of the Make matrix and commodity imports match their given totals, $\overline{\mathbf{x}}$. Thus, these two conditions will also guarantee that total input by industry is equal to total output by industry. Note that we do not explicitly consider the zero matrices of $\mathbf{A}$ in the above problem, since implicitly they are already accounted for. This is because we set $z_{ij} = 1$ for all $a_{ij} = 0$, which implies that any cell that was zero in the original matrix $\mathbf{A}$ remains zero in the estimated matrix $\mathbf{X}$ as well.

From the SUTs construction we know that the intermediate Use matrix, Make matrix, and imports vector do *not* allow for negative elements, while the final demand matrix allows for negative entries as well. Thus, let us define $\mathbf{P}_0$ as a matrix with all non-negative entries of $\overline{\mathbf{U}}_0$, and $\mathbf{N}_0 \equiv \mathbf{P}_0 - \overline{\mathbf{U}}_0$, which contains absolute values of the negative elements of $\overline{\mathbf{U}}_0$, that is those of $\mathbf{Y}_{b,0}$. Then, the associated Lagrangean of our problem is

$$\begin{aligned}\mathcal{L} \;=\; & \sum_{(i,j)\notin\mathbf{N}_0} a_{ij}\left(z_{ij}\ln(z_{ij}/e)+1\right) - \sum_{(i,j)\in\mathbf{N}_0} a_{ij}\left(z_{ij}\ln(z_{ij}/e)+1\right) \\ & + \sum_{p\in I}\lambda_p\Big(\sum_{k\in III} a_{kp}z_{kp} - \sum_{j\in II} a_{pj}z_{pj}\Big) + \sum_{j\in II}\tau_j\Big(\overline{u}_j - \sum_{k\in I} a_{kj}z_{kj}\Big) \\ & + \sum_{i\in III}\mu_i\Big(\overline{x}_i - \sum_{p\in I} a_{ip}z_{ip}\Big),\end{aligned}$$

where λ_p, τ_j and μ_i are the corresponding Lagrange multipliers of the constraints (3)-(5). The optimal solutions of this function can be easily derived as:

$$z_{pj} = \begin{cases} e^{\lambda_p}e^{\tau_j} & \text{if } a_{pj} \geq 0 \text{ for all } p \in I \text{ and all } j \in II, \\ e^{-\lambda_p}e^{-\tau_j} & \text{if } a_{pj} < 0 \text{ for all } p \in I \text{ and all } j \in II, \end{cases} \tag{6}$$

$$z_{ip} = e^{\mu_i}e^{-\lambda_p} \quad \text{for all } i \in III \text{ and all } p \in I, \tag{7}$$

Thus, expressions (6) and (7) give, respectively, the estimates of $\overline{\mathbf{U}}$ and $\overline{\mathbf{V}}$. Note that for the solution of the problem (2)-(5) it always holds that $z_{ij} > 0$, which means that the estimated matrices will preserve the signs of the original elements. For simplicity, denote $r_u(p) \equiv e^{\lambda_p}$, $s_u(j) \equiv e^{\tau_j}$ and $r_v(i) \equiv e^{\mu_i}$. Then, using the optimal solutions (6)-(7), we thus established the following result.

Theorem 1. *The solutions of the problem* (2)-(5) *of updating SUTs at basic prices are given by* $\overline{\mathbf{U}} = \hat{\mathbf{r}}_u \mathbf{P}_0 \hat{\mathbf{s}}_u - \hat{\mathbf{r}}_u^{-1} \mathbf{N}_0 \hat{\mathbf{s}}_u^{-1}$ *and* $\overline{\mathbf{V}} = \hat{\mathbf{r}}_v \overline{\mathbf{V}}_0 \hat{\mathbf{r}}_u^{-1}$.

Theorem 1 clearly shows the similarity of the joint SUTs updating to the RAS and GRAS solutions: in this case in order to get the corresponding estimates, the semi-positive Supply table $\overline{\mathbf{V}}_0$ is scaled row- and column-wise similar to RAS procedure, while the Use table $\overline{\mathbf{U}}_0$ is scaled also row- and column-wise, but the factors are different depending on whether one is updating its non-negative or strictly negative entries similar to the GRAS algorithm. Because of this closeness to the (G)RAS algorithm, we refer to our method as the SUT-RAS approach. However, the main difference now is that we join these two tables according to the SUTs framework from the outset, and do not consider RASing each matrix separately. As a result, the estimates are dependent in the sense that we use only three dependent multipliers (i.e., $\mathbf{r}_u$, $\mathbf{s}_u$ and $\mathbf{r}_v$) for the joint estimation of the SUTs components, and not four multipliers, which would be pairwise *independent*, in the case of separate updating of $\overline{\mathbf{U}}_0$ and $\overline{\mathbf{V}}_0$. The last option, however, is largely unfeasible from the practical perspective, because the totals of outputs by products, $\mathbf{q}_b$, are not available for the majority of countries around the world (except e.g., for Japan that produces annual symmetric input-output tables by products). But when the components of the SUTs are joined, we need not know this vector, which will be obtained endogenously, and furthermore, the consistency of SUTs is immediately guaranteed. Thus, in contrast, for example, to Euro method (see e.g., Eurostat 2008, Chapter 14), no further steps are needed to equalize sectoral inputs and outputs.[5]

[5]The Euro method uses the so-called fixed product sales structure model to make SUTs consistent in the second step of each iterations in its algorithm. This model is based on the assumption

Given Theorem 1, our task is now to find out how the row and column multipliers can be computed. By plugging the optimal solutions in the constraints (3)-(5), we are able to determine these vectors. First, constraint (3) implies that $\overline{\mathbf{U}}\boldsymbol{\imath} - \overline{\mathbf{V}}'\boldsymbol{\imath} = \mathbf{0}$. Using Theorem 1 we thus have $\hat{\mathbf{r}}_u \mathbf{P}_0 \mathbf{s}_u - \hat{\mathbf{r}}_u^{-1}\mathbf{N}_0\hat{\mathbf{s}}_u^{-1}\boldsymbol{\imath} - \hat{\mathbf{r}}_u^{-1}\overline{\mathbf{V}}_0'\mathbf{r}_v = \mathbf{0}$. Premultiplying the last equation by the diagonal matrix $\hat{\mathbf{r}}_u$, yields

$$\hat{\mathbf{r}}_u^2 \mathbf{P}_0 \mathbf{s}_u - \left(\mathbf{N}_0\hat{\mathbf{s}}_u^{-1}\boldsymbol{\imath} + \overline{\mathbf{V}}_0'\mathbf{r}_v\right) = \mathbf{0}.$$

This is a quadratic equation in $\mathbf{r}_u$ without a linear term that admits two solutions, but for our purposes we only need its positive root. Thus,

$$\mathbf{r}_u = \sqrt{\widehat{\mathbf{P}_0\mathbf{s}_u}^{-1}\left(\mathbf{N}_0\hat{\mathbf{s}}_u^{-1}\boldsymbol{\imath} + \overline{\mathbf{V}}_0'\mathbf{r}_v\right)}. \tag{8}$$

Constraint (4) in matrix form is $\overline{\mathbf{U}}'\boldsymbol{\imath} = \overline{\mathbf{u}}$, hence $\hat{\mathbf{s}}_u\mathbf{P}_0'\mathbf{r}_u - \hat{\mathbf{s}}_u^{-1}\mathbf{N}_0'\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath} = \overline{\mathbf{u}}$. Premultiplying the last expression by $\hat{\mathbf{s}}_u$, we obtain $\hat{\mathbf{s}}_u^2\mathbf{P}_0'\mathbf{r}_u - \hat{\mathbf{s}}_u\overline{\mathbf{u}} - \mathbf{N}_0'\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath} = \mathbf{0}$. This is a quadratic equation in $\mathbf{s}_u$, and as in (8) we are interested only in its positive root, thus

$$\mathbf{s}_u = 0.5 \times \widehat{\mathbf{P}_0'\mathbf{r}_u}^{-1}\left(\overline{\mathbf{u}} + \sqrt{\overline{\mathbf{u}} \circ \overline{\mathbf{u}} + 4 \times (\mathbf{P}_0'\mathbf{r}_u) \circ (\mathbf{N}_0'\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath})}\right), \tag{9}$$

where $\circ$ is the Hadamard product of elementwise multiplication.

Finally, condition (5) states that $\overline{\mathbf{V}}\boldsymbol{\imath} = \overline{\mathbf{x}}$, thus using again Theorem 1 we obtain $\hat{\mathbf{r}}_v\overline{\mathbf{V}}_0\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath} = \hat{\overline{\mathbf{x}}}\boldsymbol{\imath}$. This can be rewritten as $\hat{\overline{\mathbf{x}}}^{-1}\overline{\mathbf{V}}_0\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath} = \hat{\mathbf{r}}_v^{-1}\boldsymbol{\imath}$, or equivalently

$$\mathbf{r}_v = \boldsymbol{\imath} \oslash \left(\hat{\overline{\mathbf{x}}}^{-1}\overline{\mathbf{V}}_0\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath}\right), \tag{10}$$

where $\oslash$ is the Hadamard element by element division.

Note from (10) that it must be always the case that $\mathbf{x}$ is a strictly positive vector. Also it follows from (9) that $\overline{\mathbf{u}}$ should be a strictly positive vector either, otherwise

that each product has its own specific sales structure, irrespective of the industry where it is produced.

the diagonal matrix $\widehat{\mathbf{P}_0'\mathbf{r}_u}^{-1}$ is not defined.[6] We observe that equations (9) and (10), besides the exogenously given data, depend *directly* only on $\mathbf{r}_u$. Thus, we propose the following algorithm for computing the required row and column multipliers.

- *Step* $t = 0$. Initialize $\mathbf{s}_u(0) = \boldsymbol{\imath}$ and $\mathbf{r}_v(0) = \boldsymbol{\imath}$.[7]
- *Step* $t = 1, \ldots, k$. Calculate $\mathbf{r}_u(t)$ on the base of $\mathbf{s}_u(t-1)$ and $\mathbf{r}_v(t-1)$, and then use $\mathbf{r}_u(t)$ to compute $\mathbf{s}_u(t)$ and $\mathbf{r}_v(t)$.
- *Step* $t = k$. Stop when $|\mathbf{r}_u(k) - \mathbf{r}_u(k-1)| < \epsilon\boldsymbol{\imath}$ for sufficiently small $\epsilon > 0$.
- *Step* $t = k+1$. Derive the final estimates as $\overline{\mathbf{U}} = \hat{\mathbf{r}}_u(k)\mathbf{P}_0\hat{\mathbf{s}}_u(k) - \hat{\mathbf{r}}_u^{-1}(k)\mathbf{N}_0\hat{\mathbf{s}}_u^{-1}(k)$ and $\overline{\mathbf{V}} = \hat{\mathbf{r}}_v(k)\overline{\mathbf{V}}_0\hat{\mathbf{r}}_u^{-1}(k)$.

Finally, we want to briefly discuss whether the convergence of the above algorithm is guaranteed and whether there exist more than one solution to our problem. In our minimization problem, function (2) is the sum of *strictly convex* functions, hence is itself strictly convex either. Constraints (3)-(5) are *linear* equality constraints, thus are convex as well as concave functions, but not strictly so. It is well-known that the sum of a strictly convex function and a convex function is a strictly convex function (see e.g., Chiang 1984, Theorems I-III, p.342). Thus, our Lagrangian is a strictly convex function, which guarantees that there exists a *unique* solution to the problem (2)-(5). This it turn implies that the above mentioned algorithm surely converges provided that the solution to our SUT-RAS problem exists. We can say more about the necessary and sufficient conditions for the global minimum of our problem using the Kuhn-Tucker sufficiency theorem. The last states that if the following sufficient conditions hold: (a) the objective function is differentiable and convex in the non-negative orthant, (b) each constraint is differentiable and

[6] A practical note on this issue, which is true also for other settings that will follow: in real computations it is *not* necessary to delete a zero row/column from the tables. In MATLAB, for example, one might write a very simple function that derives the inverse of positive elements while desregarding (say, nullifying) the undefined ratios.

[7] Note that the two multipliers have different dimensions. In this case, $\mathbf{s}_u(0)$ has a row dimension equal to the number of industries and final demand categories, while that of $\mathbf{r}_v(0)$ is equal to the number of industries plus one (for total imports).

concave in the nonnegative orthant, and (c) the solution satisfies the Kuhn-Tucker minimum conditions, then the solution gives global minimum of the minimization function (for details see e.g., Chiang 1984, Chapter 21). The reader may confirm that the Kuhn-Tucker minimum conditions are satisfied in our SUT-RAS setting. It is well-known that the Kuhn-Tucker minimum conditions become necessary when the so-called constraint qualification is satisfied. But our setting with the *linear* constraints immediately implies that the constraint qualification is satisfied, thus if the above mentioned conditions (a) and (b) are realized, then the Kuhn-Tucker minimum conditions will be *necessary-and-sufficient* for a minimum. This is indeed the case for the SUT-RAS problem (2)-(5). And, moreover, since the minimization function is strictly convex, the solution is a unique global minimum.

2.2 SUTs estimation with additional information

Often it may happen that besides the column totals of intermediate and final uses, sectoral value-added and total outputs, and aggregate imports, there might be more data available for the projection year(s). Usually this extra information contains the vectors of exports and imports by product from international trade statistics, or, for example, households' consumption by product. It has been extensively reported that, in general, introduction of accurate exogenous information into the classical RAS updating, besides row and column sums of the projection table, improves the resulting estimates (see e.g., de Mesnard and Miller 2006), although there are cases when that does not hold. Hence, in this section we consider the joint estimation of SUTs at basic prices with extra external data availability.

Table 2 presents SUTs framework for our estimation purposes when commodity imports, $\mathbf{m}$, and commodity exports vector, denoted by $\mathbf{e}$, are exogenously available. The vector $\mathbf{e}$ can contain any other information from the final demand matrix. That is, it can also be consumption by households, investments, or the sum of any final demand categories by product. Thus, in comparison to (1), now the benchmark

Table 2: A framework of SUTs at basic prices with more exogenous information

	p	s	f^r	$\sum$
p	$\mathbf{O}$	$\mathbf{U}_b$ - Intermediate Use	$\mathbf{Y}_b^r$ - Reduced $\mathbf{Y}_b$	$\mathbf{q}_b - \mathbf{e}$
s	$\mathbf{V}_b$ - Make matrix	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{x}_b$
$\sum$	$\mathbf{q}_b' - \mathbf{m}'$	$\mathbf{u}_b' = \mathbf{x}_b' - \mathbf{v}_b'$	$\mathbf{y}_b^{r'}$	

SUTs matrix is

$$\mathbf{A} = \begin{pmatrix} \mathbf{O} & \overline{\mathbf{U}}_0 \\ \mathbf{V}_0 & \mathbf{O} \end{pmatrix},$$

where $\overline{\mathbf{U}}_0 = (\mathbf{U}_{b,0}, \mathbf{Y}_{b,0}^r)$. Note that this matrix has lower column dimension than $\overline{\mathbf{U}}_0$ in the previous section. For the sake of presentation simplicity, we do *not* introduce additional notations or sub(super)scripts, and assume that the reader understands that, for example, $\overline{\mathbf{u}} = (\mathbf{u}_b', \mathbf{y}_b^{r'})'$ in this section is different from $\overline{\mathbf{u}}$ in the previous section. Also the Make matrix is now written without the subscript b; that is, instead of $\mathbf{V}_{b,0}$ in what follows we will write simply $\mathbf{V}_0$. The corresponding sets are $I = \{p\}$, $II = \{\{s\}, \{f^r\}\}$, and $III = \{s\}$. The minimization problem will be exactly the same as in (2) subject to the constraints

$$\sum_{j \in II} a_{pj} z_{pj} - \sum_{k \in III} a_{kp} z_{kp} = m_p - e_p \quad \text{for all } p \in I, \tag{11}$$

$$\sum_{k \in I} a_{kj} z_{kj} = \overline{u}_j \quad \text{for all } j \in II, \tag{12}$$

$$\sum_{p \in I} a_{ip} z_{ip} = x_i \quad \text{for all } i \in III. \tag{13}$$

It is easy to check that Theorem 1 still holds for this case as well, but the multipliers are now computed differently. So, the first constraint (11) implies that $\overline{\mathbf{U}}\boldsymbol{\imath} - \mathbf{V}'\boldsymbol{\imath} = \mathbf{f}$, where $\mathbf{f} \equiv \mathbf{m} - \mathbf{e}$. Using Theorem 1 we thus have $\hat{\mathbf{r}}_u \mathbf{P}_0 \mathbf{s}_u - \hat{\mathbf{r}}_u^{-1} \mathbf{N}_0 \hat{\mathbf{s}}_u^{-1} \boldsymbol{\imath} - \hat{\mathbf{r}}_u^{-1} \mathbf{V}_0' \mathbf{r}_v = \mathbf{f}$. Premultiplying the last equation by the diagonal matrix $\hat{\mathbf{r}}_u$, yields

$$\hat{\mathbf{r}}_u^2 \mathbf{P}_0 \mathbf{s}_u - \hat{\mathbf{r}}_u \mathbf{f} - \left(\mathbf{N}_0 \hat{\mathbf{s}}_u^{-1} \boldsymbol{\imath} + \mathbf{V}_0' \mathbf{r}_v\right) = \mathbf{0}.$$

This is a quadratic equation in $\mathbf{r}_u$ *with* linear term, hence its positive root is

$$\mathbf{r}_u = 0.5 \times \widehat{\mathbf{P}_0\mathbf{s}_u}^{-1} \left(\mathbf{f} + \sqrt{\mathbf{f} \circ \mathbf{f} + 4 \times (\mathbf{P}_0\mathbf{s}_u) \circ (\mathbf{N}_0\hat{\mathbf{s}}_u^{-1}\boldsymbol{\imath} + \mathbf{V}_0'\mathbf{r}_v)} \right), \qquad (14)$$

which boils down to (8) whenever $\mathbf{f} = \mathbf{0}$.

Similarly, constraint (12) is $\overline{\mathbf{U}}'\boldsymbol{\imath} = \overline{\mathbf{u}}$, hence $\hat{\mathbf{s}}_u\mathbf{P}_0'\mathbf{r}_u - \hat{\mathbf{s}}_u^{-1}\mathbf{N}_0'\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath} = \overline{\mathbf{u}}$. This will yield exactly the same expression for the column multiplier $\mathbf{s}_u$ as in (9), but remember that now its row dimension is reduced due to considering exogenous information from the final demand matrix.

In the same way, condition (13) states that $\mathbf{V}\boldsymbol{\imath} = \mathbf{x}$, thus using again Theorem 1 we obtain similar expression to (10) as follows

$$\mathbf{r}_v = \boldsymbol{\imath} \oslash \left(\hat{\mathbf{x}}^{-1}\mathbf{V}_0\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath} \right). \qquad (15)$$

Here again it is true that the vectors $\overline{\mathbf{u}}$ and $\mathbf{x}$ must be strictly positive, otherwise the multipliers $\mathbf{s}_u$ and $\mathbf{r}_v$ are not defined (which, however, can be easily dealt with in practical applications, see fn. 6). The algorithm of computing the three multipliers and the final estimates in the presence of exogenous information is exactly the same that was presented in Section 2.1.

2.3 Updating SUTs when the Use tables are in purchasers' prices

Now we consider the case when the Supply table has also the valuation adjustment matrix that translates outputs at basic prices into the total supply at purchasers' prices, and the Use table is at purchasers' prices. This is currently the most common case for which SUTs are available. For example, the SUT database from Eurostat (http://epp.eurostat.ec.europa.eu) contains only Use tables at purchasers' prices, not at basic prices. This framework for our joint SUTs updating purposes is illus-

Table 3: Supply table at basic prices, including transformation into purchasers' prices, and Use table at purchasers' prices

	p	s	f	$\sum$
p	$\mathbf{O}$	$\mathbf{U}$ - Intermediate Use	$\mathbf{Y}$ - Final demand	$\mathbf{q}$
s	$\mathbf{V}_b$ - Make matrix	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{x}_b$
m	$\mathbf{m}'$ - Imports vector	$\mathbf{0}'$	$\mathbf{0}'$	M
t	$\mathbf{T}$ - Margins & net taxes	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{t}$
$\sum$	$\mathbf{q}' + \mathbf{c}'$	$\mathbf{u}' = \mathbf{x}_b' - \mathbf{v}_b'$	$\mathbf{y}'$	

trated in Table 3, which is different from Table 1 for SUTs at basic prices in only one respect, besides difference in prices. This is the inclusion of a valuation adjustment matrix $\mathbf{T}$, which includes row-wise trade margins, transportation margins, and product taxes that are net of subsidies. The only difference with the official published SUTs is that now the totals of the trade and transportation margins do not sum to zero, because the products' trade and transportation *negative* figures are nullified, and their totals in absolute value are given in the vector $\mathbf{t}$, which is a strictly positive vector. The reason for this adjustment is that with zero totals the estimation of the valuation adjustment matrix is not possible.[8] That is why, the overall sum of products in the Supply table given in Table 3 is equal to $\mathbf{q}+\mathbf{c}$, where $\mathbf{q}$ is the commodity output at purchasers' prices and $\mathbf{c}$ has the totals of margins from $\mathbf{t}$, which are distributed over the trade and transportation products, and zeros otherwise. Note that $\mathbf{U}$, $\mathbf{Y}$, $\mathbf{q}$, $\mathbf{u}$ and $\mathbf{y}$ are all expressed in purchasers' prices, thus do not have subscript b. Furthermore, the vector $\mathbf{c}$ with trade and transportation margins figures is exogenously given for the SUT-RAS method.

To give more clear picture of $\mathbf{T}$, $\mathbf{t}$ and $\mathbf{c}$, let us consider a four product economy, which is assumed to have the following valuation adjustment matrix in the official SUTs publication:

[8] Actually, one can find also the solution with zero totals, but then deriving the explicit solutions in terms of the multipliers is no longer possible. This can be implemented in a software that finds a zero of functions. But since our aim is to make the entire procedure explicit, we make the mentioned adjustment to the trade and transportation margins.

Products	Trade margins	Transport margins	Net taxes
Agriculture	12	10	-2
Manufacturing	22	15	30
Trade	-34	0	4
Transportation	0	-25	6
Total	0	0	38

For simplicity we gave only the total trade margins, but this column in reality is further divided into wholesale, retail, and motor trade margins. For the above hypothetical valuation adjustment table we then define

$$\mathbf{T} = \begin{pmatrix} 12 & 22 & 0 & 0 \\ 10 & 15 & 0 & 0 \\ -2 & 30 & 4 & 6 \end{pmatrix}, \quad \text{thus} \quad \mathbf{t} = \begin{pmatrix} 34 \\ 25 \\ 38 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 34 \\ 25 \end{pmatrix}.$$

It may be the case that for the projection year the distribution of margins over the trade and transportation products is unknown, and only their overall values are given. In that case, we suggest to distribute the totals over corresponding commodities in $\mathbf{c}$ using the constant shares from the benchmark valuation adjustment matrix. So, in contrast to the SUTs estimation setting at basic prices, now we also require the availability of totals of margins and net taxes (i.e., $\mathbf{t}$) for the projection year(s).

To estimate SUTs we again define the original matrix $\mathbf{A}$ as in (1), but with $\overline{\mathbf{U}}_0 = (\mathbf{U}_0, \mathbf{Y}_0)$ and $\overline{\mathbf{V}}_0 = (\mathbf{V}'_{b,0}, \mathbf{m}_0, \mathbf{T}')'$. Similarly, we define $\overline{\mathbf{x}} = (\mathbf{x}', M, \mathbf{t}')'$ and $\overline{\mathbf{u}} = (\mathbf{u}', \mathbf{y}')'$, and the corresponding sets are $I = \{p\}$, $II = \{\{s\}, \{f\}\}$, and $III = \{\{s\}, m, \{t\}\}$. Note that the Use table $\overline{\mathbf{U}}_0$ and its column totals $\overline{\mathbf{u}}$, and the Supply table $\overline{\mathbf{V}}_0$ and its row totals $\overline{\mathbf{x}}$ are different from those in Section 2.1 due to the price and dimension differences. However, mathematically our problem is very similar to (2)-(5), with only distinction that instead of (3) now we have

$$\sum_{j \in II} a_{pj} z_{pj} - \sum_{k \in III} a_{kp} z_{kp} = -c_p \quad \text{for all } p \in I,$$

which guarantees that the supply and use of products at purchasers' prices are equal, that is, $\overline{\mathbf{U}}\boldsymbol{\imath} = \overline{\mathbf{V}}'\boldsymbol{\imath} - \mathbf{c}$ (remember that $\mathbf{c}$ is exogenous). Thus, the commodity output at purchasers' prices, $\mathbf{q}$, is again endogenously derived. The main distinction of this setting from the SUTs at basic prices is that $\overline{\mathbf{V}}_0$ can also allow for negative entries as well: net taxes can be negative when subsidies exceed taxes. Thus, besides (6), the second optimal condition instead of (7) is

$$z_{ip} = \begin{cases} e^{\mu_i} e^{-\lambda_p} & \text{if } a_{ip} \geq 0 \text{ for all } i \in III \text{ and all } p \in I, \\ e^{-\mu_i} e^{\lambda_p} & \text{if } a_{ip} < 0 \text{ for all } i \in III \text{ and all } p \in I. \end{cases} \tag{16}$$

As in Section 2.1, define $r_u(p) \equiv e^{\lambda_p}$, $s_u(j) \equiv e^{\tau_j}$, $r_v(i) \equiv e^{\mu_i}$, and $\overline{\mathbf{U}}_0 = \mathbf{P}_0 - \mathbf{N}_0$. Further, let $\mathbf{P}_0^v$ be a matrix with all non-negative elements of $\overline{\mathbf{V}}_0$ and $\mathbf{N}_0^v \equiv \mathbf{P}_0^v - \overline{\mathbf{V}}_0$. Thus, the optimal solutions (6) and (16) imply:

Theorem 2. *The solutions of the joint estimation of SUTs, where Supply table is at basic prices and includes transformation into purchasers' prices, and Use table is at purchasers' prices, are given by* $\overline{\mathbf{U}} = \hat{\mathbf{r}}_u \mathbf{P}_0 \hat{\mathbf{s}}_u - \hat{\mathbf{r}}_u^{-1} \mathbf{N}_0 \hat{\mathbf{s}}_u^{-1}$ *and* $\overline{\mathbf{V}} = \hat{\mathbf{r}}_v \mathbf{P}_0^v \hat{\mathbf{r}}_u^{-1} - \hat{\mathbf{r}}_v^{-1} \mathbf{N}_0^v \hat{\mathbf{r}}_u$.

Theorem 2 again confirms that to jointly estimate *consistent* SUTs one needs to compute only three dependent multipliers $\mathbf{r}_u$, $\mathbf{s}_u$ and $\mathbf{r}_v$. Their dependency reflects the fact that all the components of SUTs are estimated simultaneously. Using the supply and use by products identity condition, $\overline{\mathbf{U}}\boldsymbol{\imath} - \overline{\mathbf{V}}'\boldsymbol{\imath} = -\mathbf{c}$, together with Theorem 2, yields $\hat{\mathbf{r}}_u \mathbf{P}_0 \mathbf{s}_u - \hat{\mathbf{r}}_u^{-1} \mathbf{N}_0 \hat{\mathbf{s}}_u^{-1} \boldsymbol{\imath} - \hat{\mathbf{r}}_u^{-1} \mathbf{P}_0^{v\prime} \mathbf{r}_v + \hat{\mathbf{r}}_u \mathbf{N}_0^{v\prime} \hat{\mathbf{r}}_v^{-1} \boldsymbol{\imath} = -\mathbf{c}$. Its premultiplication by the diagonal matrix $\hat{\mathbf{r}}_u$ results again in the quadratic equation in $\mathbf{r}_u$, thus

$$\mathbf{r}_u = 0.5 \times \widehat{\mathbf{p}}_u^{-1} \left(-\mathbf{c} + \sqrt{\mathbf{c} \circ \mathbf{c} + 4 \times \mathbf{p}_u \circ \mathbf{n}_u} \right), \tag{17}$$

where $\mathbf{p}_u \equiv \mathbf{P}_0 \mathbf{s}_u + \mathbf{N}_0^{v'} \hat{\mathbf{r}}_v^{-1} \boldsymbol{\imath}$ and $\mathbf{n}_u \equiv \mathbf{N}_0 \hat{\mathbf{s}}_u^{-1} \boldsymbol{\imath} + \mathbf{P}_0^{v'} \mathbf{r}_v$.

Condition (4) requires $\overline{\mathbf{U}}' \boldsymbol{\imath} = \overline{\mathbf{u}}$, hence $\hat{\mathbf{s}}_u \mathbf{P}_0' \mathbf{r}_u - \hat{\mathbf{s}}_u^{-1} \mathbf{N}_0' \hat{\mathbf{r}}_u^{-1} \boldsymbol{\imath} = \overline{\mathbf{u}}$. Its solution in terms of $\mathbf{s}_u$ is already given in (9) and is valid in the current setting as well. Finally, from (5) it follows that $\overline{\mathbf{V}} \boldsymbol{\imath} = \overline{\mathbf{x}}$. Thus, using Theorem 2, we obtain $\hat{\mathbf{r}}_v \mathbf{P}_0^v \hat{\mathbf{r}}_u^{-1} \boldsymbol{\imath} - \hat{\mathbf{r}}_v^{-1} \mathbf{N}_0^v \mathbf{r}_u = \overline{\mathbf{x}}$, or equivalently, $\hat{\mathbf{r}}_v^2 \mathbf{P}_0^v \hat{\mathbf{r}}_u^{-1} \boldsymbol{\imath} - \hat{\mathbf{r}}_v \overline{\mathbf{x}} - \mathbf{N}_0^v \mathbf{r}_u = \mathbf{0}$. Therefore,

$$\mathbf{r}_v = 0.5 \times \widehat{\mathbf{P}_0^v \hat{\mathbf{r}}_u^{-1} \boldsymbol{\imath}}^{-1} \left(\overline{\mathbf{x}} + \sqrt{\overline{\mathbf{x}} \circ \overline{\mathbf{x}} + 4 \times (\mathbf{P}_0^v \hat{\mathbf{r}}_u^{-1} \boldsymbol{\imath}) \circ (\mathbf{N}_0^v \mathbf{r}_u)} \right). \tag{18}$$

Note that mathematically the basic price SUTs estimation results are essentially a particular case of this purchasers' price setting when $\mathbf{N}_0^v$ is a zero matrix. After updating one can distribute the totals of margins in $\mathbf{c}$ over the corresponding trade and transportation products, which will result in zero totals of these margins as given in the official statistical publications.

Given our earlier discussions on the availability of extra exogenous information, the elaborations in Section 2.2 can be used in the current setting as well. Then in Table 3 we are not going to estimate the commodity imports vector $\mathbf{m}$ and a part of the final demand matrix, denoted by $\mathbf{e}$ that can be commodity exports vector, consumption vector, or the sum of any final demand categories by product. In Table 3 these exogenously given vectors will be subtracted from the corresponding totals in the table margins similar to Table 2 and the reduced final demand matrix, $\mathbf{Y}^r$, will take the place of $\mathbf{Y}$. The reader can easily confirm that in such a setting we will have $\overline{\mathbf{U}}_0 = (\mathbf{U}_0, \mathbf{Y}_0^r)$, $\overline{\mathbf{V}}_0 = (\mathbf{V}_{b,0}', \mathbf{T}')'$, $\overline{\mathbf{x}} = (\mathbf{x}', \mathbf{t}')'$, $\overline{\mathbf{u}} = (\mathbf{u}', \mathbf{y}^{r'})'$, and the sets will be $I = \{p\}$, $II = \{\{s\}, \{f^r\}\}$, and $III = \{\{s\}, \{t\}\}$. Again our problem is very similar to (2)-(5), with only exception that instead of (3) now we should have

$$\sum_{j \in II} a_{pj} z_{pj} - \sum_{k \in III} a_{kp} z_{kp} = m_p - c_p - e_p \quad \text{for all } p \in I,$$

which guarantees that the supply and use by products at purchasers' prices are equal, that is, $\overline{\mathbf{U}}\boldsymbol{\imath} + \mathbf{e} = \overline{\mathbf{V}}'\boldsymbol{\imath} + \mathbf{m} - \mathbf{c}$. Therefore, Theorem 2 holds for this setting with additional information as well, and only a small difference comes in the computation of the multiplier $\mathbf{r}_u$ in (17), where instead of $\pm\mathbf{c}$ we will have the vector $\mathbf{f} \equiv \mathbf{m}-\mathbf{c}-\mathbf{e}$ (this is due to combination of Theorem 2 and the identity $\overline{\mathbf{U}}\boldsymbol{\imath} - \overline{\mathbf{V}}'\boldsymbol{\imath} = \mathbf{m} - \mathbf{c} - \mathbf{e}$). As before, the other two multipliers $\mathbf{s}_u$ and $\mathbf{r}_v$ are defined, respectively, by (9) and (18), remembering that now they have different dimensions. In fact, this last setting is the most general of all frameworks considered above, since mathematically each of them is its particular case.[9]

The algorithm of computing the three multipliers is similar to that presented in Section 2.1, with the only difference that in the last step of $t = k+1$ the final estimate of $\overline{\mathbf{V}}$ should be derived using the corresponding expression from Theorem 2. Again our discussions in the last paragraph of Section 2.1 on uniqueness of the solution and convergence also hold in the current setting.

An important remark is that in order the estimated vectors of use and supply by products to be immediately consistent, it must be true that the exogenous given data is consistent itself. That is, it must hold that $\boldsymbol{\imath}'\overline{\mathbf{u}} + \boldsymbol{\imath}'\mathbf{y} + \boldsymbol{\imath}'\mathbf{e} = \boldsymbol{\imath}'\overline{\mathbf{x}} + \boldsymbol{\imath}'(\mathbf{m} - \mathbf{c})$ in the most general case with additional exogenous information. In words, the economy-wide use (as a scalar) should be equal to the overall production. Otherwise, the SUT-RASing might still produce the required estimates, but the error by which the mentioned identity does not hold, will appear as the overall difference of the endogenized vectors of the use and supply by products.[10] The final note is that if we were to multiply the first constraint of our optimization problem, which links Supply and Use tables and guarantees their consistency product-wise, by minus one (-1), then we would obtain *alternative* expressions of the estimates. So it can be easily shown that then the estimated Use and Supply tables instead of those given

[9] As such an algorithm written for the last case with exogenous information is immediately applicable for all other settings as well, and one needs only to nullify (ignore) its certain components.

[10] We thank Gaaitzen de Vries for "discovering" this regularity in his sensitivity check of the error threshold level on Mexican data.

in Theorem 2 would be equal to

$$\overline{\mathbf{U}} = \hat{\mathbf{r}}_u^{-1}\mathbf{P}_0\hat{\mathbf{s}}_u - \hat{\mathbf{r}}_u\mathbf{N}_0\hat{\mathbf{s}}_u^{-1}, \quad \text{and} \quad \overline{\mathbf{V}} = \hat{\mathbf{r}}_v\mathbf{P}_0^v\hat{\mathbf{r}}_u - \hat{\mathbf{r}}_v^{-1}\mathbf{N}_0^v\hat{\mathbf{r}}_u^{-1},$$

where the corresponding multipliers (in a general case of the availability of exogenous information) are

$$\begin{aligned}
\mathbf{r}_u &= 0.5 \times \widehat{\mathbf{n}}_u^{-1}\left(\mathbf{f} + \sqrt{\mathbf{f}\circ\mathbf{f} + 4\times\mathbf{p}_u\circ\mathbf{n}_u}\right),\\
\mathbf{r}_v &= 0.5 \times \widehat{\mathbf{P}_0^v\mathbf{r}_u}^{-1}\left(\overline{\mathbf{x}} + \sqrt{\overline{\mathbf{x}}\circ\overline{\mathbf{x}} + 4\times(\mathbf{P}_0^v\mathbf{r}_u)\circ(\mathbf{N}_0^v\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath})}\right),\\
\mathbf{s}_u &= 0.5 \times \widehat{\mathbf{P}_0'\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath}}^{-1}\left(\overline{\mathbf{u}} + \sqrt{\overline{\mathbf{u}}\circ\overline{\mathbf{u}} + 4\times(\mathbf{P}_0'\hat{\mathbf{r}}_u^{-1}\boldsymbol{\imath})\circ(\mathbf{N}_0'\mathbf{r}_u)}\right),
\end{aligned}$$

where $\mathbf{n}_u$ and $\mathbf{p}_u$ are defined above under (17). Certainly, the two expressions with their corresponding multipliers result in exactly equivalent estimates, thus it is only a matter of taste which one of them to use.

2.4 SUT-RAS method: the case of domestic and imported Use tables

In this section we provide a somewhat different SUTs estimation method, which distinguishes the intermediate and final Use tables into domestic and imported tables. Having Use tables separately for domestic and imported uses is important, since many economic analyses are based only on the domestic input structure of the economy, rather than its entire technology that includes also imported inputs.

If we want to use, for example, the setting analyzed in Sections 2.1–2.3, then the most obvious way of producing domestic and imported Use tables would be as follows. The SUT-RAS from the previous sections endogenously computes the vector of total outputs by product, $\mathbf{q}$, be it at basic prices or purchasers' prices. If we denote the entire domestic Use table (i.e., intermediate and final uses) by $\overline{\mathbf{U}}^d$, and the corresponding imported table by $\overline{\mathbf{U}}^m$, which are going to be estimated on the base of the available corresponding matrices of some benchmark year, then from our first SUT-RAS step we already know their column and row sums. Thus, we estimate the matrix $(\overline{\mathbf{U}}^{d'}, \overline{\mathbf{U}}^{m'})'$ such that it has the row totals of $(\mathbf{q}' - \mathbf{m}', \mathbf{m}')'$ and the column totals of $\overline{\mathbf{u}}$, which was used in the SUT-RAS approach as well. Since we know these row and column totals for the projection year, one can in the second step use the GRAS algorithm to jointly estimate $\overline{\mathbf{U}}^{d}$ and $\overline{\mathbf{U}}^{m}$. Of course, there is no guarantee that the matrix $\overline{\mathbf{U}}^{d} + \overline{\mathbf{U}}^{m}$ is equal to the total Use table $\overline{\mathbf{U}}$ from the first step SUT-RAS result. Hence, in such cases the total Use table should be taken from the second step GRAS procedure, i.e., $\overline{\mathbf{U}}^{d} + \overline{\mathbf{U}}^{m}$, implying that the first step SUT-RAS outcome for the Use table was used only to compute the vector of outputs by product and the vector of imports (if the last was not available). The projected Supply table is taken from the first step SUT-RAS approach.

Now we pose a question whether instead of the two-step procedure discussed above, can we provide a SUT-RAS setting, where Supply table and the domestic and imported Use tables are estimated within one framework. In what follows, we provide such framework of SUTs estimation, which is valid for both basic and purchasers' prices settings. Let us start with the case when Use tables are expressed in purchasers' prices. This more general SUTs setting is illustrated in Table 4, which is an adjusted version of Table 3 that makes distinction between domestic products, p^d, imported products, p^m, and also separates the intermediate and final Use tables into the domestic and imported parts. This setting divides the supply and use identity into two identities: supply of domestic products is equal to use of domestic products, and supply of imported products is equal to the use of imported products. This distinction, thus, make us to place the vector of imports separately from the Supply table of domestic products.

We define sets $I^d = \{p^d\}$, $I^m = \{p^m\}$, $II = \{\{s\}, \{f\}\}$, and $III = \{\{s\}, \{t\}\}$, and the *expanded* vectors of total outputs and total uses as $\overline{\mathbf{x}} = (\mathbf{x}_b', \mathbf{t})'$ and $\overline{\mathbf{u}} = (\mathbf{u}', \mathbf{y}')'$. Our SUT-RAS problem is minimization of (2) with respect to the following

Table 4: SUTs with domestic and imported Use tables

	p^d	p^m	s	f	$\sum$
p^d	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{U}^d$	$\mathbf{Y}^d$	$\mathbf{q}-\mathbf{m}$
p^m	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{U}^m$	$\mathbf{Y}^m$	$\mathbf{m}$
s	$\mathbf{V}_b$	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{x}_b$
t	$\mathbf{T}$	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{t}$
m	$\mathbf{0}'$	$\mathbf{m}'$	$\mathbf{0}'$	$\mathbf{0}'$	M
$\sum$	$\mathbf{q}'-\mathbf{m}'+\mathbf{c}'$	$\mathbf{m}'$	$\mathbf{u}'=\mathbf{x}_b'-\mathbf{v}_b'$	$\mathbf{y}'$	

five constraints:

$$\sum_{j\in II} a_{p^d j} z_{p^d j} - \sum_{k\in III} a_{kp^d} z_{kp^d} = -c_{p^d} \quad \text{for all } p^d \in I^d, \tag{19}$$

$$\sum_{j\in II} a_{p^m j} z_{p^m j} - a_{mp^m} z_{mp^m} = 0 \quad \text{for all } p^m \in I^m, \tag{20}$$

$$\sum_{k\in I^d} a_{kj} z_{kj} + \sum_{k\in I^m} a_{kj} z_{kj} = \overline{u}_j \quad \text{for all } j \in II, \tag{21}$$

$$\sum_{p^d\in I^d} a_{ip^d} z_{ip^d} = \overline{x}_i \quad \text{for all } i \in III, \tag{22}$$

$$\sum_{p^m\in I^m} a_{mp^m} z_{mp^m} = M. \tag{23}$$

Condition (19) ensures consistency of the supply and use of domestic products, while this identity for the imported products is reflected in (20). Constraint (21) guarantees that the column sum of total (i.e., domestic and imported) Use table is equal to the given vector $\overline{\mathbf{u}}$, while condition (22) ensures that the row totals of the Supply table corresponds to $\overline{\mathbf{x}}$. Finally, (23) ensures that the sum of imports is equal to $M > 0$. Setting the corresponding Lagrangian with multipliers λ_{p^d}, λ_{p^m}, τ_j, μ_i, and ϕ, respectively, for the above five constraints, we obtain the following first-order conditions:

$$z_{p^d j} = \begin{cases} e^{\lambda_{p^d}} e^{\tau_j} & \text{if } a_{p^d j} \geq 0 \text{ for all } p^d \in I^d \text{ and all } j \in II, \\ e^{-\lambda_{p^d}} e^{-\tau_j} & \text{if } a_{p^d j} < 0 \text{ for all } p^d \in I^d \text{ and all } j \in II, \end{cases} \tag{24}$$

$$z_{p^m j} = \begin{cases} e^{\lambda_{p^m}} e^{\tau_j} & \text{if } a_{p^m j} \geq 0 \text{ for all } p^m \in I^m \text{ and all } j \in II, \\ e^{-\lambda_{p^m}} e^{-\tau_j} & \text{if } a_{p^m j} < 0 \text{ for all } p^m \in I^m \text{ and all } j \in II, \end{cases} \tag{25}$$

$$z_{ip^d} = \begin{cases} e^{\mu_i} e^{-\lambda_{p^d}} & \text{if } a_{ip^d} \geq 0 \text{ for all } i \in III \text{ and all } p^d \in I^d, \\ e^{-\mu_i} e^{\lambda_{p^d}} & \text{if } a_{ip^d} < 0 \text{ for all } i \in III \text{ and all } p^d \in I^d, \end{cases} \tag{26}$$

$$z_{mp^m} = e^{\phi} e^{-\lambda_{p^m}}. \tag{27}$$

Similar to the previous sections, define $r_d(p^d) \equiv e^{\lambda_{p^d}}$, $r_m(p^m) \equiv e^{\lambda_{p^m}}$, $s_u(j) \equiv e^{\tau_j}$, $r_v(i) \equiv e^{\mu_i}$, and $r \equiv e^{\phi}$. Further, we denote the non-negative and absolute values of negative entries of the benchmark domestic Use table, $\overline{\mathbf{U}}_0^d$, by matrices $\mathbf{P}_0^d$ and $\mathbf{N}_0^d$, respectively, i.e., $\overline{\mathbf{U}}_0^d = \mathbf{P}_0^d - \mathbf{N}_0^d$. The corresponding matrices for the imported Use table, $\overline{\mathbf{U}}_0^m$, are $\mathbf{P}_0^m$ and $\mathbf{N}_0^m$. The benchmark Supply matrix $\overline{\mathbf{V}}_0 = (\mathbf{V}_{b,0}', \mathbf{T}_0')'$ (note that in comparison to Section 2.3, it excludes $\mathbf{m}_0$) is also separated into $\mathbf{P}_0^v$ and $\mathbf{N}_0^v$ similarly, i.e., $\overline{\mathbf{V}}_0 = \mathbf{P}_0^v - \mathbf{N}_0^v$. These distinctions are made because positive elements contribute differently to the estimation procedure of SUTs than strictly negative entries. The Use tables can have negative entries because of changes in inventories, while the existence of net taxes can result in negative entries in the Supply table. Imports vector is by definition non-negative, hence it has only one condition in (27). Using the optimal conditions (24)-(27), we thus established the following result.

Theorem 3. *The solutions of the joint estimation of SUTs that distinguishes between domestic and imported use of products are given by* $\overline{\mathbf{U}}^d = \hat{\mathbf{r}}_d \mathbf{P}_0^d \hat{\mathbf{s}}_u - \hat{\mathbf{r}}_d^{-1} \mathbf{N}_0^d \hat{\mathbf{s}}_u^{-1}$, $\overline{\mathbf{U}}^m = \hat{\mathbf{r}}_m \mathbf{P}_0^m \hat{\mathbf{s}}_u - \hat{\mathbf{r}}_m^{-1} \mathbf{N}_0^m \hat{\mathbf{s}}_u^{-1}$, $\overline{\mathbf{V}} = \hat{\mathbf{r}}_v \mathbf{P}_0^v \hat{\mathbf{r}}_d^{-1} - \hat{\mathbf{r}}_v^{-1} \mathbf{N}_0^v \hat{\mathbf{r}}_d$, *and* $\mathbf{m}' = r\mathbf{m}_0' \hat{\mathbf{r}}_m^{-1}$.

Theorem 3 shows that in order to estimate three matrices and one vector, we need to compute only five dependent multipliers, which can be derived by joint employment of the optimal conditions with the constraints (19)-(23). This principle should be clear by now to the reader, hence we only give the final results as follows:

$$\mathbf{r}_d = 0.5 \times \widehat{\mathbf{p}}_d^{-1} \left(-\mathbf{c} + \sqrt{\mathbf{c} \circ \mathbf{c} + 4 \times \mathbf{p}_d \circ \mathbf{n}_d} \right), \tag{28}$$

$$\mathbf{r}_m = \sqrt{\widehat{\mathbf{P}_0^m \mathbf{s}_u}^{-1} (\mathbf{N}_0^m \hat{\mathbf{s}}_u^{-1} \boldsymbol{\imath} + r \mathbf{m}_0)}, \tag{29}$$

$$\mathbf{r}_v = 0.5 \times \widehat{\mathbf{P}_0^v \hat{\mathbf{r}}_d^{-1} \boldsymbol{\imath}}^{-1} \left(\overline{\mathbf{x}} + \sqrt{\overline{\mathbf{x}} \circ \overline{\mathbf{x}} + 4 \times (\mathbf{P}_0^v \hat{\mathbf{r}}_d^{-1} \boldsymbol{\imath}) \circ (\mathbf{N}_0^v \mathbf{r}_d)} \right), \tag{30}$$

$$\mathbf{s}_u = 0.5 \times \widehat{\mathbf{p}}_s^{-1} \left(\overline{\mathbf{u}} + \sqrt{\overline{\mathbf{u}} \circ \overline{\mathbf{u}} + 4 \times \mathbf{p}_s \circ \mathbf{n}_s} \right), \tag{31}$$

$$r = M / (\mathbf{m}_0' \hat{\mathbf{r}}_m^{-1} \boldsymbol{\imath}), \tag{32}$$

where $\mathbf{p}_d \equiv \mathbf{P}_0^d \mathbf{s}_u + \mathbf{N}_0^{v'} \hat{\mathbf{r}}_v^{-1} \boldsymbol{\imath}$, $\mathbf{n}_d \equiv \mathbf{N}_0^d \hat{\mathbf{s}}_u^{-1} \boldsymbol{\imath} + \mathbf{P}_0^{v'} \mathbf{r}_v$, $\mathbf{p}_s = \mathbf{P}_0^{d'} \mathbf{r}_d + \mathbf{P}_0^{m'} \mathbf{r}_m$, and $\mathbf{n}_s \equiv \mathbf{N}_0^{d'} \hat{\mathbf{r}}_d^{-1} \boldsymbol{\imath} + \mathbf{N}_0^{m'} \hat{\mathbf{r}}_m^{-1} \boldsymbol{\imath}$.

From equations (28)-(32) it follows that if $\mathbf{r}_d$ and $\mathbf{r}_m$ converge, so do the other three multipliers. Thus we propose an algorithm similar to the one presented in Section 2.1 as follows:

- *Step* $t = 0$. Initialize $\mathbf{s}_u(0) = \boldsymbol{\imath}$, $\mathbf{r}_v(0) = \boldsymbol{\imath}$ and $r = 1$.
- *Step* $t = 1, \ldots, k$. Calculate $\mathbf{r}_d(t)$ and $\mathbf{r}_m(t)$ on the base of $\mathbf{s}_u(t-1)$, $\mathbf{r}_v(t-1)$ and $r(t-1)$, and then use $\mathbf{r}_d(t)$ and $\mathbf{r}_m(t)$ to compute $\mathbf{s}_u(t)$, $\mathbf{r}_v(t)$, and $r(t)$.
- *Step* $t = k$. Stop when $|\mathbf{r}_d(k) - \mathbf{r}_d(k-1)| < \epsilon \boldsymbol{\imath}$ and $|\mathbf{r}_m(k) - \mathbf{r}_m(k-1)| < \epsilon \boldsymbol{\imath}$ for sufficiently small $\epsilon > 0$.
- *Step* $t = k+1$. Derive the final estimates using Theorem 3 and multipliers from step k.

The discussions in the last paragraph of Section 2.1 on the unique solution and convergence applies to this SUT setting as well. We should note that Theorem 3 and multipliers' expressions (28)-(32) also hold for the case when Use tables are given in basic prices. In that case, one must not consider the valuation adjustment matrix $\mathbf{T}$ in the Supply table, and also set $\mathbf{c} = \mathbf{0}$. Finally, in case of availability of additional information, such as imports and exports vectors, one can easily adapt the SUT-RAS problem of this section, similar to that described in Section 2.2.

3 Empirical assessment

We apply the proposed SUT-RAS method to the Spanish benchmark SUTs for 2000 and 2005, which are available from National Statistics Institute of Spain both at basic and purchasers' prices. The data were disaggregated into 72 products and 72 industries. We symmetrized the tables because we want to compare the results of SUT-RAS algorithm to two other methods of updating SUTs. These are the Euro method (Beutel 2002, Eurostat 2008), which requires symmetric SUTs, and EUKLEMS method (Timmer et al. 2005). They require almost the same data availability for the projection year tables except for the Euro method, which does not use output vector by industry and computes it endogenously. We, however, think that since total outputs by industry are also available from the national accounts, it is more reasonable to use this extra important information rather than estimate it. It should be mentioned that the Euro method also requires that the Use table at basic prices is distinguished between domestic and imported intermediate and final uses. We do not explain the Euro and EUKLEMS methods in this paper due to space constraints, and refer the reader to Temurshoev et al. (2010) who present detailed description of both methods for updating SUTs.

In order to assess the relative performance of the methods, we use the following criteria:

1. Mean absolute percentage error (Butterfield and Mules 1980):

$$MAPE = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{|x_{ij} - x_{ij}^{true}|}{|x_{ij}^{true}|} \times 100,$$

where x_{ij}^{true} is the true element, while x_{ij} is its estimate, and $\mathbf{X}$ is the $m \times n$ matrix. Thus, $MAPE$ shows the *average* percentage by which each estimated element is larger or smaller than its true value. Note that we take the denominator in absolute value as well so that it does not allow to reduce the actual error when $x_{ij}^{true} < 0$.

2. Weighted absolute percentage error (Mínguez et al. 2009):

$$WAPE = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{|x_{ij}^{true}|}{\sum_k\sum_l x_{kl}^{true}}\right)\frac{|x_{ij} - x_{ij}^{true}|}{|x_{ij}^{true}|} \times 100,$$

which weights each percentage deviation of x_{ij} from x_{ij}^{true} by the relative size of the corresponding true element in the overall sum of the actual elements.

3. Standardized weighted absolute difference (Lahr 2001):

$$SWAD = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}^{true}| \times |x_{ij} - x_{ij}^{true}|}{\sum_k\sum_l (x_{kl}^{true})^2},$$

which is somewhat similar to $WAPE$ with the difference that the absolute deviations are weighted by the size of the true transactions.

4. The *psi statistic* (Kullback 1959, Knudsen and Fotheringham 1986):

$$\hat{\psi} = \frac{1}{\sum_k\sum_l x_{kl}^{true}}\sum_i\sum_j\left[|x_{ij}^{true}| \times \left|\ln\left(\frac{x_{ij}^{true}}{s_{ij}}\right)\right| + |x_{ij}| \times \left|\ln\left(\frac{x_{ij}}{s_{ij}}\right)\right|\right],$$

where $s_{ij} = (|x_{ij}^{true}| + |x_{ij}|)/2$. This information-based statistic has a lower limit of zero when $\mathbf{X}^{true} = \mathbf{X}$, and upper bound of $mn \ln 2$ when the non-zero elements of $\mathbf{X}^{true}$ correspond to the zero elements of $\mathbf{X}$, and vice versa. Unlike $MAPE$, $WAPE$ and $SWAD$, the psi statistic is insensitive to the change in the positions of x_{ij}^{true} and x_{ij}, and it offers the advantage of considering the case when $x_{ij}^{true} = 0$ and $x_{ij} \neq 0$ (next to the reverse situation).[11] Knudsen and Fotheringham (1986) concluded that the psi statistic is one of the most useful goodness-of-fit measures for matrix comparative purposes, as they showed a linear relation between its value and the level of error.

5. RSQ (or coefficient of determination) – the square of the correlation coefficient between the elements of the actual matrix, $\mathbf{X}^{true}$, and the predicted matrix,

[11] When $x_{ij}^{true} = 0$, we set the corresponding element of $MAPE$, $WAPE$ and $SWAD$ to zero, and when $x_{ij}^{true} = x_{ij} = 0$, the corresponding entry of $\hat{\psi}$, along other goodness-of-fit measures, is nullified as well.

X, when at least one of the elements is different from zero.

The results of the estimation of Spanish SUTs at basic prices are given in Table 5. We use the 2000 SUTs as benchmarks and the required totals vectors from 2005 SUTs in order to estimate the 2005 tables, and then compare the estimates with the true 2005 SUTs. The final demand matrix for this exercise consists of total consumption, gross capital formation, and total exports.

Table 5: Results of updating Spanish SUTs at basic prices

	MAPE	R.	WAPE	R.	SWAD	R.	$\hat{\psi}$	R.	RSQ	R.	CmR.
	Make matrix (72 × 72)										
EURO	21.62	3	8.45	3	0.109	3	0.083	3	0.9922	3	3
EUKLEMS	19.44	1	2.52	1	0.004	1	0.024	1	0.9998	1	1
SUT-RAS	20.78	2	3.03	2	0.010	2	0.029	2	0.9998	1	2
	Supply table (Make matrix + imports, 72 × 73)										
EURO	21.65	3	8.77	3	0.108	3	0.086	3	0.9916	3	3
EUKLEMS	19.47	1	3.69	1	0.007	1	0.035	1	0.9995	1	1
SUT-RAS	20.78	2	4.21	2	0.013	2	0.041	2	0.9994	2	2
	Total intermediate Use (72 × 72)										
EURO	38.99	3	20.90	3	0.349	3	0.206	3	0.9205	3	3
EUKLEMS	35.78	1	16.35	2	0.150	2	0.161	2	0.9829	2	2
SUT-RAS	36.27	2	16.07	1	0.131	1	0.158	1	0.9880	1	1
	Total final demand (72 × 3)										
EURO	254.02	2	8.40	2	0.050	3	0.085	2	0.9963	2	2
EUKLEMS	753.38	3	9.78	3	0.042	2	0.125	3	0.9955	3	3
SUT-RAS	111.10	1	7.22	1	0.028	1	0.073	1	0.9973	1	1

Note: The rank of each indicator is given in column R., while CmR. is the combined rank of the averages of all the five rankings. The indicators provide the comparison of the true 2005 SUTs with the 2005 estimates benchmarked on the 2000 SUTs.

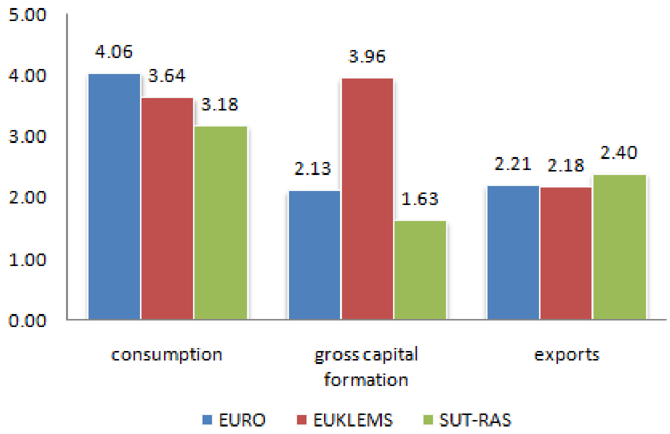
The second column of Table 5, for example, shows that the Euro method produces the estimate of the Make matrix, whose elements are, on average, 21.6% larger or smaller than the true Make matrix entries according to $MAPE$. Similarly, the EUKLEMS and SUT-RAS methods are on average, respectively, 19.4% and 20.8% 'in error' according to $MAPE$. However, we should note that $MAPE$ is not a good measure of goodness-of-fit, since it gives identical weights to all elements of a matrix of deviations of the estimated and true matrices.[12] In this respect, the

[12]This can be seen in the strange figures of $MAPE$ for the final demand matrix estimation

$WAPE$ indicator is much more relevant as it takes into account the relative size of each element of the true matrix, which, for example, strongly suggests that the Euro method is producing much worse predictions of the Make matrix and Supply table than those estimated by the EUKLEMS and SUT-RAS methods. In general, Table 5 shows that the EUKLEMS method outperforms the other two methods in estimating the Supply table *in basic prices*, although note that its average errors are quite close to those of the SUT-RAS algorithm. However, when we compare the estimates of the Use tables, we find that the SUT-RAS algorithm outperforms both the Euro and EUKLEMS methods. In particular, one can easily observe that the EUKLEMS is performing worst in the estimation of the final demand matrix. The reason for this large deviation is that EUKLEMS approach considers the column of changes in inventories as the residual between the commodity output vectors obtained from the estimated Supply and Use tables in order to make SUTs consistent. This procedure turns out to have a rather large negative impact on the overall quality of estimation of the final demand matrix. This is confirmed in Figure 1, which illustrates $WAPE$s by final demand categories, whose sum for each method equals the overall $WAPE$ given in Table 5 above. Figure 1 clearly illustrates another important issue: compared to the SUT-RAS approach, the Euro and EUKLEMS methods produce *extra* errors of 0.87% and 0.46%, respectively, in the consumption vector estimation according to $WAPE$. These percentages, in fact, do *not* indicate small errors, because total consumption accounted for 32.2% of total product use in Spain for 2005. Since the corresponding total consumption was 655,496 million euros, these additional errors roughly mean that, compared to the SUT-RAS method, the Euro and EUKLEMS wrongly estimate consumption components at the amount of, respectively, 5,714 and 3,011 million euros. For the same year, total intermediate use, gross capital

results in Tables 5 and 6. Further, we have to note that the coefficient of determination, RSQ, is a weak statistic for matrix comparison purposes either, which, in fact, has been already shown in Knudsen and Fotheringham (1986). We, however, provide the values of the $MAPE$ and RSQ measures here only for the sake of completeness, as these indicators are often reported in the studies on updating IO matrices.

Figure 1: $WAPE$s by final demand categories



formation and total exports comprised, respectively, 46.0%, 12.1%, and 9.7% of the economy-wide product use. Thus, the same reasoning also hold for gross capital formation estimation as it is the third largest consumer of product uses.

As we have mentioned earlier, the majority of countries do not have the entire SUTs available at basic prices. Thus, in our second examination, we consider the case when the Use tables are at purchasers' prices, and Supply tables include also the valuation adjustment matrix that transforms total supply at basic prices into supply at purchasers' prices. The last in our Spanish case includes trade margins, transportation margins, and taxes less subsidies on products. Further, we now consider the more detailed structure of the final demand matrix. It consists of final consumption expenditure by households, final consumption expenditure by non-profit institutions serving households (NPISH), government expenditure, gross fixed capital formation, change in inventories and valuables, and total exports.

The results of the estimation are presented in Table 6. Note that now it is the analysis in Section 2.3 that is applied to the data in order to get the estimates of our SUT-RAS method. We cannot evaluate the Euro method, since it requires the necessary data to be in basic prices and estimates SUTs at basic prices only. Again based on the 2000 SUT and using the 2005 expanded total outputs and total inputs vectors (i.e., $\overline{\mathbf{x}}$ and $\overline{\mathbf{u}}$ in Section 2.3) we estimate the 2005 SUTs, and then compare

Table 6: Results of updating Spanish SUTs: purchasers' prices setting

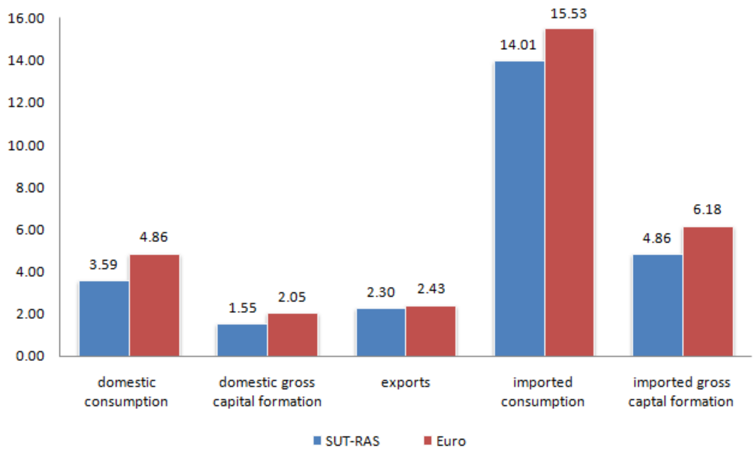
	MAPE	R.	WAPE	R.	SWAD	R.	$\hat{\psi}$	R.	RSQ	R.	CmR.
	Make matrix (72 × 72)										
EUKLEMS	19.44	1	2.52	1	0.004	1	0.024	1	0.9998	1	1
SUT-RAS	20.27	2	2.89	2	0.009	2	0.027	2	0.9998	1	2
	Supply table (Make matrix + imports + 3 valuation items, 72 × 76)										
EUKLEMS	19.85	1	6.44	2	0.009	1	0.495	2	0.9985	2	2
SUT-RAS	20.30	2	5.79	1	0.014	2	0.064	1	0.9986	1	1
	Total intermediate Use (72 × 72)										
EUKLEMS	41.23	2	19.99	2	0.071	1	0.194	2	0.9878	1	2
SUT-RAS	35.18	1	15.91	1	0.148	2	0.157	1	0.9842	2	1
	Total final demand (72 × 6)										
EUKLEMS	5989.00	2	33.75	2	0.196	2	0.361	2	0.8853	2	2
SUT-RAS	125.92	1	6.60	1	0.026	1	0.068	1	0.9977	1	1

them with the true benchmark SUTs of 2005.

Table 6 shows that in the estimation of only the 72×72 Make matrix, EUKLEMS outperforms the SUT-RAS method. Notice that the corresponding numbers for EUKLEMS are exactly those from Table 5, since the Make matrix in both considered SUTs settings is expressed in basic prices. But now if we also consider imports, trade margins, transport margins, and net taxes on products (hence the Supply table has 72 × 76 dimension), overall the SUT-RAS is performing better in predicting 2005 Supply table. The difference in estimation between the two approaches becomes more apparent in updating the Use tables. So, according to $WAPE$, SUT-RAS estimate of intermediate Use table is, on average, 15.9% 'in error', while that for the EUKLEMS estimate is 20.0%. In particular, again we can confirm that the EUKLEMS prediction of the final demand matrix is far worse than that of the SUT-RAS method (i.e., the corresponding $WAPE$s are 33.8% and 6.6%).

Next, we consider the SUT-RAS approach when Use tables are separated between domestic and imported uses (see Section 2.4), and make use of the same dataset. This distinction is made for SUTs at basic prices, thus we are also able to compare the SUT-RAS estimation with the Euro method separately for imported

Figure 2: $WAPE$s by final demand categories for domestic and imported uses



and domestic Use tables. The results of SUT-RAS vs. Euro estimation of the Make matrix and imports vector are, respectively, 3.07% vs. 8.45%, and 9.18% vs. 10.77% according to $WAPE$. The corresponding figures for the domestic and imported intermediate Use tables are, respectively, 21.39% vs. 26.86%, and 41.43% and 42.58%.[13] For the estimated domestic and imported final demand matrices the corresponding $WAPE$s of SUT-RAS vs. Euro estimation are, respectively, 7.45% vs. 9.34%, and 18.87% vs. 21.71%. Hence, in all cases the SUT-RAS method produces better estimates than those of the Euro method, and apparently there is much scope for improvement with the SUT-RAS estimation. The composition of $WAPE$s for the domestic and imported final demand categories is graphed in Figure 2. It clearly demonstrates our point made earlier on the severity of errors for final demand categories uses, since they comprise a large portion of the total product use, in particular, consumption of domestic and imported products. That is, for example, 1.28% extra error of the Euro method in comparison to the SUT-RAS approach is, in fact, significant because consumption of domestic products in Spain for 2005 accounted for 33.7% of all domestic uses. The same is true for the consumption of

[13] If we sum the derived domestic and imported intermediate Use tables and compare it to the true total intermediate Use table, we get the overall $WAPE$ of 16.15%, which is slightly higher than that presented in Table 5. The last number of 16.07% was the result of SUT-RASing when the Use tables were not distinguished between domestic and imported uses.

imported products, which overall comprised 22.6% of total imported uses, and the Euro method produces extra 1.52% error according to $WAPE$.

Often it is required to estimate taxes net of subsidies on products in the framework of SUTs at basic prices. This can be easily incorporated in the analysis of Section 2.1 by defining the expanded Supply and Use tables as follows (ignoring the subscripts 0's):

$$\overline{\mathbf{V}} = \begin{pmatrix} \mathbf{V}_b & \mathbf{0} \\ \mathbf{m}' & 0 \\ \mathbf{0}' & N \end{pmatrix} \quad \text{and} \quad \overline{\mathbf{U}} = \begin{pmatrix} \mathbf{U}_b & \mathbf{Y}_b \\ \mathbf{n}_1' & \mathbf{n}_2' \end{pmatrix},$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are, respectively, the vectors of net taxes of industries and final demand categories, and N is the overall sum of net taxes in the economy, i.e., $(\mathbf{n}_1' \, \mathbf{n}_2')\boldsymbol{\imath} = N$. Having access to the official dataset of Belgian SUTs at basic prices for the years of 1995-2004, we further examined a series of backward extrapolations using both the Euro and SUT-RAS methods.[14] Namely, using the 2004 SUTs as a benchmark, we projected SUTs for nine years of 1995-2003, and the obtained estimates were compared with the corresponding official SUTs using $WAPE$ measure. The results are reported in Table 7, which clearly suggest the superiority of SUT-RAS over the Euro method. In general, Euro method produces *extra average* error over the nine projection years ranging from 1.99% to 3.67%, and makes the largest errors in the estimation of Make matrices. The maximum and minimum individual errors are, respectively, 5.91% and 0.79% according to $WAPE$. Note that these differences are in fact quite large, because the $WAPE$ indicator, in quantifying the deviations of the estimated matrices from the actual ones, takes into account the relative weight of each individual element in the overall sum of all true elements of the corresponding matrices.

[14] This exercise was carried out together with José Manuel Rueda-Cantuche of the Joint Research Centre's Institute for Prospective and Technological Studies (IPTS) of the European Commission, who had access to the data. We are grateful to him for his collaboration in implementing the last mentioned test.

Table 7: *WAPE*s for Belgian SUTs projections at basic prices

		Make (59 × 59)	Make+ imports (60 × 59)	Inter.+ final Use (59 × 65)	Int.+final Use+net taxes (60 × 65)	Overall SUTs (60 × 124)
2003	Euro	4.0745	5.4357	6.2893	6.2884	5.8699
	SUT-RAS	2.7734	3.9253	4.8233	4.7565	4.3486
	Difference	*1.3011*	*1.5104*	*1.4660*	*1.5319*	*1.5213*
2002	Euro	6.7016	7.6829	10.0956	10.0649	8.8959
	SUT-RAS	4.0373	5.0563	7.3441	7.2780	6.1876
	Difference	*2.6643*	*2.6266*	*2.7515*	*2.7869*	*2.7083*
2001	Euro	8.3094	9.5000	12.3687	12.2646	10.9068
	SUT-RAS	4.3545	6.0907	9.6065	9.4445	7.7973
	Difference	*3.9549*	*3.4093*	*2.7622*	*2.8201*	*3.1095*
2000	Euro	9.7952	10.6351	15.3664	15.0683	12.8929
	SUT-RAS	4.9489	7.4167	12.3291	12.0878	9.7957
	Difference	*4.8463*	*3.2184*	*3.0373*	*2.9805*	*3.0972*
1999	Euro	16.2534	15.7599	28.2013	27.609	21.8006
	SUT-RAS	13.5465	14.1535	27.412	26.7434	20.5719
	Difference	*2.7069*	*1.6064*	*0.7893*	*0.8656*	*1.2287*
1998	Euro	17.5953	17.7341	28.8371	28.222	23.0784
	SUT-RAS	13.7990	14.5412	27.2773	26.5648	20.6681
	Difference	*3.7963*	*3.1929*	*1.5598*	*1.6572*	*2.4103*
1997	Euro	16.8978	16.9228	26.6086	26.1243	21.6123
	SUT-RAS	13.2177	14.4897	25.1617	24.5879	19.6362
	Difference	*3.6801*	*2.4331*	*1.4469*	*1.5364*	*1.9761*
1996	Euro	17.3751	17.3855	27.4355	27.0023	22.2867
	SUT-RAS	13.2320	14.6944	25.6383	25.0896	19.9924
	Difference	*4.1431*	*2.6911*	*1.7972*	*1.9127*	*2.2943*
1995	Euro	19.4645	19.5556	29.3423	28.923	24.3271
	SUT-RAS	13.5530	15.8796	27.0321	26.5298	21.3045
	Difference	*5.9115*	*3.6760*	*2.3102*	*2.3932*	*3.0226*
	Average error	3.6672	2.7071	1.9912	2.0538	2.3743

Note: For all projections, 2004 Belgian SUTs at basic prices are taken as benchmark tables. There are 59 products and industries, 6 final demand categories, and (two vectors of) imports and net taxes. Difference is equal to the gap between *WAPE*s values of Euro minus SUT-RAS, and the average error is the mean of these gaps over the nine projections.

The final exercise we want to consider is how introducing extra accurate information into the SUT-RAS procedure affects its final estimates. It is possible to find examples in which using additional correct exogenous information leads the traditional RAS algorithm to produce poorer estimates. However, de Mesnard and Miller (2006) state that "[a]s a general rule, introduction of accurate exogenous information into RAS improves the resulting estimates, and counterexamples should probably not be taken too seriously" (p. 517). In what follows, we would like to see whether this result also holds for the SUT-RAS algorithm. Using the Spanish SUTs

Table 8: Added information in the SUT-RAS procedure

	MAPE	WAPE	SWAD	$\hat{\psi}$
	Make matrix			
SUT-RAS	20.777	3.025	0.0102	0.0287
SUT-RAS +imports	20.932	3.024	0.0099	0.0287
SUT-RAS +imports+exports	21.307	3.070	0.0101	0.0292
	Total intermediate Use			
SUT-RAS	36.271	16.074	0.1312	0.1582
SUT-RAS +imports	35.919	15.940	0.1320	0.1569
SUT-RAS +imports+exports	36.050	15.593	0.1276	0.1534
	Total reduced final demand			
SUT-RAS	111.096	7.221	0.0283	0.0732
SUT-RAS +imports	109.190	6.754	0.0283	0.0685
SUT-RAS +imports+exports	157.315	5.778	0.0224	0.0592
	Make + intermediate and final Use			
SUT-RAS	30.209	7.460	0.0230	0.0735
SUT-RAS +imports	30.074	7.292	0.0229	0.0719
SUT-RAS +imports+exports	30.441	7.009	0.0212	0.0690

at basic prices, we consider two cases when the accurate information for imports and exports by product were used exogenously for the projection of 2005 SUTs. We consider these cases because in reality international trade statistics provide an alternative source for time series of exports and imports. The implementation of the SUT-RAS procedure here is based on Section 2.2, whose results are given in Table 8. The results show that, indeed, there are a few cases when adding true additional information produces poorer estimates. For example, having both exports and imports exogenous results in slightly poorer estimate of the Make matrix according to the $MAPE$ and $WAPE$ goodness-of-fit measures (i.e., they increase from 20.78% to 21.31% and from 3.03% to 3.07%, respectively). The same also holds for the estimated Use tables according to $MAPE$ indicator, which as we know is not a good measure anyways. However, in general, we find that adding extra true exogenous information produces better projections. So, in the overall evaluations of the Make matrix together with total intermediate and final Use matrices, we observe that all goodness-of-fit measures, except $MAPE$, constantly decrease with added true exogenous information. This also confirms the viewpoint of de Mesnard and Miller (2006) stated above for the case of our SUT-RAS algorithm in the example of Spain. Therefore, one can conclude that if for the projection years there exists more information than the minimum required exogenous data for the SUT-RAS approach, it is better to make use of them as well.

4 Conclusion

In this paper we have proposed a new method of estimation of Supply and Use tables (SUTs), which is labeled as the SUT-RAS procedure. The characterizing features of the SUT-RAS method are as follows:

- It does not require the availability of the vector of total outputs by product for the projection year(s), which is instead endogenously derived;
- It *jointly* estimates the Supply and Use tables;
- Estimated SUTs are immediately consistent, and thus, unlike the Euro method (Eurostat 2008) and EUKLEMS approach (Timmer et al. 2005), no additional assumptions are needed in order to make SUTs consistent;
- The SUT-RAS procedure is biproportional and theory-based method;
- It is general enough to handle both basic and purchasers' price settings of SUTs;
- The SUT-RAS method is also appropriate for cases when intermediate and final Use tables are distinguished between domestic and imported uses;
- One can easily consider introduction of an extra accurate information into the SUT-RAS procedure, and
- Unlike the Euro method, the Supply and Use tables do not have to be square.

Our empirical assessment of the method for the Spanish and Belgian SUTs data confirmed that the SUT-RAS method is performing quite well, where we made a detailed comparison with the outcomes of the Euro and EUKLEMS methods. Thus, we conclude that the SUT-RAS method may be used for the estimation of SUTs, and is potentially preferable, because it is theory-based approach. The economic theory behind this approach is similar to the well-known (G)RAS method. That is, one estimates the new SUTs that are as close as possible to the benchmark SUTs, but they have to satisfy certain restrictions on the SUTs structure itself and on the available information of the projection year. For interpolation when two benchmarks for the beginning and ending year are available, we suggest to use a *varying* benchmarks scheme that gives higher (resp. lower) weight to closer (resp. further) benchmark SUTs for the projection years SUTs estimation. This procedure will more or less ensure that when structural change indeed happened during the interpolation period, the varying benchmarks take it into account. We should also mention that in the search procedure for the structure similarity, large elements are given a higher weight than small transactions. From a practical point of view this feature is desirable, because statisticians always try to estimate large entries of SUTs as accurate as possible, while they might give less attention to the small transactions accuracy.

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