Matrix Homothety and GLS-based Extension of RAS Method

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The paper presents a new GLS-based method for updating economic tables within homothetic paradigm for structural similarity of rectangular matrices. The notions of angular and homothetic measures for matrix similarity are introduced. Unified analytical solution of constrained minimization problem for the homothetic measure as an objective function is derived in matrix notation. Special attention is paid to sensitivity of the solution to small changes in marginal totals of the target matrix. A decomposing procedure for updating partitioned matrices with large dimensions is described.

Keywords: RAS multiplicative pattern, matrix homothety, angular and homothetic measures of matrix similarity, Lagrange multipliers, decomposing procedure of matrix updating

JEL Classification: C61; C67; C55

> *"It is impossible to consider all updating methods, because theoretically their number is infinite."*
> *Temurshoev et al. (2011, p. 92)*

1. The formulation of problem

Using mathematical methods for updating (adjusting, projection, balancing, regionalizing etc.) national accounts, supply and use tables, input–output tables, social accounting matrices generates a class of applied algebraic problems associated with estimation of changes in economic data sets. General problem for updating a two-dimensional array (i.e. rectangular or square matrix) can be formulated as follows.

Let $\mathbf{A}$ be an (initial) matrix of dimension $N\times M$ with row and column marginal totals $\mathbf{u}_\mathbf{A} = \mathbf{A}\mathbf{e}_M$, $\mathbf{v}'_\mathbf{A} = \mathbf{e}'_N\mathbf{A}$ and further let $\mathbf{u} \neq \mathbf{u}_\mathbf{A}$ and $\mathbf{v} \neq \mathbf{v}_\mathbf{A}$ be exogenous column vectors of dimension $N\times 1$ and $M\times 1$, respectively. The problem is to estimate a target matrix $\mathbf{X}$ of dimension $N\times M$ with possibly higher level of structural similarity (or closeness etc.) to initial matrix under $N+M$ restrictions

$$\mathbf{X}\mathbf{e}_M = \mathbf{u}, \qquad \mathbf{e}'_N\mathbf{X} = \mathbf{v}', \tag{1}$$

where $\mathbf{e}_N$ and $\mathbf{e}_M$ are $N\times 1$ and $M\times 1$ summation column vectors consisting of 1's. To make the problem's formulation well defined we need to impose the evident consistency condition $\mathbf{e}'_N\mathbf{u} = \mathbf{e}'_M\mathbf{v}$. It can be easily shown that any $N+M-1$ among $N+M$ constraints (1) are mutually independent in this case.

In the sequel, the initial matrix is assumed to be not too sparse: it does not have less than $N+M$ nonzero elements, does not include any rows or columns with unique nonzero element and does not contain any pairs of rows and columns with four nonzero elements in the intersections.

Therefore it is expedient to "clear" matrix **A** of undesirable features before applying some updating methods in practice.

2. The RAS multiplicative pattern

Essential notion of the well-known RAS method is a factorization of target matrix

$$\mathbf{X} = \mathbf{RAS} = \langle \mathbf{r} \rangle \mathbf{A} \langle \mathbf{s} \rangle = \hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}}\,, \tag{2}$$

where **r** and **s** are unknown $N\times1$ and $M\times1$ column vectors. Here angled bracketing around vector's symbol or putting a "hat" over it denote a diagonal matrix, with the vector on its diagonal and zeros elsewhere (see, for example, Miller and Blair, 2009, p. 697).

Replacing **X** in (1) by (2) gives the system of nonlinear equations

$$\hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}}\mathbf{e}_M = \hat{\mathbf{r}}\mathbf{A}\mathbf{s} = \mathbf{u}\,, \qquad \mathbf{e}'_N\hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}} = \mathbf{r}'\mathbf{A}\hat{\mathbf{s}} = \mathbf{v}'\,. \tag{3}$$

Solution of system (3) can not be uniquely determined but it defines the target matrix **X** as a unique because $(k\hat{\mathbf{r}})\mathbf{A}(\hat{\mathbf{s}}/k) = \hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}}$ for any nonzero scalar *k*.

Proper transformations of system (3) lead to following pair of iterative processes:

$$\mathbf{r}_{(i)} = \left\langle \mathbf{A}\left\langle \mathbf{A}'\mathbf{r}_{(i-1)} \right\rangle^{-1} \mathbf{v} \right\rangle^{-1} \mathbf{u}\,, \qquad \mathbf{s}_{(i)} = \left\langle \mathbf{A}'\mathbf{r}_{(i)} \right\rangle^{-1} \mathbf{v}\,, \qquad i = 1 \div I; \tag{4}$$

$$\mathbf{s}_{(j)} = \left\langle \mathbf{A}'\left\langle \mathbf{A}\mathbf{s}_{(j-1)} \right\rangle^{-1} \mathbf{u} \right\rangle^{-1} \mathbf{v}\,, \qquad \mathbf{r}_{(j)} = \left\langle \mathbf{A}\mathbf{s}_{(j)} \right\rangle^{-1} \mathbf{u}\,, \qquad j = 1 \div J, \tag{5}$$

where *i* and *j* are iteration numbers. Besides, the character " ÷ " between the lower and upper bounds of index's changing range means that the index sequentially runs all integer values in the specified range.

Structural similarity between target and initial matrices is provided in RAS method by (N+M)-parametrical multiplicative pattern

$$x_{nm} = r_n s_m a_{nm}\,, \quad n = 1 \div N, \quad m = 1 \div M. \tag{6}$$

The row and column marginal totals given for target matrix **X** prevent from unreasonable mutual increasing or decreasing of the elements in **r** and **s**. So the constraints (1) restrict the scattering of factors $r_n s_m$ around some constant level. Moreover, the multiplicative pattern (6) preserves zero elements of initial matrix **A** in the same positions inside **X** that seems to be a significant contribution to structural similarity between **A** and **X**.

3. Matrix homothety

Consider an auxiliary problem for updating the initial matrix **A** in a particular case of strict proportionality between row and column marginal totals $\mathbf{u} = k\mathbf{u}_{\mathbf{A}}$ and $\mathbf{v} = k\mathbf{v}_{\mathbf{A}}$ with the same multiplier *k*. It is easy to see that under starting condition $\mathbf{r}_{(0)} = \mathbf{e}_N$ or $\mathbf{s}_{(0)} = \mathbf{e}_M$ the RAS method iterative process (4) or (5) demonstrates one-step convergence to pair of vectors $\mathbf{r} = \mathbf{e}_N$, $\mathbf{s} = k\mathbf{e}_M$ or to $\mathbf{r} = k\mathbf{e}_N$, $\mathbf{s} = \mathbf{e}_M$, respectively. Hence $r_n s_m = k$ for any n and m, $n = 1 \div N$, $m = 1 \div M$. Thus, according to RAS logic the auxiliary problem solving leads to a matrix $\mathbf{X} = k\mathbf{A}$, i.e. a homothety of initial matrix $\mathbf{A}$ with the center at null matrix and homothetic ratio (or scale factor) k.

In most practical cases, however, marginal totals proportionality can not really be observed. Nevertheless, homothetic transformation of initial matrix allows to obtain a unique solution for weakened version of auxiliary problem with given sum of all target matrix elements $\Sigma_{\mathbf{X}} = \mathbf{e}'_N \mathbf{u} = \mathbf{e}'_M \mathbf{v}$ that is sometimes called the grand total (see Dagum and Cholette, 2006) to be distinguished from (row and column) marginal totals. By combining $\mathbf{e}'_N \mathbf{X} \mathbf{e}_M = \Sigma_{\mathbf{X}}$ with matrix homothety formula $\mathbf{X} = k\mathbf{A}$ and then solving the received equation for k we obtain $k^* = \Sigma_{\mathbf{X}} / \mathbf{e}'_N \mathbf{A} \mathbf{e}_M = \Sigma_{\mathbf{X}} / \Sigma_{\mathbf{A}}$ under not difficult condition that grand total for initial matrix be nonzero. Therefore, solution of weakened auxiliary problem is given by matrix

$$\mathbf{B} = k^* \mathbf{A} = \Sigma_{\mathbf{X}} / \Sigma_{\mathbf{A}} \cdot \mathbf{A} \tag{7}$$

with zero elements in the same positions as inside initial matrix. Excellent structural similarity of matrices **B** and **A** is deepened by the fact that RAS method iterations (4) and (5) are invariant by replacing **A** with **B**. Note also that matrix **B** do correspond to a variety of marginal totals **u** and **v**.

If matrix homothety (7) is to be used in RAS method instead of initial matrix, then multiplicative pattern (6) can be rewritten in matrix notation as $\mathbf{X} = (\mathbf{r}\mathbf{s}') \circ \mathbf{B}$, where the character “ $\circ$ ” denotes the Hadamard’s product for two matrices of the same dimensions. Matrix $\mathbf{r}\mathbf{s}'$ does not contain zero elements and hence matrices **A**, **B**, **X** all have the identical location of 0’s.

Rather natural way to extend (*N*+*M*)-parametrical multiplicative pattern is to replace factors $r_n s_m$ with more common coefficients q_{nm} and to introduce (*NM*)-parametrical model

$$\mathbf{X} = \mathbf{Q} \circ \mathbf{B}, \tag{8}$$

where **Q** is $N \times M$ matrix of coefficients q_{nm}. It is important to emphasize that model (8) can be identified for any known matrices **A** and **X** with the same location of zero elements regardless of the method used for the estimation of the target matrix **X**. Thus, one can use (7) to calculate **B** and then let

$$q_{nm} = \begin{cases} x_{nm} / b_{nm}, \text{ если } b_{nm} \neq 0; \\ c, \qquad \text{ если } b_{nm} = 0, \end{cases} \qquad n = 1 \div N, \quad m = 1 \div M, \tag{9}$$

where c is positive constant chosen in advance (for example, $c = 1$). Such an approach creates certain possibility for constructing operational measure of structural similarity between target matrix and matrices from homothetic family $k\mathbf{A}$.

Applying vectorization operator *vec* (see Magnus and Neudecker, 2007), which transforms a matrix into a vector by stacking the columns of the matrix one underneath the other, to each matrix in (8) gives matrix model

$$\mathbf{x} = \hat{\mathbf{b}}\mathbf{q}\,, \tag{10}$$

where $\mathbf{x} = vec\,\mathbf{X}$, $\mathbf{b} = vec\,\mathbf{B}$ and $\mathbf{q} = vec\,\mathbf{Q}$ are column vectors with dimension $NM\times 1$. Note that *vec* operator is directly invertible, and that diagonal matrix $\hat{\mathbf{b}}$ is singular whenever **A** contains at least one zero element.

4. Angular measure for matrix similarity

Suppose **X** is the target matrix that was calculated from the initial matrix **A** by a certain method or procedure. Recall that $\mathbf{b} = k^*\mathbf{a} = k^* vec\,\mathbf{A}$, where the constant value k^* is defined by (7), so the grand totals for **B** and **X** coincide.

In geometric representation the data available is reflected by a homothetic ray $k\mathbf{b}$ at $k \ge 0$ with the collinear vectors **a** and **b** lying on this ray and by the target vector $\mathbf{x} = vec\,\mathbf{X}$, which forms an acute angle between itself and homothetic ray. Mutual location of **a**, **b** and **x** in *NM*-dimensional Euclidean space is illustrated by Figure 1.

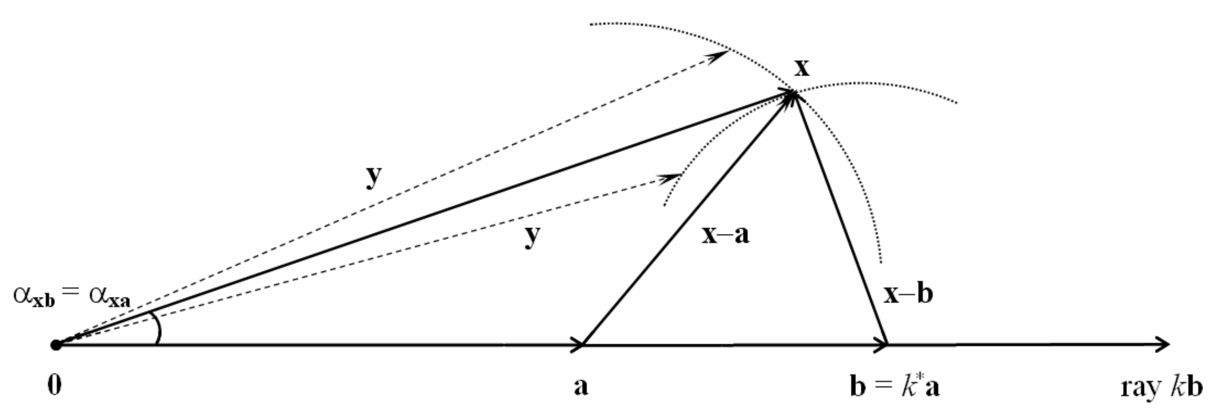

Figure 1. Vectorized representations of matrices **A**, **B** and **X** in initial coordinates

Natural measure for structural similarity between target matrix **X** and matrices from homothetic family $k\mathbf{B}$ can be defined as a value of the angle between vector **x** and homothetic ray $k\mathbf{b}$. If $(\mathbf{y}, \mathbf{z}) = \mathbf{y}'\mathbf{z} = \mathbf{z}'\mathbf{y}$ is inner product of vectors **y** and **z** in *NM*-dimensional Euclidean space, then angle $\alpha_{\mathbf{xb}}$ between target vector **x** and **b** is determined (in radians) by well-known formula

$$\alpha_{\mathbf{xb}} = \alpha_{\mathbf{xa}} = \arccos\left[\frac{(\mathbf{x},\mathbf{b})}{|\mathbf{x}|\cdot|\mathbf{b}|}\right] = \arccos\left[\frac{\mathbf{b}'\mathbf{x}}{(\mathbf{x}'\mathbf{x})^{1/2}\cdot(\mathbf{b}'\mathbf{b})^{1/2}}\right], \tag{11}$$

where $|\mathbf{x}| = (\mathbf{x}'\mathbf{x})^{1/2}$ is a length of vector **x**.

Angular measure (11) does allow to evaluate the degree of structural similarity between target matrix **X** and matrices from homothetic family $k\mathbf{B}$ by a unique scalar in reverse order of ranking. Unfortunately, angular measure is not quite operational, since it is expressed by rather cumbersome formula not suitable for further analytical computations. As a result, it seems to be more useful to create less complex metric measures of matrix similarity for operational applying in various algorithmic schemes.

At first let us study possibilities of using vectors **x**–**a** and **x**–**b** for estimating the degree of structural similarity between **X** and **A** or **B**. These measuring vectors are not orthogonal to homothetic ray, because in general $\mathbf{a'x} \neq \mathbf{a'a}$ and $\mathbf{b'x} \neq \mathbf{b'b}$. It can be shown that for a given vector's length $|\mathbf{x}-\mathbf{a}|$ or $|\mathbf{x}-\mathbf{b}|$ vector $\mathbf{y}=\mathbf{x}$ is not always a unique vector satisfying the equidistant condition $|\mathbf{y}-\mathbf{a}|=|\mathbf{x}-\mathbf{a}|$ or $|\mathbf{y}-\mathbf{b}|=|\mathbf{x}-\mathbf{b}|$ under restrictions (1). So there exists several (at least more than one) vectors **y**–**a** or **y**–**b** connecting the point **a** or **b** to spherical surface $|\mathbf{y}-\mathbf{a}|=|\mathbf{x}-\mathbf{a}|$ or $|\mathbf{y}-\mathbf{b}|=|\mathbf{x}-\mathbf{b}|$. In Figure 1 these spherical surfaces are represented by dotted arcs of two circles with their centers at points **a** and **b** and with radiuses of $|\mathbf{x}-\mathbf{a}|$ and $|\mathbf{x}-\mathbf{b}|$.

Thus, one can not establish more or less exact interrelation between vectors **x**–**a**, **x**–**b** and angle (11). It is clear from geometry (see Figure 1) that measuring vector should be chosen within orthogonal complement to the homothetic ray. Nevertheless, vector **x**–**b** demonstrates zero sum of its components in contrast to **x**–**a** (since $\mathbf{e}'_{NM}\mathbf{b}=\mathbf{e}'_{NM}\mathbf{x}\neq\mathbf{e}'_{NM}\mathbf{a}$) and therefore has more clear statistical interpretation. From the parameter estimation theory viewpoint using **x**–**b** as a measuring vector instead of **x**–**a** corresponds to refusal of the biased estimator in favor of unbiased one.

Besides, for any given $\Sigma_{\mathbf{X}}=\mathbf{e}'_N\mathbf{u}=\mathbf{e}'_M\mathbf{v}$ vector **x** tends to **b** as $\mathbf{u}\to k^*\mathbf{u}_{\mathbf{A}}$ and $\mathbf{v}\to k^*\mathbf{v}_{\mathbf{A}}$ simultaneously. So **x**–**b** can be considered as better measuring vector than **x**–**a** for applying in many algorithms of matrix updating. Unfortunately, such algorithms do not present a significant practical value because zeros' preservation in the elements of target matrix is not guaranteed.

5. Metric measure for matrix similarity

As noted above, orthogonal projecting target vector **x** onto homothetic ray $k\mathbf{b}$ may here become a useful subject of discussion. Orthogonal projection of **x** on $k\mathbf{b}$ is determined by coefficient $k^{\perp}=\mathbf{b'x}/\mathbf{b'b}$ from evident condition $\mathbf{b'}(\mathbf{x}-k^{\perp}\mathbf{b})=0$ and equals vector $k^{\perp}\mathbf{b}$. Figure 2 illustrates a case $k^{\perp}<1$, but in general the coefficient value may exceed 1. Component sum for vector

$$\mathbf{d}=\mathbf{x}-k^{\perp}\mathbf{b}=\mathbf{x}-\mathbf{b}\frac{\mathbf{b'x}}{\mathbf{b'b}}=\left(\mathbf{E}_{NM}-\frac{\mathbf{bb'}}{\mathbf{b'b}}\right)\mathbf{x} \tag{12}$$

equals $\mathbf{e}'_{NM}\mathbf{d}=(1-k^{\perp})\mathbf{e}'_{NM}\mathbf{b}$ and vanishes only at $k^{\perp}=1$, when **b** and projection of **x** coincide. It is easy to see that symmetric matrix in right-hand side of (12) is singular and idempotent.

Vector (12) represents the shortest path from the point **x** to homothetic ray $k\mathbf{b}$. Having nonzero sum of components it may be decomposed into a pair of additive items, namely $\bar{d}\mathbf{e}_{NM}$ and $\mathbf{t} = \mathbf{d} - \bar{d}\mathbf{e}_{NM}$, where $\bar{d}$ is an average value (scalar) of all elements in **d**, $\bar{d} = (1 - k^{\perp})\bar{b}$. These items are orthogonal to each other, because by definition of average value the vector **t** has zero sum of elements $\mathbf{e}'_{NM}\mathbf{t} = \mathbf{e}'_{NM}\mathbf{d} - \bar{d}\mathbf{e}'_{NM}\mathbf{e}_{NM} = 0$, so that $\bar{d}\mathbf{e}'_{NM}\mathbf{t} = 0$.

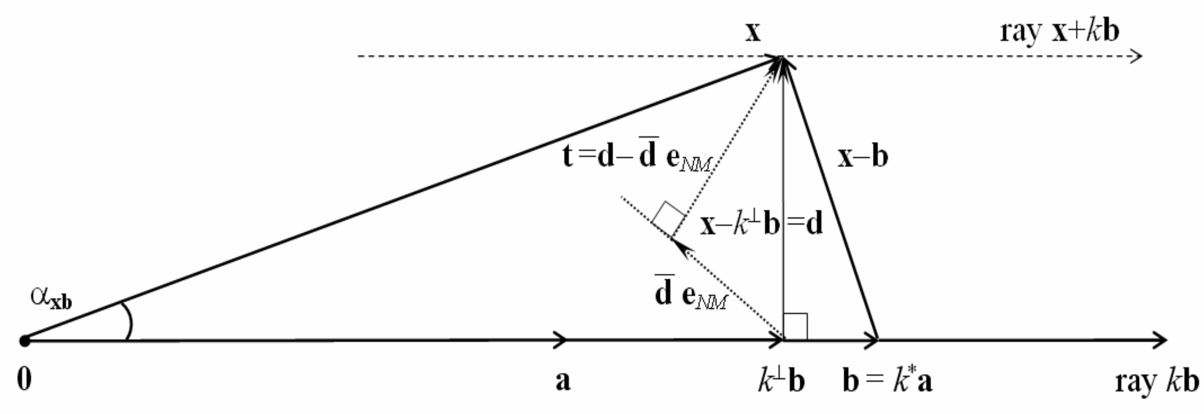

Figure 2. Projection of vector **x** on homothetic ray $k\mathbf{b}$ in initial coordinates

For a pair of non-coplanar right triangles with hypotenuses **d** and **x**–**b** and a common side **d** (see Figure 2), the non-strict double inequality $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{x} - k^{\perp}\mathbf{b}| \geq |\mathbf{x} - k^{\perp}\mathbf{b} - \bar{d}\mathbf{e}_{NM}|$ is satisfied. It becomes equality only at $k^{\perp} = 1$ when **x**–**b** is orthogonal to homothetic ray. So from the parameter estimation theory viewpoint using **d** as a measuring vector instead of **x**–**b** corresponds to refusal of the high-variance unbiased estimator in favor of biased one with less mean square error (for details, e.g., see Wackerly, Mendenhall and Scheaffer, 2008).

As a result, from evident equation $\mathbf{x} = \mathbf{b} + (\mathbf{x} - \mathbf{b}) = k^{\perp}\mathbf{b} + \mathbf{d}$ one can obtain a three-item decomposition $\mathbf{x} = k^{\perp}\mathbf{b} + \bar{d}\mathbf{e}_{NM} + \mathbf{t}$ in which first item is orthogonal to sum of the others and second item is orthogonal to the third one. In matrix notation we have $\mathbf{X} = k^{\perp}\mathbf{B} + \bar{d}\mathbf{e}_{NM}\mathbf{e}'_{NM} + \mathbf{T}$, where $\mathbf{e}_{NM}\mathbf{e}'_{NM}$ is the matrix of dimension $N \times M$ consisting of unit elements and **T** is a matrix with zero grand total.

Here, as well as earlier, the question of interrelation between vector **d** and angle (11) arises. Vector **d**, in contrast to **x**–**b**, does not have fixed origin, thus under the influence of feasible changes in **x** it may drift along the homothetic ray and, certainly, may rotate around this ray. It is easy to show that there are no other (except **x**) feasible target vectors on parallel ray $\mathbf{x} + k\mathbf{b}$ (see Figure 2). Indeed, from matrix equations $(\mathbf{X} + k\mathbf{B})\mathbf{e}_M = \mathbf{u}$ and $\mathbf{e}'_N(\mathbf{X} + k\mathbf{B}) = \mathbf{v}'$ we have $k\mathbf{u} = \mathbf{0}$ and $k\mathbf{v}' = \mathbf{0}$, so that $k = 0$.

As it follows from geometrical constructions in Figure 2, analytical interrelation between vector (12) and angle (11) is determined by

$$\mathbf{d}'\mathbf{d} = \mathbf{x}'\left(\mathbf{E}_{NM} - \frac{\mathbf{b}\mathbf{b}'}{\mathbf{b}'\mathbf{b}}\right)\mathbf{x} = \mathbf{x}'\mathbf{x}\cdot\sin^2\alpha_{\mathbf{xb}}, \qquad \sin^2\alpha_{\mathbf{xb}} = \frac{\mathbf{d}'\mathbf{d}}{\mathbf{x}'\mathbf{b}\mathbf{b}'\mathbf{x}/\mathbf{b}'\mathbf{b} + \mathbf{d}'\mathbf{d}}. \tag{13}$$

Function $f(z) = z/(c+z)$ with parameter c monotonically increases over interval $z\in[0, \infty]$, therefore it generates a bijective mapping between value sets of z and f(z) at $c = const$. Therefore, a comparison of target vectors is correct while the value $\mathbf{x}'\mathbf{b}\mathbf{b}'\mathbf{x}/\mathbf{b}'\mathbf{b}$ is being fixed, i.e. all testing target vectors must have the same orthogonal projection on homothetic ray $k\mathbf{b}$.

Thus, angular measure (11) and metric measure (12) are consistent only for any target vectors $\mathbf{x}$ and $\mathbf{y}$ satisfying the orthogonality condition $\mathbf{b}'(\mathbf{x}-\mathbf{y}) = 0$. Obvious rigidity of this condition may be considered as disadvantage of metric measuring the degree of structural similarity between target matrix and homothetic family $k\mathbf{A}$. Besides, it is important to note that multiplicative model (10), which preserves zero elements of initial matrix in the same positions inside target matrix, was not yet used in analytical computations and geometrical constructions implemented above.

6. Homothetic measure for matrix similarity

In accordance with multiplicative model (10) the operation of orthogonal projecting target vector $\mathbf{x}$ onto homothetic ray $k\mathbf{b}$ can be implemented not only in initial, but also in relative coordinates. An image of target vector $\mathbf{x}$ in system of relative coordinates is obviously coefficient vector $\mathbf{q}$, while images of $\mathbf{b}$ and $k\mathbf{b}$ are summation vector $\mathbf{e}_{NM}$ and a corresponding homothetic ray $k\mathbf{e}_{NM}$ at $k \geq 0$. Therefore, for transition to relative coordinates in right-hand side of (12) it is required to replace vector $\mathbf{b}$ by its relative equivalent $\mathbf{e}_{NM}$ and $\mathbf{x}$ by $\mathbf{q}$ as follows:

$$\boldsymbol{\delta} = \left(\mathbf{E}_{NM} - \frac{\mathbf{e}_{NM}\mathbf{e}'_{NM}}{\mathbf{e}'_{NM}\mathbf{e}_{NM}}\right)\mathbf{q} = \mathbf{q} - \bar{q}\,\mathbf{e}_{NM}, \tag{14}$$

where $\boldsymbol{\delta}$ is a difference between relative target vector $\mathbf{q}$ and its orthogonal projection on homothetic ray $k\mathbf{e}_{NM}$.

The inner product of $\boldsymbol{\delta}$ and $\mathbf{e}_{NM}$ is

$$\mathbf{e}'_{NM}\boldsymbol{\delta} = \mathbf{e}'_{NM}\mathbf{q} - \frac{\mathbf{e}'_{NM}\mathbf{e}_{NM}\mathbf{e}'_{NM}}{\mathbf{e}'_{NM}\mathbf{e}_{NM}}\mathbf{q} = \mathbf{e}'_{NM}\mathbf{q} - \mathbf{e}'_{NM}\mathbf{q} = 0,$$

so $\boldsymbol{\delta}$ is orthogonal to homothetic ray $k\mathbf{e}_{NM}$ and also has zero sum of elements. Vector $\boldsymbol{\delta}$ represents the shortest path from point $\mathbf{q}$ to homothetic ray $k\mathbf{e}_{NM}$. From (14), orthogonal projection of $\mathbf{q}$ on homothetic ray is determined by coefficient that equals average value $\bar{q}$. Figure 3 serves as an illustration for mutual location of all considered vectors in relative coordinates.

It is important to emphasize that vector $\mathbf{q}-\mathbf{e}_{NM}$, in contrast to its preimage in initial coordinates $\mathbf{x}-\mathbf{b}=\hat{\mathbf{b}}(\mathbf{q}-\mathbf{e}_{NM})$, does not demonstrates zero sum of elements, and vice versa, the component sum for δ equals zero in contrast to its preimage **d** (note, that $\mathbf{d} \neq \hat{\mathbf{b}}\boldsymbol{\delta}$). So it is clear now how to specify the scalar *c* in (9): to prevent artificial increasing of vector δ's length one must let *c* be the average ratio $x_{\mathrm{nm}}/b_{\mathrm{nm}}$ on a set of nonzero elements in **B**. From the parameter estimation theory viewpoint using δ as a measuring vector (instead of **d**) corresponds to the unbiased estimating with least variance.

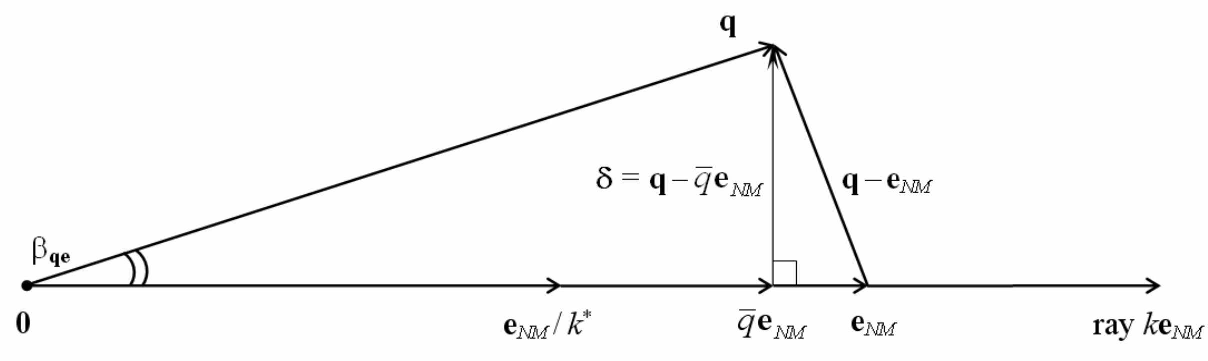

Figure 3. Projection of vector **q** on homothetic ray $k\mathbf{e}_{NM}$ in relative coordinates

Note that here the angle $\beta_{\mathbf{qe}}$ between **q** and homothetic ray $k\mathbf{e}_{NM}$, in general, does not equal the angle $\alpha_{\mathbf{xb}}$ between **x** and homothetic ray kb in Figure 2, since multiplying on diagonal matrix $\hat{\mathbf{b}}$ is not a conformal (or angle-preserving) mapping. So in addition to (11) one can introduce another angular measure of matrix similarity

$$\beta_{\mathbf{qe}} = \arccos\left[\frac{(\mathbf{q},\mathbf{e}_{NM})}{|\mathbf{q}|\cdot|\mathbf{e}_{NM}|}\right] = \arccos\left[\frac{\mathbf{e}'_{NM}\mathbf{q}}{(\mathbf{q}'\mathbf{q})^{1/2}\cdot(\mathbf{e}'_{NM}\mathbf{e}_{NM})^{1/2}}\right], \tag{15}$$

that is defined in system of relative coordinates of *NM*-dimensional Euclidean space. Compared with (11) this measure is more universal in nature because it does not depend on initial matrix elements and, for this reason, it can be applied for comparison of target vectors (and matrices) for various initial matrices with different marginal totals.

As it follows from geometrical constructions in Figure 3, analytical interrelation between vector (14) and angle (15) is determined by

$$\cos^2\beta_{\mathbf{qe}} = \frac{(\bar{q}\mathbf{e}_{NM})'\bar{q}\mathbf{e}_{NM}}{\mathbf{q}'\mathbf{q}} = \frac{\mathbf{q}'\mathbf{e}_{NM}\mathbf{e}'_{NM}\mathbf{q}}{\mathbf{e}'_{NM}\mathbf{e}_{NM}\cdot\mathbf{q}'\mathbf{q}}, \qquad \sin^2\beta_{\mathbf{qe}} = \frac{\boldsymbol{\delta}'\boldsymbol{\delta}}{\mathbf{q}'\mathbf{q}} = \frac{\boldsymbol{\delta}'\boldsymbol{\delta}}{\bar{q}^2\mathbf{e}'_{NM}\mathbf{e}_{NM}+\boldsymbol{\delta}'\boldsymbol{\delta}}. \tag{16}$$

By analogy with the analysis of expressions (13), one may conclude that angular measure (15) and homothetic measure (14) are consistent only for any relative target vectors **q** and **p** satisfying the average equality condition $\overline{q} = \overline{p}$, which is equivalent to the orthogonality condition $\mathbf{e}'_{NM}(\mathbf{q}-\mathbf{p}) = 0$. Under these conditions the homothetic measure (14) does allow to estimate the degree of structural similarity between target and initial matrices by a unique scalar in reverse order of ranking. Note that orthogonality condition $\mathbf{e}'_{NM}(\mathbf{q}-\mathbf{p}) = 0$ is, certainly, rather less rigid than $\mathbf{b}'(\mathbf{x}-\mathbf{y}) = 0$ but also highly restrictive.

As a conclusion, the angle between coefficient vector $\mathbf{q}$ and homothetic ray $k\mathbf{e}_{NM}$ at $k \geq 0$ can be considered as a universal measure of structural similarity between target and initial matrices. Main "technical" disadvantage of angular measure appears to be the complexity of expressions (16) along with arising difficulties of using (16) to construct competing (in particular, with the RAS method) algorithms of matrix updating. Based on orthogonal projecting operation, homothetic measure (14) is the simplified version of an angular measure with some shortcomings. Nevertheless, proposed homothetic measure demonstrates a row of useful properties and may become operational in various algorithmic schemes.

7. Optimization problems of matrix updating

To apply the results obtained above for constructing certain algorithms of matrix updating, one needs to rewrite left-hand sides of the equations (1) in vector notation. It is easy to see that in this context the $N \times NM$ matrix $\mathbf{G} = \mathbf{e}'_M \otimes \mathbf{E}_N$, which consists of M identity matrix $\mathbf{E}_N$ located horizontally, and the $M \times NM$ matrix $\mathbf{H} = \mathbf{E}_M \otimes \mathbf{e}'_{\mathrm{N}}$ – N-fold successive replication of each column from identity matrix $\mathbf{E}_M$ – are the proper substitutes of summation vectors $\mathbf{e}_M$ and $\mathbf{e}'_N$ respectively. Note that each column of $\mathbf{G}$ and $\mathbf{H}$ includes exactly one nonzero (unit) element such that $\mathbf{e}'_N\mathbf{G} = \mathbf{e}'_M\mathbf{H} = \mathbf{e}'_{NM}$. Thus, the system of equations (1) and multiplicative model (10) can be combined as follows:

$$\mathbf{X}\mathbf{e}_M = \mathbf{G}\mathbf{x} = \mathbf{G}\hat{\mathbf{b}}\mathbf{q} = \mathbf{u}, \qquad \mathbf{e}'_N\mathbf{X} = \mathbf{H}\mathbf{x} = \mathbf{H}\hat{\mathbf{b}}\mathbf{q} = \mathbf{v}. \tag{17}$$

Recall that under condition $\mathbf{e}'_N\mathbf{u} = \mathbf{e}'_M\mathbf{v}$ any N+M–1 among N+M constraints (17) are mutually independent.

First expression in (16) generates the following mathematical programming problem: to maximize the quadratic fractional objective function

$$\mathrm{F}_{\mathrm{cosine}}(\mathbf{q}) = \frac{1}{\mathbf{q}'\mathbf{q}}\mathbf{q}'\frac{\mathbf{e}_{NM}\mathbf{e}'_{NM}}{\mathbf{e}'_{NM}\mathbf{e}_{NM}}\mathbf{q} \tag{18}$$

subject to linear constraints (17). Symmetric idempotent matrix $\mathbf{e}_{NM}\mathbf{e}'_{NM}/\mathbf{e}'_{NM}\mathbf{e}_{NM}$ in (18) has unit eigenvalue with unit multiplicity and corresponding eigenvector $\mathbf{e}_{NM}$ and also has zero eigenvalue with multiplicity NM–1 and corresponding eigenvector $\mathbf{z}$ from the hyperplane $\mathbf{e}'_{NM}\mathbf{z}=0$ that is orthogonal to homothetic ray $k\mathbf{e}_{NM}$. So the Rayleigh quotient (18) takes its values between the least and the most eigenvalues, i.e. over interval [0, 1] (for details, e.g., see Wackerly, Mendenhall and Scheaffer, 2008). Further, the matrix in (18) is singular and as being product of two vectors has rank 1.

Second expression in (16) and formula (14) allow to invert optimizing direction in problem (18), (17) and to represent it in the following form: to minimize the Rayleigh quotient

$$\mathrm{F}_{\mathrm{sine}}(\mathbf{q}) = 1 - F(\mathbf{q}) = \frac{\boldsymbol{\delta}'\boldsymbol{\delta}}{\mathbf{q}'\mathbf{q}} = \frac{1}{\mathbf{q}'\mathbf{q}}\mathbf{q}'\left(\mathbf{E}_{NM} - \frac{\mathbf{e}_{NM}\mathbf{e}'_{NM}}{\mathbf{e}'_{NM}\mathbf{e}_{NM}}\right)\mathbf{q} \tag{19}$$

subject to linear constraints (17).

Symmetric idempotent matrix in round brackets has zero eigenvalue with unit multiplicity and corresponding eigenvector $\mathbf{e}_{NM}$ and also has unit eigenvalue with multiplicity NM–1 and corresponding eigenvector $\mathbf{z}$ from the hyperplane $\mathbf{e}'_{NM}\mathbf{z}=0$ that is orthogonal to homothetic ray. It can be shown that this matrix in (19) is singular and has rank NM–1.

It is difficult to solve equivalent constrained nonlinear problems (18), (17) and (19), (17) in analytical manner. Numerical optimization methods implemented in various well-known software packages (MATLAB etc.) provide solving of constrained nonlinear problems with not so high number of variables ($NM \sim 10^3$). However, in statistical practice of compiling national accounts, supply and use tables, input–output tables one can meet economic data arrays to be updated with total quantity of elements up to 10^4 and even to 10^5.

In this connection the developing of a matrix updating method on the base of proposed homothetic measure (14) seems to be useful. So let us formulate the following mathematical programming problem: to minimize the quadratic objective function

$$\mathrm{f}(\mathbf{q}) = \boldsymbol{\delta}'\boldsymbol{\delta} = \mathbf{q}'\left(\mathbf{E}_{NM} - \frac{\mathbf{e}_{NM}\mathbf{e}'_{NM}}{\mathbf{e}'_{NM}\mathbf{e}_{NM}}\right)\mathbf{q} = (\mathbf{q} - \bar{q}\mathbf{e}_{NM})'(\mathbf{q} - \bar{q}\mathbf{e}_{NM}) \tag{20}$$

subject to linear constraints (17). Objective function (20) expresses the length's square for the shortest path from the point $\mathbf{q}$ to homothetic ray $k\mathbf{e}_{NM}$ and vanishes only if target vector $\mathbf{q}$ and homothetic ray are collinear. From the mathematical statistics theory viewpoint the function (20) is proportional to a sample variance for the scattering of matrix $\mathbf{Q}$'s elements around their average value.

Above-mentioned singularity of symmetric idempotent matrix in round brackets serves as an obvious technical obstacle for the analytical solving of constrained minimization problems (20), (17), but this obstacle can be bypassed in special manner discussed below.

8. Constrained minimization of homothetic measure

Consider a parametric family of functions

$$\mathrm{f}(\mathbf{q};\gamma) = (\mathbf{q}-\gamma)'\mathbf{W}(\mathbf{q}-\gamma), \qquad (21)$$

where $\boldsymbol{\gamma}$ is an exogenous parameter vector of dimension $NM\times 1$, and $\mathbf{W} = \hat{\mathbf{w}}$ is a nonsingular diagonal matrix with the relative reliability (relative confidence) factors for elements of vector $\mathbf{q}$. In terms of GLS $\boldsymbol{\gamma}$ can be interpreted as a mean of random vector $\mathbf{q}$ and $\mathbf{W}$ – as a inverse covariance matrix for $\mathbf{q}$ in case of zero autocorrelations.

It is easy to detect a linkage between (20) and (21) by putting $\boldsymbol{\gamma} = c\mathbf{e}_{NM}$ in (21), where c is an unknown scalar, and comparing the obtained expression with right-hand side of (20). However, the matrix of quadratic form in (21) is nonsingular in contrast with the matrix in (20). It allows to get an analytical solution for minimization of objective function (21) subject to linear constraints (17) in general form $\mathbf{q}(\boldsymbol{\gamma}) = \boldsymbol{\gamma} + \tilde{\mathbf{q}} + \tilde{\mathbf{Q}}\boldsymbol{\gamma}$, where $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{Q}}$ are computable vector and matrix of proper dimensions. Then unknown scalar c can be determined from orthogonality condition $\boldsymbol{\gamma}'\mathbf{W}(\mathbf{q}-\boldsymbol{\gamma}) = 0$ that delivers non-full quadratic equation $\boldsymbol{\gamma}'\mathbf{W}(\tilde{\mathbf{q}}+\tilde{\mathbf{Q}}\boldsymbol{\gamma}) = c\mathbf{e}'_{NM}\mathbf{W}\tilde{\mathbf{q}} + c^2\mathbf{e}'_{NM}\mathbf{W}\tilde{\mathbf{Q}}\mathbf{e}_{NM} = 0$ with the unique nonzero root.

Objective function (21) appears to be similar to the one proposed by Harthoorn and van Dalen (1987). Nevertheless, there are at least two significant distinctions between them. First, Harthoorn and van Dalen have used metric measure $\mathbf{x}-\mathbf{a}$ and, second, they have not used the operation of orthogonal projecting.

The Lagrange function for constrained minimization problem (21), (17) with parameter vector $\boldsymbol{\gamma}$ is

$$\mathrm{L}_{\mathrm{f}}(\mathbf{q};\boldsymbol{\gamma},\boldsymbol{\lambda},\boldsymbol{\mu}) = (\mathbf{q}-\boldsymbol{\gamma})'\mathbf{W}(\mathbf{q}-\boldsymbol{\gamma}) - \boldsymbol{\lambda}'(\mathbf{G}\hat{\mathbf{b}}\mathbf{q}-\mathbf{u}) - \boldsymbol{\mu}'(\mathbf{H}\hat{\mathbf{b}}\mathbf{q}-\mathbf{v}), \qquad (22)$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are vectors of Lagrange multipliers with dimension $N\times 1$ and $M\times 1$. By setting the partial derivatives of (22) with respect to $\mathbf{q}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ equal to zero, we obtain the system of $NM+N+M$ linear equations

$$2\mathbf{W}(\mathbf{q}-\boldsymbol{\gamma}) - \hat{\mathbf{b}}\mathbf{G}'\boldsymbol{\lambda} - \hat{\mathbf{b}}\mathbf{H}'\boldsymbol{\mu} = \mathbf{0}, \qquad \mathbf{G}\hat{\mathbf{b}}\mathbf{q}-\mathbf{u} = \mathbf{0}, \qquad \mathbf{H}\hat{\mathbf{b}}\mathbf{q}-\mathbf{v} = \mathbf{0}.$$

While $\mathbf{W}$ is nonsingular matrix, the first equation can be resolved with respect to $\mathbf{q}$ as

$$\mathbf{q} = \boldsymbol{\gamma} + \frac{1}{2}\mathbf{W}^{-1}\hat{\mathbf{b}}\left(\mathbf{G}'\boldsymbol{\lambda} + \mathbf{H}'\boldsymbol{\mu}\right). \qquad (23)$$

The second and the third equations from system above and (23) can be combined into $N+M$ equations with $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ as unknown variables:

$$\mathbf{G}\hat{\mathbf{b}}\mathbf{W}^{-1}\hat{\mathbf{b}}\mathbf{G}'\boldsymbol{\lambda} + \mathbf{G}\hat{\mathbf{b}}\mathbf{W}^{-1}\hat{\mathbf{b}}\mathbf{H}'\boldsymbol{\mu} = \boldsymbol{\pi}_{11}\boldsymbol{\lambda} + \boldsymbol{\pi}_{12}\boldsymbol{\mu} = 2\left(\mathbf{u} - \mathbf{G}\hat{\mathbf{b}}\boldsymbol{\gamma}\right), \qquad (24)$$

$$\mathbf{H}\hat{\mathbf{b}}\mathbf{W}^{-1}\hat{\mathbf{b}}\mathbf{G}'\boldsymbol{\lambda}+\mathbf{H}\hat{\mathbf{b}}\mathbf{W}^{-1}\hat{\mathbf{b}}\mathbf{H}'\boldsymbol{\mu}=\boldsymbol{\pi}_{21}\boldsymbol{\lambda}+\boldsymbol{\pi}_{22}\boldsymbol{\mu}=2(\mathbf{v}-\mathbf{H}\hat{\mathbf{b}}\boldsymbol{\gamma})\,. \qquad (25)$$

For a nonsingular diagonal matrix $\mathbf{W}=\hat{\mathbf{w}}$ it can be shown that:

$\boldsymbol{\pi}_{12}$ is matrix with elements $(\boldsymbol{\pi}_{12})_{nm}=b_{nm}^2 / w_{n+N(m-1)}$, $n = 1\div N$, $m = 1\div M$;

$\boldsymbol{\pi}_{11}$ is diagonal matrix with elements $(\boldsymbol{\pi}_{11})_{nn}=\sum_{j=1}^{M} b_{nj}^2 / w_{n+N(j-1)}$, $n = 1\div N$;

$\boldsymbol{\pi}_{21}$ is matrix with elements $(\boldsymbol{\pi}_{21})_{mn}=(\boldsymbol{\pi}_{12})_{nm}=b_{nm}^2 / w_{n+N(m-1)}$, $n = 1\div N$, $m = 1\div M$;

$\boldsymbol{\pi}_{22}$ is diagonal matrix with elements $(\boldsymbol{\pi}_{22})_{mm}=\sum_{j=1}^{N} b_{jm}^2 / w_{j+N(m-1)}$, $m = 1\div M$.

It is easy to see that $\boldsymbol{\pi}_{11}=\langle\boldsymbol{\pi}_{12}\mathbf{e}_M\rangle$, $\boldsymbol{\pi}_{22}=\langle\boldsymbol{\pi}_{21}\mathbf{e}_N\rangle=\langle\boldsymbol{\pi}_{12}'\mathbf{e}_N\rangle$ and hence, $\boldsymbol{\pi}_{11}\mathbf{e}_N-\boldsymbol{\pi}_{12}\mathbf{e}_M=\mathbf{0}$, $\boldsymbol{\pi}_{21}\mathbf{e}_N-\boldsymbol{\pi}_{22}\mathbf{e}_M=\mathbf{0}$, i.e. the columns of matrix $\boldsymbol{\pi}$ formed by blocks $\boldsymbol{\pi}_{11},\boldsymbol{\pi}_{12},\boldsymbol{\pi}_{21},\boldsymbol{\pi}_{22}$ are linearly dependent. Thus, a general solution to corresponding homogeneous system (24), (25) is $\boldsymbol{\lambda}^{(0)}=k\mathbf{e}_N$, $\boldsymbol{\mu}^{(0)}=-k\mathbf{e}_M$ with any scalar constant k.

As a general solution to nonhomogeneous linear system equals the sum of a general solution to corresponding homogeneous system and any particular solution to nonhomogeneous system, let $\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(0)}+\boldsymbol{\lambda}^{(1)}$ and $\boldsymbol{\mu}=\boldsymbol{\mu}^{(0)}+\boldsymbol{\mu}^{(1)}$, where $\boldsymbol{\lambda}^{(1)}$, $\boldsymbol{\mu}^{(1)}$ is particular solution to system (24), (25). Recall that $\mathbf{e}_N'\mathbf{G}=\mathbf{e}_N'\mathbf{H}=\mathbf{e}_{NM}'$, so putting these formulas into round-bracketed expression in the right-hand side (23) gives

$$\mathbf{G}'\boldsymbol{\lambda}+\mathbf{H}'\boldsymbol{\mu}=\mathbf{G}'(k\mathbf{e}_N+\boldsymbol{\lambda}^{(1)})+\mathbf{H}'(-k\mathbf{e}_M+\boldsymbol{\mu}^{(1)})=k\mathbf{e}_{NM}-k\mathbf{e}_{NM}+\mathbf{G}'\boldsymbol{\lambda}^{(1)}+\mathbf{H}'\boldsymbol{\mu}^{(1)}=\mathbf{G}'\boldsymbol{\lambda}^{(1)}+\mathbf{H}'\boldsymbol{\mu}^{(1)}.$$

Therefore, to find any particular solution of system (24), (25) means to solve constrained minimization problem (21), (17). The particular solution can be found by numerical methods, and also in analytical form.

9. Numerical solution of system (24), (25)

Matrix $\boldsymbol{\pi}$ is singular but its square blocks $\boldsymbol{\pi}_{11}$ and $\boldsymbol{\pi}_{22}$ are not if matrix **A** (and **B**) does not have zero rows and columns. So one can resolve (24) with respect to $\boldsymbol{\lambda}$ and (25) with respect to $\boldsymbol{\mu}$ in forms

$$\boldsymbol{\lambda}=2\boldsymbol{\pi}_{11}^{-1}(\mathbf{u}-\mathbf{G}\hat{\mathbf{b}}\boldsymbol{\gamma})-\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\mu}\,, \qquad (26)$$

$$\boldsymbol{\mu}=2\boldsymbol{\pi}_{22}^{-1}(\mathbf{v}-\mathbf{H}\hat{\mathbf{b}}\boldsymbol{\gamma})-\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\lambda}\,. \qquad (27)$$

The crossing substitutions (27) in (26) and (26) in (27) give two equations as follows:

$$\boldsymbol{\lambda}=2\boldsymbol{\pi}_{11}^{-1}(\mathbf{u}-\mathbf{G}\hat{\mathbf{b}}\boldsymbol{\gamma})-2\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}(\mathbf{v}-\mathbf{H}\hat{\mathbf{b}}\boldsymbol{\gamma})+\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\lambda}=\mathbf{c}_N(\boldsymbol{\gamma})+\boldsymbol{\Pi}_N\boldsymbol{\lambda}\,, \qquad (28)$$

$$\boldsymbol{\mu}=-2\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\pi}_{11}^{-1}(\mathbf{u}-\mathbf{G}\hat{\mathbf{b}}\boldsymbol{\gamma})+2\boldsymbol{\pi}_{22}^{-1}(\mathbf{v}-\mathbf{H}\hat{\mathbf{b}}\boldsymbol{\gamma})+\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\mu}=\mathbf{c}_M(\boldsymbol{\gamma})+\boldsymbol{\Pi}_M\boldsymbol{\mu}\,. \qquad (29)$$

Further,

$$\mathbf{\Pi}_N\mathbf{e}_N=\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\mathbf{e}_N=\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{22}\mathbf{e}_M=\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\mathbf{e}_M=\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{11}\mathbf{e}_N=\mathbf{e}_N,\quad (\mathbf{E}_N-\mathbf{\Pi}_N)\mathbf{e}_N=\mathbf{e}_N-\mathbf{e}_N=0,$$

$$\mathbf{\Pi}_M\mathbf{e}_M=\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\mathbf{e}_M=\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{11}\mathbf{e}_N=\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\mathbf{e}_N=\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{22}\mathbf{e}_M=\mathbf{e}_M,\quad (\mathbf{E}_M-\mathbf{\Pi}_M)\mathbf{e}_M=\mathbf{e}_M-\mathbf{e}_M=0,$$

so that matrices Π_N, Π_M are stochastic, and matrices $\mathbf{E}_N-\Pi_N$, $\mathbf{E}_M-\Pi_M$ with linearly dependent columns are singular.

Nevertheless, equations (28), (29) may be useful for getting a numerical solution of system (24), (25) because they are represented in the form suitable for iterations by formulas

$$\boldsymbol{\lambda}_{(i)}=\mathbf{c}_N(\boldsymbol{\gamma})+\mathbf{\Pi}_N\boldsymbol{\lambda}_{(i-1)},\ i=1\div I,\quad \boldsymbol{\lambda}_{(0)}=\mathbf{c}_N(\boldsymbol{\gamma});\qquad \boldsymbol{\mu}_{(j)}=\mathbf{c}_M(\boldsymbol{\gamma})+\mathbf{\Pi}_M\boldsymbol{\mu}_{(j-1)},\ j=1\div J,\quad \boldsymbol{\mu}_{(0)}=\mathbf{c}_M(\boldsymbol{\gamma}),$$

where i and j are iteration numbers. By the successive substitutions as $I\to\infty$ and $J\to\infty$ we get

$$\boldsymbol{\lambda}=\sum_{i=0}^{\infty}\mathbf{\Pi}_N^i\cdot\mathbf{c}_N(\boldsymbol{\gamma}),\qquad \boldsymbol{\mu}=\sum_{j=0}^{\infty}\mathbf{\Pi}_M^j\cdot\mathbf{c}_M(\boldsymbol{\gamma}).\tag{30}$$

Since $\mathbf{\Pi}_N^i\mathbf{e}_N=\mathbf{e}_N$ and $\mathbf{\Pi}_M^j\mathbf{e}_M=\mathbf{e}_M$, the row marginal totals in partial sums of matrix power series in (30) increase unboundedly. So the convergence of considered iterative processes is questionable and needs to be studied.

From the theory of homogeneous Markov chains one may conclude that stochastic matrices $\mathbf{\Pi}_N^i$ and $\mathbf{\Pi}_M^j$ have properties as follows:

$$\lim_{i\to\infty}\mathbf{\Pi}_N^i=\lim_{i\to\infty}(\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21})^i=\mathbf{e}_N\mathbf{v}_N'=\mathbf{\Pi}_N^{\infty},\qquad \lim_{j\to\infty}\mathbf{\Pi}_M^j=\lim_{j\to\infty}(\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\pi}_{11}^{-1}\pi_{12})^j=\mathbf{e}_M\mathbf{v}_M'=\mathbf{\Pi}_M^{\infty},$$

where $\mathbf{v}_N'=\mathbf{v}_N'\mathbf{\Pi}_N$ and $\mathbf{v}_M'=\mathbf{v}_M'\mathbf{\Pi}_M$ are the left eigenvectors of Π_N and Π_M both corresponding to unit eigenvalues (for more details in transposed case of right eigenvector, see Bellman, 1960, pp. 256 – 258). Using these matrix algebra results it can be shown that $\mathbf{\Pi}_N^{\infty}\mathbf{c}_N(\boldsymbol{\gamma})$ and $\mathbf{\Pi}_M^{\infty}\mathbf{c}_M(\boldsymbol{\gamma})$ are null vectors with proper dimensions.

Let $\mathbf{v}_N'=\mathbf{z}_N'\boldsymbol{\pi}_{11}$, where $\mathbf{z}_N'$ is an unknown row vector. Putting this relation into the left eigenvector definition $\mathbf{v}_N'=\mathbf{v}_N'\mathbf{\Pi}_N$ gives homogeneous equation $(\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}-\boldsymbol{\pi}_{11})\mathbf{z}_N=0$. As $\mathbf{\Pi}_N\mathbf{e}_N=\mathbf{e}_N$, its solution is $\mathbf{z}_N=k\mathbf{e}_N$ with any scalar constant k. Indeed, from $\mathbf{v}_N'\boldsymbol{\pi}_{11}^{-1}=\mathbf{z}_N'=k\mathbf{e}_N'$ we have $\mathbf{v}_N'\mathbf{\Pi}_N=k\mathbf{e}_N'\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}=k\mathbf{e}_M'\boldsymbol{\pi}_{22}\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}=k\mathbf{e}_M'\boldsymbol{\pi}_{21}=(k\mathbf{e}_N')\boldsymbol{\pi}_{11}=\mathbf{v}_N'$.

Consider the product of matrix $\mathbf{\Pi}_N^{\infty}=\mathbf{e}_N\mathbf{v}_N'$ and vector $\mathbf{c}_N(\boldsymbol{\gamma})$ by regrouping relevant summands in right-hand side of (28) as

$$\mathbf{\Pi}_{\mathrm{N}}^{\infty}\mathbf{c}_N(\boldsymbol{\gamma})=2k\mathbf{e}_N\mathbf{e}_N'(\mathbf{u}-\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\mathbf{v})+2k\mathbf{e}_N\mathbf{e}_N'(\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\mathbf{H}-\mathbf{G})\hat{\mathbf{b}}\boldsymbol{\gamma}.$$

Since $\mathbf{e}_N'\boldsymbol{\pi}_{12}=\mathbf{e}_M'\boldsymbol{\pi}_{22}$, we get $\mathbf{e}_N'(\mathbf{u}-\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\mathbf{v})=\mathbf{e}_N'\mathbf{u}-\mathbf{e}_M'\mathbf{v}=0$ and

$$\mathbf{e}_N'(\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\mathbf{H}-\mathbf{G})\hat{\mathbf{b}}\boldsymbol{\gamma}=(\mathbf{e}_M'\mathbf{H}-\mathbf{e}_N'\mathbf{G})\hat{\mathbf{b}}\boldsymbol{\gamma}=(\mathbf{e}_{NM}'-\mathbf{e}_{NM}')\hat{\mathbf{b}}\boldsymbol{\gamma}=0.$$

Therefore, vector $\mathbf{\Pi}_N^{\infty}\mathbf{c}_N(\boldsymbol{\gamma})$ is a sum of two null vectors. The statement $\mathbf{\Pi}_M^{\infty}\mathbf{c}_M(\boldsymbol{\gamma}) = \mathbf{0}$ can be proved by analogy.

Thus, the convergence of considered iterative processes is provided by orthogonality between $\mathbf{c}_N(\boldsymbol{\gamma})$ and $\mathbf{v}_N$ and also between $\mathbf{c}_M(\boldsymbol{\gamma})$ and $\mathbf{v}_M$. In practice it is expedient to calculate the partial sums of matrix power series in (30) subject to terminal criteria resembling $\left|\mathbf{\Pi}_N^{I}\mathbf{c}_N(\boldsymbol{\gamma})\right| \le \varepsilon$ and $\left|\mathbf{\Pi}_M^{J}\mathbf{c}_M(\boldsymbol{\gamma})\right| \le \varepsilon$, where ε is a small positive value.

As a result, the numerical solving of system (24), (25) consists of finding vector $\boldsymbol{\lambda}$ in iterative fashion and calculating $\boldsymbol{\mu}$ from (27) or finding vector $\boldsymbol{\mu}$ in iterative fashion and calculating $\boldsymbol{\lambda}$ from (26). Note that these iteration-based solutions do not coincide among themselves, as well as RAS solutions (4) and (5). Computational complexity of such an algorithm is not high because it requires the diagonal matrix inverses only.

10. Analytical solution of system (24), (25)

As repeatedly noted above, any $N+M-1$ among $N+M$ constraints (17) are mutually independent while $\mathbf{e}'_N\mathbf{u} = \mathbf{e}'_M\mathbf{v}$. Therefore, without loss of generality any one of them can be eliminated. Let $\underline{\mathbf{G}}, \underline{\mathbf{H}}$ and $\underline{\mathbf{u}}, \underline{\mathbf{v}}$ be matrices and vectors obtaining from **G**, **H** and **u**, **v** by deleting a one row either out N-row expanded matrix $(\mathbf{G}\,|\,\mathbf{u})$ or out M-row expanded matrix $(\mathbf{H}\,|\,\mathbf{v})$. So reduced system of linear constraints (17) in new notation can be written as

$$\underline{\mathbf{G}}\hat{\mathbf{b}}\mathbf{q} = \underline{\mathbf{u}}, \qquad \underline{\mathbf{H}}\hat{\mathbf{b}}\mathbf{q} = \underline{\mathbf{v}}. \tag{31}$$

From the first-order conditions for constrained minimization problem (21), (31) with parameter vector $\boldsymbol{\gamma}$ by analogy with (23) – (25) we get

$$\mathbf{q} = \boldsymbol{\gamma} + \frac{1}{2}\mathbf{W}^{-1}\hat{\mathbf{b}}\left(\underline{\mathbf{G}}'\underline{\boldsymbol{\lambda}} + \underline{\mathbf{H}}'\underline{\boldsymbol{\mu}}\right), \tag{32}$$

$$\underline{\mathbf{G}}\hat{\mathbf{b}}\mathbf{W}^{-1}\hat{\mathbf{b}}\underline{\mathbf{G}}'\underline{\boldsymbol{\lambda}} + \underline{\mathbf{G}}\hat{\mathbf{b}}\mathbf{W}^{-1}\hat{\mathbf{b}}\underline{\mathbf{H}}'\underline{\boldsymbol{\mu}} = \mathbf{P}_{11}\underline{\boldsymbol{\lambda}} + \mathbf{P}_{12}\underline{\boldsymbol{\mu}} = 2(\underline{\mathbf{u}} - \underline{\mathbf{G}}\hat{\mathbf{b}}\boldsymbol{\gamma}), \tag{33}$$

$$\underline{\mathbf{H}}\hat{\mathbf{b}}\mathbf{W}^{-1}\hat{\mathbf{b}}\underline{\mathbf{G}}'\underline{\boldsymbol{\lambda}} + \underline{\mathbf{H}}\hat{\mathbf{b}}\mathbf{W}^{-1}\hat{\mathbf{b}}\underline{\mathbf{H}}'\underline{\boldsymbol{\mu}} = \mathbf{P}_{21}\underline{\boldsymbol{\lambda}} + \mathbf{P}_{22}\underline{\boldsymbol{\mu}} = 2(\underline{\mathbf{v}} - \underline{\mathbf{H}}\hat{\mathbf{b}}\boldsymbol{\gamma}), \tag{34}$$

where $\underline{\boldsymbol{\lambda}}$ and $\underline{\boldsymbol{\mu}}$ are reduced vectors of Lagrange multipliers with dimension $N\times 1$ or $(N-1)\times 1$ and $M\times 1$ or $(M-1)\times 1$, respectively.

Using well-known formulas for the inverse of a partitioned matrix (for details, see Miller and Blair, 2009, Appendix A) the solution of linear system (33), (34) with partitioned matrix **P** can be written in two analytical forms as

$$\begin{bmatrix} \underline{\boldsymbol{\lambda}} \\ \underline{\boldsymbol{\mu}} \end{bmatrix} = 2\begin{bmatrix} (\mathbf{P}_{11} - \mathbf{P}_{12}\mathbf{P}_{22}^{-1}\mathbf{P}_{21})^{-1} & -(\mathbf{P}_{11} - \mathbf{P}_{12}\mathbf{P}_{22}^{-1}\mathbf{P}_{21})^{-1}\mathbf{P}_{12}\mathbf{P}_{22}^{-1} \\ -\mathbf{P}_{22}^{-1}P_{21}(\mathbf{P}_{11} - \mathbf{P}_{12}\mathbf{P}_{22}^{-1}\mathbf{P}_{21})^{-1} & \mathbf{P}_{22}^{-1} + \mathbf{P}_{22}^{-1}\mathbf{P}_{21}(\mathbf{P}_{11} - \mathbf{P}_{12}\mathbf{P}_{22}^{-1}\mathbf{P}_{21})^{-1}\mathbf{P}_{12}\mathbf{P}_{22}^{-1} \end{bmatrix}\begin{bmatrix} \underline{\mathbf{u}} - \underline{\mathbf{G}}\hat{\mathbf{b}}\boldsymbol{\gamma} \\ \underline{\mathbf{v}} - \underline{\mathbf{H}}\hat{\mathbf{b}}\boldsymbol{\gamma} \end{bmatrix}$$

or as

$$\begin{bmatrix}\underline{\boldsymbol{\lambda}}\\ \underline{\boldsymbol{\mu}}\end{bmatrix}=2\begin{bmatrix}\mathbf{P}_{11}^{-1}+\mathbf{P}_{11}^{-1}\mathbf{P}_{12}(\mathbf{P}_{22}-\mathbf{P}_{21}\mathbf{P}_{11}^{-1}\mathbf{P}_{12})^{-1}\mathbf{P}_{21}\mathbf{P}_{11}^{-1} & -\mathbf{P}_{11}^{-1}\mathbf{P}_{12}(\mathbf{P}_{22}-\mathbf{P}_{21}\mathbf{P}_{11}^{-1}\mathbf{P}_{12})^{-1}\\ -(\mathbf{P}_{22}-\mathbf{P}_{21}\mathbf{P}_{11}^{-1}\mathbf{P}_{12})^{-1}\mathbf{P}_{21}\mathbf{P}_{11}^{-1} & (\mathbf{P}_{22}-\mathbf{P}_{21}\mathbf{P}_{11}^{-1}\mathbf{P}_{12})^{-1}\end{bmatrix}\begin{bmatrix}\underline{\mathbf{u}}-\underline{\mathbf{G}}\hat{\mathbf{b}}\boldsymbol{\gamma}\\ \underline{\mathbf{v}}-\underline{\mathbf{H}}\hat{\mathbf{b}}\boldsymbol{\gamma}\end{bmatrix}.$$

First notation is based on inverse of symmetric matrix $\mathbf{P}_{11}-\mathbf{P}_{12}\mathbf{P}_{22}^{-1}\mathbf{P}_{21}$ of order N or N–1, while the second one is founded on inverse of symmetric matrix $\mathbf{P}_{22}-\mathbf{P}_{21}\mathbf{P}_{11}^{-1}\mathbf{P}_{12}$ of order M or M–1. In practice, therefore, if $N < M$, it is better to delete a row out N-row expanded matrix $(\mathbf{G}\,|\,\mathbf{u})$ and to choose the first notation. Vice versa, if $N > M$, eliminating a row out M-row expanded matrix $(\mathbf{H}\,|\,\mathbf{v})$ with choice of the second notation is more preferable.

Thus, the reduced Lagrange multipliers are expressed in terms of the vectors in right-hand sides of (33) and (34) as their linear functions

$$\underline{\boldsymbol{\lambda}}=2\mathbf{D}_{11}(\underline{\mathbf{u}}-\underline{\mathbf{G}}\hat{\mathbf{b}}\boldsymbol{\gamma})+2\mathbf{D}_{12}(\mathbf{v}-\underline{\mathbf{H}}\hat{\mathbf{b}}\boldsymbol{\gamma}),\qquad \underline{\boldsymbol{\mu}}=2\mathbf{D}_{21}(\underline{\mathbf{u}}-\underline{\mathbf{G}}\hat{\mathbf{b}}\boldsymbol{\gamma})+2\mathbf{D}_{22}(\underline{\mathbf{v}}-\underline{\mathbf{H}}\hat{\mathbf{b}}\boldsymbol{\gamma}),\qquad (35)$$

where $\mathbf{D}_{11}$, $\mathbf{D}_{12}$, $\mathbf{D}_{21}$ and $\mathbf{D}_{22}$ are the corresponding blocks of the inverse matrix $\mathbf{D}=\mathbf{P}^{-1}$ (note that $\mathbf{D}_{12}=\mathbf{D}'_{21}$).

11. Analytical solution of constrained minimization problem (21), (31)

Transformation of the round-bracketed expression in right-hand side (32) using formulas (35) gives the analytical solution of constrained minimization problem (21), (31) with an unknown parameter vector $\boldsymbol{\gamma}$ as

$$\mathbf{q}=\boldsymbol{\gamma}+\mathbf{W}^{-1}\hat{\mathbf{b}}\left[(\underline{\mathbf{G}}'\mathbf{D}_{11}+\underline{\mathbf{H}}'\mathbf{D}_{21})(\underline{\mathbf{u}}-\underline{\mathbf{G}}\hat{\mathbf{b}}\boldsymbol{\gamma})+(\underline{\mathbf{G}}'\mathbf{D}_{12}+\underline{\mathbf{H}}'\mathbf{D}_{22})(\underline{\mathbf{v}}-\underline{\mathbf{H}}\hat{\mathbf{b}}\boldsymbol{\gamma})\right].\qquad (36)$$

To find vector $\boldsymbol{\gamma}$, as noted earlier, the orthogonality condition $\boldsymbol{\gamma}'\mathbf{W}(\mathbf{q}-\boldsymbol{\gamma})=0$ may be used. So let us write (36) in more compact form $\mathbf{q}-\boldsymbol{\gamma}=\mathbf{W}^{-1}\hat{\mathbf{b}}(\mathbf{z}-\mathbf{Y}\hat{\mathbf{b}}\boldsymbol{\gamma})$, where $\mathbf{z}$ is a vector of dimension $NM\times 1$, and $\mathbf{Y}$ is a symmetric matrix of order NM. It is easy to see that $\mathbf{z}$ and $\mathbf{Y}$ do not depend on $\boldsymbol{\gamma}$ and, besides, $\mathbf{Y}$ do not depend on $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$. By combining the orthogonality condition $\boldsymbol{\gamma}'\hat{\mathbf{b}}\mathbf{z}-\boldsymbol{\gamma}'\hat{\mathbf{b}}\mathbf{Y}\hat{\mathbf{b}}\boldsymbol{\gamma}=0$ and a uniparametric specification $\boldsymbol{\gamma}=c\mathbf{e}_{NM}$ we have non-full quadratic equation with the unique nonzero root $c^*=\mathbf{b}'\mathbf{z}/\mathbf{b}'\mathbf{Y}\mathbf{b}$.

Hence, $\boldsymbol{\gamma}^*=\mathbf{e}_{NM}\cdot\mathbf{b}'\mathbf{z}/\mathbf{b}'\mathbf{Y}\mathbf{b}$ and, furthermore,

$$\mathbf{q}^*=\mathbf{W}^{-1}\hat{\mathbf{b}}\mathbf{z}+\boldsymbol{\gamma}^*-\mathbf{W}^{-1}\hat{\mathbf{b}}\mathbf{Y}\hat{\mathbf{b}}\,\boldsymbol{\gamma}^*=\mathbf{W}^{-1}\hat{\mathbf{b}}\mathbf{z}+\frac{\mathbf{b}'\mathbf{z}}{\mathbf{b}'\mathbf{Y}\mathbf{b}}\mathbf{e}_{NM}-\frac{\mathbf{b}'\mathbf{z}}{\mathbf{b}'\mathbf{Y}\mathbf{b}}\mathbf{W}^{-1}\hat{\mathbf{b}}\mathbf{Y}\mathbf{b}.$$

The latter result seems to be very hopeful because the dependency of its right-hand side on huge matrix $\mathbf{Y}$ of order NM is represented here as one on $NM\times 1$ vector $\mathbf{y}=\mathbf{Y}\mathbf{b}$ with much less number of the elements.

As a result, the analytical solution of constrained minimization problem (21), (31) is given by

$$\mathbf{q}^* = \mathbf{W}^{-1}\hat{\mathbf{b}}\mathbf{z} + \frac{\mathbf{b}'\mathbf{z}}{\mathbf{b}'\mathbf{y}}(\mathbf{e}_{NM} - \mathbf{W}^{-1}\hat{\mathbf{b}}\mathbf{y}), \tag{37}$$

where $NM\times 1$ columns **z** and **y** are determined as

$$\mathbf{z} = (\underline{\mathbf{G}}'\mathbf{D}_{11} + \underline{\mathbf{H}}'\mathbf{D}_{21})\underline{\mathbf{u}} + (\underline{\mathbf{G}}'\mathbf{D}_{12} + \underline{\mathbf{H}}'\mathbf{D}_{22})\underline{\mathbf{v}}, \qquad \mathbf{y} = (\underline{\mathbf{G}}'\mathbf{D}_{11} + \underline{\mathbf{H}}'\mathbf{D}_{21})\underline{\mathbf{G}}\mathbf{b} + (\underline{\mathbf{G}}'\mathbf{D}_{12} + \underline{\mathbf{H}}'\mathbf{D}_{22})\underline{\mathbf{H}}\mathbf{b}. \tag{38}$$

Attendant calculations include an estimation of the inverse for one of the symmetric matrices $\mathbf{P}_{11} - \mathbf{P}_{12}\mathbf{P}_{22}^{-1}\mathbf{P}_{21}$ or $\mathbf{P}_{22} - \mathbf{P}_{21}\mathbf{P}_{11}^{-1}\mathbf{P}_{12}$ of order $\min\{N, M\} - 1$ and computing NM elements for each vector **z**, **y**, $\mathbf{q}^*$ at high sparsity of matrices $\underline{\mathbf{G}}, \underline{\mathbf{H}}$ and diagonality of matrices $\mathbf{W}, \hat{\mathbf{b}}$.

12. Lagrange multipliers and solution sensitivity analysis

Lagrange multipliers in the constrained minimum point of problem (21), (31) are determined by putting the vector $\boldsymbol{\gamma}^* = \mathbf{e}_{NM} \cdot \mathbf{b}'\mathbf{z}/\mathbf{b}'\mathbf{Y}\mathbf{b}$ into the right-hand sides of (35) as

$$\underline{\boldsymbol{\lambda}} = 2(\mathbf{D}_{11}\underline{\mathbf{u}} + \mathbf{D}_{12}\underline{\mathbf{v}}) - 2(\mathbf{D}_{11}\underline{\mathbf{G}} + \mathbf{D}_{12}\underline{\mathbf{H}})\frac{\mathbf{b}\mathbf{b}'\mathbf{z}}{\mathbf{b}'\mathbf{y}}, \qquad \underline{\boldsymbol{\mu}} = 2(\mathbf{D}_{21}\underline{\mathbf{u}} + \mathbf{D}_{22}\underline{\mathbf{v}}) - 2(\mathbf{D}_{21}\underline{\mathbf{G}} + \mathbf{D}_{22}\underline{\mathbf{H}})\frac{\mathbf{b}\mathbf{b}'\mathbf{z}}{\mathbf{b}'\mathbf{y}}. \tag{39}$$

It is easy to see that they are linear functions of the arguments $\underline{\mathbf{u}}, \underline{\mathbf{v}}$.

The analytical solution (37), (38) can be obtained by N+M equivalent ways in dependence on the choice of an excessive linear constraint to be deleted from (17) to get the reduced system of linear constraints (31). In each of these cases the united (i.e. concatenated) reduced vector of Lagrange multipliers $\underline{\boldsymbol{\rho}}$ has dimensions (N+M–1)×1, while Lagrange function (22) and the system (24), (25) involve the united multiplier vector $\boldsymbol{\rho}$ of dimensions NM×1. Relationship between $\boldsymbol{\rho}$ and $\underline{\boldsymbol{\rho}}$ is set by

$$\boldsymbol{\rho} = \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} = \overline{\mathbf{E}}^{(j)}_{N+M-1}\begin{bmatrix} \underline{\boldsymbol{\lambda}} \\ \underline{\boldsymbol{\mu}} \end{bmatrix} + k\begin{bmatrix} \mathbf{e}_N \\ -\mathbf{e}_M \end{bmatrix} = \overline{\mathbf{E}}^{(j)}_{N+M-1}\underline{\boldsymbol{\rho}} + k\begin{bmatrix} \mathbf{e}_N \\ -\mathbf{e}_M \end{bmatrix}, \tag{40}$$

where j is a number of the excessive constraint thrown out (17) in transition from **G**, **H, u**, **v** to $\underline{\mathbf{G}}, \underline{\mathbf{H}}$, $\underline{\mathbf{u}}, \underline{\mathbf{v}}$, j = 1÷(N+M), and $\overline{\mathbf{E}}^{(j)}_{N+M-1}$ is a rectangular (N+M)×(N+M–1) matrix, which is obtained by incorporating a zero row into identity matrix of order N+M–1 on position j. Relation (40) is constructed on the basis of the above-getting general solution to corresponding homogeneous system (24), (25) and reflects the obvious fact that the eliminated constraint corresponds to zero Lagrange multiplier. An equality in (40) is provided by a proper choice of the constant k for each pair of $\boldsymbol{\rho}$ and $\underline{\boldsymbol{\rho}}$.

From the theory of mathematical programming is known that Lagrange multipliers in the optimal solution of an extremal problem with equality constraints are the components of the objective function's gradient with respect to the right-hand sides of constraints – for example, see Magnus and Neudecker (2007), pp. 160, 161. However, in this case Lagrange multipliers $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(1)} + k\mathbf{e}_N$ and $\boldsymbol{\mu} = \boldsymbol{\mu}^{(1)} - k\mathbf{e}_M$ can not be uniquely identified. So their using for the sensitivity analysis of the objective function (21) at a conditional minimum point under impact of changes in the target marginal totals **u** and **v** is questionable and needs to be clarified.

Any disturbance of vector **u** through the frame of consistency condition $\Sigma_{\mathbf{X}} = \mathbf{e}'_N\mathbf{u} = \mathbf{e}'_M\mathbf{v}$ generates some compensating changes in the elements of **v**, and vice versa. In particular, the increasing of an element u_n by a certain value entails the adding to **v** any vector with this value as the algebraic sum of its elements. Clearly, some disturbances lead to the increased constrained minimum of objective function (21), while others contribute to decrease it.

As stated earlier, linear function $\mathbf{G}'\boldsymbol{\lambda} + \mathbf{H}'\boldsymbol{\mu}$ is invariant under any change of parameter k from general solution of corresponding homogeneous system (24), (25). Another similar invariant is matrix $\mathbf{L} = \boldsymbol{\lambda}\mathbf{e}'_M + \mathbf{e}_N\boldsymbol{\mu}' = \boldsymbol{\lambda}\mathbf{e}'_M + \mathbf{e}_N\boldsymbol{\mu}' + k\mathbf{e}_N\mathbf{e}'_M - k\mathbf{e}_N\mathbf{e}'_M = (\boldsymbol{\lambda} + k\mathbf{e}_N)\mathbf{e}'_M + \mathbf{e}_N(\boldsymbol{\mu}' - k\mathbf{e}'_M)$ with dimensions $N\times M$ and elements $l_{nm} = \lambda_n + \mu_m$, $n = 1 \div N$, $m = 1 \div M$. It is easy to see that each l_{nm} may be considered as a coefficient of the constrained minimum's sensitivity under impact of the simultaneous increasing u_n and v_m by the same small value ε. So if one replaces u_n and v_m by $u_n + \varepsilon$ and $v_m + \varepsilon$ respectively, then the increment of constrained minimum $\mathrm{f}^*(\mathbf{u},\mathbf{v}) = \mathrm{f}(\mathbf{q}^*;\boldsymbol{\gamma}^*)$ should equal $\mathrm{f}^*(\mathbf{u},\mathbf{v}\,|\,u_n + \varepsilon, v_m + \varepsilon) - \mathrm{f}^*(\mathbf{u},\mathbf{v}) = \varepsilon l_{nm}$. Thus, to decrease the minimum $\mathrm{f}^*(\mathbf{u},\mathbf{v})$ a small scalar ε is to be chosen with the sign reversed from the sign of l_{nm}.

In this context the larger absolute values of matrix **L**'s elements are of great interest. Let l_{nm} be an element with the largest absolute value of any one in matrix **L**. Then the best strategy for a local enhancing of constrained minimum $\mathrm{f}^*(\mathbf{u},\mathbf{v})$ is to disturb u_n and v_m by the same small value $-\varepsilon\cdot\mathrm{sgn}(l_{nm})$, where $\varepsilon > 0$ and $\mathrm{sgn}(\cdot)$ is a signum function.

Further, let $l_+ > 0$ and $l_- < 0$ be a maximal and a minimal elements of **L** respectively. Then the best two-component strategy for a local enhancing $\mathrm{f}^*(\mathbf{u},\mathbf{v})$ at the fixed grand total $\Sigma_{\mathbf{X}}$ is to decrease the elements of **u** and **v** corresponding to l_+ by –ε and to increase the elements of **u** and **v** corresponding to l_- by ε simultaneously.

It can be easily shown that, in general, total sensitivity effect is formulated as

$$\Delta_{\mathrm{f}}(\boldsymbol{\Delta}_{\mathbf{u}}, \boldsymbol{\Delta}_{\mathbf{v}}) = \mathrm{f}^*(\mathbf{u} + \boldsymbol{\Delta}_{\mathbf{u}}, \mathbf{v} + \boldsymbol{\Delta}_{\mathbf{v}}) - \mathrm{f}^*(\mathbf{u},\mathbf{v}) = \boldsymbol{\Delta}'_{\mathbf{u}}\boldsymbol{\lambda} + \boldsymbol{\mu}'\boldsymbol{\Delta}_{\mathbf{v}}, \quad (41)$$

where vectors $\boldsymbol{\Delta}_{\mathbf{u}}$ and $\boldsymbol{\Delta}_{\mathbf{v}}$ are exogenous disturbances for **u** and **v** respectively satisfying the consistency condition $\mathbf{e}'_N\boldsymbol{\Delta}_{\mathbf{u}} = \mathbf{e}'_M\boldsymbol{\Delta}_{\mathbf{v}} = \Sigma_{\Delta\mathbf{X}}$. To express the right-hand side of (41) in matrix **L** terms, it is necessary to consider two cases, namely, $\Sigma_{\Delta\mathbf{X}} = 0$ and $\Sigma_{\Delta\mathbf{X}} \neq 0$.

The disturbances $\boldsymbol{\Delta}_\mathbf{u}$ and $\boldsymbol{\Delta}_\mathbf{v}$ with zero sums $\mathbf{e}'_N\boldsymbol{\Delta}_\mathbf{u} = \mathbf{e}'_M\boldsymbol{\Delta}_\mathbf{v} = 0$ play an important role in statistical practice. They entail the redistributions of **u**'s and **v**'s components while the grand total $\Sigma_\mathbf{X} + \Sigma_{\Delta\mathbf{X}}$ is being fixed. It is easy to see from (41) that the total redistribution effect depends on the marginal totals of matrix **L** and is estimated by

$$\Delta_\mathrm{f}(\boldsymbol{\Delta}_\mathbf{u}, \boldsymbol{\Delta}_\mathbf{v} | \Sigma_{\Delta\mathbf{X}} = 0) = \boldsymbol{\Delta}'_\mathbf{u}(\boldsymbol{\lambda} + \overline{\mu}\mathbf{e}_N) + (\boldsymbol{\mu} + \overline{\lambda}\mathbf{e}_M)'\boldsymbol{\Delta}_\mathbf{v} = \frac{1}{M}\boldsymbol{\Delta}'_\mathbf{u}\mathbf{L}\mathbf{e}_M + \frac{1}{N}\mathbf{e}'_N\mathbf{L}\boldsymbol{\Delta}_\mathbf{v}. \qquad (42)$$

Here the first summand implies that in the total effect calculation an each value $(\boldsymbol{\Delta}_\mathbf{u})_n$ is uniformly distributed among *M* components of $\boldsymbol{\Delta}_\mathbf{v}$ and generates *M* elementary effects, sum of which is proportional to a row marginal total *n* for **L** divided by *M*. By analogy, the second summand in (42) implies that an each value $(\boldsymbol{\Delta}_\mathbf{v})_m$ is uniformly distributed among *N* components of $\boldsymbol{\Delta}_\mathbf{u}$ and generates *N* simple effects, sum of which is proportional to a column marginal total *m* for **L** divided by *N*.

On the other hand, the numerical function of disturbances $\boldsymbol{\Delta}'_\mathbf{u}\mathbf{L}\boldsymbol{\Delta}_\mathbf{v}$ can be transformed as follows:

$$\boldsymbol{\Delta}'_\mathbf{u}\mathbf{L}\boldsymbol{\Delta}_\mathbf{v} = \boldsymbol{\Delta}'_\mathbf{u}(\boldsymbol{\lambda}\mathbf{e}'_M + \mathbf{e}_N\boldsymbol{\mu}')\boldsymbol{\Delta}_\mathbf{v} = \boldsymbol{\Delta}'_\mathbf{u}\boldsymbol{\lambda}(\mathbf{e}'_M\boldsymbol{\Delta}_\mathbf{v}) + (\boldsymbol{\Delta}'_\mathbf{u}\mathbf{e}_N)\boldsymbol{\mu}'\boldsymbol{\Delta}_\mathbf{v} = \Sigma_{\Delta\mathbf{X}}(\boldsymbol{\Delta}'_\mathbf{u}\boldsymbol{\lambda} + \boldsymbol{\mu}'\boldsymbol{\Delta}_\mathbf{v}).$$

Hence, the total sensitivity effect may be represented as

$$\Delta_\mathrm{f}(\boldsymbol{\Delta}_\mathbf{u}, \boldsymbol{\Delta}_\mathbf{v} | \Sigma_{\Delta\mathbf{X}} \neq 0) = \boldsymbol{\Delta}'_\mathbf{u}\boldsymbol{\lambda} + \boldsymbol{\mu}'\boldsymbol{\Delta}_\mathbf{v} = \frac{1}{\Sigma_{\Delta\mathbf{X}}}\boldsymbol{\Delta}'_\mathbf{u}\mathbf{L}\boldsymbol{\Delta}_\mathbf{v}, \qquad (43)$$

where the disturbance grand total $\Sigma_{\Delta\mathbf{X}}$ is assumed to be nonzero. Recall, that in contrast to (43) formula (42) is well defined only for the redistribution case $\Sigma_{\Delta\mathbf{X}} = 0$.

13. Decomposing procedure for large-scale problems

The constrained minimization problem (21), (31) contains *NM* unknown variables, so its dimension rapidly grows while *N* and *M* increases. At a lack of resources for computing one can refuse the optimal solution in favor of an suboptimal one, which is obtained by handling the chain of subproblems with rather less dimensions.

Basic notion for decomposition of the problem (21), (31) consists of constructing a family of multiplicative models $x_{ij} = q_{ij}b_{ij}$, $i = 1 \div N$, $j = 1 \div M$, for a initial matrix of dimension $N \times M$ divided into rectangular submatrices (blocks). For convenience (but without loss of generality), let us assume that all the blocks have the same dimension $n \times m$, where *n* and *m* are divisors of *N* and *M*, respectively. Thus, the total number of blocks in a initial matrix partition equals $(NM)/(nm)$. The developed decomposing procedure for updating a matrix of large dimension is implementing in accordance with the following four-stage algorithm.

Stage 1. The elements of each block of initial matrix **A** is aggregating into an element of matrix $\mathbf{A}_{00}=(\mathbf{E}_{N/n}\otimes\mathbf{e}'_n)\mathbf{A}(\mathbf{E}_{M/m}\otimes\mathbf{e}_m)$ with dimensions $(N/n)\times(M/m)$. The marginal totals **u** and **v** are replacing by the aggregated vectors $\mathbf{u}_{00}=(\mathbf{E}_{N/n}\otimes\mathbf{e}'_n)\mathbf{u}$ of dimension $(N/n)\times 1$ and $\mathbf{v}_{00}=(\mathbf{E}_{M/m}\otimes\mathbf{e}'_m)\mathbf{v}$ of dimension $(M/m)\times 1$. Then the matrix $\mathbf{B}_{00}=k^*\mathbf{A}^{00}$ is calculating, where scalar multiplier k^* is determined from (7). Further, using an analytical solution (37) and (38), it is required to update the matrix $\mathbf{A}_{00}$ with marginal totals $\mathbf{u}_{00}$ and $\mathbf{v}_{00}$, i.e. to estimate matrix $\mathbf{X}_{00}$ with dimensions $(N/n)\times(M/m)$ that satisfies $\mathbf{X}_{00}\mathbf{e}_{M/m}=\mathbf{u}_{00}$ and $\mathbf{e}'_{N/n}\mathbf{X}_{00}=\mathbf{v}'_{00}$.

Stage 2. (For each $i=1\div N/n$.) Here we need to construct the chain of N/n matrices $\mathbf{A}^i_{\rightarrow}=(\mathbf{e}'_{N/n,i}\otimes\mathbf{E}_n)\mathbf{A}(\mathbf{E}_{M/m}\otimes\mathbf{e}_m)$ of dimension $n\times(M/m)$ and corresponding chain of N/n subvectors $\mathbf{u}^i_{\rightarrow}=(\mathbf{e}'_{N/n,i}\otimes\mathbf{E}_n)\mathbf{u}\subset\mathbf{u}$ of dimension $n\times 1$, where $\mathbf{e}_{N/n,i}$ is a column vector from the natural basis of N/n-dimensional space with 1 in a position i. It is easy to see that the columns of each matrix $\mathbf{A}^i_{\rightarrow}$ are formed by the row sums for the initial matrix blocks located along horizontal line i. Further, using formulas (7), (37) and (38), it is required to sequentially update N/n matrices $\mathbf{A}^i_{\rightarrow}$ with marginal totals $\mathbf{u}^i_{\rightarrow}$ and $\mathbf{v}^i_{\rightarrow}=\mathbf{X}'_{00}\mathbf{e}_{N/n,i}$, i.e. to estimate N/n matrices $\mathbf{X}^i_{\rightarrow}$ of dimension $n\times(M/m)$. Note that $\mathbf{e}'_{N/n,i}\mathbf{X}_{00}$ is a row i in the matrix $\mathbf{X}_{00}$ calculated at the stage 1, $i=1\div N/n$.

Stage 3. (For each $j=1\div M/m$.) At this stage we need to construct the chain of M/m matrices $\mathbf{A}^j_{\downarrow}=(\mathbf{E}_{N/n}\otimes\mathbf{e}'_n)\mathbf{A}(\mathbf{e}_{M/m,j}\otimes\mathbf{E}_m)$ of dimension $(N/n)\times m$ and corresponding chain of M/m subvectors $\mathbf{v}^j_{\downarrow}=(\mathbf{e}'_{M/m,j}\otimes\mathbf{E}_m)\mathbf{v}\subset\mathbf{v}$ of dimension $m\times 1$, where $\mathbf{e}_{M/m,j}$ is a column vector from the natural basis of M/m-dimensional space with 1 in a position j. It is easy to see that the rows of each matrix $\mathbf{A}^j_{\downarrow}$ are formed by the column sums for the initial matrix blocks located along vertical line j. Then, using formulas (7), (37) and (38), it is required to estimate M/m matrices $\mathbf{X}^j_{\downarrow}$ of dimension $(N/n)\times m$ by sequentially updating M/m matrices $\mathbf{A}^j_{\downarrow}$ with marginal totals $\mathbf{u}^j_{\downarrow}=\mathbf{X}_{00}\mathbf{e}_{M/m,j}$ and $\mathbf{v}^j_{\downarrow}$. Note that $\mathbf{X}_{00}\mathbf{e}_{M/m,j}$ is a column j in the matrix $\mathbf{X}_{00}$ calculated at the stage 1, $j=1\div M/m$.

Stage 4. At this final stage the initial matrix **A** is partitioning into $(NM)/(nm)$ rectangular blocks $\mathbf{A}_{ij}=(\mathbf{e}'_{N/n,i}\otimes\mathbf{E}_n)\mathbf{A}(\mathbf{e}_{M/m,j}\otimes\mathbf{E}_m)$ of dimension $n\times m$. (Further, for each pair i and j, $i=1\div N/n$, $j=1\div M/m$.) For a block $\mathbf{A}_{ij}$ the column vector $\mathbf{X}^i_{\rightarrow}\mathbf{e}_{M/m,j}$, namely the column j of $n\times(M/m)$-dimensional matrix $\mathbf{X}^i_{\rightarrow}$ calculated at the stage 2, is using as the row marginal total, and the row vector $\mathbf{e}'_{N/n,i}\mathbf{X}^j_{\downarrow}$, namely the row i of $(N/n)\times m$-dimensional matrix $\mathbf{X}^j_{\downarrow}$ calculated at the stage 3, is using as the column marginal total. By using formulas (7), (37) and (38), one needs to sequentially update (*NM*)/(*nm*) matrices $\mathbf{A}_{ij}$ and so to estimate (*NM*)/(*nm*) matrices $\mathbf{X}_{ij}$. The target matrix **X** of dimension *N*×*M* is formed by natural integration of all the blocks $\mathbf{X}_{ij}$, $i = 1 \div N/n$, $j = 1 \div M/m$. It is easy to see that target matrix obtained in such manner satisfies the marginal total conditions $\mathbf{X}\mathbf{e}_M = \mathbf{u}$ and $\mathbf{e}'_N\mathbf{X} = \mathbf{v}'$.

Thus, the decomposing procedure described above involves a solving of constrained minimization problem (21), (31) in multiple fashion for one matrix of dimension (*N/n*)×(*M/m*) at the stage 1, for *N/n* matrices of dimension *n*×(*M/m*) at the stage 2, for *M/m* matrices of dimension (*N/n*)×*m* at the stage 3 and for (*NM*)/(*nm*) blocks of dimension *n*×*m* at the stage 4. To increase the computational efficiency it is preferable to have all the updating matrices with approximately same dimensions. Therefore, it is advisable to choose integer algorithm's parameters *n* and *m* on the base of conditions $N/n \approx n$ and $M/m \approx m$ that imply $\mathrm{n} \approx \sqrt{\mathrm{N}}$ and $\mathrm{m} \approx \sqrt{\mathrm{M}}$.

14. Numerical examples and concluding remarks

Consider the Eurostat input–output data set given in "Box 14.2: RAS procedure" (see Eurostat, 2008, p. 452) for compiling several numerical examples. The 3×4-dimensional initial matrix **A** combines the entries in intersections of the columns "Agriculture", "Industry", "Services", "Final d." with the rows "Agriculture", "Industry", "Services" in "Table 1: Input-output data for year 0". Note, that all the elements of this matrix are nonzero. The row marginal total vector **u** of dimension 3×1 is the proper part of the column "Output" in "Table 2: Input-output data for year 1", and the column marginal total vector $\mathbf{v}'$ of dimension 1×4 involves the proper entries of the row "Total" in the near-mentioned data source.

Initial matrix **A**, marginal totals **u**, $\mathbf{v}'$, and also corresponding matrix **B** calculated by formula (7) with $k^* = 1.0290$ are presented in Table 1.

Table 1. Initial matrix **A** with nonzero elements

	A				$\mathbf{u_A}$	**u**	k^*		**B**				$\mathbf{u_B}$	**u**
	20.00	34.00	10.00	36.00	100.00	94.78	1.0290		20.58	34.99	10.29	37.05	102.90	94.78
	20.00	152.00	40.00	188.00	400.00	412.86			20.58	156.41	41.16	193.46	411.61	412.86
	10.00	72.00	20.00	98.00	200.00	212.68			10.29	74.09	20.58	100.84	205.81	212.68
$\mathbf{v}'_\mathbf{A}$	50.00	258.00	70.00	322.00	700.00			$\mathbf{v}'_\mathbf{B}$	51.45	265.49	72.03	331.35	720.32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32		$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

The first numerical example is to handle the data set in Table 1 by RAS method with iterative processes (4) or (5) and by analytical approach (37), (38) for solving the constrained minimization problem (21), (31) – briefly, by GLS method. The computation results grouped in

Table 1a seem to be very similar among themselves.

Table 1a. RAS and GLS results for updating data set in Table 1

RAS	**X**				$\mathbf{u_X}$	$\mathbf{u}$
	17.94	32.77	9.76	34.31	94.78	94.78
	19.36	158.08	42.12	193.30	412.86	412.86
	9.98	77.17	21.70	103.84	212.68	212.68
$\mathbf{v}'_{\mathbf{X}}$	47.28	268.02	73.58	331.44	720.32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

GLS	**X**				$\mathbf{u_X}$	$\mathbf{u}$
	18.35	32.41	10.03	33.99	94.78	94.78
	19.07	158.82	42.60	192.37	412.86	412.86
	9.86	76.79	20.95	105.08	212.68	212.68
$\mathbf{v}'_{\mathbf{X}}$	47.28	268.02	73.58	331.44	720.32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

Nevertheless, GLS method demonstrates the stable 5-percentage advantage over RAS method both in homothetic measure of matrix similarity (14) and in angular measure (15) as follows:

$$\left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 0{,}1847, \qquad \left|\boldsymbol{\delta}^{\mathrm{GLS}}\right| = 0{,}1756, \qquad \left|\boldsymbol{\delta}^{\mathrm{GLS}}\right| \big/ \left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 95{,}10\%;$$

$$\beta_{\mathbf{qe}}^{\mathrm{RAS}} = 3{,}1161°, \qquad \beta_{\mathbf{qe}}^{\mathrm{GLS}} = 2{,}9677°, \qquad \beta_{\mathbf{qe}}^{\mathrm{GLS}} \big/ \beta_{\mathbf{qe}}^{\mathrm{RAS}} = 95{,}24\%.$$

The next numerical example is assigned to test the methods' response to zero elements in the initial matrix. So let us disturb one element of our data set, say (3, 1), by putting it equal to zero for years 0 and 1. After recalculation of the marginal totals we get Table 2.

Table 2. Initial matrix **A** with zero element

	A				$\mathbf{u_A}$	$\mathbf{u}$
	20.00	34.00	10.00	36.00	100.00	94.78
	20.00	152.00	40.00	188.00	400.00	412.86
	0.00	72.00	20.00	98.00	190.00	202.88
$\mathbf{v}'_{\mathbf{A}}$	40.00	258.00	70.00	322.00	690.00	
$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52

k^*	**B**				$\mathbf{u_B}$	$\mathbf{u}$
1.0297	20.59	35.01	10.30	37.07	102.97	94.78
	20.59	156.52	41.19	193.59	411.90	412.86
	0.00	74.14	20.59	100.91	195.65	202.88
$\mathbf{v}'_{\mathbf{B}}$	41.19	265.67	72.08	331.58	710.52	
$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52

The results of computations by RAS and GLS methods collected in Table 2a, as earlier, seem to be very similar among themselves.

Table 2a. RAS and GLS results for updating data set in Table 2

RAS	**X**				$\mathbf{u_X}$	$\mathbf{u}$
	18.02	32.74	9.75	34.27	94.78	94.78
	19.46	158.05	42.11	193.25	412.86	412.86
	0.00	77.23	21.72	103.92	202.88	202.88
$\mathbf{v}'_{\mathbf{X}}$	37.48	268.02	73.58	331.44	710.52	
$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52

GLS	**X**				$\mathbf{u_X}$	$\mathbf{u}$
	18.36	32.40	10.04	33.98	94.78	94.78
	19.12	158.80	42.58	192.37	412.86	412.86
	0.00	76.82	20.96	105.10	202.88	202.88
$\mathbf{v}'_{\mathbf{X}}$	37.48	268.02	73.58	331.44	710.52	
$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52

To estimate the degree of matrix similarity more correctly in the presence of zero elements one can apply an adjustment procedure for the scalar c in (9) described above. But even in this case GLS method still keeps on the 5-percentage advantage over RAS method both in homothetic and angular measures as follows:

$$\left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 0{,}1826, \qquad \left|\boldsymbol{\delta}^{\mathrm{GLS}}\right| = 0{,}1736, \qquad \left|\boldsymbol{\delta}^{\mathrm{GLS}}\right| \big/ \left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 95{,}08\%;$$

$$\beta_{\mathbf{qe}}^{\mathrm{RAS}} = 3{,}0778°, \qquad \beta_{\mathbf{qe}}^{\mathrm{GLS}} = 2{,}9291°, \qquad \beta_{\mathbf{qe}}^{\mathrm{GLS}} \big/ \beta_{\mathbf{qe}}^{\mathrm{RAS}} = 95{,}17\%.$$

An advantage of GLS method observed here is not so impressive because of small number of "free" variables $NM-(N+M)$ in our numerical examples. However, if the dimensions of updating matrix tend to grow, then this advantage rapidly increases. At the dimensions more than 3×7 (7×3) and 4×5 (5×4) a total amount of free variables starts to exceed total number of RAS variables, so flexibility of GLS method substantially grows. Computational experiments with 15×20-dimensional matrices indicates that GLS method seems to be almost twice more effective than RAS in the sense of homothetic measure (14) and angular measure (15).

As it is well-known, "… RAS can only handle non-negative matrices, which limits its application to SUTs that often contain negative entries…" – see Temurshoev et al. (2011, p. 92). So the final numerical example is assigned to test the methods' response to negative elements in the initial matrix. Let us disturb three elements of our data set, say (1,3), (3, 1) and (3,3), by reversing their sign for years 0 and 1. After proper recalculation of the marginal totals we obtain Table 3.

Table 3. Initial matrix **A** with some negative elements

	A				$\mathbf{u}_{\mathbf{A}}$	$\mathbf{u}$	k^*		**B**				$\mathbf{u}_{\mathbf{B}}$	$\mathbf{u}$
	20.00	34.00	**-10.00**	36.00	80.00	74.50	1.0263		20.53	34.89	**-10.26**	36.95	82.10	74.50
	20.00	152.00	40.00	188.00	400.00	412.86			20.53	155.99	41.05	192.94	410.50	412.86
	-10.00	72.00	**-20.00**	98.00	140.00	148.92			**-10.26**	73.89	**-20.53**	100.57	143.68	148.92
$\mathbf{v}'_{\mathbf{A}}$	30.00	258.00	10.00	322.00	620.00			$\mathbf{v}'_{\mathbf{B}}$	30.79	264.77	10.26	330.46	636.28	
$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28		$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28

The results of computations by RAS and GLS method grouped in Table 3a now demonstrates wide differences in the elements of two target matrices calculated, especially in x_{13}, x_{23}, x_{24} and x_{33} .

Table 3a. RAS and GLS results for updating data set in Table 3

RAS	**X**				$\mathbf{u}_{\mathbf{X}}$	$\mathbf{u}$	**GLS**	**X**				$\mathbf{u}_{\mathbf{X}}$	$\mathbf{u}$
	17.09	31.06	**-6.18**	32.53	74.50	74.50		18.55	32.30	**-10.21**	33.87	74.50	74.50
	20.13	163.54	29.12	200.07	412.86	412.86		19.27	159.99	39.34	194.26	412.86	412.86
	-9.54	73.42	**-13.80**	98.84	148.92	148.92		**-10.13**	75.73	**-19.99**	103.31	148.92	148.92
$\mathbf{v}'_{\mathbf{X}}$	27.68	268.02	9.14	331.44	636.28		$\mathbf{v}'_{\mathbf{X}}$	27.68	268.02	9.14	331.44	636.28	
$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28	$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28

An advantage of GLS method in this case seems to be overwhelming. Indeed, the received estimates of homothetic and angular measures are

$$\left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 0,4906, \qquad \left|\boldsymbol{\delta}^{\mathrm{GLS}}\right| = 0,1479, \qquad \left|\boldsymbol{\delta}^{\mathrm{GLS}}\right| \big/ \left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 30,14\%;$$

$$\beta_{\mathbf{qe}}^{\mathrm{RAS}} = 9,1437°, \qquad \beta_{\mathbf{qe}}^{\mathrm{GLS}} = 2,5102°, \qquad \beta_{\mathbf{qe}}^{\mathrm{GLS}} \big/ \beta_{\mathbf{qe}}^{\mathrm{RAS}} = 27,45\%.$$

Thus, one can conclude that this method is especially effective under the complicated circumstances because of its immanent flexibility. In practice the proposed GLS-based method allows to generate much more compact the multiplicative model's factor distributions in comparison with RAS method.

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