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Productivity Gain and the Structural Propagation

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Abstract A new technology is assessed by the state of welfare in the economy ex-post of implementing that new technology to an economy-wide production system. We model the transition of technological structure (substitution of technology), due to the cost changes initiated by the introduction of a new technology. This structural propagation is quantified by using a system of cost functions compatible with the CES, Cobb-Douglas, and Leontief production functions whose parameters we estimate via two timely distant input-output accounts. The economy-wide welfare gain obtainable by introducing a new technology will be hence quantified via the technological structure ex-post of structural propagation. Welfare gain of productivity doubling in the port operation industry is studied as an example.

Keywords Technology assessment · Productivity · CES production function

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1 Introduction

Input-output analysis is now a standard tool for assessing new technology, whereby estimating the direct and indirect economic costs and benefits, through an observed technological structure of an economy-wide production. At the same time, we may argue that an input-output analysis is retrospective in the sense that a technology subject to an assessment has no feedback upon the structure of production that the assessment is based upon. In other words, input-output analysis is backward-looking, using previously observed structure of production (i.e., the input-output coefficient) for assessing the costs and benefits of marginal changes in the final demand that characterize the new technology in question.

In this study, we present a methodology that fully takes into account the feedback effect of introducing new technologies into the system economy-wide production. More specifically, we take the technological substitution into account. While it is known that technology will not substitute, (hence, technological structure will maintain) as far as the change in the final demand is concerned,[1] this will not apply in the case when any new technology is actually introduced in an industry; technology of any industry can substitute according to the cost/price change in commodities initiated by the introduction of a new technology in any industry.

The structural propagation analysis we outline in this paper is prospective (or, forward looking) in the sense that a new technology is assessed based on the projected technological structure ex-post of the technological substitution initiated by the introduction of that new technology. Upon modeling structural propagation we use a class of production function that is less restricted in regard to the elasticities of substitution. We estimate the elasticity of substitution for all industrial sectors using two timely distant input-output accounts, which enables us to model a relevant structural propagation, rather than using more restricted substitution patterns such as Leontief and Cobb-Douglas.[2]

More specifically, we estimate the parameters for the system of multi-factor multi-sector CES (constant elasticity of substitution) production functions, and therefore, model the technological substitution using compatible system of unit cost functions which enable us to handle the structural propagation in a recursive fashion. Sector-wise CES parameters, namely, the share parameters and the elasticity, will be measured using two timely distant input-output accounts. The elasticity is measured so as to minimize the potential discrepancies between the tow consistent share parameters while the share parameters are measured so as to meet the latest cost share accounts. CES production function also requires the sector-wise productivity to be estimated, while this is measured by way of the two input-output accounts.

We may apply structural propagation analysis for any given new technology to assess the prospective gains in the economy-wide costs and benefits. In this study we use exogenous productivity doubling (as a proxy for introducing new technology)

[1] This non-substitution theorem will hold under the conditions of constant returns to scale technology, one-to-one correspondences between commodity and industry, and the oneness of the number of primary inputs.

[2] Nishimura (2002) studied structural propagation in regard to the structural viability (i.e., Hawkins-Simon condition), based on a system of Cobb-Douglas technologies.

in the port operation industry, for the sake of experiment. We will use the estimated CES elasticity parameters while we also set them as unity and zero to compare the outcomes based on Cobb-Douglas and Leontief production functions.

The remainder of the paper is organized as follows. In the next section we first measure the sector-wise total factor productivity gain using input-output accounts. In so doing, we aggregate labor and capital inputs so that there be a single primary input besides intermediate inputs. Then, we measure the parameters for the multi-factor multi-sector CES production function using two timely distant input-output accounts and the measured sector-wise productivity gain. In section 3 we formulate the structural propagation under the system of multi-factor multi-sector CES production, and demonstrate structural propagations triggered by some exogenously given productivity alteration. Section 4 is reserved for concluding remarks.

2 Production Function

2.1 Productivity gain

We start with the production function of an industry (the index j is omitted):

$$y = zf(x_0, x_1, \ldots, x_n) = zf(\mathbf{x}), \tag{1}$$

where we denote the output of this production by y, and the ith input by x_i. Here, z denotes the absolute productivity, which reflects the technology level of the industry in question. Also, $f(\mathbf{x})$ is assumed to be homogeneous of degree one with respect to the inputs (i.e., constant returns to scale). Taking the log and time derivative, we have

$$\frac{\dot{y}}{y} = \frac{\dot{z}}{z} + \sum_{i=0}^{n} \left(\frac{\partial f(\mathbf{x})}{\partial x_i} \frac{x_i}{f(\mathbf{x})} \right) \frac{\dot{x}_i}{x_i}. \tag{2}$$

The term in parenthesis will be the cost share, which we denote by α_i. This will be true under the following monetary balance of constant returns to scale production:

$$py = pzf(\mathbf{x}) = \sum_{i=0}^{n} p_i x_i, \qquad pz\frac{\partial f(\mathbf{x})}{\partial x_i} = p_i$$

for which we may describe the cost share of input i as follows:

$$\frac{p_i x_i}{py} = \frac{\partial f(\mathbf{x})}{\partial x_i} \frac{x_i}{f(\mathbf{x})} = \alpha_i \tag{3}$$

Thus, (2) can be reduced as the following formula for productivity growth,

$$\Delta \ln z = \Delta \ln y - \sum_{i=0}^{n} \alpha_i \Delta \ln x_i, \tag{4}$$

where Δ indicates the observed differences between two periods. (4) can also be described by way of monetary output $Y = py$ and inputs $X_i = p_i x_i$ i.e.,

$$\Delta \ln z = (\Delta \ln Y - \Delta \ln p) - \sum_{i=0}^{n} \alpha_i (\Delta \ln X_i - \Delta \ln p_i) \tag{5}$$

Hence, productivity gain observed between two periods $t = 0$ and 1 for an industrial sector j i.e., $z_j^1/z_j^0 = \exp(\Delta \ln z_j)$ can be calculated by way of input-output accounts X_{ij} and Y_j, cost share accounts α_{ij}, and the deflator for all commodity prices $p_i^1/p_i^0 = \exp(\Delta \ln p_i)$, using (5). For subsequent study we estimated total factor productivity gain for 395 industrial sectors using the Japanese input-output tables (coefficients and transactions) and the official deflators for 2000–2005 (MIAC, 2009). Note that we aggregated fixed capital with labor inputs for simplicity so that there be only one primary factor ($i = 0$). We used two-period average of monetary input-output coefficients for the cost share accounts i.e., $\alpha_{ij} = (a_{ij}^0 + a_{ij}^1)/2$. Figure 1 illustrates the estimated values of productivity gain z_j^1/z_j^0 of sector j.[3]

[3] The highest value corresponds to personal computers and related appliances.

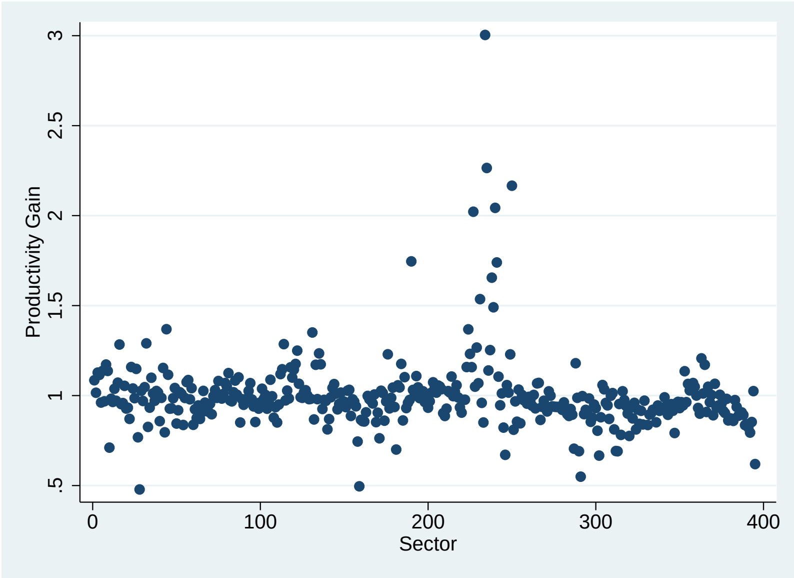

Fig. 1: Estimates of total factor productivity gain for various industrial sectors (z_j^1/z_j^0) based on the official input-output accounts and deflators, for 2000–2005, in Japan. Notable sectors with large numbers: Personal Computers (3.00); Electronic computing equipment (ex. PCs) (2.26); Ships (ex. steel ships) (2.17); Video recording and playback equipment (2.02); Turbines (1.74).

2.2 Multi-factor CES production function

Our purpose here is to estimate the following multi-factor CES production function of an industrial sector (index j omitted) such that,

$$y = zf(\mathbf{x}) = z\left(\delta_0^{\frac{1}{\sigma}} x_0^{\frac{\sigma-1}{\sigma}} + \delta_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \cdots + \delta_n^{\frac{1}{\sigma}} x_n^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \tag{6}$$

where we need the estimates for the share parameters ($\delta_i > 0, \sum_i \delta_i = 1$) and the elasticity of substitution $\sigma \geq 0$. Note that, while share parameters are dependent on the kind of inputs i, the elasticity of substitution is unique for each sector (Uzawa, 1962). As we take for granted that productivity gain is available via the previous discussion, we set the benchmark ($t = 0$) absolute productivity $z^0 = 1$ and the ex-post ($t = 1$) absolute productivity $z^1 = \exp(\Delta \ln z)$ in regard to (5).

The cost shares for the ith input under CES, while obtainable by applying (3) on (6), can be monitored for two periods $t = 0$ and 1, are displayed below:

$$\alpha_i^0 = \delta_i \left(z^0 p^0 / p_i^0\right)^{\sigma-1}, \quad \alpha_i^1 = \delta_i \left(z^1 p^1 / p_i^1\right)^{\sigma-1}. \tag{7}$$

The parameters δ_i and σ are assumed to be constant over time, but there is only a small chance that these identities are simultaneously true. So, we take the strategy to find the parameters that are most fitting to the two observations. We first rewrite (7) to describe the share parameter δ_i as a function of σ that is consistent with the observations for two periods. That is,

$$\delta_i(\sigma; t=0) \equiv \alpha_i^0 \left(z^0 p^0 / p_i^0\right)^{1-\sigma}, \quad \delta_i(\sigma; t=1) \equiv \alpha_i^1 \left(z^1 p^1 / p_i^1\right)^{1-\sigma}.$$

These parameters are constant *per se*, so we search for the σ that these two parameters are as close as possible. That is,

$$\sigma = \arg\max_{\sigma \geq 0} \text{Corr}\left(\boldsymbol{\delta}(\sigma; t=0), \boldsymbol{\delta}(\sigma; t=1)\right). \tag{8}$$

where Corr stands for the Pearson correlation coefficient. Here, we use Corr for assessing the similarities between two vectors. Note that cosine similarity, which is the uncentered version of Pearson correlation, or many other vector distance metrics can be used as a measure of vector similarity. Pearson correlation coefficient between two vectors $\mathbf{r} = (r_1, \cdots, r_n)$ and $\mathbf{s} = (s_1, \cdots, s_n)$ is defined as

$$\text{Corr}(\mathbf{r}, \mathbf{s}) = \frac{\sum_i (r_i - \bar{r})(s_i - \bar{s})}{\sqrt{\sum_i (r_i - \bar{r})^2} \sqrt{\sum_i (s_i - \bar{s})^2}}$$

where, $\bar{r} = \sum_i r_i / n$ and $\bar{s} = \sum_i s_i / n$.

Figure 2 shows the estimated values for 395 industrial sectors using the Japanese input–output tables for 2000 and 2005 (MIAC, 2009). For about 100 sectors the production functions were estimated to be Leontief ($\sigma = 0$), while the remaining sectors were evenly divided for being sub-Cobb–Douglas ($\sigma < 1$) and meta-Cobb–Douglas ($\sigma > 1$).[4] In Figure 3 we display the maximized Corr for all j sectors (Corr_j), upon estimating the elasticity of substitutions displayed in Figure 2. Note that Pearson correlation coefficient will reach unity when the two vectors are proportional (i.e., $\mathbf{r} = \theta \mathbf{s}$ with θ being a scalar). However, since $\sum_i \delta_{ij} = 1$ for all j, reflecting constant returns to scale, proportionality of two vectors will indicate accordance in this case.

Further, for the subsequent analysis of structural propagation, we calibrated the sector-wise CES parameters, namely δ_{ij}, to meet the latest technological structure (i.e., 2005 input-output coefficients) under the estimated marginal elasticity of substitution σ_j, while resetting the relative productivity gain z_j to unity. In other words, we set the parameters according to the latter equilibrium price p_j, and cost shares a_{ij} (or input-output coefficients), for the reference period as they satisfy the following identity:

$$\delta_{ij} = a_{ij} \left(p_j / p_i\right)^{1-\sigma_j} \tag{9}$$

Note that because CES comprehends both Cobb–Douglas ($\sigma_j = 1$) and Leontief ($\sigma_j = 0$) with regard to the elasticities, δ_{ij} equals the monetary input-output coefficient (a_{ij}) for Cobb–Douglas, and the physical input-output coefficient ($\xi_{ij} = a_{ij} \left(p_j / p_i\right)$) for Leontief, in light of (9).

[4] For asymptotic equivalences between CES, Cobb-Douglas and Leontief production functions, see for example, Saito (2012).

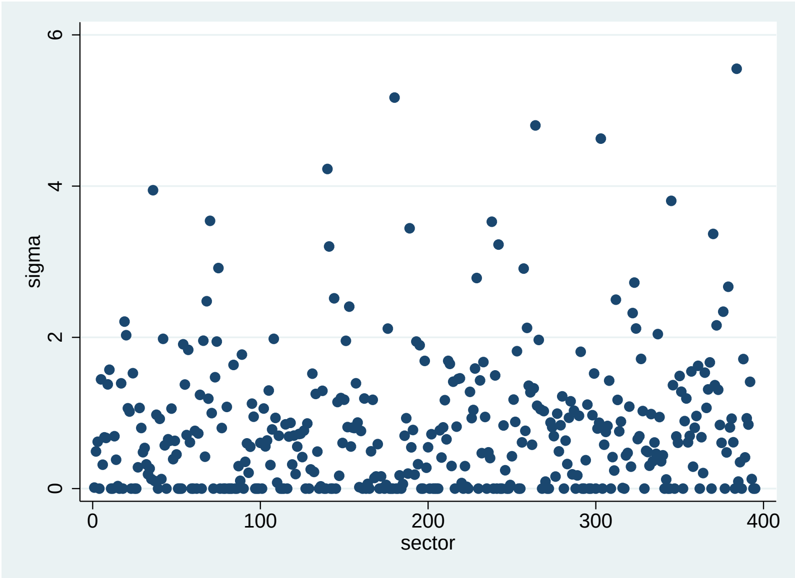

Fig. 2: Estimates of CES marginal elasticity of substitution for various industrial sectors (σ_j) based on the official deflator, for 2000–2005, in Japan.

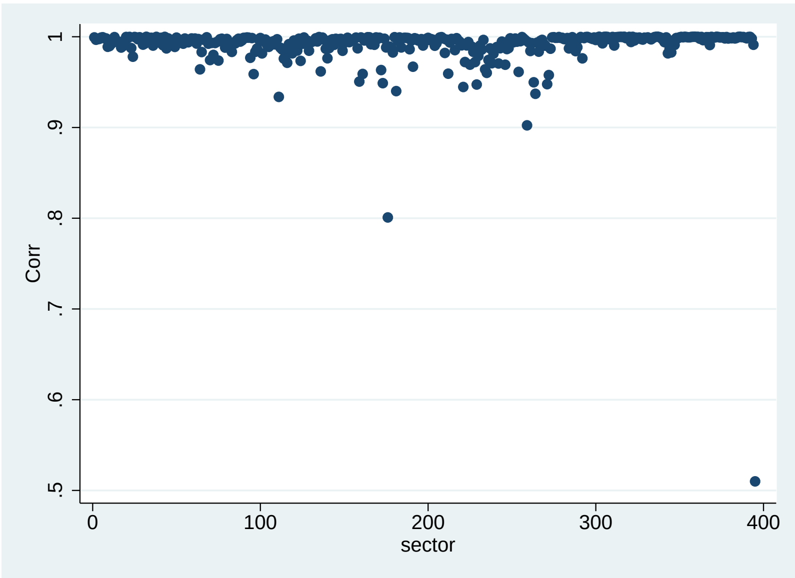

Fig. 3: Vector similarity (Corr_j) upon estimating σ_j via (8).

3 Propagation Analysis

3.1 Technological structure

Below is the unit cost function for the multi-factor CES production function for an industrial sector (index j omitted), compatible with (6):

$$h\left(p_0, p_1, \cdots, p_n; z\right) = \frac{1}{z}\left(\delta_0 p_0^{1-\sigma} + \delta_1 p_1^{1-\sigma} + \cdots + \delta_n p_n^{1-\sigma}\right)^{1/(1-\sigma)}$$

We abbreviate the system of above unit cost functions by

$$\mathbf{h}\left(p_0, \mathbf{p}; \mathbf{z}\right) = \left(h_1\left(p_0, \mathbf{p}; z_1\right), \cdots, h_n\left(p_0, \mathbf{p}; z_n\right)\right). \tag{10}$$

Applying Shephard's Lemma on $\mathbf{h}\left(p_0, \mathbf{p}; \mathbf{z}\right)$ we have,

$$\begin{bmatrix} \frac{\partial h_1(p_0,\mathbf{p};z_1)}{\partial p_0} & \frac{\partial h_2(p_0,\mathbf{p};z_2)}{\partial p_0} & \cdots & \frac{\partial h_n(p_0,\mathbf{p};z_n)}{\partial p_0} \\ \frac{\partial h_1(p_0,\mathbf{p};z_1)}{\partial p_1} & \frac{\partial h_2(p_0,\mathbf{p};z_2)}{\partial p_1} & \cdots & \frac{\partial h_n(p_0,\mathbf{p};z_n)}{\partial p_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_1(p_0,\mathbf{p};z_1)}{\partial p_n} & \frac{\partial h_2(p_0,\mathbf{p};z_2)}{\partial p_n} & \cdots & \frac{\partial h_n(p_0,\mathbf{p};z_n)}{\partial p_n} \end{bmatrix} = \begin{bmatrix} \nabla_0 \mathbf{h}\left(p_0, \mathbf{p}; \mathbf{z}\right) \\ \nabla \mathbf{h}\left(p_0, \mathbf{p}; \mathbf{z}\right) \end{bmatrix} \tag{11}$$

Note that $\nabla_0 \mathbf{h}\left(p_0, \mathbf{p}; \mathbf{z}\right)$ is the *ex-post* physical primary input coefficients vector, and $\nabla \mathbf{h}\left(p_0, \mathbf{p}; \mathbf{z}\right)$ is the *ex-post* physical input-output coefficient matrix, for which we otherwise call technological structure. Moreover, it should be worthwhile mentioning that innovation (as represented by the productivity gain $\mathbf{z}$) has the influence of changing the technological structure, according to (11). Structural propagation designates this influence in particular.

3.2 Structural propagation

For obvious reasons, the *ex-post* equilibrium price under given $\mathbf{z}$ is needed, to examine the *ex-post* technological structure of (11). As equilibrium price will coincide with the unit cost under prefect competition, we have the following identity:

$$\mathbf{p} = \mathbf{h}\left(p_0, \mathbf{p}; \mathbf{z}\right) \tag{12}$$

Let $\boldsymbol{\pi}\left(\mathbf{z}\right) = \left(\pi_1(\mathbf{z}), \cdots, \pi_n(\mathbf{z})\right)$ be the solution for (12), given the numéraire price p_0. The *ex-post* propagated equilibrium technological structure is the technological structure (11) evaluated at this equilibrium solution, as stated below:

$$\boldsymbol{\xi}_0\left(\mathbf{z}\right) \equiv \nabla_0 \mathbf{h}\left(p_0, \mathbf{p}; \mathbf{z}\right)\big|_{\mathbf{p}=\boldsymbol{\pi}(\mathbf{z})}, \qquad \boldsymbol{\Xi}\left(\mathbf{z}\right) \equiv \nabla_0 \mathbf{h}\left(p_0, \mathbf{p}; \mathbf{z}\right)\big|_{\mathbf{p}=\boldsymbol{\pi}(\mathbf{z})}. \tag{13}$$

Also, note that *ex-post* element-wise physical input-output coefficients can be derived for CES production function as follows:

$$\xi_{ij}\left(\mathbf{z}\right) = \frac{\partial h_j\left(p_0, \mathbf{p}; z_j\right)}{\partial p_i}\bigg|_{\mathbf{p}=\boldsymbol{\pi}(\mathbf{z})} = \delta_{ij} z_j^{\sigma_j - 1}\left(\frac{\pi_j\left(\mathbf{z}\right)}{\pi_i\left(\mathbf{z}\right)}\right)^{\sigma_j} = a_{ij}\left(\mathbf{z}\right)\frac{\pi_j\left(\mathbf{z}\right)}{\pi_i\left(\mathbf{z}\right)} \tag{14}$$

We may then use (13) to perform *ex-post* input-output analysis, for example, in the following way:

$$\mathbf{L}(\mathbf{z}) = p_0 \boldsymbol{\xi}_0(\mathbf{z}) \left\langle [\mathbf{I} - \boldsymbol{\Xi}(\mathbf{z})]^{-1} \bar{\mathbf{d}}' \right\rangle = \mathbf{a}_0(\mathbf{z}) \left\langle [\mathbf{I} - \mathbf{A}(\mathbf{z})]^{-1} \langle \boldsymbol{\pi}(\mathbf{z}) \rangle \bar{\mathbf{d}}' \right\rangle \tag{15}$$

where $\mathbf{L}(\mathbf{z}) = (L_1(\mathbf{z}), \cdots, L_n(\mathbf{z}))$ denotes the sector-wise primary factor (in monetary terms) required for the economy to be able to consume a fixed amount (vector) of final demand which we denote by $\bar{\mathbf{d}} = (\bar{d}_1, \cdots, \bar{d}_n)$. Note that the second identity is due to the third identity for (14), and that angle brackets indicate diagonalization.

So the question finally comes down on how to solve (12). Although we may have analytical solution for specific cases i.e., $\boldsymbol{\delta} = 1$ (Cobb–Douglas) and $\boldsymbol{\delta} = 0$ (Leontief), for which we present in the Appendix, there are no analytical solution otherwise. Still, we can use the recursive methodology, since the system of unit cost functions (10) is strictly concave with respect to the entries $\mathbf{p}$. In other words, we may apply (12) recursively, whereby feeding back the output into the input iteratively, to eventually reach at the equilibrium solution. That is,

$$\mathbf{p}^{t+1} = \mathbf{h}\left(p_0, \mathbf{p}^t; \mathbf{z}\right), \qquad \lim_{t\to\infty} \mathbf{p}^t = \boldsymbol{\pi}(\mathbf{z}) \tag{16}$$

where $\mathbf{p}^t$ denotes the price vector for the tth iteration.

Below we present the results obtained for calculating $\mathbf{L}(\mathbf{z})$ where we used $\mathbf{z} = \mathbf{z}_{\mathrm{PO}}$, or a doubling of port operation productivity ($\mathbf{z}_{\mathrm{PO}} = (1, \cdots, 1, z_{\mathrm{PO}}, 1, \cdots, 1)$, where $z_{\mathrm{PO}} = 2$), as the trigger of structural propagation. We have obtained *ex-post* equilibrium price via (16) with 20 iterations.[5] Figures 6, 4, and 5, display the primary input saved i.e., $\Delta\mathbf{L} = \mathbf{L}(\mathbf{1}) - \mathbf{L}(\mathbf{z}_{\mathrm{PO}})$ for CES, Cobb–Douglas, and Leontief productions, respectively. Naturally, we used the estimated sector-wise elasticity of substitution (Figure 2) for CES productions while setting all the elasticities unity for Cobb–Douglas and zero for Leontief productions. The sum of the saved primary factor i.e., $\Delta\mathbf{L}\mathbf{1}'$ is displayed in Table 1. Also, displayed is the kurtosis that measures the degree of polarization of the sector-wise distribution of the savings $\Delta\mathbf{L}$.

Table 1: Saved primary input by port operation productivity doubling in different functional forms. (unit: Million JPY)

	Cobb–Douglas	Leontief	CES
$\Delta\mathbf{L}\mathbf{1}'$	927,494	726,101	875,729
kurtosis	(114)	(342)	(182)

As it is observed from the numbers displayed in Table 1, the magnitude of propagation is relatively large for Cobb-Douglas than Leontief, whereas the sector-wise distribution is polarized for Leontief than Cobb-Douglas. It is imaginable that inflexibility of technology (zero elasticity) can consolidate the potential propagation effects

[5] Note that the final differential (numbers between the 19th and the 20th iteration) was negligibly small.

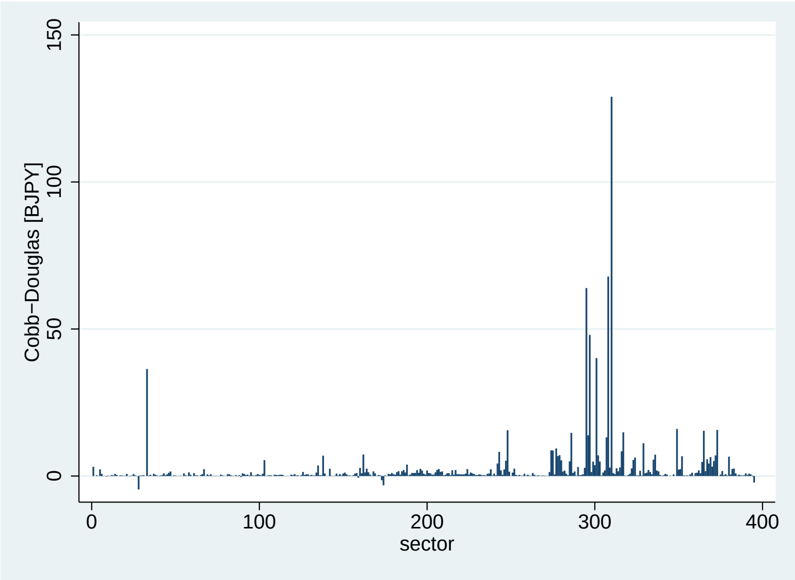

Fig. 4: Propagation of port operation productivity doubling under Cobb-Douglas production.

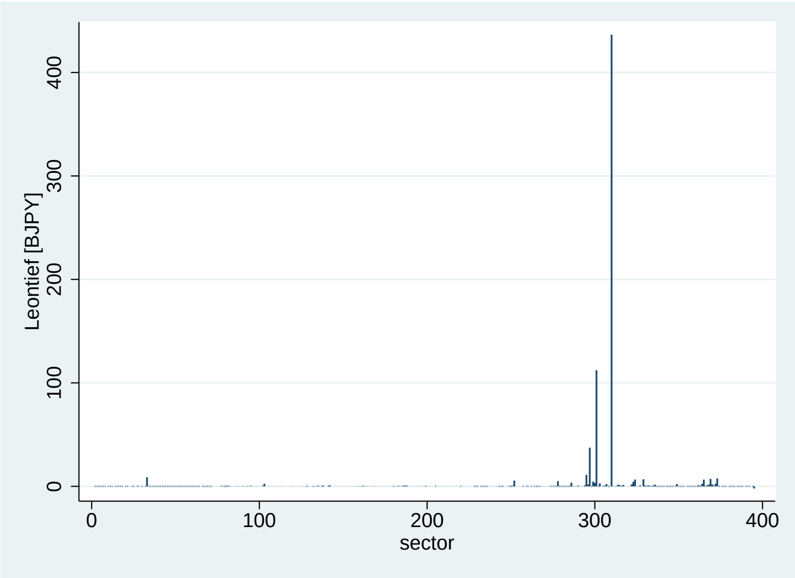

Fig. 5: Propagation of port operation productivity doubling under Leontief production.

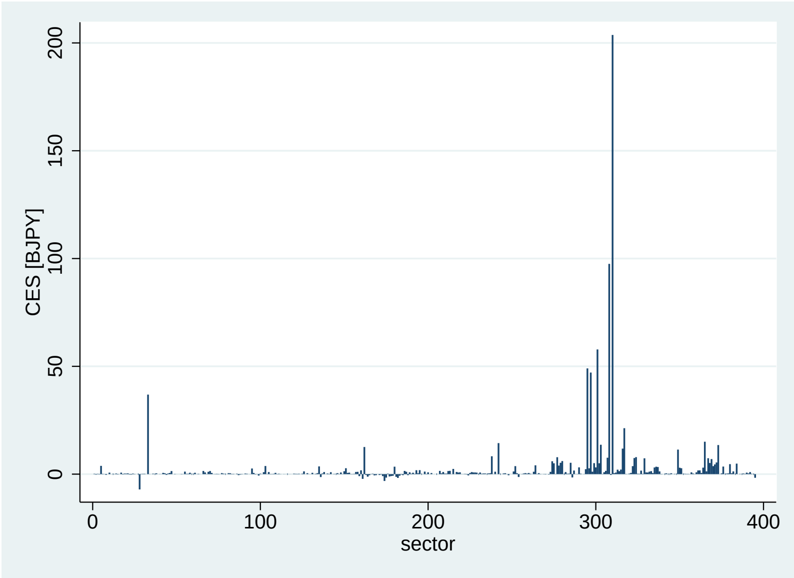

Fig. 6: Propagation of port operation productivity doubling under CES production.

while flexibility of technology (nonzero elasticity) can do the opposite. Our estimates on CES production indicate that the propagation effects, both in terms of magnitude and distribution, lie in between for Cobb-Douglas and Leontief. This result is closely related to our estimates on the elasticities whose sector-wide average was 0.784, which is sub-Cobb-Douglas and meta-Leontief, on average.

4 Concluding Remarks

To this date input-output analysis has been extensively used for assessing the costs and benefits of new goods and new innovations. Taken for granted is that these studies has relied upon the non-substitution theorem that allows the investigator to study under a fixed technological structure, while restricting the subjects of the analyses to the transformations within the final demand. Nevertheless, substitution of technology will prevail in any industry when a new technology/innovation is actually introduced into any component (industry) of the economy. Larger influence is typically foreseeable for intermediate industry's technologies, as they have much larger and wider feedback on the economy-wide system of production.

In order to take full technology substitution possibilities into account, we proposed in this study a methodology to measure the sector-wise elasticity of substitution for CES production function, in stead of using uniform *a priori* elasticity of substitution (such as zeros and ones), when modeling the economy-wide multi-sector multi-factor production system. Recursive method in the dual (i.e., unit cost functions) was used to evaluate the influences upon the general equilibrium technological substitutions and eventually upon the social costs and benefits, called structural propagation, initiated by the introduction of new technology/innovation for which we treat it as the gain in productivity.

We have found that more elastic production functions (Cobb-Douglas in this case) have more significant and wider propagation effects, whereas those for inelastic production functions (Leontief) were relatively less and polarized; and those for the measured CES production functions laid in between. After all, the reliability of this analytical framework comes down to the measurement of sector-wise technological elasticities, for which we obtained in this study as the maximizer of the correlation between the two observation-consistent share parameters. Naturally, different metrics (e.g., Euclidean distances, cosine similarity, and others) can be tested for vector similarity evaluation. Application and extensions of structural propagation analysis can be immense, including internationalization, dynamicalization, quality consideration, structural viability assessment and so on, all remaining for future investigations.

Appendix

Here, we present analytical solution to (12) for solvable two cases, namely, Cobb–Douglas ($\sigma = 1$) and Leontief ($\sigma = 0$).

Cobb–Douglas Production

We write down below the Cobb–Douglas unit cost functions of jth industry for both *ex-post* and benchmark. That is,

$$\pi_j(\mathbf{z}) = \frac{1}{z_j} \prod_{i=0}^{n} \left(\frac{\pi_i(\mathbf{z})}{a_{ij}} \right)^{a_{ij}}, \qquad \bar{p}_j = \prod_{i=0}^{n} \left(\frac{\bar{p}_i}{a_{ij}} \right)^{a_{ij}} \tag{17}$$

Here, a_{ij} denotes jth industry's output elasticity for the ith input, which is assumed to be constant under Cobb–Douglas production. Note that a_{ij} is also identical to the benchmark cost share of ith input for jth industry's output (or the benchmark monetary input-output coefficient). Also, we note that $\bar{p}_i$ denotes benchmark (i.e., $\pi_i(\mathbf{1}) = \bar{p}_i$) equilibrium price.

By taking the log and subtraction on (17), we obtain

$$\ln \pi_j(\mathbf{z}) - \ln \bar{p}_j = \sum_{i=0}^{n} a_{ij} \left(\ln \pi_j(\mathbf{z}) - \ln \bar{p}_j - \ln z_i \right) \tag{18}$$

Rewriting (18) for an $n \times n$ multiple-industry setting we have,

$$\ln \boldsymbol{\pi}(\mathbf{z}) - \ln \bar{\mathbf{p}} = [\ln \boldsymbol{\pi}(\mathbf{z}) - \ln \bar{\mathbf{p}} - \ln \mathbf{z}] \, \mathbf{A} \tag{19}$$

where we abbreviate, for example, $\ln \boldsymbol{\pi} = (\ln \pi_1, \cdots, \ln \pi_n)$, etc. Then we may solve (19) for $\boldsymbol{\pi}(\mathbf{z})$ to obtain the analytical solution to (12). That is,

$$\boldsymbol{\pi}(\mathbf{z}) = \bar{\mathbf{p}} \left\langle \exp \left(-(\ln \mathbf{z}) \, [\mathbf{I} - \mathbf{A}]^{-1} \right) \right\rangle \tag{20}$$

Furthermore, following identities must hold for $\sigma_j = 1$ and $\mathbf{z} = \mathbf{1}$ in regard to (14).

$$\delta_{ij} = a_{ij}(\mathbf{z}), \qquad \delta_{ij} = a_{ij}(\mathbf{1}) = a_{ij} \tag{21}$$

Thus, we see that $a_{ij}(\mathbf{z})$ will remain unchanged. In other words, we may substitute *ex-post* input-output coefficients with those of the benchmark i.e.,

$$\mathbf{a}_0(\mathbf{z}) = \mathbf{a}_0, \qquad \mathbf{A}(\mathbf{z}) = \mathbf{A} \tag{22}$$

Hence, for Cobb–Douglas production, (15) can be evaluated as follows:

$$\mathbf{L}(\mathbf{z}) = \mathbf{a}_0 \left\langle [\mathbf{I} - \mathbf{A}]^{-1} \left\langle \exp \left(-(\ln \mathbf{z}) \, [\mathbf{I} - \mathbf{A}]^{-1} \right) \right\rangle \langle \bar{\mathbf{p}} \rangle \, \bar{\mathbf{d}}' \right\rangle \tag{23}$$

Leontief Production

Below we write down the *ex-post* equilibrium monetary balance for jth industry:

$$y_j \pi_j(\mathbf{z}) = \pi_0(\mathbf{z}) x_{0j} + \pi_1(\mathbf{z}) x_{1j} + \cdots + \pi_n(\mathbf{z}) x_{nj}$$

Let us arrange this formula for further investigation:

$$\begin{aligned} \pi_j(\mathbf{z}) &= \pi_0(\mathbf{z}) \frac{x_{0j}}{y_j} + \pi_1(\mathbf{z}) \frac{x_{1j}}{y_j} + \cdots + \pi_n(\mathbf{z}) \frac{x_{nj}}{y_j} \\ &= \pi_0(\mathbf{z}) \xi_{0j}(\mathbf{z}) + \pi_1(\mathbf{z}) \xi_{1j}(\mathbf{z}) + \cdots + \pi_n(\mathbf{z}) \xi_{nj}(\mathbf{z}) \\ &= \pi_0(\mathbf{z}) \frac{\xi_{0j}}{z_0} + \pi_1(\mathbf{z}) \frac{\xi_{1j}}{z_1} + \cdots + \pi_n(\mathbf{z}) \frac{\xi_{nj}}{z_n} \end{aligned} \tag{24}$$

Note that the last identity can be derived by applying $\sigma_j = 0$ and $\mathbf{z} = \mathbf{1}$ in (14), which we describe below:

$$\xi_{ij}(\mathbf{z}) = \delta_{ij} z_j^{-1}, \qquad \xi_{ij}(\mathbf{1}) = \xi_{ij} = \delta_{ij}$$

Thus, (24) can be reduced as follows:

$$\boldsymbol{\pi}(\mathbf{z}) \langle \mathbf{z} \rangle = \boldsymbol{\xi}_0 + \boldsymbol{\pi}(\mathbf{z}) \boldsymbol{\Xi} = \mathbf{a}_0 \bar{\mathbf{p}} + \boldsymbol{\pi}(\mathbf{z}) \langle \bar{\mathbf{p}} \rangle^{-1} \mathbf{A} \langle \bar{\mathbf{p}} \rangle \tag{25}$$

where we normalized prices using $\pi_0 = \bar{p}_0 = 1$. For the second identity we used $\xi_{ij} = a_{ij} \bar{p}_j / \bar{p}_i$. Now, (25) can be solved for $\boldsymbol{\pi}(\mathbf{z})$ as follows:

$$\boldsymbol{\pi}(\mathbf{z}) = \mathbf{a}_0 [\mathbf{z} - \mathbf{A}]^{-1} \langle \bar{\mathbf{p}} \rangle \tag{26}$$

Hence, for Leontief production, (15) can be evaluated as follows:

$$\mathbf{L}(\mathbf{z}) = \mathbf{a}_0 \left\langle [\mathbf{z} - \mathbf{A}]^{-1} \langle \bar{\mathbf{p}} \rangle \bar{\mathbf{d}}' \right\rangle \tag{27}$$

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