A Unified Framework of Trade in Value Added; Physical, Monetary, Exchange Rates, and GHG Emissions*

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Abstract

Koopman et al. (2014) developed a method to consistently decompose gross exports in value added terms, which accommodates infinite repercussions of international and inter-sector transactions. This helps us better understand the Trade in Value Added (TiVA) in global value chains (GVCs) as compared to the conventional gross exports accompanied by double counting problems. However, the framework based on the monetary I-O tables cannot distinguish prices from quantities, and thus unable to consider financial adjustments through the exchange market. This paper proposes a framework based on a physical I-O system, in its linear programming equivalent, which can clarify the various complexities relevant to the existing indicators, and is proved to be consistent with the Koopman's results when the physical decompositions are evaluated in monetary terms. While the international monetary tables are generally described in current U.S. dollars, the physical framework can elucidate the impact of price adjustments through the exchange market. An iterative procedure to calculate the exchange rates is proposed, and some numerical exercises with hypothetical data are conducted to demonstrate the significance of local wages and capital flows, which are exogenous to the I-O system. The physical framework is also convenient to consider the indicators associated with GHG emissions.

1 Introduction

The rise of global value chains (GVCs) during the last two decades has significantly changed the nature and structure of international trade, with many new implications for policy making (Baldwin and Robert-Nicoud, 2014; Timmer et al., 2013). One of the most important features of GVCs is the transition of trade pattern from "trade in goods" to "trade in tasks" (see Grossman and Rossi-Hansberg, 2008) in global production networks. This phenomenon has also been explained as the so-called "the second great unbundling" (see Baldwin, 2012). The theoretical background is that the reduction of communication cost due to the IT revolution has enabled the international unbundling of factories and offices, which means that tasks can

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also be traded globally. In other words, countries no longer have to build or host the entire production chain, since they can integrate by developing or attracting productive capacity in one link of the chain through fragmentation production where their comparative advantages fit the best. As a result, more and more intermediate goods, such as parts and components, are produced in sequential substages in different countries and then exported to other countries for further production. This in turn, has significantly increased the complexity and sophistication of international production networks, thus brought many new challenges on how to better understand the creation, transfer and distribution of value added, income and job opportunities in GVCs.

Policy-makers require well conceptualized indicators that can reveal the degree and nature of the interaction of their country with her major economic partners, the degree of GVC participation, and the location of their country in GVCs (see OECD-WTO-UNCTAD, 2013). Along this line, a lot of new indictors and measures based on input-output (I-O) data have been proposed. For example, Hummels et al. (2001) used the "import content of exports" indicator to measure a country's participation level in vertical specialization trade; Johnson and Noguera (2012) proposed the "Trade in Value-added" (TiVA) indictor to measure how a country's value-added is directly and indirectly absorbed by other country's final demand through GVCs; Antràs et al. (2012) developed a concept of "distance" which is the number of stages that the product goes through before reaching the final demand to measure the position of country or industry in GVCs; Timmer et al. (2013) presented a new indicator for measuring the level of fragmentation production; Koopman et al. (2014) developed a method to consistently decompose gross exports in value added terms, which helps better understanding on value-added trade in GVCs as compared to the conventional gross exports accompanied by double counting problems. Wang et al. (2014) further extended the work of Koopman et al. for consistently measuring value added trade at bilateral and industrial levels.

However, the above efforts in developing the measurement of GVCs are all based on the monetary I-O tables which cannot distinguish prices from quantities, and thus unable to consider financial adjustments through the exchange market. This paper aims to propose a more general framework based on a physical I-O system to clarify the various complexities relevant to the existing I-O based GVCs indicators. Since the international monetary I-O tables are generally described in current U.S. dollars, the physical framework can elucidate the impact of price adjustments through the exchange market. An iterative procedure to calculate the exchange rates is also proposed in the paper, and some numerical exercises with hypothetical data are conducted to demonstrate the significance of local wages and capital flows, which are exogenous to the I-O system. The physical framework proposed is convenient to consider the indicators associated with GHG emissions as well.

In following, after reviewing the linear programming problem of the one country physical input-output system, the problem for the world complete with two countries is formulated. With the physical system, it is easy to calculate the contributions of individual sectors and countries on each commodity price after considering infinite repercussions of intermediate trade. Once such contributions are evaluated, it is easy to decompose the GDP of each country, which is the sum of values added. In section 4, the system is generalized to include n sectors and m countries. In section 5, the correspondence between physical and monetary systems are discussed to show that our results are essentially the same as the ones in Koopman et al. Returning to the physical system, the iterative process to endogenize the exchange rates is discussed in section 6. However, it presupposes the existence of the outside systems to determine the wage levels and capital flows; such as the labor and international financial markets. The greenhouse gases (GHG) can also be incorporated in the commodities traded. Similar approach can be used to determine who are ultimately responsible for emission after considering infinite repercussions, which is the subject to be discussed in sections 7 and 8.

2 One-country physical table

The linear programming problem proposed by Dorfman et al. (1958) is convenient to formalize the physical input-output system, and it might be beneficial to review the single country case to start with. The problem is to find the output schedule x that minimize the labor cost to satisfy the final demand requirement y:

$$\min_x \{wa_0 x | (I - A)x \geq y, x \geq 0\}, \tag{1}$$

where w, a_0, and A, respectively, are the prevailing wage, labor (value-added) input coefficient vector, and input coefficient matrix.

The Lagrangian function for the problem can be written with the row vector of multipliers p as follows:

$$L = wa_0 x + p(y - (I - A)x). \tag{2}$$

Then one of the first-order conditions would become:

$$\frac{\partial L}{\partial x} = wa_0 - p(I - A) \geq 0, \tag{3}$$

where p can be interpreted as the price vector.

When p is positive, then the usual output equation is obtained as the optimal solution, viz., $x = (I - A)^{-1} y$. Conversely, when the output vector x is positive, the row vector of commodity prices can be solved.

$$p = wa_0(I - A)^{-1}, \tag{4}$$

which is positive when the wage is positive, the labor inputs are non-negative but non-zero, and the Leontief inverse in positive definite.

Since (4) can be decomposed as the sum of geometric series,

$$p = wa_0 + wa_0 A + wa_0 A^2 + wa_0 A^3 + \cdots,$$

The first term represents the direct labor cost included in the product price while the second term represents the first-round repercussion as intermediate inputs to another commodity, and so forth.

When b_{ij} denotes the (i, j) element of the Leontief inverse, the price of commodity i can be written as a weighted sum of the labor costs in all the sectors: $p_i = w \sum_j a_{0j} b_{ji}$. Then the portion of the price of commodity i attributable to commodity j as an intermediate input can be calculated as follows:

$$c_{ji} = \frac{wa_{oj} b_{ji}}{p_i}. \tag{5}$$

3 Two-country physical table

Similar discussion applies to the case when the commodity price composition in terms of origins of intermediate inputs is considered. In this case, country 1's problem is to minimize the costs of labor and imported intermediate inputs required to produce the domestic outputs:

$$\min_{x^1}\{(w^1a_0^1 + p^2A^{21})x^1 | (I - A^{11})x^1 - A^{12}x^2 \geq y^{11} + y^{12}, x^1 \geq 0\}, \tag{6}$$

and the problem of country 2 would become as follows:

$$\min_{x^2}\{(w^2a_0^2 + p^1A^{12})x^2 | - A^{21}x^1 + (I - A^{22})x^2 \geq y^{21} + y^{22}, x^2 \geq 0\}, \tag{6'}$$

where the superscripts 1 and 2 indicate the respective countries. y^{rs} represents the amounts of country r's products consumed as the final demand in country s, and A^{rs} denotes the submatrix of interregional input coefficient matrix. Each country regards the price of imports p^s as well as the domestic wage w^r as being exogenous.

In these problems, each country regards the outputs of other country as being exogenous to her. Then the problem can be described as a Nash problem, and x^1 and x^2 at the Nash equilibrium are determined by solving (6) and (6') simultaneously. The same output schedule can be obtained from the *world problem* combining both countries.

$$\begin{array}{ll} \min_{x^1,x^2} & w^1a_0^1x^1 + w^2a_0^2x^2, \\ \text{s.t.} & (I - A^{11})x^1 - A^{12}x^2 \geq y^{11} + y^{12}, \\ & -A^{21}x^1 + (I - A^{22})x^2 \geq y^{21} + y^{22}, \\ & x^1 \geq 0, \quad \text{and} \quad x^2 \geq 0, \end{array} \tag{7}$$

and the Lagrangian function for the problem can be written with the multipliers p^1 and p^2 for respective countries:

$$L = w^1a_0^1x^1 + w^2a_0^2x^2 + p^1(y^{11} + y^{12} - (I - A^{11})x^1 + A^{12}x^2) + p^2(y^{21} + y^{22} + A^{21}x^1 - (I - A^{22})x^2). \tag{8}$$

Some of the first-order conditions are obtained as follows:

$$\begin{aligned} \frac{\partial L}{\partial x^1} &= w^1a_0^1 - p^1(I - A^{11}) + p^2A^{21} \geq 0 \\ \frac{\partial L}{\partial x^2} &= w^2a_0^2 + p^1A^{12} - p^2(I - A^{22}) \geq 0 \end{aligned} \tag{9}$$

If the output vectors are positive, the price vectors can be solved in the matrix form as follows:

$$(p^1 \quad p^2) = (w^1a_0^1 \quad w^2a_0^2)\begin{pmatrix} I - A^{11} & -A^{12} \\ -A^{21} & I - A^{22} \end{pmatrix}^{-1} = (w^1a_0^1 \quad w^2a_0^2)\begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix} \tag{10}$$

Denoting the transaction between sectors i and j in the submatrix B^{rs} by b_{ij}^{rs}, the price of commodity i produced in country 1 can be decomposed as follows:

$$p_i^1 = w^1 \sum_j a_{0j}^1 b_{ji}^{11} + w^2 \sum_j a_{0j}^2 b_{ji}^{21}. \tag{11}$$

Then, in the two-country framework, the portions of the prices of country 1's product that is attributed to the values added originated in countries 1 and 2 can easily be calculated:

$$c_i^{11} = \frac{w^1 \sum_j a_{0j}^1 b_{ji}^{11}}{p_i^1} \quad \text{and} \quad c_i^{21} = \frac{w^2 \sum_j a_{0j}^2 b_{ji}^{21}}{p_i^1}.$$

Note that these expressions can directly be extended to the multi-country case. Since the values added can be attributed to each industry in each region, the portions of commodity i's price produced in country s that is attributable to industry j in country r, and their aggregation by the originating country can be written, respectively, as follows:

$$c_{ji}^{rs} = \frac{w^r a_{0j}^r b_{ji}^{rs}}{p_i^s} \quad \text{and} \quad c_i^{rs} = \frac{w^r \sum_j a_{0j}^r b_{ji}^{rs}}{p_i^s} \tag{12}$$

4 Decomposition of GDPs

By definition, the GDP of country 1 is given by:

$$\begin{aligned} Y^1 &= p^1 y^{11} + p^2 y^{21} + p^1(y^{12} + A^{12}x^2) - p^2(y^{21} + A^{21}x^1) \\ &= \sum_i p_i^1 (y_i^{11} + y_i^{12} + \sum_j a_{ij}^{12} x_j^2) - \sum_i p_i^2 \sum_j a_{ij}^{21} x_j^1, \end{aligned} \tag{13}$$

where the first line represents the final demand for the domestic product $p^1 y^{11}$ plus the exports subtracted by the imports. The exports and imports include both final and intermediate demands, and are evaluated by the prices of their origins. Likewise the GDP of country 2 can be written as follows:

$$\begin{aligned} Y^2 &= p^2 y^{22} + p^1 y^{12} + p^2(y^{21} + A^{21}x^1) - p^1(y^{12} + A^{12}x^2) \\ &= \sum_i p_i^2 (y_i^{22} + y_i^{21} + \sum_j a_{ij}^{21} x_j^1) - \sum_i p_i^1 \sum_j a_{ij}^{12} x_j^2. \end{aligned}$$

Utilizing the portions c_i^{rs} defined in (12), these GDPs can be decomposed into the contributions of respective countries, which considers the infinite repercussions of intermediate transactions. That is, $Y^1 = Y^{11} + Y^{21}$ and $Y^2 = Y^{22} + Y^{12}$, where Y^{rs} represents the part of country s's GDP that is eventually attributable to country r.

$$\begin{aligned} Y^{11} &= \sum_i c_i^{11} p_i^1 (y_i^{11} + y_i^{12} + \sum_j a_{ij}^{12} x_j^2) - \sum_i (1 - c_i^{12}) p_i^2 \sum_j a_{ij}^{21} x_j^1, \\ Y^{21} &= \sum_i c_i^{21} p_i^1 (y_i^{11} + y_i^{12} + \sum_j a_{ij}^{12} x_j^2) - \sum_i c_i^{12} p_i^2 \sum_j a_{ij}^{21} x_j^1, \\ Y^{22} &= \sum_i c_i^{22} p_i^2 (y_i^{22} + y_i^{21} + \sum_j a_{ij}^{21} x_j^1) - \sum_i (1 - c_i^{21}) p_i^1 \sum_j a_{ij}^{12} x_j^2, \\ Y^{12} &= \sum_i c_i^{12} p_i^2 (y_i^{22} + y_i^{21} + \sum_j a_{ij}^{21} x_j^1) - \sum_i c_i^{21} p_i^1 \sum_j a_{ij}^{12} x_j^2. \end{aligned} \tag{14}$$

The second terms in the right hand sides of the above represent the imports. In the case of Y^{11}, only the share of c^{22} is subtracted because the remainder, c^{12} is the portion attributable to its own, which need not be subtracted. Likewise, the second term of Y^{21} subtracts country

1's contribution from the import from country 2, since that part must be accounted for country 1 rather than country 2.[1]

Suppose there are $i, j = 1, \ldots, n$ commodities and $r, s = 1, \ldots, m$ countries, and p^s denotes the $(1 \times n)$ vector of f.o.b. prices in country s. Further introduce a diagonal matrix C^{rs} comprising $\{c_1^{rs}, \ldots, c_n^{rs}\}$ obtained in (12). Then the general formulae of the GDP decompositions can be written as follows:

$$\begin{aligned} Y^{ss} &= p^s C^{ss}(\sum_r y^{sr} + \sum_{r \neq s} A^{sr} x^r) - \sum_{r \neq s} p^r (I_n - C^{sr}) A^{rs} x^s \\ Y^{rs} &= p^s C^{rs}(\sum_r y^{sr} + \sum_{r \neq s} A^{sr} x^r) - p^r C^{sr} A^{rs} x^s, \quad (r \neq s). \end{aligned} \tag{15}$$

It is difficult to describe (15) in a simple matrix expression. For example, when there are three countries, the below provides one of such expressions.

$$\begin{aligned} &\begin{pmatrix} Y^{11} & Y^{12} & Y^{13} \\ Y^{21} & Y^{22} & Y^{23} \\ Y^{31} & Y^{32} & Y^{33} \end{pmatrix} \\ &= P^0 \begin{pmatrix} C^{11} & C^{12} & C^{13} \\ C^{21} & C^{22} & C^{23} \\ C^{31} & C^{32} & C^{33} \end{pmatrix} \begin{pmatrix} \sum_r y^{1r} + \sum_{r \neq 1} A^{1r} x^r & 0 & 0 \\ 0 & \sum_r y^{2r} + \sum_{r \neq 2} A^{2r} x^r & 0 \\ 0 & 0 & \sum_r y^{3r} + \sum_{r \neq 3} A^{3r} x^r \end{pmatrix} \\ &- \begin{pmatrix} 0 & p^2(I - C^{12}) & p^3(I - C^{13}) \\ 0 & p^2 C^{12} & 0 \\ 0 & 0 & p^3 C^{13} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ A^{21} x^1 & 0 & 0 \\ A^{31} x^1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} p^1 C^{21} & 0 & 0 \\ p^1(I - C^{21}) & 0 & p^3(I - C^{23}) \\ 0 & 0 & p^3 C^{23} \end{pmatrix} \begin{pmatrix} 0 & A^{12} x^2 & 0 \\ 0 & 0 & 0 \\ 0 & A^{32} x^2 & 0 \end{pmatrix} \\ &- \begin{pmatrix} p^1 C^{31} & 0 & 0 \\ 0 & p^2 C^{32} & 0 \\ p^1(I - C^{31}) & p^2(I - C^{32}) & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & A^{13} x^3 \\ 0 & 0 & A^{23} x^3 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{16}$$

where P^0 denotes the $(3 \times 3n)$ matrix of price vectors, viz. $P^0 = \begin{pmatrix} p^1 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & p^3 \end{pmatrix}$.

Using the commodity based coefficients c_{ji}^{rs} in (12), the decomposition (15) may be rewritten at the commodity level as follows:

$$\begin{aligned} Y_{ji}^{ss} &= p_i^s c_{ji}^{ss}(\sum_r y_i^{sr} + \sum_{r \neq s} \sum_{j'} a_{ij'}^{sr} x_{j'}^r) - \sum_{r \neq s} p^r (1 - c_{ji}^{sr}) \sum_{j'} a_{ij'}^{rs} x_{j'}^s, \\ Y_{ji}^{rs} &= p_i^s c_{ji}^{rs}(\sum_r y_i^{sr} + \sum_{r \neq s} \sum_{j'} a_{ij'}^{sr} x_{j'}^r) - p_i^r c_{ji}^{sr} \sum_{j'} a_{ij'}^{rs} x_{j'}^s, \quad (r \neq s), \end{aligned} \tag{17}$$

where Y_{ji}^{rs} is the part of country s's income from producing commodity i that is eventually attributable to sector j in country r.

5 The monetary representation

Koopman et al. (2014) demonstrated a similar measure that evaluates the value added attributable to each country after infinite repercussions of trade. Since their results are derived

[1] It can readily be seen $Y^1 = Y^{11} + Y^{21}$, since $c^{11} + c^{21} = 1$ and $c^{22} + c^{12} = 1$ by definition.

from the monetary table, it is important to confirm that our results are consistent with theirs when our formulas are transformed into the monetary terms.

The relationship between the physical and monetary inter-regional input coefficients, a_{ij}^{rs} and α_{ij}^{rs}, can be established as follows:

$$\alpha_{ij}^{rs} = \frac{p_i^r x_{ij}^{rs}}{p_j^s X_j^s} = \frac{p_i^r}{p_j^s} a_{ij}^{rs}. \tag{18}$$

Similarly, with the physical primary input L_j^s, the relationship for the value-added input coefficients, a_{0j}^s and α_{0j}^s, can also be established.

$$\alpha_{0j}^s = \frac{w^s L_j^s}{p_j^s X_j^s} = \frac{w^s}{p_j^s} a_{0j}^s \tag{19}$$

For simplicity, let us consider a two-country, two-commodity monetary table as shown in Table 1. With the physical and monetary input coefficients, the output equation for the first line of the table can be written as follows:

$$\begin{aligned} p_1^1 X_1^1 &= p_1^1 a_{11}^{11} X_1^1 + p_1^1 a_{12}^{11} X_2^1 + p_1^1 a_{11}^{12} X_1^2 + p_1^1 a_{12}^{12} X_2^2 + p_1^1 y_1^{11} + p_1^1 y_1^{12} \\ &= p_1^1 \alpha_{11}^{11} X_1^1 + p_2^1 \alpha_{12}^{11} X_2^1 + p_1^2 \alpha_{11}^{12} X_1^2 + p_2^2 \alpha_{12}^{12} X_2^2 + p_1^1 y_1^{11} + p_1^1 y_1^{12} \\ &= \alpha_{11}^{11} \hat{X}_1^1 + \alpha_{12}^{11} \hat{X}_2^1 + \alpha_{11}^{12} \hat{X}_1^2 + \alpha_{12}^{12} \hat{X}_2^2 + \hat{y}_1^{11} + \hat{y}_1^{12} = \hat{X}_1^1, \end{aligned}$$

where $\hat{X}_i^r = p_i^r X_i^r$ and $\hat{y}_i^{rs} = p_i^r y_i^{rs}$ represent the monetary values of X_i^r and y_i^{rs}, respectively.

Table 1: Framework of a two-country, two-commodity monetary table.

	country 1		country 2		final demand
country 1	$p_1^1 x_{11}^{11}$	$p_1^1 x_{12}^{11}$	$p_1^1 x_{11}^{12}$	$p_1^1 x_{12}^{12}$	$p_1^1 y_1^{11} + p_1^1 y_1^{12}$
	$p_2^1 x_{21}^{11}$	$p_2^1 x_{22}^{11}$	$p_2^1 x_{21}^{12}$	$p_2^1 x_{22}^{12}$	$p_2^1 y_2^{11} + p_2^1 y_2^{12}$
country 2	$p_1^2 x_{11}^{21}$	$p_1^2 x_{12}^{21}$	$p_1^2 x_{11}^{22}$	$p_1^2 x_{12}^{22}$	$p_1^2 y_1^{21} + p_1^2 y_1^{22}$
	$p_2^2 x_{21}^{21}$	$p_2^2 x_{22}^{21}$	$p_2^2 x_{21}^{22}$	$p_2^2 x_{22}^{22}$	$p_2^2 y_2^{21} + p_2^2 y_2^{22}$
values added	$w^1 a_{01}^1 X_1^1$	$w^1 a_{02}^1 X_2^1$	$w^2 a_{01}^2 X_1^2$	$w^2 a_{02}^2 X_2^2$	

Denoting the 4×4 matrix of inter-regional monetary input coefficients by $\hat{A}$, the system of output equations in the above table can be summarized as:

$$\hat{A}\hat{X} + \hat{y} = \hat{X}, \tag{20}$$

where $\hat{X}$ and $\hat{y}$ are the column vectors of monetary outputs and final demands, respectively. To clarify the relationship between monetary and physical expressions, it is necessary to establish the relationship between Leontief inverse matrices in monetary and physical terms. In the two-country and two-commodity setting, the monetary inverse can be transformed as follows:

$$(I - \hat{A})^{-1} = \begin{pmatrix} \beta_{11}^{11} & \beta_{12}^{11} & \beta_{11}^{12} & \beta_{12}^{12} \\ \beta_{21}^{11} & \beta_{22}^{11} & \beta_{21}^{12} & \beta_{22}^{12} \\ \beta_{11}^{21} & \beta_{12}^{21} & \beta_{11}^{22} & \beta_{12}^{22} \\ \beta_{21}^{21} & \beta_{22}^{21} & \beta_{21}^{22} & \beta_{21}^{22} \end{pmatrix} = \begin{pmatrix} 1 - \alpha_{11}^{11} & -\alpha_{12}^{11} & -\alpha_{11}^{12} & -\alpha_{12}^{12} \\ -\alpha_{21}^{11} & 1 - \alpha_{22}^{11} & -\alpha_{21}^{12} & -\alpha_{22}^{12} \\ -\alpha_{11}^{21} & -\alpha_{12}^{21} & 1 - \alpha_{11}^{22} & -\alpha_{12}^{22} \\ -\alpha_{21}^{21} & -\alpha_{22}^{21} & -\alpha_{21}^{22} & 1 - \alpha_{21}^{22} \end{pmatrix}^{-1}$$

$$
= \begin{pmatrix} 1-\frac{p_1^1}{p_1^1}a_{11}^{11} & -\frac{p_1^1}{p_2^1}a_{12}^{11} & -\frac{p_1^1}{p_1^2}a_{11}^{12} & -\frac{p_1^1}{p_2^2}a_{12}^{12} \\ -\frac{p_2^1}{p_1^1}a_{21}^{11} & 1-\frac{p_2^1}{p_2^1}a_{22}^{11} & -\frac{p_2^1}{p_1^2}a_{21}^{12} & -\frac{p_2^1}{p_2^2}a_{22}^{12} \\ -\frac{p_1^2}{p_1^1}a_{11}^{21} & -\frac{p_1^2}{p_2^1}a_{12}^{21} & 1-\frac{p_1^2}{p_1^2}a_{11}^{22} & -\frac{p_1^2}{p_2^2}a_{12}^{22} \\ -\frac{p_2^2}{p_1^1}a_{21}^{21} & -\frac{p_2^2}{p_2^1}a_{22}^{21} & -\frac{p_2^1}{p_1^2}a_{21}^{22} & 1-\frac{p_2^2}{p_2^2}a_{21}^{22} \end{pmatrix}^{-1}
$$

$$
= \left(\begin{pmatrix} p_1^1 & 0 & 0 & 0 \\ 0 & p_2^1 & 0 & 0 \\ 0 & 0 & p_1^2 & 0 \\ 0 & 0 & 0 & p_2^2 \end{pmatrix} \begin{pmatrix} 1-a_{11}^{11} & -a_{12}^{11} & -a_{11}^{12} & -a_{12}^{12} \\ -a_{21}^{11} & 1-a_{22}^{11} & -a_{21}^{12} & -a_{22}^{12} \\ -a_{11}^{21} & -a_{12}^{21} & 1-a_{11}^{22} & -a_{12}^{22} \\ -a_{21}^{21} & -a_{22}^{21} & -a_{21}^{22} & 1-a_{21}^{22} \end{pmatrix} \begin{pmatrix} \frac{1}{p_1^1} & 0 & 0 & 0 \\ 0 & \frac{1}{p_2^1} & 0 & 0 \\ 0 & 0 & \frac{1}{p_1^2} & 0 \\ 0 & 0 & 0 & \frac{1}{p_2^2} \end{pmatrix} \right)^{-1}
$$

$$
= \left(P(I-A)P^{-1}\right)^{-1} = P(I-A)^{-1}P^{-1} = P \begin{pmatrix} b_{11}^{11} & b_{12}^{11} & b_{11}^{12} & b_{12}^{12} \\ b_{21}^{11} & b_{22}^{11} & b_{21}^{12} & b_{22}^{12} \\ b_{11}^{21} & b_{12}^{21} & b_{11}^{22} & b_{12}^{22} \\ b_{21}^{21} & b_{22}^{21} & b_{21}^{22} & b_{21}^{22} \end{pmatrix} P^{-1}, \tag{21}
$$

where P denotes the diagonal matrix of 2×2 prices.

Reciprocally, the physical inverse matrix B can also be written in terms of monetary inverse matrix $\hat{B}$.[2]

$$
B = (I-A)^{-1} = P^{-1}(I-\hat{A})^{-1}P = P^{-1}\hat{B}P \tag{21$'$}
$$

Accordingly, the expressions in (12) can easily be rewritten with monetary coefficients:

$$
c_{ji}^{rs} = \frac{w^r a_{0j}^r b_{ji}^{rs}}{p_i^s} = \frac{w^r}{p_i^s}(\frac{p_j^r}{w^r})\alpha_{0j}^r(\frac{p_i^s}{p_j^r})\beta_{ji}^{rs} = \alpha_{0j}^r \beta_{ji}^{rs} \quad \text{and} \quad c_i^{rs} = \sum_j \alpha_{0j}^r \beta_{ji}^{rs}. \tag{22}
$$

Then the GDP decompositions may be calculated by plugging these coefficients into (17).

While Koopman et al. (2014) illustrates the case with single commodity, it can easily be extended to the case with multiple commodities. The domestic value-added coefficient v_j^s for sector j corresponds to α_{0j}^s in our notation. Recalling that an element of the monetary Leontief inverse is denoted by β_{ij}^{rs}, their country shares of values added are calculated for the two commodity case as follows:

$$
\begin{pmatrix} v_1^1 & 0 & 0 & 0 \\ 0 & v_2^1 & 0 & 0 \\ 0 & 0 & v_1^2 & 0 \\ 0 & 0 & 0 & v_2^2 \end{pmatrix} \begin{pmatrix} \beta_{11}^{11} & \beta_{12}^{11} & \beta_{11}^{12} & \beta_{12}^{12} \\ \beta_{21}^{11} & \beta_{22}^{11} & \beta_{21}^{12} & \beta_{22}^{12} \\ \beta_{11}^{21} & \beta_{12}^{21} & \beta_{11}^{22} & \beta_{12}^{22} \\ \beta_{21}^{21} & \beta_{22}^{21} & \beta_{21}^{22} & \beta_{22}^{22} \end{pmatrix} = \begin{pmatrix} \alpha_{01}^1\beta_{11}^{11} & \alpha_{01}^1\beta_{12}^{11} & \alpha_{01}^1\beta_{11}^{12} & \alpha_{01}^1\beta_{12}^{12} \\ \alpha_{02}^1\beta_{21}^{11} & \alpha_{02}^1\beta_{22}^{11} & \alpha_{02}^1\beta_{21}^{12} & \alpha_{02}^1\beta_{22}^{12} \\ \alpha_{01}^2\beta_{11}^{21} & \alpha_{01}^2\beta_{12}^{21} & \alpha_{01}^2\beta_{11}^{22} & \alpha_{01}^2\beta_{12}^{22} \\ \alpha_{02}^2\beta_{21}^{21} & \alpha_{02}^2\beta_{22}^{21} & \alpha_{02}^2\beta_{21}^{22} & \alpha_{02}^2\beta_{22}^{22} \end{pmatrix} \tag{23}
$$

Let $\hat{p}_j^s$ denote the dual variable for the monetary system. Then by definition, it will become unity, and is calculated as follows:

$$
\hat{p}_j^s = \sum_i \sum_r \alpha_{0i}^r \beta_{ij}^{rs} = 1.
$$

Hence, the column sums of (23) must equal to one, and each element represents the share of the value added eventually attributable to the relevant sector and country.

[2]Considering that $\hat{X} = PX$ and $\hat{y} = Py$, the monetary output equation (20) can be written as

$$
P(I-A)^{-1}P^{-1}Py = P(I-A)^{-1}y = PX.
$$

By pre-multiplying P^{-1}, this becomes equivalent to its physical counterpart.

6 Exchange rate

Return to the physical system, it is possible to calculate the effective exchange rate from the balance of payments. If there are m countries, one currency must be regarded as the numéraire, and other currencies are valued relative to it. In the two country case, it is reasonable to regard the currency of country 1 as the numéraire, and let μ denote the exchange rate for country 2. Then the price equations for each country can be written as follows:

$$\begin{aligned} p^1 A^{11} + \mu p^2 A^{21} + w^1 a_0^1 &= p^1 \\ p^1 A^{12} + \mu p^2 A^{22} + \mu w^2 a_0^2 &= \mu p^2 \end{aligned} \tag{24}$$

By limiting the number of sectors to 2, for simplicity, the trade balance of country 1 can be written as follows:

$$\begin{aligned} & p_1^1 a_{11}^{12} x_1^2 + p_2^1 a_{21}^{12} x_1^2 + p_1^1 a_{12}^{12} x_2^2 + p_2^1 a_{22}^{12} x_2^2 + p_1^1 y_1^{12} + p_2^1 y_2^{12} \\ - & \mu (p_1^2 a_{11}^{21} x_1^1 + p_2^2 a_{21}^{21} x_1^1 + p_1^2 a_{12}^{21} x_2^1 + p_2^2 a_{22}^{21} x_2^1 + p_1^2 y_1^{21} + p_2^2 y_2^{21}) = 0 \end{aligned}$$

If there is no income transfer and capital flows between two countries, the exchange rate μ is determined solely from the above. However, it is unlikely so that the net capital flow F into country 1 is introduced. Then the equation is modified to include F.[3]

$$\begin{aligned} & p_1^1 a_{11}^{12} x_1^2 + p_2^1 a_{21}^{12} x_1^2 + p_1^1 a_{12}^{12} x_2^2 + p_2^1 a_{22}^{12} x_2^2 + p_1^1 y_1^{12} + p_2^1 y_2^{12} + F \\ - & \mu (p_1^2 a_{11}^{21} x_1^1 + p_2^2 a_{21}^{21} x_1^1 + p_1^2 a_{12}^{21} x_2^1 + p_2^2 a_{22}^{21} x_2^1 + p_1^2 y_1^{21} + p_2^2 y_2^{21}) = 0 \end{aligned} \tag{25}$$

In the world complete with two countries, the balance of payments for country 2, where the capital flow is given by $-\mu F$, brings no additional information. Then the exchange rate is directly calculated from (25).

$$\mu = \frac{p_1^1 a_{11}^{12} x_1^2 + p_2^1 a_{21}^{12} x_1^2 + p_1^1 a_{12}^{12} x_2^2 + p_2^1 a_{22}^{12} x_2^2 + p_1^1 y_1^{12} + p_2^1 y_2^{12} + F}{p_1^2 a_{11}^{21} x_1^1 + p_2^2 a_{21}^{21} x_1^1 + p_1^2 a_{12}^{21} x_2^1 + p_2^2 a_{22}^{21} x_2^1 + p_1^2 y_1^{21} + p_2^2 y_2^{21}} \tag{26}$$

In the present framework, where the final demands in physical units are given exogenously, the physical outputs can be determined independent of the price system. Thus the solution to the problem (7) can readily be calculated.

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} I - A^{11} & -A^{12} \\ -A^{21} & I - A^{22} \end{pmatrix}^{-1} \begin{pmatrix} y^{11} + y^{12} \\ y^{21} + y^{22} \end{pmatrix} \tag{27}$$

However, monetary variables w^r and p_i^r are to be determined through an iterative process. When wages (w^1, w^2) are appropriately given, the corresponding price vectors (p^1, p^2) are calculated by (10). Then given capital flow F, the initial exchange rate $\tilde{\mu}^{(0)}$ is determined by (26). While the wages must be evaluated in the local currency, our initial setup is denominated in the common currency. Hence, the wage in country 2 must be revised to reflect the provisional exchange rate $\mu = \tilde{\mu}^{(0)}$, in step $k = 1$.

$$(p^1 \quad p^2) = (w^1 a_0^1 \quad \mu w^2 a_0^2) \begin{pmatrix} I - A^{11} & -A^{12} \\ -A^{21} & I - A^{22} \end{pmatrix}^{-1} = \begin{pmatrix} w^1 a_0^1 B^{11} & w^1 a_0^1 B^{12} \\ \mu w^2 a_0^2 B^{21} & \mu w^2 a_0^2 B^{22} \end{pmatrix}.$$

[3] To be exact, the balance of payments is obtained as the sum of trade balance, income transfer, and capital flows. Here the sum of the latter two is simply called "capital flow".

When these revised price vectors are plugged into (26), the incremental exchange rate $\tilde{\mu}^{(k)}$ is obtained. The convergence is reached when $|\tilde{\mu}^{(k)} - 1| < \epsilon$ is satisfied with sufficiently small $\epsilon > 0$. Otherwise, the above process must be repeated with the exchange rate $\mu = \prod_{i=0}^{k} \tilde{\mu}^{(i)}$ in step $k+1$. If the process converged at step ℓ, the exchange rate and corresponding country 2's wage in the local currency are obtained, respectively, as follows:

$$\mu = \prod_{k=1}^{\ell} \tilde{\mu}^{(i)} \quad \text{and} \quad \hat{w}^2 = \mu w^2.$$

When there exist $m > 2$ countries, $m-1$ independent exchange rates are determined. The balance of payments for country r can be written as follows:

$$\mu^r \sum_i p_i^r \sum_{s \neq r} (\sum_j a_{ij}^{rs} x_j^s + y_i^{rs}) = \sum_{s \neq r} \mu^s \sum_i p_i^s (\sum_j a_{ij}^{sr} x_j^r + y_i^{sr}) \tag{28}$$

By letting $\mu^1 = 1$, the exchange rates $\mu^r (r = 2, \ldots, m)$ can be solved from $m-1$, out of the sum of m, equations (28) using the similar iterative process as described above. In any case, it must be emphasized that the exchange rates crucially depend on how the wage levels in individual countries and capital flows among them are specified.

7 GHG emissions

Consider a world of two countries where GHG emissions are not priced. The output system can be written exactly as the constraints in problem (7).

$$\begin{aligned} (I - A^{11})x^1 - A^{12}x^2 &= y^{11} + y^{12}, \\ -A^{21}x^1 + (I - A^{22})x^2 &= y^{21} + y^{22}. \end{aligned}$$

Let a_g^r be the unit emission vector from production activities, and e_g^r be the same from consumption of final products in country r.[4] Then the emission in each country is calculated as follows:[5]

$$g^1 = a_g^1 x^1 + e_g^1 (y^{11} + y^{21}) \quad \text{and} \quad g^2 = a_g^2 x^2 + e_g^2 (y^{12} + y^{22})$$

Since the Leontief inverse represents the infinite repercussions of inter-sector and international transactions, it is straightforward to assess the impact of each final demand segment on GHG emission of each country.

$$\begin{aligned} \begin{pmatrix} g^1 \\ g^2 \end{pmatrix} &= \begin{pmatrix} a_g^1 & 0 \\ 0 & a_g^2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} e_g^1 & e_g^1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y^{11} \\ y^{21} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ e_g^2 & e_g^2 \end{pmatrix} \begin{pmatrix} y^{12} \\ y^{22} \end{pmatrix} \\ &= \begin{pmatrix} a_g^1 B^{11} + e_g^1 & a_g^1 B^{12} + e_g^1 \\ a_g^2 B^{21} & a_g^2 B^{22} \end{pmatrix} \begin{pmatrix} y^{11} \\ y^{21} \end{pmatrix} + \begin{pmatrix} a_g^1 B^{11} & a_g^1 B^{12} \\ a_g^2 B^{21} + e_g^2 & a_g^2 B^{22} + e_g^2 \end{pmatrix} \begin{pmatrix} y^{12} \\ y^{22} \end{pmatrix} \end{aligned} \tag{29}$$

[4] The seminal article by Leontief (1970) only considers pollutants from production sectors. However, the emission from the final demand sectors cannot be ignored when GHG is concerned.

[5] When the gas in concern is unique, a_g^r becomes a row vector of size n, but it can easily be extended to cover k kinds of gases. In that case, a_g^r becomes a $k \times n$ matrix. Moreover, the same formulation can also be applied to water resources. In that case, a_g^r and e_g^r are interpreted as the unit water demand associated with the production process, and with the final demand consumption, respectively.

Each country is responsible for the emissions accrued from her final demand. For example, country 1's emission g^1 can be decomposed into the two parts, viz. g^{11} and g^{12}, for which country 1 and 2 are responsible, respectively.

$$g^{11} = (a_g^1 B^{11} + e_g^1)y^{11} + (a_g^1 B^{12} + e_g^1)y^{21} \quad \text{and} \quad g^{12} = a_g^1 B^{11} y^{12} + a_g^1 B^{12} y^{22}.$$

Likewise, country 2's emission g^2 can also be decomposed:

$$g^{21} = a_g^2 B^{21} y^{11} + a_g^2 B^{22} y^{21} \quad \text{and} \quad g^{22} = (a_g^2 B^{21} + e_g^2)y^{12} + (a_g^2 B^{22} + e_g^2)y^{22}.$$

Similarly as c_i^{rs} in (12), it is possible to define the ratio f^{rs} of gas emission in country r, for which country s is responsible, in a multi-country setting as follows:[6]

$$f^{rr} = \frac{\sum_\ell (a_g^r B^{r\ell} + e_g^r) y^{\ell r}}{a_g^r x^r + e_g^r \sum_\ell y^{\ell r}} \quad \text{and} \quad f^{rs} = \frac{a_g^r \sum_\ell B^{r\ell} y^{\ell s}}{a_g^r x^r + e_g^r \sum_\ell y^{\ell r}} \quad (r \neq s) \tag{30}$$

with $\sum_s f^{rs} = 1$ being satisfied by definition. Alternately, the above expressions can be detailed to the commodity level:

$$f^{rr} = \frac{\sum_i (a_{gi}^r \sum_\ell \sum_j b_{ij}^{r\ell} y_j^{\ell r} + e_{gi}^r \sum_\ell y_i^{\ell r})}{\sum_i (a_{gi}^r x_i^r + e_{gi}^r \sum_\ell y_i^{\ell r})} \quad \text{and} \quad f^{rs} = \frac{\sum_i a_{gi}^r \sum_\ell \sum_j b_{ij}^{r\ell} y_j^{\ell s}}{\sum_i (a_{gi}^r x_i^r + e_{gi}^r \sum_\ell y_i^{\ell r})} \quad (r \neq s) \tag{31}$$

In the matrix form, equation (29) can easily be extended to the multi-country case by defining the following matrices:

$$A_g = \begin{pmatrix} a_g^1 & 0 & \cdots & 0 \\ 0 & a_g^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_g^m \end{pmatrix}, B = \begin{pmatrix} B^{11} & \cdots & B^{1m} \\ \vdots & \ddots & \vdots \\ B^{m1} & \cdots & B^{mm} \end{pmatrix}, E_g = \begin{pmatrix} e_g^1 & e_g^1 & \cdots & e_g^1 \\ e_g^2 & e_g^2 & \cdots & e_g^2 \\ \vdots & \vdots & \ddots & \vdots \\ e_g^m & e_g^m & \cdots & e_g^m \end{pmatrix},$$

where A_g, B and E_g are the matrices of size $m \times mn$, $mn \times mn$, and $m \times mn$, respectively. Further define

$$X = \begin{pmatrix} x^1 \\ \vdots \\ x^m \end{pmatrix}, Y = \begin{pmatrix} y^{11} & \cdots & y^{1m} \\ \vdots & \ddots & \vdots \\ y^{m1} & \cdots & y^{mm} \end{pmatrix}, g = \begin{pmatrix} g^1 \\ \vdots \\ g^m \end{pmatrix}, \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

which are a column vector of size mn, a matrix of size $mn \times m$, a column vector of size m, and the all-one vector of the same size, respectively. Then the decomposition of GHG emissions from the production process can be written in the following matrix formula:

$$A_g X = A_g B Y \mathbf{1}.$$

With the operator $Diag(\bullet)$ to extract the diagonal elements of square matrices, the emissions from final demand consumption can be written as $Diag(E_g Y)\mathbf{1}$. Thus the emission vector G can be written, in a matrix form, as follows:

$$g = (A_g B Y + Diag(E_g Y))\mathbf{1}. \tag{32}$$

[6] In the case of price decomposition, c_i^{rs} represents the share of country s's product i that comes from country r. When comparing the sums $\sum_r c_i^{rs} = 1$ and $\sum_s f^{rs} = 1$, the superscripts in f^{rs} may seem confusing as they represent the opposite direction. This reflects the fact that TiVA represents the backward linkage while emission responds to the forward linkage.

The decomposition of GHG emissions over the countries can then be obtained using (30):

$$G = \begin{pmatrix} g^{11} & g^{12} & \cdots & g^{1m} \\ g^{21} & g^{22} & \cdots & g^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ g^{m1} & g^{m2} & \cdots & g^{mm} \end{pmatrix} = \begin{pmatrix} g^{1} & 0 & \cdots & 0 \\ 0 & g^{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g^{m} \end{pmatrix} \begin{pmatrix} f^{11} & f^{12} & \cdots & f^{1m} \\ f^{21} & f^{22} & \cdots & f^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ f^{m1} & f^{m2} & \cdots & f^{mm} \end{pmatrix} = diag(g)F. \tag{33}$$

8 The problem with GHG abatement

While the model in the previous section is open-ended in the sense that it simply calculate the GHG emissions and clarify the responsibility of each country without considering environmental restrictions. In contrast, when such restrictions along with the pollution abatement activity are introduced, it is possible to assess the fair penalty for the GHG discharged to the environment. In the case of an isolated country with two industrial and one abatement sectors, the environmental restriction is normally given by the following form:

$$a_{g1}x_1 + a_{g2}x_2 + a_{gg}x_g + e_{g1}y_1 + e_{g2}y_2 - x_g \leq g, \tag{34}$$

where g is the amount of GHG permitted to the environment, x_g is the amount of GHG eliminated, and a_{gg} is the GHG emission by the abatement activity. Likewise the output requirement for industrial sectors can be written with a_{ig}, the input requirement for a unit reduction of GHG, as follows:

$$x_i - a_{i1}x_1 - a_{i2}x_2 - a_{ig}x_g \geq y_i \quad (i = 1, 2).$$

Considering the direction of inequalities, the linear programming problem similar to (1) can be formulated with the labor input in the abatement sector a_{0g}.

$$\begin{array}{ll} \min_{x_1,x_2,x_g} & w(a_{01}x_1 + a_{02}x_2 + a_{0g}x_g), \\ \text{s.t.} & (1 - a_{11})x_1 - a_{12}x_2 - a_{1g}x_g \geq y_1, \\ & -a_{21}x_1 + (1 - a_{22})x_2 - a_{2g}x_g \geq y_2, \\ & -a_{g1}x_1 - a_{g2}x_2 + (1 - a_{gg})x_g \geq e_{g1}y_1 + e_{g2}y_2 - g, \\ & x_1, x_2, x_g \geq 0. \end{array} \tag{35}$$

The solution to this problem is readily be obtained.[7]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_g \end{pmatrix} = \begin{pmatrix} 1 - a_{11} & -a_{12} & -a_{1g} \\ -a_{21} & 1 - a_{22} & -a_{2g} \\ -a_{g1} & -a_{g2} & 1 - a_{gg} \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \\ e_{g1}y_1 + e_{g2}y_2 - g \end{pmatrix} \tag{36}$$

[7] According to the weak-solvability, the solution to the Leontief model, $x = (I - A)^{-1}y$, is guaranteed non-negative when the Leontief matrix $(I - A)$ is positive definite and y is non-negative (see e.g. Nikaido, 1968). In this case, however, such conditions does not necessarily apply since $e_{g1}y_1 + e_{g2}y_2 - g$ could be negative in an unrealistic case where the environmental restriction is very loose and no need for abatement.

When λ denotes the Lagrangian multiplier assigned to (34), it is interpreted as the price of unit GHG emission. All the price variables, including λ, are obtained from the dual system.

$$(p_1 \quad p_2 \quad \lambda) = w(a_{01} \quad a_{02} \quad a_{0g}) \begin{pmatrix} 1-a_{11} & -a_{12} & -a_{1g} \\ -a_{21} & 1-a_{22} & -a_{2g} \\ -a_{g1} & -a_{g2} & 1-a_{gg} \end{pmatrix}^{-1} \tag{37}$$

In order to extend the problem with the GHG abatement sector to the world problem comprising two countries, the output equations are formulated for individual countries as follows:

$$\begin{pmatrix} x_1^1 \\ x_2^1 \\ x_g^1 \end{pmatrix} - \begin{pmatrix} a_{11}^{11} & a_{12}^{11} & a_{1g}^{11} \\ a_{21}^{11} & a_{22}^{11} & a_{2g}^{11} \\ a_{g1}^{1} & a_{g2}^{1} & a_{gg}^{1} \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_g^1 \end{pmatrix} - \begin{pmatrix} a_{11}^{12} & a_{12}^{12} & a_{1g}^{12} \\ a_{21}^{12} & a_{22}^{12} & a_{2g}^{12} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_g^2 \end{pmatrix} = \begin{pmatrix} y_1^{11} + y_1^{12} \\ y_2^{11} + y_2^{12} \\ e_{g1}^1(y_1^{11} + y_1^{21}) + e_{g2}^1(y_2^{11} + y_2^{21}) - g^1 \end{pmatrix}$$

$$\begin{pmatrix} x_1^2 \\ x_2^2 \\ x_g^2 \end{pmatrix} - \begin{pmatrix} a_{11}^{21} & a_{12}^{21} & a_{1g}^{21} \\ a_{21}^{21} & a_{22}^{21} & a_{2g}^{21} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_g^1 \end{pmatrix} - \begin{pmatrix} a_{11}^{22} & a_{12}^{22} & a_{1g}^{22} \\ a_{21}^{22} & a_{22}^{22} & a_{2g}^{22} \\ a_{g1}^{2} & a_{g2}^{2} & a_{gg}^{2} \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_g^2 \end{pmatrix} = \begin{pmatrix} y_1^{21} + y_1^{22} \\ y_2^{21} + y_2^{22} \\ e_{g1}^2(y_1^{12} + y_1^{22}) + e_{g2}^2(y_2^{12} + y_2^{22}) - g^2 \end{pmatrix}$$

Here transportation of GHG across countries is precluded; i.e., production activity in a country does not discharge GHG in the other country.[8]

For convenience sake, let $\tilde{A}^{11}, \tilde{A}^{12}, \tilde{A}^{21}$, and $\tilde{A}^{22}$ respectively denote the matrices of input coefficients in the order as they appeared in the above two equations. Also let u^1 and u^2 denote the column vectors in the right hand side of the above equations. Moreover, the augmented column vector of outputs and row vector of labor inputs are defined as follows:

$$\tilde{x}^r = (x_1^r \quad x_2^r \quad x_g^r)' \quad \text{and} \quad \tilde{a}_0^r = (a_{01}^r \quad a_{02}^r \quad a_{0g}^r).$$

Then the world problem with GHG abatement activity can be formulated in a matrix form.

$$\begin{aligned} \min_{\tilde{x}^1, \tilde{x}^2} \quad & w^1\tilde{a}_0^1\tilde{x}^1 + w^2\tilde{a}_0^2\tilde{x}^2 \\ \text{s.t.} \quad & (I - \tilde{A}^{11})\tilde{x}^1 - \tilde{A}^{12}\tilde{x}^2 \geq u^1 && (38) \\ & -\tilde{A}^{21}\tilde{x}^1 + (I - \tilde{A}^{22})\tilde{x}^2 \geq u^2 && (39) \\ & \tilde{x}^1, \tilde{x}^2 \geq 0, \end{aligned}$$

By denoting the row vectors of Lagrange multipliers attached to (38) and (39) as q^1 and q^2, respectively, the Lagrangian function for the problem can be written as follows:

$$L = w^1\tilde{a}_0^1\tilde{x}^1 + w^2\tilde{a}_0^2\tilde{x}^2 + q^1(u^1 - (I - \tilde{A}^{11})\tilde{x}^1 + \tilde{A}^{12}\tilde{x}^2) + q^2(u^2 + \tilde{A}^{21}\tilde{x}^1 - (I - \tilde{A}^{22})\tilde{x}^2). \tag{40}$$

With the non-negative constraints, the first-order conditions would become:

$$\begin{aligned} \frac{\partial L}{\partial \tilde{x}^1} &= w^1\tilde{a}_0^1 - q^1(I - \tilde{A}^{11}) + q^2\tilde{A}^{21} \geq 0 \\ \frac{\partial L}{\partial \tilde{x}^2} &= w^2\tilde{a}_0^2 + q^1\tilde{A}^{12} - q^2(I - \tilde{A}^{22}) \geq 0 && (41) \end{aligned}$$

[8] It must be noted that the combination of traded final demands is different for the GHG line from the rest of commodities.

Thus the multipliers are determined as follows:

$$(q^1 \quad q^2) = (p_1^1 \;\; p_2^1 \;\; \lambda^1 \;\; p_1^2 \;\; p_2^2 \;\; \lambda^2) \leq (w^1\tilde{a}_0^1 \quad w^2\tilde{a}_0^2) \begin{pmatrix} I - \tilde{A}^{11} & -\tilde{A}^{12} \\ -\tilde{A}^{21} & I - \tilde{A}^{22} \end{pmatrix}^{-1} \tag{42}$$

When the vector $(\tilde{x}^1, \tilde{x}^2)$ are positive, (42) holds in equality. However, it is not necessarily true with very loose environmental restrictions since the GHG emission may become a "free good" ($\lambda^r = 0$) in that case.

Suppose all the constraints are binding, and $\tilde{B}^{rs}$ denotes the element of Leontief inverse in (42). Then the responsibilities of GHG emissions are distributed over the countries similarly as in the case without abatement activity.

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} (\tilde{a}_g^1\tilde{B}^{11} + \tilde{e}_g^1)\tilde{y}^{11} + (\tilde{a}_g^1\tilde{B}^{12} + \tilde{e}_g^1)\tilde{y}^{21} & \tilde{a}_g^1\tilde{B}^{11}\tilde{y}^{12} + \tilde{a}_g^1\tilde{B}^{12}\tilde{y}^{22} \\ \tilde{a}_g^2\tilde{B}^{21}\tilde{y}^{11} + \tilde{a}_g^2\tilde{B}^{22}\tilde{y}^{21} & (\tilde{a}_g^2\tilde{B}^{21} + \tilde{e}_g^2)\tilde{y}^{12} + (\tilde{a}_g^2\tilde{B}^{22} + \tilde{e}_g^2)\tilde{y}^{22} \end{pmatrix}, \tag{43}$$

where vectors $\tilde{a}_g^r, \tilde{e}_g^r$, and $\tilde{y}^{rs}$ are also augmented to include the abatement.

$$\tilde{a}_g^r = (a_{g1}^r \;\; a_{g2}^r \;\; a_{gg}^r), \quad \tilde{e}_g^r = (e_{g1}^r \;\; e_{g2}^r \;\; 0), \quad \text{and} \quad \tilde{y}^{rs} = (y_1^{rs} \;\; y_2^{rs} \;\; 0)'.$$

By the same token, the values added can also be decomposed as follows:

$$\begin{pmatrix} Y^{11} & Y^{12} \\ Y^{21} & Y^{22} \end{pmatrix} =$$
$$\begin{pmatrix} q^1\tilde{C}^{11}(\tilde{y}^{11} + \tilde{y}^{12} + \tilde{A}^{12}\tilde{x}^2) - q^2\tilde{C}^{22}\tilde{A}^{21}\tilde{x}^1 & q^1\tilde{C}^{12}(\tilde{y}^{21} + \tilde{y}^{22} + \tilde{A}^{21}\tilde{x}^1) - q^1\tilde{C}^{21}\tilde{A}^{12}\tilde{x}^2 \\ q^2\tilde{C}^{21}(\tilde{y}^{11} + \tilde{y}^{12} + \tilde{A}^{12}\tilde{x}^2) - q^2\tilde{C}^{12}\tilde{A}^{21}\tilde{x}^1 & q^2\tilde{C}^{22}(\tilde{y}^{21} + \tilde{y}^{22} + \tilde{A}^{21}\tilde{x}^1) - q^1\tilde{C}^{11}\tilde{A}^{12}\tilde{x}^2 \end{pmatrix}, \tag{44}$$

where the diagonal matrix $\tilde{C}^{rs}$ is augmented to include the decomposition of GHG abatement cost λ^s.

In this article, we demonstrated that the linear programming equivalent of the physical input-output system can decompose not only values added, but also GHG emissions to the ultimate beneficiaries or causes in a consistent manner. The GHG emissions are likely proportional to the physical amounts produced or consumed than their monetary values. For example, fuel efficiency would better be evaluated by the liters rather than dollars of gasoline burned, and thus, the use of physical system seems more appropriate. When the GHG abatement activity is introduced, the price for emission right can be endogenized. Then the question is how to determine the fair allocation of emission permits g (see Uzawa, 2003). The tradable emission permits introduce income transfer among countries. Besides when the domestic labor market and international financial market are properly combined, the physical framework can also endogenize the exchange rates. Annexing several markets outside the input-output system, the system becomes closer to the spatial computable general equilibrium (SCGE) model (see e.g. Ando and Meng, 2014).

Although the physical input-output system has several desirable properties, the problem is that the physical (international) tables are not available. Then our task will be to compile a non-survey physical table from existing monetary tables, and derive some meaningful analytical results. However, such a task is beyond the scope of this article, and has been left in the future.

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