Angular Measure for Matrix Similarity and Hadamard-multiplicative Generalization of RAS Method

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The paper presents a new GLS-based method for updating economic tables within angular paradigm for structural similarity of rectangular matrices. The notion of a homothetic ray for vectorized initial matrix in system of relative coordinates in multidimensional Euclidean space is introduced. The ensuing notions of angular and homothetic measures for matrix similarity are studied. Unified analytical solution of constrained minimization problem for the angular measure as an objective function is derived in matrix notation. Special attention is paid to sensitivity of minimization problem's solution to small changes in marginal totals of the target matrix. A chain of illustrative numerical examples is given.

Keywords: matrix updating methods, RAS multiplicative pattern, matrix homothety, angular and homothetic measures of matrix similarity, constrained minimization, Lagrange multipliers

JEL Classification: C61; C67

1. An introduction

The subject of this study is a general problem for updating rectangular (or square) matrices, which can be formulated as follows. Let **A** be an initial matrix of dimension $N\times M$ with row and column marginal totals $\mathbf{u_A} = \mathbf{A}\mathbf{e}_M$, $\mathbf{v}'_\mathbf{A} = \mathbf{e}'_N\mathbf{A}$ where $\mathbf{e}_N$ and $\mathbf{e}_M$ are $N\times 1$ and $M\times 1$ summation column vectors with unit elements. Further, let $\mathbf{u} \neq \mathbf{u_A}$ and $\mathbf{v} \neq \mathbf{v_A}$ be exogenous column vectors of dimension $N\times 1$ and $M\times 1$, respectively. The problem is to estimate a target matrix **X** of dimension $N\times M$ at the highest possible level of structural similarity (or closeness, etc.) to initial matrix under $N+M$ equality constraints

$$\mathbf{X}\mathbf{e}_M = \mathbf{u}, \qquad \mathbf{e}'_N\mathbf{X} = \mathbf{v}' \tag{1}$$

and the consistency condition

$$\mathbf{e}'_N\mathbf{u} = \mathbf{e}'_M\mathbf{v}. \tag{2}$$

Clearly, the system of equations (1) is dependent at consistency condition (2). However, any $N+M-1$ among $N+M$ constraints (1) are mutually independent.

It is assumed that initial matrix does not include any zero rows or zero columns, does not have less than $N+M$ nonzero elements, does not include any rows or columns with a unique nonzero element, and does not contain any pairs of rows and columns with four nonzero elements in the intersections. Otherwise, it is expedient to "clear" matrix **A** of undesirable features before applying any matrix updating method in practice.

The aim of this paper is to make more operational and to advance the notion of angular measure for structural similarity between target and initial matrices introduced in Motorin (2014).

2. The RAS multiplicative pattern

The key idea of the well-known and widely used RAS method is a factorization of target matrix

$$\mathbf{X} = \mathbf{RAS} = \langle \mathbf{r} \rangle \mathbf{A} \langle \mathbf{s} \rangle = \hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}} \tag{3}$$

where **r** and **s** are unknown $N\times1$ and $M\times1$ column vectors. Here angled bracketing around a vector's symbol or putting a "hat" over it denotes a diagonal matrix, with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697).

Putting (3) into (1), we have the system of nonlinear equations

$$\hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}}\mathbf{e}_M = \hat{\mathbf{r}}\mathbf{A}\mathbf{s} = \langle \mathbf{A}\mathbf{s} \rangle \mathbf{r} = \mathbf{u}, \qquad \mathbf{e}'_N \hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}} = \mathbf{r}'\mathbf{A}\hat{\mathbf{s}} = \mathbf{s}' \langle \mathbf{A}'\mathbf{r} \rangle = \mathbf{v}'. \tag{4}$$

Proper transformations of system (4) lead to following pair of iterative processes:

$$\mathbf{r}_{(i)} = \left\langle \mathbf{A} \left\langle \mathbf{A}'\mathbf{r}_{(i-1)} \right\rangle^{-1} \mathbf{v} \right\rangle^{-1} \mathbf{u}, \quad i = 1 \div I; \qquad \mathbf{s}_{(I)} = \left\langle \mathbf{A}'\mathbf{r}_{(I)} \right\rangle^{-1} \mathbf{v}; \tag{5}$$

$$\mathbf{s}_{(j)} = \left\langle \mathbf{A}' \left\langle \mathbf{A}\mathbf{s}_{(j-1)} \right\rangle^{-1} \mathbf{u} \right\rangle^{-1} \mathbf{v}, \quad j = 1 \div J; \qquad \mathbf{r}_{(J)} = \left\langle \mathbf{A}\mathbf{s}_{(J)} \right\rangle^{-1} \mathbf{u} \tag{6}$$

where i and j are iteration numbers.

Thus, in RAS method the structural similarity between target and initial matrices is provided by $(N+M)$-parametrical multiplicative pattern

$$x_{nm} = r_n s_m a_{nm}, \quad n = 1 \div N, \quad m = 1 \div M \tag{7}$$

where the character " ÷ " between the lower and upper bounds of index's changing range means that the index sequentially runs all integer values in the specified range. Note that pair of constraints (1) restricts the scattering of factors $r_n s_m$ around some constant level. Further, the multiplicative pattern (7) preserves zero elements of matrix **A** in the same positions inside **X** that seems to be a significant contribution to structural similarity between the initial and target matrices.

In accordance with (7), the factorization (3) can be written as

$$\mathbf{X} = (\mathbf{r}\mathbf{s}') \circ \mathbf{A} \tag{8}$$

where the character " ∘ " denotes the Hadamard's product for two matrices of the same dimensions. Since the matrix $\mathbf{r}\mathbf{s}'$ does not contain any zero elements, matrices **A** and **X** all have the same inner location of zeros.

3. A general Hadamard-multiplicative model

A natural way to generalize $(N+M)$-parametrical multiplicative pattern (8) is to replace factors

$r_n s_m$ with more common coefficients q_{nm} and to consider (*NM*)-parametrical model

$$\mathbf{X} = \mathbf{Q} \circ \mathbf{A} \tag{9}$$

where **Q** is *N*×*M* matrix of unknown coefficients q_{nm}.

It is easy to see that multiplicative model (9) is not strictly (or just) identifiable if the initial and target matrices do contain one or more zero elements. To illustrate this statement let us assume that matrices **A** and **X** are known both. Nevertheless, it does not allow to identify coefficient matrix **Q** as a unique one because

$$q_{nm} = \begin{cases} x_{nm} / a_{nm}, & \text{if } a_{nm} \neq 0; \\ \text{any scalar}, & \text{if } a_{nm} = 0; \end{cases} \qquad n = 1 \div N, \quad m = 1 \div M. \tag{10}$$

In the context of model's identifiability consider a particular case of strict proportionality between row and column marginal totals $\mathbf{u} = k\mathbf{u}_{\mathbf{A}}$ and $\mathbf{v} = k\mathbf{v}_{\mathbf{A}}$ with the same multiplier *k*. It can be easily shown that under starting condition $\mathbf{r}_{(0)} = \mathbf{e}_N$ or $\mathbf{s}_{(0)} = \mathbf{e}_M$ the RAS method iterative process (5) or (6) demonstrates one-step convergence to pair of vectors $\mathbf{r} = \mathbf{e}_N$, $\mathbf{s} = k\mathbf{e}_M$ or to $\mathbf{r} = k\mathbf{e}_N$, $\mathbf{s} = \mathbf{e}_M$, respectively. Hence $r_n s_m = k$ for any *n* and *m*, $n = 1 \div N$, $m = 1 \div M$, or in matrix notation $\mathbf{rs}' = k\mathbf{e}_N\mathbf{e}'_M$ and **X** = *k***A**. Further, it is easy to see that the replacing initial matrix **A** with its homothety *k***A** leaves the RAS method iterations (5) and (6) invariant.

From above, one can establish the fact of an excellent structural similarity of all matrices from homothetic family $k\mathbf{A} = k(\mathbf{e}_N\mathbf{e}'_M) \circ \mathbf{A}$, $k \geq 0$ in accordance with RAS logic. This conclusion may serve as a criterion base for constructing the operational method to estimate of unknown coefficients in multiplicative model (9) strictly. Indeed, setting a goal to dispose the target matrix as close to homothetic family *k***A** as possible, we obtain a uniparametrical (with parameter *k*) optimization problem that prescribe to minimize a certain norm of matrix $\mathbf{X} - k\mathbf{A} = (\mathbf{Q} - k\mathbf{e}_N\mathbf{e}'_M) \circ \mathbf{A}$ subject to the pair of constraints $(\mathbf{Q} \circ \mathbf{A})\mathbf{e}_M = \mathbf{u}$ and $\mathbf{e}'_N(\mathbf{Q} \circ \mathbf{A}) = \mathbf{v}'$. Note that this problem is represented here in preliminary formulation and is not quite operational yet.

4. A common approach to model identification

The handling of optimization problem formulated above becomes more operational with its vectorization. Applying vectorization operator *vec* (see Magnus and Neudecker, 2007), which transforms a matrix into a vector by stacking the columns of the matrix one underneath the other, to each matrix in (9) gives multiplicative model

$$\mathbf{x} = \hat{\mathbf{a}}\mathbf{q} \tag{11}$$

where $\mathbf{x} = vec\,\mathbf{X}$, $\mathbf{a} = vec\,\mathbf{A}$ and $\mathbf{q} = vec\,\mathbf{Q}$ are column vectors with dimension *NM*×1. In general, one can choose any other operator for vectorization of matrices **X**, **Q** and **A**. Note that diagonal matrix $\hat{\mathbf{a}}$ is singular whenever **A** contains at least one zero element.

A quite explainable requirement to dispose the target vector **x** as close to vector homothety $k\mathbf{a}$ as possible entails uniparametrical optimization problem that prescribes to minimize a certain norm of vector $\mathbf{x} - k\mathbf{a} = \hat{\mathbf{a}}(\mathbf{q} - k\mathbf{e}_{NM})$ subject to constraints (1) properly transformed into vectorized forms (see Section 6 below). In order to make an objective function of this problem independent on initial data, it is expedient to transit into a system of relative coordinates by omitting diagonal matrix $\hat{\mathbf{a}}$ in right-hand side of latter equality and to minimize (subject to vectorized constraints) a norm of vector $\mathbf{q} - k\mathbf{e}_{NM}$ which measures a deviation of relative target vector **q** from relative homothetic ray $k\mathbf{e}_{NM}$ at $k \geq 0$.

Optimization problem under consideration contains *NM* unknown variables **q**, one unknown scalar parameter *k*, and *N*+*M* constraints from which *N*+*M*–1 constraints are mutually independent. The main associated question is how to properly define a measure for similarity between a vector and a ray.

5. Angular and homothetic measures for matrix similarity

There are two essential ways to estimate a deviation of relative target vector **q** from homothetic ray $k\mathbf{e}_{NM}$. First, most natural measure for similarity between a vector and a ray can be defined as a value of the angle between **q** and $k\mathbf{e}_{NM}$, which is assumed to be acute. Secondly, the shortest path from the point **q** to the ray $k\mathbf{e}_{NM}$ can serve as an alternative measure that is further called homothetic one.

If $(\mathbf{y}, \mathbf{z}) = \mathbf{y}'\mathbf{z} = \mathbf{z}'\mathbf{y}$ is an inner product of vectors **y** and **z** in *NM*-dimensional Euclidean space, then angle $\beta_{\mathbf{qe}}$ between target vector **q** and ray $k\mathbf{e}_{NM}$ is determined (in radians) by well-known formula

$$\beta_{\mathbf{qe}} = \arccos\left[\frac{(\mathbf{q}, \mathbf{e}_{NM})}{|\mathbf{q}| \cdot |\mathbf{e}_{NM}|}\right] = \arccos\left[\frac{\mathbf{e}'_{NM}\mathbf{q}}{(\mathbf{q}'\mathbf{q})^{1/2} \cdot (\mathbf{e}'_{NM}\mathbf{e}_{NM})^{1/2}}\right] \tag{12}$$

where $|\mathbf{q}| = (\mathbf{q}'\mathbf{q})^{1/2}$ is a length of vector **q**.

Orthogonal projection of **q** on the ray $k\mathbf{e}_{NM}$ is determined by coefficient $k^{\perp} = \mathbf{e}'_{NM}\mathbf{q}/\mathbf{e}'_{NM}\mathbf{e}_{NM}$ from evident condition $\mathbf{e}'_{NM}(\mathbf{q} - k^{\perp}\mathbf{e}_{NM}) = 0$ and equals vector $k^{\perp}\mathbf{e}_{NM}$. Hence, the shortest path from the point **q** to the ray $k\mathbf{e}_{NM}$ is lying along the vector

$$\boldsymbol{\delta} = \mathbf{q} - k^{\perp}\mathbf{e}_{NM} = \left(\mathbf{E}_{NM} - \frac{\mathbf{e}_{NM}\mathbf{e}'_{NM}}{\mathbf{e}'_{NM}\mathbf{e}_{NM}}\right)\mathbf{q} \tag{13}$$

where $\mathbf{E}_{NM}$ is identity matrix of order *NM*. Note that the inner product of vectors δ and $\mathbf{e}_{NM}$ equals $\mathbf{e}'_{NM}\boldsymbol{\delta} = \mathbf{e}'_{NM}\mathbf{q} - \mathbf{e}'_{NM}\mathbf{q} = 0$, so δ is orthogonal to homothetic ray $k\mathbf{e}_{NM}$ and besides has zero sum of elements.

Thus, it is clear now how to specify a scalar at $a_{nm} = 0$ in (10): to prevent an artificial increasing of vector δ's length one must let this scalar be the average ratio $x_{\mathrm{nm}} / a_{\mathrm{nm}}$ on the subset of nonzero elements in **A**. From the parameter estimation theory viewpoint using (13) as a measuring vector corresponds to an unbiased estimation of **q** with least variance (see, e.g., Wackerly, Mendenhall and Scheaffer, 2008).

It can be shown that symmetric idempotent matrix in parentheses in (13) has zero eigenvalue with unit multiplicity and corresponding eigenvector $\mathbf{e}_{NM}$, and also has unit eigenvalue with multiplicity *NM*–1 and corresponding eigenvector **z** from the hyperplane $\mathbf{e}'_{NM}\mathbf{z} = 0$, which is orthogonal to homothetic ray. So this singular matrix has rank *NM*–1.

It is easy to detect a linkage between angular and homothetic measures for matrix similarity introduced above because a solution of the right triangle with the sides $\mathbf{q}, k^{\perp}\mathbf{e}_{NM}$ and **δ** gives

$$\sin^2 \beta_{\mathbf{qe}} = \frac{\boldsymbol{\delta}'\boldsymbol{\delta}}{\mathbf{q}'\mathbf{q}}. \tag{14}$$

From geometrical viewpoint one can conclude that angular measure (14) and homothetic measure (13) are consistent only for any pair of relative target vectors **q** and **p** satisfying orthogonality condition $\mathbf{e}'_{NM}(\mathbf{q} - \mathbf{p}) = 0$, i.e., all testing target vectors must have the same orthogonal projection onto homothetic ray.

As a conclusion, an angle between target vector **q** and homothetic ray $k\mathbf{e}_{NM}$ at $k \geq 0$ can be considered as a universal measure of structural similarity between target and initial matrices. Main "technical" disadvantage of angular measure appears to be the complexity of formulae (12) and (14) along with arising difficulties of using (14) to construct competing (in particular, with the RAS method) algorithms of matrix updating. Based on orthogonal projecting operation, homothetic measure (13) is the simplified version of an angular measure with some shortcomings. Nevertheless, homothetic measure demonstrates a row of helpful properties and may become operational in various algorithmic schemes.

6. Vectorization of the linear constraints

To apply the results obtained above for constructing certain algorithms of matrix updating, one needs to rewrite left-hand sides of the equations (1) in vector notation. It is easy to see that in this context the $N \times NM$ matrix $\mathbf{G} = \mathbf{e}'_M \otimes \mathbf{E}_N$, which consists of *M* identity matrix $\mathbf{E}_N$ located horizontally, and the $M\times NM$ matrix $\mathbf{H}=\mathbf{E}_M\otimes\mathbf{e}'_{\mathrm{N}}$ – N-fold successive replication of each column from identity matrix $\mathbf{E}_M$ – are the proper substitutes of summation vectors $\mathbf{e}_M$ and $\mathbf{e}'_N$ respectively. Note that each column of **G** and **H** includes exactly one nonzero (unit) element such that $\mathbf{e}'_N\mathbf{G}=\mathbf{e}'_M\mathbf{H}=\mathbf{e}'_{NM}$. Thus, the system of equations (1) and multiplicative model (11) can be combined as follows:

$$\mathbf{X}\mathbf{e}_M=\mathbf{G}\mathbf{x}=\mathbf{G}\hat{\mathbf{a}}\mathbf{q}=\mathbf{u}\,,\qquad \mathbf{e}'_N\mathbf{X}=\mathbf{H}\mathbf{x}=\mathbf{H}\hat{\mathbf{a}}\mathbf{q}=\mathbf{v}\,. \tag{15}$$

Recall that under consistency condition (2) any N+M–1 among N+M constraints (15) are mutually independent.

7. Minimization of angular measure via homothetic measure

The expression (14) in conjunction with monotonicity of function $\sin^2 x$ at acute angles x generates the following nonlinear programming problem: to minimize the fractional quadratic objective function or, as it is sometimes called, Rayleigh quotient

$$\mathrm{F}(\mathbf{q})=\frac{1}{\mathbf{q}'\mathbf{q}}\mathbf{q}'\left(\mathbf{E}_{NM}-\frac{\mathbf{e}_{NM}\mathbf{e}'_{NM}}{\mathbf{e}'_{NM}\mathbf{e}_{NM}}\right)\mathbf{q}=\frac{\mathrm{f}(\mathbf{q})}{\mathbf{q}'\mathbf{q}} \tag{16}$$

subject to linear constraints (15). Note that angular measure (16) has the same value F(**q**) along a straight line $k\mathbf{q}$ at any $k\neq 0$. Recall that symmetric idempotent matrix in parentheses has rank NM–1. Singularity of this matrix serves as an obvious technical obstacle for the analytical solving of constrained minimization problems (16), (15), but this obstacle can be bypassed in a special way proposed below.

The function f(**q**) in the numerator of Rayleigh quotient (16) can be rewritten as $\mathrm{f}(\mathbf{q})=(\mathbf{q}-\bar{q}\mathbf{e}_{NM})'(\mathbf{q}-\bar{q}\mathbf{e}_{NM})$ where $\bar{q}=k^{\perp}=\mathbf{e}'_{NM}\mathbf{q}/\mathbf{e}'_{NM}\mathbf{e}_{NM}$ is average value of elements in **q**. As shown above, this function expresses the length's square for the shortest path from the point **q** to homothetic ray and vanishes whenever target vector **q** and homothetic ray tend to be collinear. Hence, nonlinear programming problem (16), (15) with auxiliary constraint $\bar{q}=k$ (where k is assumed to be an arbitrary constant) is equivalent to quadratic optimization problem that prescribes to minimize uniparametrical objective function

$$\mathrm{f}\,(\mathbf{q};k)=(\mathbf{q}-k\mathbf{e}_{NM})'(\mathbf{q}-k\mathbf{e}_{NM}) \tag{17}$$

subject to constraints (15) and orthogonality condition

$$\mathbf{e}'_{NM}(\mathbf{q}-k\mathbf{e}_{NM})=0 \tag{18}$$

in which k is playing the role of an instrumental variable. Clearly, the solution point **q** for this quadratic optimization problem is lying on the hyperplane (18), which is orthogonal to homothetic ray and crosses it at the point $k\mathbf{e}_{NM}$. As established earlier, angular measure (14) and homothetic measure (13) are consistent on this orthogonal hyperplane.

Thus, the solution of nonlinear programming problem (16), (15) can be obtained in two stages. At first stage one needs to solve quadratic optimization problem (17), (15), (18) for every k and to find uniparametrical vector family $\mathbf{q}_*(k)$ that provides a local constrained minimum of homothetic measure

$$\mathrm{f}_*(k) = \min_{\mathbf{q}} \left\{ \mathrm{f}\,(\mathbf{q}; k) \;\middle|\; \mathbf{G}\hat{\mathbf{a}}\mathbf{q} = \mathbf{u},\;\; \mathbf{H}\hat{\mathbf{a}}\mathbf{q} = \mathbf{v},\;\; \mathbf{e}'_{NM}(\mathbf{q} - k\mathbf{e}_{NM}) = 0 \right\} \tag{19}$$

on each hyperplane (18). As a result, we obtain a geometric place of feasible points located at a minimal distance from points $k\mathbf{e}_{NM}$ on homothetic ray at various values of parameter k.

At second stage the unconstrained minimum

$$\mathrm{F}^* = \min_k \left\{ \mathrm{F}(k) = \frac{\mathrm{f}_*(k)}{\mathbf{q}'_*(k)\,\mathbf{q}_*(k)} \right\} \tag{20}$$

is to be found together with corresponding vector $\mathbf{q}^*$ as the optimal solution of angular measure minimization problem (16), (15). Besides, the other unconstrained minimum

$$\mathrm{f}_* = \min_k \left\{ \mathrm{f}_*(k) \right\} \tag{21}$$

corresponds to global minimization of homothetic measure along homothetic ray.

8. Uniparametrical constrained minimization of homothetic measure

In conjunction with (17) and general least squares (GLS) principles consider a uniparametrical family of functions

$$\mathrm{f}\,(\mathbf{q}; k) = (\mathbf{q} - k\mathbf{e}_{NM})'\mathbf{W}(\mathbf{q} - k\mathbf{e}_{NM}) \tag{22}$$

where k is unknown scalar parameter, and $\mathbf{W} = \hat{\mathbf{w}}$ is a nonsingular diagonal matrix of order NM with the relative reliability (relative confidence) factors for elements of vector $\mathbf{q}$. In terms of GLS $k\mathbf{e}_{NM}$ can be interpreted as a mean of random vector $\mathbf{q}$, and $\mathbf{W}$ – as a inverse covariance matrix for $\mathbf{q}$ in case of zero autocorrelations. Usually vector $\mathbf{w}$ is assumed to be normalized by multiplying it on a proper factor, i.e., $\mathbf{e}'_{NM}\mathbf{w} = 1$.

The objective function (22) appears to be similar to the one proposed by Harthoorn and van Dalen (1987). Nevertheless, there are at least two significant distinctions between them. First, Harthoorn and van Dalen have used metric measure based on vector $\mathbf{x} - \mathbf{a}$, and secondly, they have not used the operation of orthogonal projecting onto a homothetic ray.

The Lagrange function for problem to minimize objective function (22) subject to linear constraints (15) and GLS-analog of (18) with scalar parameter k is

$$\mathrm{L}_{\mathrm{f}}(\mathbf{q}; \boldsymbol{\lambda}, \boldsymbol{\mu}, \nu; k) = (\mathbf{q} - k\mathbf{e}_{NM})'\mathbf{W}(\mathbf{q} - k\mathbf{e}_{NM}) - \boldsymbol{\lambda}'(\mathbf{G}\hat{\mathbf{a}}\mathbf{q} - \mathbf{u}) - \boldsymbol{\mu}'(\mathbf{H}\hat{\mathbf{a}}\mathbf{q} - \mathbf{v}) - \gamma(\mathbf{e}'_{NM}\mathbf{W}\mathbf{q} - k\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}) \tag{23}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are vectors of Lagrange multipliers with dimension $N\times1$ and $M\times1$, and γ is a scalar Lagrange multiplier. By setting the partial derivatives of (23) with respect to $\mathbf{q}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, γ equal to zero, we obtain the system of $NM+N+M+1$ linear equations

$$2\mathbf{W}(\mathbf{q}-k\mathbf{e}_{NM})-\hat{\mathbf{a}}\mathbf{G}'\boldsymbol{\lambda}-\hat{\mathbf{a}}\mathbf{H}'\boldsymbol{\mu}-\gamma\mathbf{W}\mathbf{e}_{NM}=\mathbf{0}_{NM},\quad \mathbf{G}\hat{\mathbf{a}}\mathbf{q}-\mathbf{u}=\mathbf{0}_N,\quad \mathbf{H}\hat{\mathbf{a}}\mathbf{q}-\mathbf{v}=\mathbf{0}_M,\quad \mathbf{e}'_{NM}\mathbf{W}\mathbf{q}-k\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}=0.$$

While $\mathbf{W}$ is nonsingular matrix, the first equation can be resolved with respect to $\mathbf{q}$ as

$$\mathbf{q}=k\mathbf{e}_{NM}+\frac{1}{2}\mathbf{W}^{-1}\hat{\mathbf{a}}\left(\mathbf{G}'\boldsymbol{\lambda}+\mathbf{H}'\boldsymbol{\mu}\right)+\frac{1}{2}\gamma\mathbf{e}_{NM}.$$

Putting this expression into fourth equation gives

$$\gamma\mathbf{e}_{NM}=-\frac{\mathbf{e}_{NM}\mathbf{a}'}{\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}}\left(\mathbf{G}'\boldsymbol{\lambda}+\mathbf{H}'\boldsymbol{\mu}\right),$$

and after backward substitution we obtain

$$\mathbf{q}=k\mathbf{e}_{NM}+\frac{1}{2}\left(\mathbf{W}^{-1}\hat{\mathbf{a}}-\frac{\mathbf{e}_{NM}\mathbf{a}'}{\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}}\right)\left(\mathbf{G}'\boldsymbol{\lambda}+\mathbf{H}'\boldsymbol{\mu}\right). \qquad (24)$$

The second and the third equations from system above and (24) can be combined into $N+M$ equations with Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ as unknown variables:

$$\left(\mathbf{G}\hat{\mathbf{a}}\mathbf{W}^{-1}\hat{\mathbf{a}}\mathbf{G}'-\frac{\mathbf{G}\mathbf{a}\mathbf{a}'\mathbf{G}'}{\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}}\right)\boldsymbol{\lambda}+\left(\mathbf{G}\hat{\mathbf{a}}\mathbf{W}^{-1}\hat{\mathbf{a}}\mathbf{H}'-\frac{\mathbf{G}\mathbf{a}\mathbf{a}'\mathbf{H}'}{\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}}\right)\boldsymbol{\mu}=\boldsymbol{\pi}_{11}\boldsymbol{\lambda}+\boldsymbol{\pi}_{12}\boldsymbol{\mu}=2\left(\mathbf{u}-k\mathbf{G}\mathbf{a}\right), \qquad (25)$$

$$\left(\mathbf{H}\hat{\mathbf{a}}\mathbf{W}^{-1}\hat{\mathbf{a}}\mathbf{G}'-\frac{\mathbf{H}\mathbf{a}\mathbf{a}'\mathbf{G}'}{\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}}\right)\boldsymbol{\lambda}+\left(\mathbf{H}\hat{\mathbf{a}}\mathbf{W}^{-1}\hat{\mathbf{a}}\mathbf{H}'-\frac{\mathbf{H}\mathbf{a}\mathbf{a}'\mathbf{H}'}{\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}}\right)\boldsymbol{\mu}=\boldsymbol{\pi}_{21}\boldsymbol{\lambda}+\boldsymbol{\pi}_{22}\boldsymbol{\mu}=2\left(\mathbf{v}-k\mathbf{H}\mathbf{a}\right). \qquad (26)$$

It can be shown that $\boldsymbol{\pi}_{11}\mathbf{e}_N-\boldsymbol{\pi}_{12}\mathbf{e}_M=\mathbf{0}_N$, $\boldsymbol{\pi}_{21}\mathbf{e}_N-\boldsymbol{\pi}_{22}\mathbf{e}_M=\mathbf{0}_M$, i.e., the columns of symmetric matrix $\boldsymbol{\pi}$, which is formed by blocks $\boldsymbol{\pi}_{11},\boldsymbol{\pi}_{12},\boldsymbol{\pi}_{21},\boldsymbol{\pi}_{22}$, are linearly dependent. Thus, the general solution to homogeneous system corresponding (25), (26) is $\boldsymbol{\lambda}^{(0)}=c\mathbf{e}_N$, $\boldsymbol{\mu}^{(0)}=-c\mathbf{e}_M$ with the same scalar constant c.

Since general solution to nonhomogeneous linear system equals the sum of general solution to corresponding homogeneous system and any particular solution to nonhomogeneous system, let $\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(0)}+\boldsymbol{\lambda}^{(1)}$ and $\boldsymbol{\mu}=\boldsymbol{\mu}^{(0)}+\boldsymbol{\mu}^{(1)}$, where $\boldsymbol{\lambda}^{(1)}$, $\boldsymbol{\mu}^{(1)}$ is particular solution to system (25), (26). Recall that $\mathbf{e}'_N\mathbf{G}=\mathbf{e}'_N\mathbf{H}=\mathbf{e}'_{NM}$, so putting these formulae into round-bracketed expression in the right-hand side of (24) gives

$$\mathbf{G}'\boldsymbol{\lambda}+\mathbf{H}'\boldsymbol{\mu}=\mathbf{G}'\left(k\mathbf{e}_N+\boldsymbol{\lambda}^{(1)}\right)+\mathbf{H}'\left(-k\mathbf{e}_M+\boldsymbol{\mu}^{(1)}\right)=k\mathbf{e}_{NM}-k\mathbf{e}_{NM}+\mathbf{G}'\boldsymbol{\lambda}^{(1)}+\mathbf{H}'\boldsymbol{\mu}^{(1)}=\mathbf{G}'\boldsymbol{\lambda}^{(1)}+\mathbf{H}'\boldsymbol{\mu}^{(1)}.$$

Therefore, to find any particular solution of system (25), (26) means to solve constrained minimization problem with objective function (22) subject to linear constraints (15) and GLS-analog of (18).

9. Analytical solutions for Lagrange multipliers

The Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ can be found from system (25), (26) in two ways.

Since any N+M–1 among N+M constraints in set (15) are mutually independent under consistency condition (2), without loss of generality any one of them can be eliminated from the system. The details of analytical solution to similar reduced systems of linear constraints are discussed in Motorin (2014).

Another way is based on the "easy-to-check" fact that matrix $\boldsymbol{\pi}$ is singular, but its square blocks $\boldsymbol{\pi}_{11}$ and $\boldsymbol{\pi}_{22}$ are not if matrix **A** does not have zero rows and columns. So one can resolve (25) with respect to $\boldsymbol{\lambda}$ and (26) with respect to $\boldsymbol{\mu}$ as

$$\boldsymbol{\lambda} = 2\boldsymbol{\pi}_{11}^{-1}(\mathbf{u} - k\mathbf{Ga}) - \boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\mu}, \qquad \boldsymbol{\mu} = 2\boldsymbol{\pi}_{22}^{-1}(\mathbf{v} - k\mathbf{Ha}) - \boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\lambda}, \tag{27}$$

and after that the crossing substitutions give two equations as follows:

$$\boldsymbol{\lambda} = 2\boldsymbol{\pi}_{11}^{-1}(\mathbf{u} - k\mathbf{Ga}) - 2\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}(\mathbf{v} - k\mathbf{Ha}) + \boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\lambda} = \mathbf{c}_N(k) + \boldsymbol{\Pi}_N\boldsymbol{\lambda}, \tag{28}$$

$$\boldsymbol{\mu} = -2\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\pi}_{11}^{-1}(\mathbf{u} - k\mathbf{Ga}) + 2\boldsymbol{\pi}_{22}^{-1}(\mathbf{v} - k\mathbf{Ha}) + \boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\mu} = \mathbf{c}_M(k) + \boldsymbol{\Pi}_M\boldsymbol{\mu}. \tag{29}$$

Further, the square matrices $\boldsymbol{\Pi}_N$ and $\boldsymbol{\Pi}_M$ have the properties as

$$\boldsymbol{\Pi}_N\mathbf{e}_N = \boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\mathbf{e}_N = \boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{22}\mathbf{e}_M = \boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\mathbf{e}_M = \boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{11}\mathbf{e}_N = \mathbf{e}_N,$$

$$\boldsymbol{\Pi}_M\mathbf{e}_M = \boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\mathbf{e}_M = \boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{11}\mathbf{e}_N = \boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\mathbf{e}_N = \boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{22}\mathbf{e}_M = \mathbf{e}_M,$$

but they are not stochastic because may have some negative entries. Since $(\mathbf{E}_N - \boldsymbol{\Pi}_N)\mathbf{e}_N = \mathbf{0}_N$ and $(\mathbf{E}_M - \boldsymbol{\Pi}_M)\mathbf{e}_M = \mathbf{0}_M$, the matrices $\mathbf{E}_N - \boldsymbol{\Pi}_N$ and $\mathbf{E}_M - \boldsymbol{\Pi}_M$ with linearly dependent columns are singular, so that we can not solve the matrix equations (28) and (29) in the usual way.

Formally, one can write the solutions to (28) and (29) as

$$\boldsymbol{\lambda} = \sum_{i=0}^{\infty} \boldsymbol{\Pi}_N^i \cdot \mathbf{c}_N(k), \qquad \boldsymbol{\mu} = \sum_{j=0}^{\infty} \boldsymbol{\Pi}_M^j \cdot \mathbf{c}_M(k). \tag{30}$$

However, the row marginal totals in partial sums of matrix power series in (30) increase unboundedly, since $\boldsymbol{\Pi}_N^i\mathbf{e}_N = \mathbf{e}_N$ and $\boldsymbol{\Pi}_M^j\mathbf{e}_M = \mathbf{e}_M$. Thus, an existence of solutions (30) is questionable and needs to be studied.

From the theory of homogeneous Markov chains it is known that stochastic matrix $\boldsymbol{\Pi}$ has a marginal property $\lim_{i\to\infty}\boldsymbol{\Pi}^i = \boldsymbol{\Pi}^\infty = \mathbf{ev}'$, where $\mathbf{v}' = \mathbf{v}'\boldsymbol{\Pi}$ is the left eigenvector of matrix $\boldsymbol{\Pi}$ corresponding to unit eigenvalue (for more details in transposed case of right eigenvector, see Bellman, 1960, pp. 256 – 258). Despite the matrices $\boldsymbol{\Pi}_N$ and $\boldsymbol{\Pi}_M$ are not stochastic because of negative entries, they have matrix norms with unit upper bound and demonstrate similar features:

$$\lim_{i\to\infty}\boldsymbol{\Pi}_N^i = \boldsymbol{\Pi}_N^\infty = \mathbf{e}_N\mathbf{v}_N', \qquad \lim_{j\to\infty}\boldsymbol{\Pi}_M^j = \boldsymbol{\Pi}_M^\infty = \mathbf{e}_M\mathbf{v}_M'$$

where $\mathbf{v}'_N = \mathbf{v}'_N \mathbf{\Pi}_N$ and $\mathbf{v}'_M = \mathbf{v}'_M \mathbf{\Pi}_M$ are the left eigenvectors of $\mathbf{\Pi}_N$ and $\mathbf{\Pi}_M$ both corresponding to unit eigenvalues. It is easy to show that $\mathbf{v}'_N = c\mathbf{e}'_N \boldsymbol{\pi}_{11}$ and $\mathbf{v}'_M = c\mathbf{e}'_M \boldsymbol{\pi}_{22}$ where c is an arbitrary constant. Indeed, e.g., $\mathbf{v}'_N \mathbf{\Pi}_N = c\mathbf{e}'_N \boldsymbol{\pi}_{11} \cdot \boldsymbol{\pi}_{11}^{-1} \boldsymbol{\pi}_{12} \boldsymbol{\pi}_{22}^{-1} \boldsymbol{\pi}_{21} = c\mathbf{e}'_M \boldsymbol{\pi}_{22} \boldsymbol{\pi}_{22}^{-1} \boldsymbol{\pi}_{21} = c\mathbf{e}'_N \boldsymbol{\pi}_{11} = \mathbf{v}'_N$. Using these matrix algebra results it can be establish that $\mathbf{\Pi}^i_N \mathbf{c}_N(k)$ and $\mathbf{\Pi}^j_M \mathbf{c}_M(k)$ tend to nill vectors with proper dimensions as $i \to \infty$ and $j \to \infty$.

Consider the marginal vectors for matrix power series (30) by regrouping relevant summands in right-hand sides of (28) and (29) as

$$\mathbf{\Pi}^{\infty}_{\mathrm{N}} \mathbf{c}_N(k) = \mathbf{e}_N \mathbf{v}'_N \mathbf{c}_N(k) = 2c\mathbf{e}_N \mathbf{e}'_N (\mathbf{u} - \boldsymbol{\pi}_{12} \boldsymbol{\pi}_{22}^{-1} \mathbf{v}) + 2ck\mathbf{e}_N \mathbf{e}'_N (\boldsymbol{\pi}_{12} \boldsymbol{\pi}_{22}^{-1} \mathbf{H} - \mathbf{G}) \mathbf{a} \,,$$

$$\mathbf{\Pi}^{\infty}_{\mathrm{M}} \mathbf{c}_M(k) = \mathbf{e}_M \mathbf{v}'_M \mathbf{c}_M(k) = 2c\mathbf{e}_M \mathbf{e}'_M (\mathbf{v} - \boldsymbol{\pi}_{21} \boldsymbol{\pi}_{11}^{-1} \mathbf{u}) + 2ck\mathbf{e}_M \mathbf{e}'_M (\boldsymbol{\pi}_{21} \boldsymbol{\pi}_{11}^{-1} \mathbf{G} - \mathbf{H}) \mathbf{a} \,.$$

Since $\mathbf{e}'_N \boldsymbol{\pi}_{12} = \mathbf{e}'_M \boldsymbol{\pi}_{22}$ and $\mathbf{e}'_M \boldsymbol{\pi}_{21} = \mathbf{e}'_N \boldsymbol{\pi}_{11}$, we have $\mathbf{e}'_N (\mathbf{u} - \boldsymbol{\pi}_{12} \boldsymbol{\pi}_{22}^{-1} \mathbf{v}) = -\mathbf{e}'_M (\mathbf{v} - \boldsymbol{\pi}_{21} \boldsymbol{\pi}_{11}^{-1} \mathbf{u}) = \mathbf{e}'_N \mathbf{u} - \mathbf{e}'_M \mathbf{v} = 0$ in accordance with the consistency condition (2). Further, the statement

$$\mathbf{e}'_N (\boldsymbol{\pi}_{12} \boldsymbol{\pi}_{22}^{-1} \mathbf{H} - \mathbf{G}) = -\mathbf{e}'_M (\boldsymbol{\pi}_{21} \boldsymbol{\pi}_{11}^{-1} \mathbf{G} - \mathbf{H}) = \mathbf{e}'_M \mathbf{H} - \mathbf{e}'_N \mathbf{G} = \mathbf{e}'_{NM} - \mathbf{e}'_{NM} = \mathbf{0}'_{NM}$$

immediately follows from the definitions of matrices **G** and **H** (see Section 6 above).

Hence, the eigenvectors of matrices $\mathbf{\Pi}_N$ and $\mathbf{\Pi}_M$ are orthogonal to $\mathbf{c}_N(k)$ and $\mathbf{c}_M(k)$ respectively, so $\mathbf{\Pi}^{\infty}_{\mathrm{N}} \mathbf{c}_N(k) = \mathbf{0}_N$ and $\mathbf{\Pi}^{\infty}_{\mathrm{M}} \mathbf{c}_M(k) = \mathbf{0}_M$. Of course, the statements proved serve as necessary conditions for an existence of solutions (30). Formally, they are not sufficient to provide a convergence of matrix power series in (30). However, in practice such series appear to converge rather fast, and it is expedient to calculate the partial sums in (30) subject to terminal criteria resembling $\left|\mathbf{\Pi}^i_N \mathbf{c}_N(k)\right| \le \varepsilon$ and $\left|\mathbf{\Pi}^j_M \mathbf{c}_M(k)\right| \le \varepsilon$ where ε is a small positive value.

So the solutions of equations (25) and (26) can be represented by vector-valued linear functions of k as

$$\boldsymbol{\lambda}(k) = 2\boldsymbol{\varphi}_N + 2k\boldsymbol{\psi}_N \,, \qquad \boldsymbol{\mu}(k) = 2\boldsymbol{\varphi}_M + 2k\boldsymbol{\psi}_M \tag{31}$$

where

$$\boldsymbol{\varphi}_N = \lim_{I\to\infty} \left\{ \sum_{i=0}^{I} \mathbf{\Pi}^i_N \cdot \boldsymbol{\pi}_{11}^{-1} (\mathbf{u} - \boldsymbol{\pi}_{12} \boldsymbol{\pi}_{22}^{-1} \mathbf{v}) \right\}, \qquad \boldsymbol{\psi}_N = \lim_{I\to\infty} \left\{ \sum_{i=0}^{I} \mathbf{\Pi}^i_N \cdot \boldsymbol{\pi}_{11}^{-1} (\boldsymbol{\pi}_{12} \boldsymbol{\pi}_{22}^{-1} \mathbf{H} - \mathbf{G}) \mathbf{a} \right\},$$

$$\boldsymbol{\varphi}_M = \lim_{J\to\infty} \left\{ \sum_{j=0}^{J} \mathbf{\Pi}^j_M \cdot \boldsymbol{\pi}_{22}^{-1} (\mathbf{v} - \boldsymbol{\pi}_{21} \boldsymbol{\pi}_{11}^{-1} \mathbf{u}) \right\}, \qquad \boldsymbol{\psi}_M = \lim_{J\to\infty} \left\{ \sum_{j=0}^{J} \mathbf{\Pi}^j_M \cdot \boldsymbol{\pi}_{22}^{-1} (\boldsymbol{\pi}_{21} \boldsymbol{\pi}_{11}^{-1} \mathbf{G} - \mathbf{H}) \mathbf{a} \right\}$$

are the recursively computable vectors with dimensions $N\times1$ and $M\times1$ respectively.

Note that for finding Lagrange multipliers **λ** and **μ** by solving system (25), (26) each formula in (31) must be used together with a complimentary formula from (27). In particular, if $N < M$, it is better to calculate $\boldsymbol{\lambda}(k) = 2\boldsymbol{\varphi}_N + 2k\boldsymbol{\psi}_N$ as in (31) and then to determine $\boldsymbol{\mu}(k)$ by second formula (27), which becomes

$$\boldsymbol{\mu}(k) = 2\boldsymbol{\pi}_{22}^{-1}(\mathbf{v} - k\mathbf{Ha}) - \boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21}\boldsymbol{\lambda}(k) = 2\boldsymbol{\pi}_{22}^{-1}(\mathbf{v} - \boldsymbol{\pi}_{21}\boldsymbol{\varphi}_N) - 2k\boldsymbol{\pi}_{22}^{-1}(\mathbf{Ha} + \boldsymbol{\pi}_{21}\boldsymbol{\psi}_N). \qquad (32)$$

Vice versa, if $N > M$, the choice of $\boldsymbol{\mu}(k) = 2\boldsymbol{\varphi}_M + 2k\boldsymbol{\psi}_M$ from (31) is more preferable with successive applying first formula (27), which becomes

$$\boldsymbol{\lambda}(k) = 2\boldsymbol{\pi}_{11}^{-1}(\mathbf{u} - k\mathbf{Ga}) - 2\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12}\boldsymbol{\mu}(k) = 2\boldsymbol{\pi}_{11}^{-1}(\mathbf{u} - \boldsymbol{\pi}_{12}\boldsymbol{\varphi}_M) - 2k\boldsymbol{\pi}_{11}^{-1}(\mathbf{Ga} + \boldsymbol{\pi}_{12}\boldsymbol{\psi}_M). \qquad (33)$$

These iteration-based results express, in general, two different particular solutions of system (25), (26) and do not coincide among themselves, as well as RAS solutions (5) and (6). It is important to note that in constrained minimization problem with objective function (22) subject to linear constraints (15) and GLS-analog of (18) the Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are represented by vector-valued linear functions of the instrumental variable k.

10. The solutions of minimization problems for homothetic measure

As it was shown in Section 8, the round-bracketed vector in the right-hand side of (24) does not depend on a choice of certain particular solution to system (25), (26). Applying the first formula (31) in a pair with (32) gives

$$\mathbf{G}'\boldsymbol{\lambda}(k) + \mathbf{H}'\boldsymbol{\mu}(k) = 2\left[\mathbf{H}'\boldsymbol{\pi}_{22}^{-1}\mathbf{v} + (\mathbf{G}' - \mathbf{H}'\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21})\boldsymbol{\varphi}_N\right] + 2k\left[(\mathbf{G}' - \mathbf{H}'\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21})\boldsymbol{\psi}_N - \mathbf{H}'\boldsymbol{\pi}_{22}^{-1}\mathbf{Ha}\right],$$

whereas mutual using the second formula (31) and (33) leads to

$$\mathbf{G}'\boldsymbol{\lambda}(k) + \mathbf{H}'\boldsymbol{\mu}(k) = 2\left[\mathbf{G}'\boldsymbol{\pi}_{11}^{-1}\mathbf{u} + (\mathbf{H}' - \mathbf{G}'\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12})\boldsymbol{\varphi}_M\right] + 2k\left[(\mathbf{H}' - \mathbf{G}'\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12})\boldsymbol{\psi}_M - \mathbf{G}'\boldsymbol{\pi}_{11}^{-1}\mathbf{Ga}\right].$$

Hence, this vector can be represented by vector-valued linear function of k as follows:

$$\mathbf{G}'\boldsymbol{\lambda}(k) + \mathbf{H}'\boldsymbol{\mu}(k) = 2\mathbf{z} + 2k\mathbf{y} \qquad (34)$$

where

$$\mathbf{z} = \mathbf{H}'\boldsymbol{\pi}_{22}^{-1}\mathbf{v} + (\mathbf{G}' - \mathbf{H}'\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21})\boldsymbol{\varphi}_N = \mathbf{G}'\boldsymbol{\pi}_{11}^{-1}\mathbf{u} + (\mathbf{H}' - \mathbf{G}'\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12})\boldsymbol{\varphi}_M,$$

$$\mathbf{y} = (\mathbf{G}' - \mathbf{H}'\boldsymbol{\pi}_{22}^{-1}\boldsymbol{\pi}_{21})\boldsymbol{\psi}_N - \mathbf{H}'\boldsymbol{\pi}_{22}^{-1}\mathbf{Ha} = (\mathbf{H}' - \mathbf{G}'\boldsymbol{\pi}_{11}^{-1}\boldsymbol{\pi}_{12})\boldsymbol{\psi}_M - \mathbf{G}'\boldsymbol{\pi}_{11}^{-1}\mathbf{Ga}$$

are both NM-dimensional vectors. It is important to note that according to (31) vector **y** does not depend on **u** and **v** (marginal totals for target matrix) in contrast to the vector **z**.

Now a general solution of constrained minimization problem for homothetic measure (22), (15) with GLS-analog of (18) can be derived by putting (34) into (24). Thus, we have uniparametrical vector family

$$\mathbf{q}_*(k) = k\mathbf{e}_{NM} + \left(\mathbf{W}^{-1}\hat{\mathbf{a}} - \frac{\mathbf{e}_{NM}\mathbf{a}'}{\mathbf{e}_{NM}'\mathbf{W}\mathbf{e}_{NM}}\right)(\mathbf{z} + k\mathbf{y}) = k\mathbf{e}_{NM} + \mathbf{D}(\mathbf{z} + k\mathbf{y}) \qquad (35)$$

which is obtained in accordance with a requirement formulated in (19). Note that **D** is a square matrix of order NM. Along with (22) and (35) scalar function (19) becomes polynomial of second order as

$$\mathrm{f}_*(k) = (\mathbf{z} + k\mathbf{y})'\mathbf{D}'\mathbf{W}\mathbf{D}(\mathbf{z} + k\mathbf{y}) = \mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z} + 2k\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z} + k^2\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y} \qquad (36)$$

These analytical results complete a first stage of solving process for nonlinear programming problem (16), (15).

Formula (35) describes a geometric place of feasible points $\mathbf{q}_*(k)$ located at a minimal distance from points $k\mathbf{e}_{NM}$ on homothetic ray at various values of parameter *k*. It is important to emphasize that vector $\mathbf{q}_*(k) - k\mathbf{e}_{NM} = \mathbf{D}(\mathbf{z} + k\mathbf{y})$ is orthogonal to homothetic ray at any *k* by model construction because of GLS-analog of (18) among constraints. Indeed,

$$\mathbf{e}'_{NM}\mathbf{W}\mathbf{D} = \mathbf{e}'_{NM}\mathbf{W}\left(\mathbf{W}^{-1}\hat{\mathbf{a}} - \frac{\mathbf{e}_{NM}\mathbf{a}'}{\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}}\right) = \mathbf{a}' - \mathbf{a}' = \mathbf{0}'_{NM} \tag{37}$$

so that $\mathbf{e}'_{NM}\mathbf{W}\left[\mathbf{q}_*(k) - k\mathbf{e}_{NM}\right] = \mathbf{e}'_{NM}\mathbf{W}\mathbf{D}(\mathbf{z} + k\mathbf{y}) = 0$ at any *k*.

The first and the second derivatives of quadratic function (36) with respect to *k* are defined as

$$\frac{\mathrm{d}\mathrm{f}_*(k)}{\mathrm{d}k} = 2\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z} + 2k\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y}, \qquad \frac{\mathrm{d}^2\mathrm{f}_*(k)}{\mathrm{d}k^2} = 2\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y} \geq 0.$$

Clearly, this convex function of *k* has a unique minimum at the parameter value that equals $k_* = -\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z} / \mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y}$ (here the first derivative vanishes).

As a result, the global minimum of homothetic measure along homothetic ray

$$\mathrm{f}_* = \min_k \min_{\mathbf{q}} \left\{ \mathrm{f}(\mathbf{q}; k) \mid \mathbf{G}\hat{\mathbf{a}}\mathbf{q} = \mathbf{u},\ \ \mathbf{H}\hat{\mathbf{a}}\mathbf{q} = \mathbf{v},\ \mathbf{e}'_{NM}\mathbf{W}(\mathbf{q} - k\mathbf{e}_{NM}) = 0 \right\},$$

which corresponds to a requirement formulated in (21), is achieved according to (35) at the point

$$\mathbf{q}_* = \mathbf{D}\mathbf{z} + k_*(\mathbf{e}_{NM} + \mathbf{D}\mathbf{y}) = \mathbf{D}\mathbf{z} - \frac{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y}}(\mathbf{e}_{NM} + \mathbf{D}\mathbf{y}) \tag{38}$$

with the objective function value

$$\mathrm{f}_* = \mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z} - 2\frac{(\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z})^2}{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y}} + \frac{(\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z})^2}{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y}} = \mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z} - \frac{(\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z})^2}{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y}} \tag{39}$$

where **z** and **y** are *NM*-dimensional vectors defined below (34). It can be shown that formulae (38) and (39) also define the solution of a quadratic programming problem, in which it is required to minimize the objective function (22) subject to linear constraints (15) and unknown parameter *k* is determined from a statement that vector $\mathbf{q}_*(k) - k\mathbf{e}_{NM}$ is to be orthogonal to homothetic ray. This problem is considered in details in Motorin (2014).

11. The solution of unconstrained minimization problem for angular measure

At second stage of solving process for nonlinear programming problem (16), (15) we have a new objective function

$$\mathrm{F}(k)=\frac{\mathrm{f}_*(k)}{\mathbf{q}'_*(k)\mathbf{W}\,\mathbf{q}_*(k)}=\frac{(\mathbf{z}+k\mathbf{y})'\mathbf{D}'\mathbf{W}\mathbf{D}(\mathbf{z}+k\mathbf{y})}{[k\mathbf{e}_{NM}+\mathbf{D}(\mathbf{z}+k\mathbf{y})]'\mathbf{W}[k\mathbf{e}_{NM}+\mathbf{D}(\mathbf{z}+k\mathbf{y})]} \tag{40}$$

that is obtained by substituting (36) and (35) into the function in braces from (20). In accordance with a requirement formulated in (20) it is to be minimized. However, this function of k is fractional quadratic and, hence, can have more than one minimum.

Putting (37) into the denominator of (40) gives

$$[k\mathbf{e}_{NM}+\mathbf{D}(\mathbf{z}+k\mathbf{y})]'\mathbf{W}[k\mathbf{e}_{NM}+\mathbf{D}(\mathbf{z}+k\mathbf{y})]=k^2\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}+(\mathbf{z}+k\mathbf{y})'\mathbf{D}'\mathbf{W}\mathbf{D}(\mathbf{z}+k\mathbf{y})$$

so $\mathrm{F}(k)\le 1$ at any values of k. Note that the maximum of $\mathrm{F}(k)$ equals 1 and is achieved at zero value of k.

It can be shown that first derivative of the fractional quadratic function (40) with respect to k is defined as

$$\frac{\mathrm{dF}(k)}{\mathrm{d}k}=\frac{-2k\cdot\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}(\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}+k\cdot\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y})}{[k^2\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}+(\mathbf{z}+k\mathbf{y})'\mathbf{D}'\mathbf{W}\mathbf{D}(\mathbf{z}+k\mathbf{y})]^2}\,.$$

It seems clear that function (40) has a unique maximum at $k=0$ and a unique minimum at $k^*=-\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}/\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y}=-\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}/\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}$. The first case concerns an orthogonality of the relative target vector to homothetic ray whereas the second one is associated with a minimal angle between the relative target vector and homothetic ray.

As a result, the global minimum of angular measure $\mathrm{F}^*=\min\limits_k\{\,\mathrm{F}(k)\,\}$, which corresponds to a requirement formulated in (20), is achieved at the point

$$\mathbf{q}^*=\mathbf{D}\mathbf{z}+k^*(\mathbf{e}_{NM}+\mathbf{D}\mathbf{y})=\mathbf{D}\mathbf{z}-\frac{\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}(\mathbf{e}_{NM}+\mathbf{D}\mathbf{y}) \tag{41}$$

with the objective function value

$$\mathrm{F}^*=\frac{\left(\mathbf{z}'-\frac{\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}\mathbf{y}'\right)\mathbf{D}'\mathbf{W}\mathbf{D}\left(\mathbf{z}-\frac{\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}\mathbf{y}\right)}{\left(\frac{\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}\right)^2\mathbf{e}'_{NM}\mathbf{W}\mathbf{e}_{NM}+\left(\mathbf{z}'-\frac{\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}\mathbf{y}'\right)\mathbf{D}'\mathbf{W}\mathbf{D}\left(\mathbf{z}-\frac{\mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}{\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z}}\mathbf{y}\right)} \tag{42}$$

where $\mathbf{z}$ and $\mathbf{y}$ are NM-dimensional vectors defined below (34). Thus, formulae (41) and (42) define the solution of a nonlinear programming problem in which it is required to minimize the objective function (16) subject to linear constraints (15).

12. Notes on sensitivity analysis of the obtained solutions

Three optimization problem is considered in this paper:

- to minimize the quadratic objective function (22) with scalar parameter k subject to linear constraints (15) and GLS-analog of (18), i.e., uniparametrical minimization problem for homothetic measure;

- to minimize the quadratic objective function (22) subject to linear constraints (15) with finding unknown parameter k from a minimum condition for univariate function (36) or (as in Motorin, 2014) from an orthogonality condition for vector $\mathbf{q}_*(k) - k\mathbf{e}_{NM}$ and homothetic ray, i.e., global minimization problem for homothetic measure;
- to minimize the fractional quadratic objective function (16) subject to linear constraints (15) with finding unknown parameter k from a minimum condition for univariate function (40) , i.e., global minimization problem for angular measure.

Applying a technique of partial derivatives for sensitivity analysis in this context represents a quite complicated task because of mutually dependent linear constraints (15). In such cases the Lagrange multipliers can not be uniquely identified. For sensitivity analysis in first and second above-mentioned problems one can use simple instrumental approach developed in Motorin (2014).

Any disturbance of marginal total vector **u** through the frame of consistency condition (2) generates some compensating changes in the elements of **v**, and vice versa. Clearly, some disturbances lead to an increasing of constrained minimum for objective function (22), while others contribute to decrease it.

Along with vector-valued linear function $\mathbf{G}'\boldsymbol{\lambda} + \mathbf{H}'\boldsymbol{\mu}$, $N \times M$-dimensional matrix

$$\mathbf{L} = \boldsymbol{\lambda}\mathbf{e}'_M + \mathbf{e}_N\boldsymbol{\mu}' = \boldsymbol{\lambda}\mathbf{e}'_M + \mathbf{e}_N\boldsymbol{\mu}' + k\mathbf{e}_N\mathbf{e}'_M - k\mathbf{e}_N\mathbf{e}'_M = (\boldsymbol{\lambda} + k\mathbf{e}_N)\mathbf{e}'_M + \mathbf{e}_N(\boldsymbol{\mu}' - k\mathbf{e}'_M)$$

is invariant under any change of parameter k. It is easy to see that its element l_{nm} can be considered as a coefficient of the constrained minimum's sensitivity under impact of the simultaneous increasing u_n and v_m by the same small value ε. Thus, to decrease the minimum $\mathrm{f}_*(\mathbf{u},\mathbf{v}) = \mathrm{f}_*(k)$ a small scalar ε is to be chosen with the sign reversed from the sign of l_{nm}.

In this context the larger absolute values of matrix **L**'s elements are of great interest. Let l_{nm} be an element with the largest absolute value of any one in matrix **L**. Then the best strategy for a local enhancing of constrained minimum is to disturb u_n and v_m by the same small value $-\varepsilon \cdot \mathrm{sgn}(l_{nm})$ where $\varepsilon > 0$ and $\mathrm{sgn}(\cdot)$ is a signum function.

Further, let $l_+ > 0$ and $l_- < 0$ be a maximal and a minimal elements of **L** respectively. Then the best two-component strategy for a local enhancing of constrained minimum at the fixed grand total $\sigma_{\mathbf{X}} = \mathbf{e}'_N\mathbf{u} = \mathbf{e}'_M\mathbf{v}$ is to decrease the elements of **u** and **v** corresponding to l_+ by –ε and to increase the elements of **u** and **v** corresponding to l_- by ε simultaneously.

In general, total sensitivity effect is formulated as

$$\Delta_{\mathrm{f}}\left(\mathbf{\Delta_u},\mathbf{\Delta_v}\right)=\mathrm{f}_*\left(\mathbf{u}+\mathbf{\Delta_u},\mathbf{v}+\mathbf{\Delta_v}\right)-\mathrm{f}_*\left(\mathbf{u},\mathbf{v}\right)=\mathbf{\Delta}'_{\mathbf{u}}\boldsymbol{\lambda}+\boldsymbol{\mu}'\mathbf{\Delta_v}, \tag{43}$$

where vectors $\mathbf{\Delta_u}$ and $\mathbf{\Delta_v}$ are exogenous disturbances for **u** and **v** respectively satisfying the consistency condition $\mathbf{e}'_N\mathbf{\Delta_u}=\mathbf{e}'_M\mathbf{\Delta_v}=\sigma_{\mathbf{\Delta X}}$. To express the right-hand side of (43) in matrix **L** terms it is necessary to consider two cases, namely, $\sigma_{\mathbf{\Delta X}}=0$ and $\sigma_{\mathbf{\Delta X}}\neq 0$.

The disturbances $\mathbf{\Delta_u}$ and $\mathbf{\Delta_v}$ with zero sums $\mathbf{e}'_N\mathbf{\Delta_u}=\mathbf{e}'_M\mathbf{\Delta_v}=\sigma_{\mathbf{\Delta X}}=0$ play an important role in statistical practice. They entail the redistributions of **u**’s and **v**’s components while the grand total $\sigma_{\mathbf{X}}+\sigma_{\mathbf{\Delta X}}$ is being fixed. It is easy to see from (43) that the total redistribution effect depends on the marginal totals of matrix **L** and is estimated by

$$\Delta_{\mathrm{f}}\left(\mathbf{\Delta_u},\mathbf{\Delta_v}\middle|\sigma_{\mathbf{\Delta X}}=0\right)=\mathbf{\Delta}'_{\mathbf{u}}\left(\boldsymbol{\lambda}+\bar{\mu}\mathbf{e}_N\right)+\left(\boldsymbol{\mu}+\bar{\lambda}\mathbf{e}_M\right)'\mathbf{\Delta_v}=\frac{1}{M}\mathbf{\Delta}'_{\mathbf{u}}\mathbf{L}\mathbf{e}_M+\frac{1}{N}\mathbf{e}'_N\mathbf{L}\mathbf{\Delta_v}. \tag{44}$$

Here the first summand implies that in the total effect calculation an each value $(\mathbf{\Delta_u})_n$ is uniformly distributed among *M* components of $\mathbf{\Delta_v}$ and generates *M* elementary effects, sum of which is proportional to a row marginal total *n* for **L** divided by *M*. By analogy, the second summand in (44) implies that an each value $(\mathbf{\Delta_v})_m$ is uniformly distributed among *N* components of $\mathbf{\Delta_u}$ and generates *N* simple effects, sum of which is proportional to a column marginal total *m* for **L** divided by *N*.

On the other hand, the bilinear function of disturbances $\mathbf{\Delta}'_{\mathbf{u}}\mathbf{L}\mathbf{\Delta_v}$ can be transformed as follows:

$$\mathbf{\Delta}'_{\mathbf{u}}\mathbf{L}\mathbf{\Delta_v}=\mathbf{\Delta}'_{\mathbf{u}}\left(\boldsymbol{\lambda}\mathbf{e}'_M+\mathbf{e}_N\boldsymbol{\mu}'\right)\mathbf{\Delta_v}=\mathbf{\Delta}'_{\mathbf{u}}\boldsymbol{\lambda}\left(\mathbf{e}'_M\mathbf{\Delta_v}\right)+\left(\mathbf{\Delta}'_{\mathbf{u}}\mathbf{e}_N\right)\boldsymbol{\mu}'\mathbf{\Delta_v}=\sigma_{\mathbf{\Delta X}}\left(\mathbf{\Delta}'_{\mathbf{u}}\boldsymbol{\lambda}+\boldsymbol{\mu}'\mathbf{\Delta_v}\right).$$

Hence, the total sensitivity effect may be represented as

$$\Delta_{\mathrm{f}}\left(\mathbf{\Delta_u},\mathbf{\Delta_v}\middle|\sigma_{\mathbf{\Delta X}}\neq 0\right)=\mathbf{\Delta}'_{\mathbf{u}}\boldsymbol{\lambda}+\boldsymbol{\mu}'\mathbf{\Delta_v}=\frac{1}{\sigma_{\mathbf{\Delta X}}}\mathbf{\Delta}'_{\mathbf{u}}\mathbf{L}\mathbf{\Delta_v} \tag{45}$$

where the disturbance grand total $\sigma_{\mathbf{\Delta X}}$ is assumed to be nonzero. Recall, that in contrast to (45) formula (44) is well defined only for the redistribution case $\sigma_{\mathbf{\Delta X}}=0$.

Sensitivity analysis in the third above-mentioned problem can not be implemented within described approach. However, since the solutions of second and third problems are rather close to each other, the sensitivity analysis results for global minimization of homothetic measure may be delivered to global minimization of angular measure without a significant loss of accuracy.

13. Numerical examples and concluding remarks

Consider the Eurostat input–output data set given in “Box 14.2: RAS procedure” (see Eurostat, 2008, p. 452) for compiling several numerical examples. The 3×4-dimensional initial matrix **A** combines the entries in intersections of the columns “Agriculture”, “Industry”, “Services”, “Final d.” with the rows “Agriculture”, “Industry”, “Services” in “Table 1: Input-output data for year 0”. Note that all the elements of this matrix are nonzero. The row marginal total vector **u** of dimension 3×1 is the proper part of the column “Output” in “Table 2: Input-output data for year 1”, and the column marginal total vector $\mathbf{v}'$ of dimension 1×4 involves the proper entries of the row “Total” in the near-mentioned data source.

Initial matrix **A** and marginal totals **u**, $\mathbf{v}'$ are presented in the left half of Table 1. The first numerical example is to handle the data set available by RAS method with iterative processes (5) or (6) and by methods (38), (39) and (41), (42) proposed to solve the constrained minimization problem for homothetic and angular measures (22), (15) and (16), (15) – briefly, by HOM and ANG methods respectively. The computation results at $\mathbf{W} = \mathbf{E}_{NM} / \mathbf{e}'_{NM}\mathbf{E}_{NM}\mathbf{e}_{NM}$ are grouped in the right half of Table 1 for RAS method and in Table 1a for HOM and ANG methods; they seem to be very similar among themselves.

Table 1. Initial matrix **A** with nonzero elements and RAS results for its updating

	A				$\mathbf{u}_\mathbf{A}$	**u**
	20.00	34.00	10.00	36.00	100.00	94.78
	20.00	152.00	40.00	188.00	400.00	412.86
	10.00	72.00	20.00	98.00	200.00	212.68
$\mathbf{v}'_\mathbf{A}$	50.00	258.00	70.00	322.00	700.00	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

RAS	**X**				$\mathbf{u}_\mathbf{X}$	**u**
	17.94	32.77	9.76	34.31	94.78	94.78
	19.36	158.08	42.12	193.30	412.86	412.86
	9.98	77.17	21.70	103.84	212.68	212.68
$\mathbf{v}'_\mathbf{X}$	47.28	268.02	73.58	331.44	720.32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

Table 1a. HOM and ANG results for updating of data set from Table 1

HOM	**X**				$\mathbf{u}_\mathbf{X}$	**u**
	18.35	32.41	10.03	33.99	94.78	94.78
	19.07	158.82	42.60	192.37	412.86	412.86
	9.86	76.79	20.95	105.08	212.68	212.68
$\mathbf{v}'_\mathbf{X}$	47.28	268.02	73.58	331.44	720.32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

ANG	**X**				$\mathbf{u}_\mathbf{X}$	**u**
	18.33	32.41	10.04	34.00	94.78	94.78
	19.08	158.81	42.58	192.40	412.86	412.86
	9.87	76.80	20.96	105.04	212.68	212.68
$\mathbf{v}'_\mathbf{X}$	47.28	268.02	73.58	331.44	720.32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

Nevertheless, HOM and ANG methods demonstrate the stable 5-percentage advantage over RAS method both in homothetic measure of matrix similarity (13) and in angular measure (14) as follows:

$$\left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 0.0549, \qquad \left|\boldsymbol{\delta}^{\mathrm{HOM}}\right| = 0.0522, \qquad \left|\boldsymbol{\delta}^{\mathrm{ANG}}\right| = 0.0522, \qquad \left|\boldsymbol{\delta}^{\mathrm{HOM}}\right| \big/ \left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 95.10\%;$$

$$\beta_{\mathbf{qe}}^{\mathrm{RAS}} = 3.1161°, \qquad \beta_{\mathbf{qe}}^{\mathrm{HOM}} = 2.9677°, \qquad \beta_{\mathbf{qe}}^{\mathrm{ANG}} = 2.9675°, \qquad \beta_{\mathbf{qe}}^{\mathrm{ANG}} \big/ \beta_{\mathbf{qe}}^{\mathrm{RAS}} = 95.23\%.$$

The next numerical example is assigned to test the methods’ response to zero elements in the initial matrix. So let us disturb one element of our data set, say (3, 1), by putting it equal to zero for years 0 and 1. After recalculation of the marginal totals we get the data set in the left half of Table 2.

The results of computations are collected in the right half of Table 2 for RAS method and in Table 2a for HOM and ANG methods; as earlier, they seem to be very similar among themselves.

Table 2. Initial matrix **A** with zero element and RAS results for its updating

	A				$\mathbf{u_A}$	**u**
	20.00	34.00	10.00	36.00	100.00	94.78
	20.00	152.00	40.00	188.00	400.00	412.86
	0.00	72.00	20.00	98.00	190.00	202.88
$\mathbf{v}'_{\mathbf{A}}$	40.00	258.00	70.00	322.00	690.00	
$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52

RAS	**X**				$\mathbf{u_X}$	**u**
	18.02	32.74	9.75	34.27	94.78	94.78
	19.46	158.05	42.11	193.25	412.86	412.86
	0.00	77.23	21.72	103.92	202.88	202.88
$\mathbf{v}'_{\mathbf{X}}$	37.48	268.02	73.58	331.44	710.52	
$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52

Table 2a. HOM and ANG results for updating of data set from Table 2

HOM	**X**				$\mathbf{u_X}$	**u**
	18.36	32.40	10.04	33.98	94.78	94.78
	19.12	158.80	42.58	192.37	412.86	412.86
	0.00	76.82	20.96	105.10	202.88	202.88
$\mathbf{v}'_{\mathbf{X}}$	37.48	268.02	73.58	331.44	710.52	
$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52

ANG	**X**				$\mathbf{u_X}$	**u**
	18.35	32.40	10.05	33.98	94.78	94.78
	19.13	158.78	42.55	192.39	412.86	412.86
	0.00	76.84	20.98	105.07	202.88	202.88
$\mathbf{v}'_{\mathbf{X}}$	37.48	268.02	73.58	331.44	710.52	
$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52

Again, HOM and ANG methods still keep on the 5-percentage advantage over RAS method both in homothetic and angular measures as follows:

$$\left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 0.0543, \quad \left|\boldsymbol{\delta}^{\mathrm{HOM}}\right| = 0.0516, \quad \left|\boldsymbol{\delta}^{\mathrm{ANG}}\right| = 0.0516, \quad \left|\boldsymbol{\delta}^{\mathrm{HOM}}\right| / \left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 95.04\%;$$

$$\beta_{\mathbf{qe}}^{\mathrm{RAS}} = 3.0805°, \quad \beta_{\mathbf{qe}}^{\mathrm{HOM}} = 2.9291°, \quad \beta_{\mathbf{qe}}^{\mathrm{ANG}} = 2.9286°, \quad \beta_{\mathbf{qe}}^{\mathrm{ANG}} / \beta_{\mathbf{qe}}^{\mathrm{RAS}} = 95.07\%.$$

An advantage of HOM and ANG methods observed here is not so impressive because of small number of “free” variables $NM-(N+M)$ in our numerical examples. However, if the dimensions of updating matrix tend to grow, then this advantage rapidly increases. At the dimensions more than 3×7 (7×3) and 4×5 (5×4) a total amount of free variables starts to exceed total number of RAS variables, so flexibility of HOM and ANG methods substantially grows. Computational experiments with 15×20-dimensional matrices indicates that HOM and ANG methods seem to be almost twice more effective than RAS in the sense of homothetic measure (13) and angular measure (14).

As it is well-known, “… RAS can only handle non-negative matrices, which limits its application to SUTs that often contain negative entries…” – see Temurshoev et al. (2011, p. 92). So the final numerical example is assigned to test the methods’ response to negative elements in the initial matrix. Let us disturb three elements of our data set, say (1,3), (3, 1) and (3,3), by reversing their sign for years 0 and 1. After proper recalculation of the marginal totals we obtain the data set in the left half of Table 3.

The results of computations are grouped in the right half of Table 3 for RAS method and in Table 3a for HOM and ANG methods; now they demonstrate wide differences in the elements of three target matrices calculated, especially in x_{13}, x_{23}, x_{24} and x_{33} .

Table 3. Initial matrix **A** with zero element and RAS results for its updating

	A				$\mathbf{u_A}$	**u**
	20.00	34.00	**-10.00**	36.00	80.00	74.50
	20.00	152.00	40.00	188.00	400.00	412.86
	-10.00	72.00	**-20.00**	98.00	140.00	148.92
$\mathbf{v}'_\mathbf{A}$	30.00	258.00	10.00	322.00	620.00	
$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28

RAS	**X**				$\mathbf{u_X}$	**u**
	17.09	31.06	**-6.18**	32.53	74.50	74.50
	20.13	163.54	29.12	200.07	412.86	412.86
	-9.54	73.42	**-13.80**	98.84	148.92	148.92
$\mathbf{v}'_\mathbf{X}$	27.68	268.02	9.14	331.44	636.28	
$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28

Table 3a. HOM and ANG results for updating of data set from Table 3

HOM	**X**				$\mathbf{u_X}$	**u**
	18.55	32.30	**-10.21**	33.87	74.50	74.50
	19.27	159.99	39.34	194.26	412.86	412.86
	-10.13	75.73	**-19.99**	103.31	148.92	148.92
$\mathbf{v}'_\mathbf{X}$	27.68	268.02	9.14	331.44	636.28	
$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28

ANG	**X**				$\mathbf{u_X}$	**u**
	18.56	32.31	**-10.26**	33.89	74.50	74.50
	19.30	159.91	39.47	194.18	412.86	412.86
	-10.18	75.80	**-20.07**	103.37	148.92	148.92
$\mathbf{v}'_\mathbf{X}$	27.68	268.02	9.14	331.44	636.28	
$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28

An advantage of HOM and ANG methods in this case seems to be overwhelming. Indeed, the received estimates of homothetic and angular measures are

$$|\boldsymbol{\delta}^{\mathrm{RAS}}| = 0.1453, \qquad |\boldsymbol{\delta}^{\mathrm{HOM}}| = 0.0438, \qquad |\boldsymbol{\delta}^{\mathrm{ANG}}| = 0.0438, \qquad |\boldsymbol{\delta}^{\mathrm{HOM}}| / |\boldsymbol{\delta}^{\mathrm{RAS}}| = 30.14\%;$$

$$\beta_{\mathbf{qe}}^{\mathrm{RAS}} = 9.1437°, \qquad \beta_{\mathbf{qe}}^{\mathrm{HOM}} = 2.5102°, \qquad \beta_{\mathbf{qe}}^{\mathrm{ANG}} = 2.5081°, \qquad \beta_{\mathbf{qe}}^{\mathrm{ANG}} / \beta_{\mathbf{qe}}^{\mathrm{RAS}} = 27.43\%.$$

Thus, one can conclude that HOM and ANG methods are especially effective under the complicated circumstances because of its immanent flexibility. In practice the proposed GLS-based methods allow to generate much more compact distributions of the multiplicative model's factors in comparison with RAS method.

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