# Meeting Eigenvectors during a Square Dance on a Hilbert Cube with Interindustry Transactions: A Complex Story.

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In recent years we have witnessed many papers using Social Network Analysis, relying on methods of Graph Theory, in order to explore and evaluate input-output matrices. This has enabled the introduction of new tools for the input-output analysis such as measures of centrality of the nodes (sectors) of a graph; identification and evaluation of main paths who spread economic impulses within the industrial network; cohesion of economic networks, and other measures that help to know the characteristics of individual branches, groups of branches, and finally the topology of an inter-industrial network as a whole. However, some of these measures face problems of information loss due to the use of algorithms from graph theory that usually cannot cope with asymmetric information inherent in the inter industry tables, so the analysts tend to filter, dichotomize or symmetrize the input-output matrices.

To tackle those limitations, in this article it is shown how to encode the input-output matrices in a complex numbers space. Each entry of the data table is recorded as a complex number, i.e. with two dimensions; one axis (real) for inputs and the other (imaginary) for outputs. Stated in graph theory terms, any node (sector) is linked to the others by two arcs, one for purchases and other for sales.

As a next step, the complex numbers matrix is rotated in order to get its conjugate, so it has been constructed a Hermitian matrix inserted in a Hilbert Space. This construction lead to a metrizable and separable space, so it is homeomorphic to a subspace of the **Hilbert Cube**. Every Hermitian matrix is a normal matrix. The finite-dimensional spectral theorem says that any Hermitian matrix can be diagonalized by a unitary matrix, and that the resulting diagonal matrix has only real entries. This implies that all eigenvalues of a Hermitian matrix *A* are real, and that *A* has *n* linearly independent eigenvectors. Moreover, it is possible to find an orthonormal basis of $C^n$ consisting of *n* eigenvectors of *A*. Also, the matrix *A* can be written as a linear combination of orthogonal paired projectors, i.e. spectrally decomposed. The level or intensity of purchases and sales is deployed in the spectrum (set of characteristic values) and its associated eigenvectors show subgroups of related economic sectors through its commercial ties with the most influential sector of each group. This allows interpretation of the system beyond the main eigenpair. In fact, an interpretation for the totality of eigenvalues and associated eigenvectors is achieved. An extension of the methodology give us a set of clusters for the economy.

This paper is organized in four parts. The first one provides an introduction to the asymmetry issue; in a second part the methodology is developed; in the third part we show an application to the Mexican economy; the fourth part is devoted to some remarks and final considerations

Keywords: Input-Output Matrix, Hilbert Space, Complex Numbers, Asymmetry, Influential Sectors, Bilateral Trade.

## Introduction.

The approach to the input-output tables (IOT) through graph theory and social networks analysis starts with Hubbell (1965) using indicators that were originally developed in the field of sociology during the fifties of the last century Bavelas (1948), Harary and Norman (1953). Applications to economics of these tools, with relative autonomy, were developed by French researchers since the late sixties and until 1990, Ponsard (1969), Lantner (1972), Gazon (1976), Auray, Duru, Mougeot (1977). More recently, Spanish researchers driven by Morillas (1983), Garcia Perez (1999), García Muñiz (2006), Garcia and Ramos (2003), nearly faded out the borders in the use of social networks analysis tools and input output analysis. This has enabled the introduction of new tools for the input-output analysis such as measures of centrality of the nodes (sectors) of a graph; identification and evaluation of main paths who spread economic impulses within the industrial network; cohesion of economic networks, and other measures that help to know the characteristics of individual branches, groups of branches, and finally the topology of an inter-industrial network as a whole. However, some of these measures face problems of information loss due to the use of algorithms from graph theory that usually cannot cope with asymmetric information inherent in the inter industry tables, so the analysts tend to filter, dichotomize or symmetrize the input-output matrices (often not knowing it!, due to the usage of software "ready to use").

Asymmetry is a fact that occurs in many phenomena in social life and nature. It refers to the circumstance in which relationships between pairs of objects occur unevenly. One case, evident in the economic analysis, is the foreign trade between countries where it is common to find exports and imports between countries, in which many of them have a very large trade deficit. There are other economic phenomena such as income distribution, size of business, labor absorption capacity, and many others that are characterized by great asymmetry. The asymmetry is embedded in the information we collect to analyze a particular discipline.

In the traditional economic input-output analysis, the metric characterization of their tables is not analyzed, i.e., it is assumed that they are Euclidean; and there are no questions about the asymmetry in the relations described in these tables. However, if Social Networks Analysis and Multivariate Statistics are going to be used as tools for studying economic input out tables, as we will see later, there is no way but analyze explicitly the asymmetry of the data Solis and Garcia (2009).

There are several ways to cope with asymmetric relationships, as will be apparent in the next paragraphs, but we decided to encode the input-output matrices in a complex numbers space due to the fact that: a) We can treat asymmetry without loss of information derived of pre-processing the input-output matrices through its normalization (vertically or horizontally), filtering or dichotomization; b) Complex Eigenanalysis will reveal the structural characteristics of the input-output matrices, allowing us to interpret the whole spectrum (eigenvalues) and associated eigenvectors, beyond the practice of many economists that only interpret the leading eigenvalue and its associated eigenvector; c) We will contribute to overcome the limitations derived from the use of two different models (Leontief and Gosh) for studying allocation and production of goods and services within the interindustrial relationships.

## 1.1 A basic framework for studying asymmetric input-output relationships.

When it is known that data is skewed[1], we can perform a decomposition of the matrix into two parts, one symmetrical and one asymmetrical:

Let us define a symmetric matrix as

$$S=\frac{1}{2}(X+X')$$

And a skew-symmetric matrix as

$$A=\frac{1}{2}(X-X'),$$

Then we have a unique decomposition of X as

X=S+A,

Noting that $tr(SA)=\sum_{i=1}^{n}\sum_{j=1}^{n}s_{ij}a_{ij}=0$

So we can derive $\sum_{i=1}^{n}\sum_{j=1}^{n}X_{ij}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}s_{ij}^2+\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^2$

Which could be expressed in terms of the matrix norm as

$$\|X\|^2=\|S\|^2+\|A\|^2$$

The average of $\{X_{jk}\}$ is equal to the average of $\{S_{jk}\}$ and the average of $\{a_{jk}\}$ is zero, then it follows that

$$Var(x_{jk})=Var(s_{jk})+Var(a_{jk}) \ .$$

There are two approaches to addressing this decomposition. In the first we deal with S and A separately, applying a method to measure and analyze the symmetric structure, and other method for analyzing the asymmetric structure. In a second approach, we work out with S and A simultaneously. In one case it is assumed that there are two parts which reflect different processes that can be distinguished by appropriate analysis; on the other hand, one considers the symmetric part and skew-symmetric part of the data to be inseparables parts of the same fundamental processes. The former approach was initially developed by Gower (1977), the later approach was developed by Escoufier and Grorud (1980). These works were seminal and a huge literature appeared in the following years. We shall follow the Escoufier-Grorud approach in a wider

[1] There are several test on this subject. See Chino N. (2008)

framework. For an exhaustive account of those developments see the book by Saito and Yadohisa (2005) and the vast survey by Chino (2012).

Originally, Escoufier and Grorud, introducing Hermitian Matrices, developed their work in order to generalize the Principal Components to analyze asymmetric data, however a close technique, Multi-Dimensional Scaling (MDS), is more flexible and a best way to expand our view of the asymmetry of the relations within an input-output matrix, because MDS requires an explicit treatment of the metric properties of the data involved in a particular study (input output analysis in our case). Another advantage of approaching the asymmetry via MDS is its close relationship with the visualization of Social Networks Analysis. Recently it has been used Multi-Dimensional Scaling (MDS) and Graph Layout (GL) techniques to explore matrices on a two-dimensional space, Borgatti S et al. (2013). One of the most robust Graph Layout algorithm is due to Kamada and Kawai (1989), whose final calculation formula is equivalent to a non-metric multidimensional scaling.

The MDS is formally supported on two fundamental theorems due to Young and Householder (1938) for the case of symmetric matrices and Chino and Shiraiwa (1993) for asymmetric matrices. In both cases, symmetric and asymmetric (dis)similarities, those theorems provide us with necessary and sufficient conditions that must be fulfilled to assert that the coordinates of objects in a multidimensional space are real points in the Euclidean space (for symmetric data) or a (complex) Hilbert space. The main condition in both cases is that the data matrix must be positive semi definite. Although those theorems by themselves do not describe the way of determining the number of dimensions of the space in which the data is embedded their authors refer to a solution utilizing the famous Singular Value Decomposition, applied by Eckart and Young (1936) for the symmetric case, and the use of the generalization by Shmidt-Mirsky for the complex (asymmetric) case. It has to be noted that many properties of the singular value decomposition hold for the complex spectral decomposition. As we shall see later, the eigenvalues of a Hermitian matrix and their multiplicities are unique, and their corresponding eigenvectors to a multiple eigenvalue span a unique subspace, and the eigenvectors can be chosen as any orthonormal basis for that subspace.

### 1.2 Some notes on Vertical and Horizontal Models.

In the input-output model the basic identity which equals total demand and total supply is met by the addition of records per row (demand) and column (supply). These identities leaded to the construction of two models closely related. In specialized literature those models are known as the Leontief´s model (LM) and Gosh´s model (GM). The interpretations of the relationship between the two models has been the subject of extensive controversy[2]. Actually those models are two sides of the same coin, so modeling simultaneously horizontal identities and vertically ones is achieved closing to some extent the model. In this way, there is a link between the variables considered exogenous in each of the models. Final demand in the case of LM model and the value added in the case of GM model. There are several alternatives to close the models Guerra and Sancho (2010), however we shall follow N. Adamou (2007) approach due to the fact that he applies extensively a solution based on the Eigensystem of both models, and will be useful to interpret the

[2] See, De Mesnard 2009, Dietzenbacher 1989, 1997, Miller 1989, Oosterhaven 1988, 1989, 1996 and many others.

Complex Eigensystem that we shall develop in the next section. In this approach, the closure of the models is based on the fact that for a given direct requirement and allocation coefficient matrices A (LM) and B (GM) , and gross output x , one may derive the similarity transformations $A=\hat{x}B\hat{x}^{-1}$ and $B=\hat{x}^{-1}A\hat{x}$ . These transformations indicate that the requirement coefficient matrix at a given time of an I-O table may be transformed into the allocation coefficient matrix, and ***they change together*** as interindustrial transactions and gross output change. It must be stressed that the eigenvalues, the trace and determinants, for both matrices, are the same.

Whenever one has a finite dimensional vector space with two bases, there is a unique *transition matrix* translating form one space into the other. The sectoral gross output in its diagonal form is such a transition matrix, from value of total of production to the value of total demand. The matrix of direct requirement coefficients A is relative to the total value of production (vertical proportionality). At the same time, the matrix of direct allocation coefficients B is relative to the value of total demand (horizontal proportionality), while the diagonal matrix of gross output x provides the *transition* from total value of production to the total value of demand, and its inverse, $x^{-1}$, transforms in the opposite direction. The Leontief and Gosh inverse matrices, as Adamou shows[3], may provide the same output multiplier whenever the Leontief inverse is weighted appropriately by the distribution of final use and the Ghoshian inverse by the distribution of value added. This identical output multiplier may then be decomposed in such a way that indicates either the detailed impact of final use or value added. The appropriate output multipliers and their decomposition provide the magnitude effect of an eventual disturbance, while the eigenvalues indicate the impact's mode and the corresponding eigenvectors furnish the spatial directions of the disturbances.

It is interesting the interpretation of the full Eigensystem provided by N. Adamou: "In linear systems, eigenvalues indicate the limits of their vibrations whenever the system is disturbed by an outside force, while eigenvectors denote the directions of such vibrations. Positive eigenvalues mark potential expansion, while negative eigenvalues mark potential contraction. Complex eigenvalues suggest directional change… The directions of eigenvalues are given by their corresponding eigenvectors. All eigenvalues must be taken into account, not only the dominant ones."[4]

This approach does have three main features: a) the departure of the traditional estimates of the horizontal and vertical models which only display results on a Cartesian plane where there is no intrinsic causality in the graphic display but simply an interpretation from the use of two models; b) The interpretation of the full Eigensystem; c) A model that simultaneously take relations of supply and demand and whose solution allows us to clearly evaluate the functional position of that sector have within the economic network as a whole.

In this paper we face the issue of vertical and horizontal models from the perspective of the asymmetric nature of the economics involved in the input output model, and instead of dealing

[3] Op cit p. 5 note 16.

[4] Adamou N. unpublished paper: "Determinants and Eigensystems in Similar Input-Output Matrices: 'Supply' & 'Demand' Driven I-O Structures of the Irish Economy" pp. 23-24. There is a internet version in www.academia.edu

with two models, we merge the dual direction of the interindustrial flow matrix data. To do this, we need to combine, into a single number, purchases and sales. This is achieved through a representation of the input output matrices into complex numbers. Indeed in each cell of the input-output model can be encoded the value of the sales and simultaneously the value of the purchases. The sales can be the real part of a complex number, and the purchases its imaginary part. A complex number is not the simple addition of two numbers but a number with two dimensions. We shall perform a Complex Eigenanalysis, interpreting the full set of its results.

In order to show the whole picture, in the next section we will introduce a self-contained introduction to Complex Eigenspectral Analysis.

## 2. Methodology: Hilbert Spaces, complex Hermitian matrices and spectral analysis of adjacent complex Hermitian matrices.

### 2.1. Notation and basic definitions.

#### Hilbert Spaces.

Let a and b be two real numbers and i = $\sqrt{-1}$ the imaginary unit. A **complex number** z can be represented in algebraic form (or binomial) as z = a + bi and its exponential form as

$$z = a + bi = |z|\, e^{i\phi} \tag{1}$$

Where a = *Re(z)* its real part, b = *Im(z)* its imaginary part, $|z| = \sqrt{a^2 + b^2}$ the module and $\phi = \arccos \frac{a}{|z|}$ its argument[5]. We represent by X the complex numbers field.

The conjugate of z is the complex number $\bar{z} = a - bi$ and for any z, $z_1$ and $z_2 \in X$, it does occur that

$$z + \bar{z} = 2Re\,(z) \tag{2}$$

$$z = \bar{z} \text{ si y solo si } z \in P \tag{3}$$

$$z_1 z_2 = |z_1||z_2| e^{i(\phi_1 + \phi_2)} \tag{4}$$

$$z\bar{z} = |z|^2 \tag{5}$$

$$\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2} \tag{6}$$

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[5] In other manuals it is frequent to calculate the argument $\phi$ of z as $\phi = \mathrm{artg}\frac{b}{a}$ , as well as to use the polar form of z = $|z|_{\phi}$.

From now on, unless otherwise indicated, all numbers will be complex. Let us consider the vector field $X^n$, in a way that any of its elements $\vec{x}$ represent a column vector whose components $x_{ij}$ = 1, 2, ...., n are complex. Matrices will be denoted by upper-case roman letters and their elements $a_{ij}$ by lower case letters de la. $\lambda_j$ represents the j-tuple eigenvalue of a matrix. The expression $\vec{x}^t$ indicates the transposed vector of $\vec{x}$ (row vector) and $\vec{x}^*$ represents the complex transposed vector, i.e the conjugate transposed of $\vec{x}$.

Let us define the **outer product** of two vectors $\vec{x}$ e $\vec{y}$ $\in X^n$ as follows:

$$\vec{x}\,\vec{y}^* = \begin{pmatrix} x_1\overline{y_1} & x_1\overline{y_2} & \cdots\cdots & x_1\overline{y_n} \\ x_2\overline{y_1} & x_2\overline{y_2} & \cdots\cdots & x_2\overline{y_n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ x_n\overline{y_1} & x_n\overline{y_2} & & x_n\overline{y_n} \end{pmatrix} \tag{7}$$

And also the **inner product** of two vectors $\vec{x}$ e $\vec{y}$ $\in X^n$ as a bilinear[6] form in that vector space, such as:

$$\langle \vec{x} | \vec{y} \rangle = \vec{x}^*\,\vec{y} = \sum_{k=1}^{n} \overline{x_k}\, y_k \tag{8}$$

From now on, **the vector module** $\vec{x} \in X^n$ that we notate as $\|\vec{x}\|$ is defined as:

$$\|\vec{x}\| = \sqrt{\langle \vec{x} | \vec{x} \rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2} \tag{9}$$

With the inner product of two vectors and the norm of a vector defined in that way, we can say that the vector space $X^n$ is a complete **Hilbert Space**[7].

Besides, $\forall$ $\vec{x}$, $\vec{y}$, $\vec{z} \in X^n$ fulfill the following properties:

$$\langle \vec{x} | \vec{x} \rangle \geq 0 \;\text{ And }\; \langle \vec{x} | \vec{x} \rangle = 0 \;\text{ iff }\; \vec{x} = \vec{0} \tag{10}$$

$$\langle \vec{x} | \vec{y} \rangle = \overline{\langle \vec{y} | \vec{x} \rangle} \tag{11}$$

$$\langle \vec{x} | \vec{0} \rangle = \langle \vec{0} | \vec{y} \rangle = 0 \tag{12}$$

$$\forall\ \alpha \in X \text{ It is true that } \langle \alpha\vec{x} | \vec{y} \rangle = \overline{\alpha}\, \langle \vec{x} | \vec{y} \rangle$$

[6] Given a vectorial space X, it is said that a **function** f defined in X is **linear** if for any couple of vectors u, v of X and for any scalar pair $\alpha$, $\beta$ it is true that $f(\alpha u + \beta v) = \alpha f(u)+\beta f(v)$; and it is said that f is a bilinear function or linear-conjugated if it happens that $f(\alpha u+\beta v) = \overline{\alpha} f(u) + \overline{\beta} f(v)$

[7] Es It is complete because the Condition of Cauchy is verified that says that all succession of Cauchy of vectors $\{\vec{x}_1, \vec{x}_2,, \ \ , \vec{x}_n, .........\}$ is convergent.

$$\text{And also that } \left\langle \vec{x} \middle| \alpha \vec{y} \right\rangle = \alpha \left\langle \vec{x} \middle| \vec{y} \right\rangle \tag{13}$$

$$\left\langle \vec{x} + \vec{y} \middle| \vec{z} \right\rangle = \left\langle \vec{x} \middle| \vec{z} \right\rangle + \left\langle \vec{y} \middle| \vec{z} \right\rangle \tag{14}$$

On the other hand, the following three identities are verified in this Hilbert Space:

- Cauchy-Schwarz Inequality:

$$\forall\ \vec{x},\ \vec{y} \in X^n \text{ and holds } \left| \left\langle \vec{x} \middle| \vec{y} \right\rangle \right| \leq \left\| \vec{x} \right\| . \left\| \vec{y} \right\| \tag{15}$$

  The equality is true when $\vec{x} = \alpha \vec{y}$ with $\alpha \in X$

- Triangle Inequality:

$$\forall\ \vec{x},\ \vec{y} \in X^n \text{ happens that } \left\| \vec{x} + \vec{y} \right\| \leq \left\| \vec{x} \right\| + \left\| \vec{y} \right\| \tag{16}$$

  With equality when $\vec{x} = \alpha \vec{y}$ with $\alpha \in P$

- Bessel's Inequality:

  For any vector $\vec{x}$ and any set of an orthonormal[8] sequence $\{ \overrightarrow{x_1},\ \overrightarrow{x_2},\ .......,\overrightarrow{x_n} \}$, it is true that

$$\sum_{k=1}^{n} \left| \left\langle \vec{x} \middle| \overrightarrow{e_k} \right\rangle \right|^2 \leq \left\| \vec{x} \right\|^2 \tag{17}$$

- Parseval's Identity:

  For any vector $\vec{x}$ and any complete[9] orthonormal vector basis $\{ \overrightarrow{e_1},\ \overrightarrow{e_2},\ .......,\overrightarrow{e_n} \}$, it is true that

$$\sum_{k=1}^{n} \left| \left\langle \vec{x} \middle| \overrightarrow{e_k} \right\rangle \right|^2 = \left\| \vec{x} \right\|^2 \tag{18}$$

This is really the Pythagorean Theorem for inner product spaces.

**Hermite Matrices.**

[8] Two vectors $\vec{x}$ and $\vec{y}$ are orthogonal if happens that $\left\langle \vec{x} \middle| \vec{y} \right\rangle = 0$ and often is written as $\vec{x} \perp \vec{y}$. A group of vectors is said that it is orthogonal when the vectors are orthogonal two at two. If it also happens that all the vectors of the group are unitary, i.e. , $\left\| \vec{x} \right\| = 1$, then, it is said that the set is **orthonormal**. This condition can also be expressed as $\left\langle \vec{x} \middle| \vec{y} \right\rangle = \delta_{ij} = \begin{cases} 1 & \text{si } \vec{x} = \vec{y} \\ 0 & \text{si } \vec{x} \neq \vec{y} \end{cases}$ .

[9] An orthonormal family of vectors $\{x_1, x_2, ......., x_n\}$ of H is **complete** if happens that n = dimH. In this case, it is a base called orthonormal.

In a Hilbert Space H, any continuous linear mapping T: H -----> H is an operator[10]. In particular, **I** is the identity operator, **0** the zero operator, **λI** the scaling operator. It can be shown that for any operador T, there is one and only one continuous linear mapping $T^*$: H -----> H such that for every pair of elements x, y∈ H, it is true that $\langle Tx | y \rangle = \langle x | T^* y \rangle$. This new operator $T^*$ is named **adjoint operator of T**. When T * = T we say that this is a Hermite operator (or Hermitian). Under these conditions, a square matrix of complex numbers H is said to be a Hermitian matrix, if verifies that:

$$H^* = H \tag{19}$$

A Hermitian Matrix is equal to its conjugate[11]transpose. In this case, the diagonal elements must be real numbers and $h_{ij} = \overline{h_{ji}}$ ∀i, j. By its very definition, we can say that every Hermitian matrix is normal since $H^*H = HH^*$.

A square matrix with complex entries, i.e. $A \in X^{n,}$ is said to be **skew-Hermitian** or **antihermitian** if its conjugate transpose is equal to its negative. That is, the matrix *A* is skew-Hermitian if it satisfies the relation

$$A^* = -A. \tag{20}$$

Any matrix M can be decomposed as follows:

$$M = \frac{1}{2}\left(M + M^*\right) + \frac{1}{2}\left(M - M^*\right) \tag{21}$$

Where the first term is a hermitian matrix and the second antihermitian.

For any vector space X and any operator defined in X, we say that a complex number λ is an eigenvalue, an eigenvalue or a characteristic value of T if there exists a nonzero vector x∈X such that:

$$Tx = \lambda x \tag{22}$$

The vector x is called **eigenvector, or characteristic vector of T** associated to the eigenvalue λ. The set $V_\lambda$ = {v∈X / Tx = λx} is a vector sub-space called eigenspace **for the eigenvalue λ** whose dimension is the multiplicity of λ. The set of distinct eigenvalues, denoted by σ(A), is called spectrum of A and can be proved that X is the orthonormal sum of the eigenspaces $V_\lambda$.

In the particular case of a Hermitian matrix H, being a normal matrix always can be transformed to a diagonal matrix, i.e., it is similar to a diagonal matrix D (∃ a unary matrix X such as $X^*HX = D$). That is, for a normal matrix the eigenvalues associated to distinct eigenvalues constitute an orthogonal system.

## 2.2. Properties and interpretation of the values and eigenvectors of a Hermitian matrix.

Let us recapitulate some features of Hermitian matrices that will be useful in this paper:

- All their eigenvalues are real numbers.

[10] Ver S. K. Berberian (1970, pp. 129-130).

[11] This condition is equivalent to $\langle Hx | y \rangle = \langle x | Hy \rangle$ ∀x, y.

On one hand, it is true that $\langle Hx|x\rangle = \langle \lambda x|x\rangle = \bar{\lambda}\langle x|x\rangle$ and in the other hand, $\langle x|Hx\rangle = \langle x|\lambda x\rangle = \lambda\langle x|x\rangle$, then $\lambda = \bar{\lambda}$ which means that $\lambda \in R$.

Then we can order the absolute value of the eigenvalues

$$|\lambda_1| = \max_k > |\lambda_2| > .......... > |\lambda_n| = \min_k |\lambda_k|$$

- All their eigenvalues are simple due to the fact that H does have maximum[12]rank and it is diagonalizable. Therefore, the dimension of each subspace is 1, which means that for each eigenvalue there is just an eigenvector.

- We can choose an orthonormal basis, with eigenvectors associated to different eigenvalues:

$$\langle \overrightarrow{x_k} | \overrightarrow{x_l} \rangle = \delta_{kl} \tag{23}$$

This result is also valid in the case of arbitrary rotations:

$$\langle e^{i\varphi_k}\overrightarrow{x_k} | e^{i\varphi_{lk}}\overrightarrow{x_l} \rangle = e^{-i\varphi_k} e^{i\varphi_l} \langle \overrightarrow{x_k} | \overrightarrow{x_l} \rangle = e^{i(\varphi_l - \varphi_k)} \langle \overrightarrow{x_k} | \overrightarrow{x_l} \rangle = e^{i(\varphi_l - \varphi_k)} \delta_{kl} \tag{24}$$

- The spectral decomposition theorem holds that:

$$H = \sum_{k=1}^{n} \lambda_k P_k \tag{25}$$

Where $P_k = \begin{pmatrix} x_1\overline{x_1} & \cdots\cdots & x_1\overline{x_n} \\ \cdots & \cdots\cdots & \cdots \\ x_n\overline{x_1} & \cdots\cdots & x_{n1}\overline{x_n} \end{pmatrix}$ are orthogonal projectors such as $\sum_{k=1}^{n} P_k = I$, $P_k = P_k^*$ and that $P_k^2 = P_k$.

Chino asserts on this decomposition the following: "The spectral decomposition of a Hermitian Matrix (or more generally, normal matrix) is nothing but a special case of the Fourier expansion. Moreover, each eigenvalue appearing in the expansion is considered as coordinate in an inner product space (or pre-Hilbert Space) spanned by the Hermitian matrix…"[13]. In our context, the eigenvectors elements are ordered, according to their importance, for each member of a sub-group. In addition, each member has, for each sector of the eigenspace in the spectral representation, a different importance. This ratio depends on the relationship with the respective anchor of the subgroup.

- The square sum of the eigenvalues is the data variance.

  As it happens that[14] $\|H\|^2 = \sum_{k=1}^{n} \lambda_k^2$, then if we add up to an index m ≤ n and divide it by the sum of all $\lambda_k^2$, the result will show us the variance included by the first m sub-spaces.

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[12] Meyer, C. D. (2000, p. 548).
[13] Chino N (1998, note 7)
[14] Hoser, B. (2004, p. 45) or Chino (1998, p. 58)

For a complex Hermitian matrix whose trace is zero, some eigenvalues have to be negative as $H = UDU^{-1}$, then $\sum_{k=1}^{n} \lambda_k = tr(H) = \sum_{k=1}^{n} h_{kk} = 0$.

- A graph with star's form has two characteristic same values in magnitude, but opposed sign[15]:

$$\sigma(B) = \{+\lambda_1, -\lambda_2\} \quad (26)$$

With $|\lambda_1| = |\lambda_2|$.

Within the social network analysis Seary and Richards (2000, section F), Chino (1998) and Barnett and Rice (1985) provide some suggestions on the interpretation of negative eigenvalues.

- A star-shaped graph has two eigenvectors with equal magnitudes but opposite direction. Also can be seen as eigenvectors belonging to two equal eigenvalues in absolute magnitude but differ by $\pi$ in their phase.
- **Automatic determination of clusters,** Hoser and Schroder (2007). The eigenvectors matrix associated with positive characteristic values of the Hermitian matrix may generate, by the inner product operation, a matrix of n x n size, whose diagonal elements give us the norm of each row of it. In addition, the minimum distance between two vertices, can be seen as the maximum of the real part of the inner product between the two; thus, the maximum of the real part of the elements of the columns identify part of a cluster (when a vertex not belong to any cluster, the maximum corresponds to the main diagonal of the matrix).

## 2.3. Construction of an adjacent complex Hermitian matrix.

From now on we will consider a valued and directed graph *G= {N, E}* where N is the set of vertices members and E the set of connections, links or relationships between different members. The relationship of each node are excludes himself.

We followed[16] the construction of an adjacent matrix H, associated to this graph, in two steps:

1. For each vertex we consider the number of links m that start from k and also the number of p connections leading to that node k. A complex square matrix A defined by adjacency is then constructed:

   $$a_{kl} = m + ip$$

   So, it is assured that $a_{kl} = i\,\overline{a_{lk}}$.

2. To get the Hermitian matrix *H*, we rotate *A* multiplying, each one of its members, by $e^{-i\frac{\pi}{4}}$ i.e.:

[15] Ver Meyer, C. D. (2000, p. 555)
[16] Hoser, B. and Geyer-Schutz, A. (2005)

$$H = Ae^{-i\frac{\pi}{4}}$$

Demonstration/

Let $a_{kl} = re^{i\phi}$, then $a_{lk} = i\,\overline{a_{kl}} = ire^{-i\phi}$.

$a_{kk} = 0$ non reflexivity

Let $h_{kl} = a_{kl}e^{i\psi} = re^{i\phi}e^{i\psi} = re^{i(\phi+\psi)}$, then

$$h_{lk} = a_{lk}e^{i\psi} = ire^{-i\phi}e^{i\psi} = e^{i\frac{\pi}{2}}re^{i(\psi-\phi)} = re^{i(\frac{\pi}{2}+\psi-\phi)}$$

As it is Hermitian $h_{kl} = \overline{h_{lk}}$, so

$$re^{i(\phi+\psi)} = re^{-i(\frac{\pi}{2}+\psi-\phi)}$$

Or, $\phi + \psi = -\frac{\pi}{2} - \psi + \phi$

And we conclude that $\psi = -\frac{\pi}{4}$

3. Under this similarity transformation, Meyer (2000, p. 256) shows that the independent features of the communication model are maintained, i.e., there is no loss of information.

This H construction is related to Chino´s construction[17] of $H_C$ by $H_C = \frac{\sqrt{2}}{2}\overline{H}$, with $\overline{H}$ representing the complex conjugated matrix of $H_H$

Depending on the number of relationships between different vertices, the following table shows how would look the elements of this new matrix H, once rotation has been performed:

| Relationships | $a_{kl} = m + ip$ | $h_{kl} = m_r + ip_r$ |
|---|---|---|
| No self- reference | $a_{kk} = 0$ | $h_{kk} = 0$ |
| $k \to l > l \to k$ | $m > p$ | $p_r < 0$ |
| $k \to l < l \to k$ | $m < p$ | $p_r > 0$ |
| $k \to l = l \to k$ | $m = p$ | $p_r = 0$, $m_r > 0$ |

It is, after this rotation, the diagonal elements of H remain 0. If we consider two vertices k and l and if it happens that there is more flow from k to l, this element $h_{kl}$ has a negative imaginary part. In the opposite case, the considered element $h_{kl}$ has a positive imaginary part. When both flows are equal, the $h_{kl}$ element is a positive real number.

We should note that in this new standard matrix H is invariant under rotation. The absolute value, which is what actually measures the amount of flow exchanged does not change.

Due to the rotational invariance of a system full orthonormal eigenvectors, we can improve the visibility of specific components of an eigenvector, applying a rotation to make real these

[17] Hoser, B. (2005, p. 274).

components. For example, the Mathematica program automatically rotates the eigenvector for this component with the largest absolute value becoming real and positive. In this work, this element will be considered as the anchor or support, i.e., the most influential member of a subgroup.

## 3. An application to the Mexican economy.

In this study the Input Output Matrix of the Mexican economy employed is for the year 2012. This matrix was published by Mexico's National Institute of Statistics and Geography. The main results are shown for the total transactions matrix, although their spectra and phases are compared with domestic transactions matrix.

The matrix considered 259 sectors, so its entire presentation is difficult; however, as will be seen below, the first 25 characteristic values recorded over 98 percent of the total variation in the data, so most of the tables and graphs shown considering between 25 and 30 sectors only.

### 3.1 Solution method: SVD and Eigensystem procedures.

As already pointed out, the solutions for a obtaining a solution to the MDS problem was developed for both, metric and Hilbert spaces, employing a Singular Value Decomposition; let us quote to Chino and Shiraiwa: "HCM (Hermitian Canonical Form, *parenthesis added by me*) can have some metric properties such as a finite-dimentional complex Hilbert space structure under a general condition…; however, in a practical situation, these matrices may be fallible and not necessarily be measured at the ratio level. Furthermore, we can neither observe nor estimate the special distances in a Hilbert space… in marked contrast to the distance in classical MDS. In such a case we must estimate them from the data. If the proximity judgments are measured at the ratio level, there are no missing observations, and the matrix H is positive semi-definite, we can use HCM as one such method. For in such a case HCM is also solved by a singular value decomposition and thus has a least squares (LS) property according to the Schmidt-Mirsky theorem, which is a generalization of the famous Eckart-Young theorem. Otherwise, we must estimate them using some scaling procedure." [18]

In our case, the conditions in order to apply a SVD are met, so we had not the necessity of a scaling procedure; we applied a conventional routine of the Mathematica package (version 10.01) and obtained the SVD, which shows the full set of eigenvalues and eigenvectors we were looking for; however the eigenvalues were all real and positive, due to the fact that the nonzero singular values of a matrix A (real or Hermitian) are the positive square roots of the nonzero eigenvalues[19] of $A^*A$ (and $AA^*$). Alternatively, we adopted the Eigensystem routine of the package and preserved the signs of the eigenvalues (whose sum is equal to zero, as already showed). Our interest preserving the eigenvalues signs is to assess the presence of a star shapes structures, and the extension of the eigenvectors phases.

### 3.1 Data encoding and the empirical Hermitian Matrix

[18] See….Chino Shirawa op. cit page 46.
[19] See Meyer, op. cit. pag 555

The first step in this analysis consists on presenting the matrix of inter-industry flows, coded in complex numbers. For each vertex we consider the links m that start from k and also the number of p connections leading to that node k. The complex square adjacent matrix A, derived from the original two way data is constructed as $a_{kl} = m + ip$, where m is the real part of a complex number and ip its imaginary part. In this way, it is assured that $a_{kj} = i\bar{a}_{lk}.a$ .To get the Hermitian matrix H, we rotate A multiplying, each one of its members by $e^{-i\frac{\pi}{4}}$. We show a sample of a small matrices (5 by 5) taken from original data. It must be remembered that the relationship considered is not reflexive, i.e. $a_{ii} = 0$ :

**TABLE 1 ORIGINAL DATA**

| | Sector 1 | Sector 2 | Sector 3 | Sector 4 | Sector 5 |
|---|---|---|---|---|---|
| Sector 1 | 0 | 0.032 | 0.33 | 0.004 | 0 |
| Sector 2 | 0.003 | 0 | 2.928 | 0.01 | 0.002 |
| Sector 3 | 0.066 | 6.134 | 0 | 0.184 | 0.044 |
| Sector 4 | 0.02 | 747.598 | 49.51 | 0 | 9.013 |
| Sector 5 | 2652.88 | 2287.28 | 0.064 | 84.266 | 0 |

Source: INEGI Matriz Insumo Producto 2102

**TABLE 2 COMPLEX NUMBERS MATRIX**

| | | | | |
|---|---|---|---|---|
| 0 | 0.032+0.003i | 0.33+0.066i | 0.004+0.002i | 0.+2652.88i |
| 0.003+0.032i | 0 | 2.928+6.134i | 0.01+747.598i | 0.002+2287.28i |
| 0.066+0.33i | 6.134+2.928i | 0 | 0.184+49.51i | 0.044+0.064i |
| 0.02+0.004i | 747.598+0.01i | 49.51+0.184i | 0 | 9.013+84.266i |
| 2652.88+0i | 2287.28+0.002i | 0.064+0.044i | 84.266+9.103i | 0 |

**TABLE 3 HERMITIAN MATRIX**

| | | | | |
|---|---|---|---|---|
| 0 | 0.0247-0.02050i | 0.2800-0.1866i | 0.0169+0.0113i | 1875.87+1875.87i |
| 0.0247+0.02050i | 0 | 6.4078+2.2669i | 528.639+528.625i | 1617.35+1617.35i |
| 0.2800+0.1866i | 6.4078-2.2669i | 0 | 35.139+34.8787i | 0.0763+0.0141i |
| 0.0169-0.0113i | 528.639-528.625i | 35.139-34.8787i | 0 | 65.9582+53.2119i |
| 1875.87-1875.87i | 1617.35-1617.35i | 0.0763-0.0141i | 65.9582-53.2119i | 0 |

## 3.2 Spectrum.

First, consider the spectrum generated by the eigenvalues of the Hermitian matrix. Ordering its eigenvalues high to low and drawing the spectrum in Figure No. 1 we can see that the data show a certain symmetry. A better display shown in Figure No. 2, which is obtained by arranging eigenvalues, ordered by absolute value and keeping their original sign, where we can see the symmetry of the spectrum. In this figure, pairs of eigenvalues have similar magnitude and opposite sign. This indicates that the network tends a star shape with multiple centers. The most important are those that explain the greater variation in the data, as discussed below.

To the extent that transactions between nodes in the periphery of a star-shaped graph form are perturbed, the symmetry of its eigenvalues is lost. Yet, it is possible to identify whether the structure tends to form a star, as higher eigenvalues of the Hermitian matrix would be relatively close and they will have opposite signs and absolute magnitudes; the eigenvectors of these leading roots would have the same characteristics, that is, the major components in both vectors would occupy the same position and their phases would be zero. The distribution of these eigenvector components would show significantly higher anchors than that all other components in absolute value. In our case, the presence of several stars shows us that some few key sectors dominate the entire economy. When there are key sectors in the network, you can properly speak of a hierarchy, i.e., a tree whose structure, when considered from the viewpoint of a graph, is analogous to the geometry of the flakes of snow.

**Figure 1**

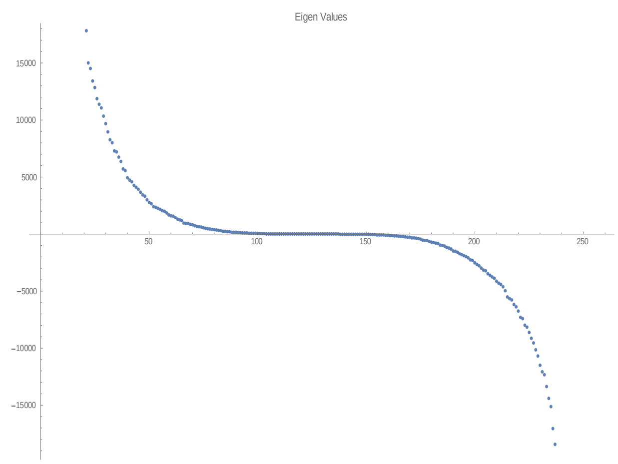
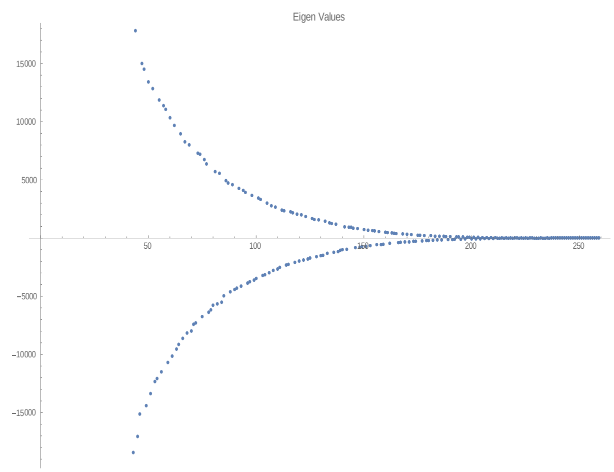
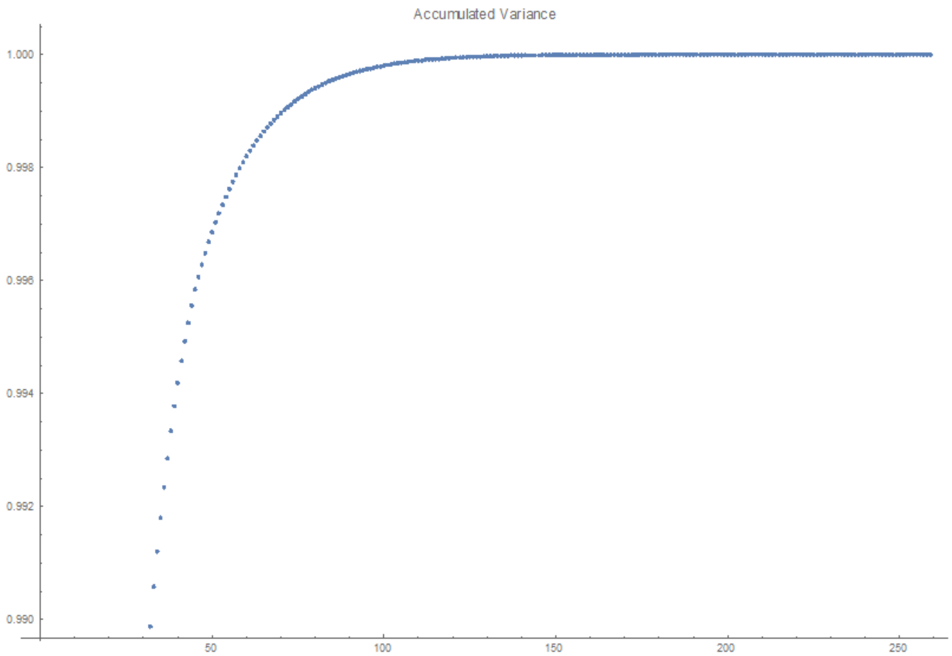



**Figure 2**




Observing the eigenvalues spectrum, we can assess the variance of the Eigensystem. In Figure 3 the cumulative variation of the spectrum is shown, where we can see that the first 25 eigenvalues explain more than 98 % of the total variance of the system. This result shows that the Mexican economy is highly concentrated in certain branches, which will be identified through the eigenvectors obtained from the Hermitian matrix.

**Figure 3. Accumulated Variance**




| | |
|---|---|
| 1 | 0.3090 |
| 2 | 0.6092 |
| 3 | 0.7099 |
| 4 | 0.7705 |
| 5 | 0.8019 |
| 6 | 0.8324 |
| 7 | 0.8494 |
| 8 | 0.8653 |
| 9 | 0.8787 |
| 10 | 0.8906 |
| 11 | 0.9016 |
| 12 | 0.9118 |
| 13 | 0.9214 |
| 14 | 0.9306 |
| 15 | 0.9393 |
| 16 | 0.9478 |
| 17 | 0.9546 |
| 18 | 0.9608 |
| 19 | 0.9652 |
| 20 | 0.9687 |
| 21 | 0.9719 |
| 22 | 0.9744 |
| 23 | 0.9767 |
| 24 | 0.9788 |
| 25 | 0.9810 |

### 3.3 Eigenvectors.

The components of the eigenvectors are presented in absolute numbers, together with its arguments (phases) that have been rotated so that its largest absolute value (anchor) has a zero-phase component and the other phases are related to that anchor. This allows easy identification of the presence of components of the vectors having the same anchor, on the one hand, and the pattern of directions of flows of purchases and sales, on the other.

In the following 5 eigenvectors can be seen between 25 and 32 of its components, ordered from largest to smallest components. As shown in tables 4, 5 and 6, the highest values of the characteristic vectors have zero phase. In the first case, the anchor correspond to the branch of “Production of Petroleum and coal”, which is the leading sector of the whole economy, followed by a hierarchy of the 31 main sectors of the Mexican economy. As can be seen from this tables, the first ranked members of subspaces 1 and 2 are the suggesting centers of star-like patterns. The distribution in the first subspace is a little bit more uniform distribution than the second. This implies a better connected flow pattern in the first sub space. The rapid decrease of figures in the second eigenvector distribution on the other hand, indicates a strong star-like pattern (around the Petroleum and coke production), giving a more specialized set of sectors. The distribution of the phase in the two subspaces is given in Figures 4 and 5, as can be seen those is such that the first phase only varies between $\frac{1}{4}\pi$ and $-\frac{1}{4}\pi$, while the second between $+\pi$ and $-\pi$. This confirm that the flows between sectors in the first vector is better balanced than those in the second.

# Table 4. Leading Eigenvectors

**First Eigenvector**

| | Vector | Phase |
|---|---|---|
| Manufacture of petroleum and coal | 0.6925 | 0 |
| Oil and Gas | 0.6610 | -0.7566 |
| General cargo transportations | 0.1526 | 0.7197 |
| Manufacture of basic chemicals | 0.1343 | -0.4123 |
| Generation, transmission and distribution of electricity | 0.1066 | 0.6413 |
| Foreign collective passenger transport of fixed routes | 0.1020 | 0.7948 |
| Wholesale Grocery and food | 0.0643 | 0.1409 |
| Taxi and limousine services | 0.0513 | 0.7881 |
| Manufacture of parts for motor vehicles | 0.0413 | 0.3661 |
| Scheduled air transport | 0.0410 | 0.7741 |
| Urban public transport and suburban passenger fixed route | 0.0355 | 0.7955 |
| Lessors of real estate | 0.0338 | 0.5088 |
| Production of automobiles and trucks | 0.0293 | 1.1247 |
| Law enforcement and maintenance of security and public order | 0.0233 | 0.8168 |
| Elaboration of bakery products and tortillas | 0.0198 | 0.8916 |
| Corporate | 0.0197 | -1.1893 |
| Residential building | 0.0193 | 0.9882 |
| Nonresidential building | 0.0187 | 0.9256 |
| Employment services | 0.0180 | -0.4867 |
| Slaughter, meat packing and processing of livestock, poultry and other eatable animals | 0.0162 | 0.8662 |
| Metalworking and manufacture of screws | 0.0154 | -0.8572 |
| Manufacture of cement and concrete products | 0.0150 | 0.6960 |
| Other construction works | 0.0148 | 0.5944 |
| Manufacture of plastic products | 0.0146 | 0.2311 |
| Manufacture of resins and synthetic rubbers and chemical fibers | 0.0132 | 0.3217 |
| Basic industries of iron and steel | 0.0125 | 0.5753 |
| Beverage industry | 0.0117 | 0.8885 |
| General public administration | 0.0116 | 0.8726 |
| Construction of facilities for water supply , oil, gas , electricity and telecommunications | 0.0115 | 0.8715 |
| Rail transport | 0.0114 | 0.7318 |
| Production of soaps, cleaners and dressing room preparations | 0.0109 | 0.5224 |
| Cultivation of oleaginous, leguminous seeds and cereals | 0.0109 | 0.5670 |

**Second Eigenvector**

| | Vector | Phase |
|---|---|---|
| Production of petroleum and coal | 0.6997 | 0.0000 |
| Oil and Gas | 0.6722 | 2.3808 |
| General cargo transportations | 0.1507 | -2.3645 |
| Foreign collective passenger transport of fixed routes | 0.1000 | -2.3655 |
| Generation, transmission and distribution of electricity | 0.0960 | -2.3739 |
| Manufacture of basic chemicals | 0.0887 | -0.3223 |
| Taxi and limousine services | 0.0515 | -2.3595 |
| Scheduled air transport | 0.0408 | -2.3630 |
| Urban public transport and suburban passenger fixed route | 0.0346 | -2.3684 |
| Lessors of real estate | 0.0224 | -2.3425 |
| Law enforcement and maintenance of security and public order | 0.0216 | -2.3915 |
| Manufacture of parts for motor vehicles | 0.0167 | 0.1719 |
| Corporate | 0.0135 | -1.7596 |
| Wholesale Grocery and food | 0.0124 | -2.3797 |
| Rail transport | 0.0113 | -2.3697 |
| Manufacture of cement and concrete products | 0.0112 | -2.3250 |
| Non-residential building | 0.0102 | -2.5890 |
| Metalworking and manufacture of screws | 0.0096 | 2.6933 |
| Elaboration of bakery and tortilla | 0.0092 | -2.6921 |
| General public administration | 0.0086 | -2.4983 |
| Other construction works | 0.0080 | -2.1397 |
| Construction of roads | 0.0078 | -2.4403 |
| Construction of facilities for water supply , oil, gas , electricity and telecommunications | 0.0074 | -2.5216 |
| Manufacture of resins and synthetic rubbers and chemical fibers | 0.0070 | -2.7912 |
| Residential building | 0.0070 | -2.8617 |

A representation in a complex plane, given in figure 6, for the main sectors linked to the extraction of petroleum and coke, allow us to visualize the proximity among clients and suppliers of this prominent sector of the Mexican economy. In this drawing we just include a few names of the representative sectors. In this case we employed directly the second complex eigenvector, including both its real and imaginary parts. The complex plane is a geometric representation of the complex numbers established by the real axis and the orthogonal imaginary axis. It can be thought of as a modified Cartesian plane, with the real part of a complex number represented by a displacement along the x-axis, and the imaginary part by a displacement along the y-axis. In this representation the oil and gas production is the closest sector to the extraction of petroleum, followed by transport sectors and the production of electricity, which coincide with the information provided in table 4.

**Figure 4 First Phase**

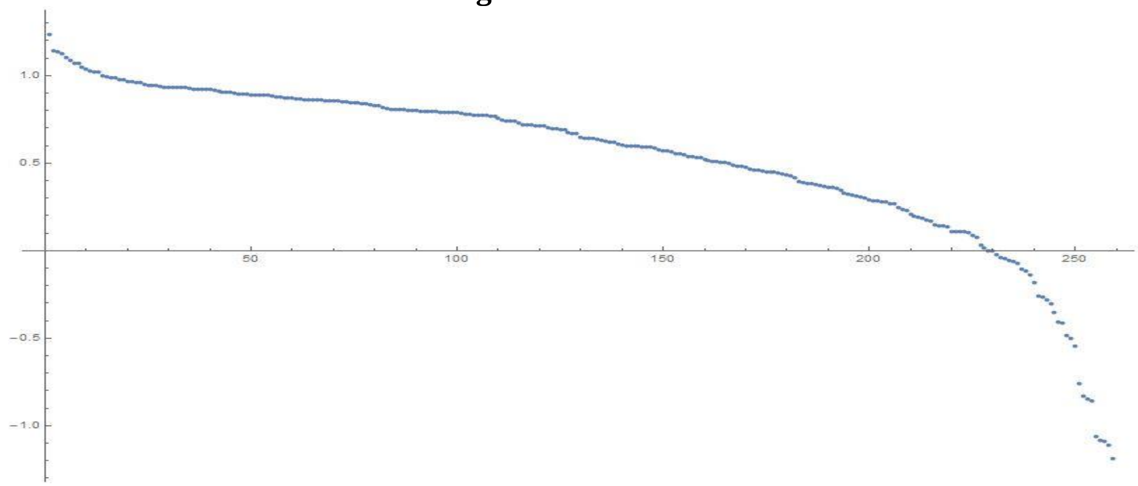



**Figure 5 Second Phase**

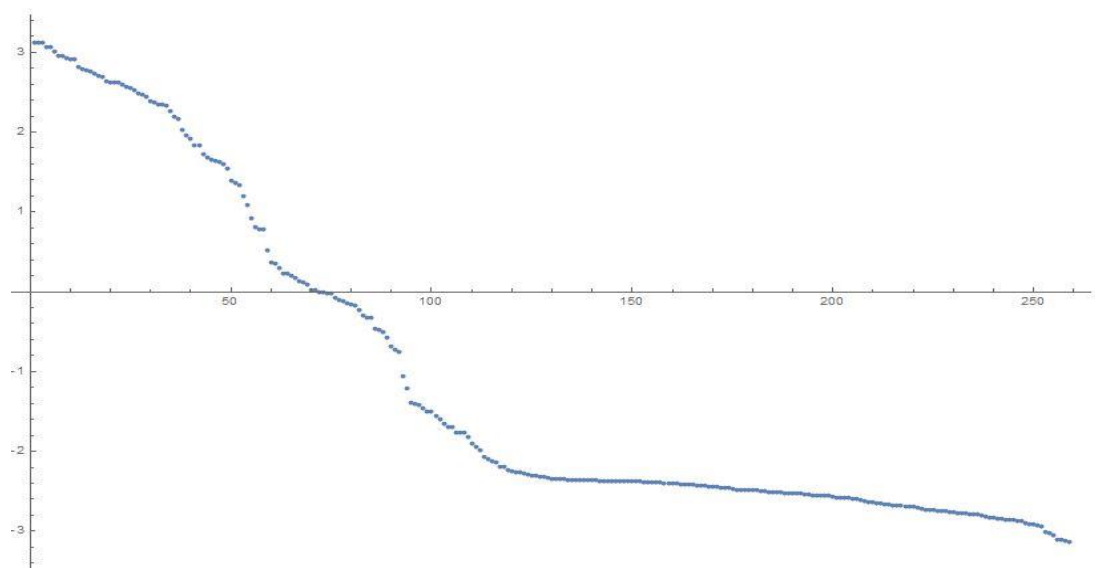
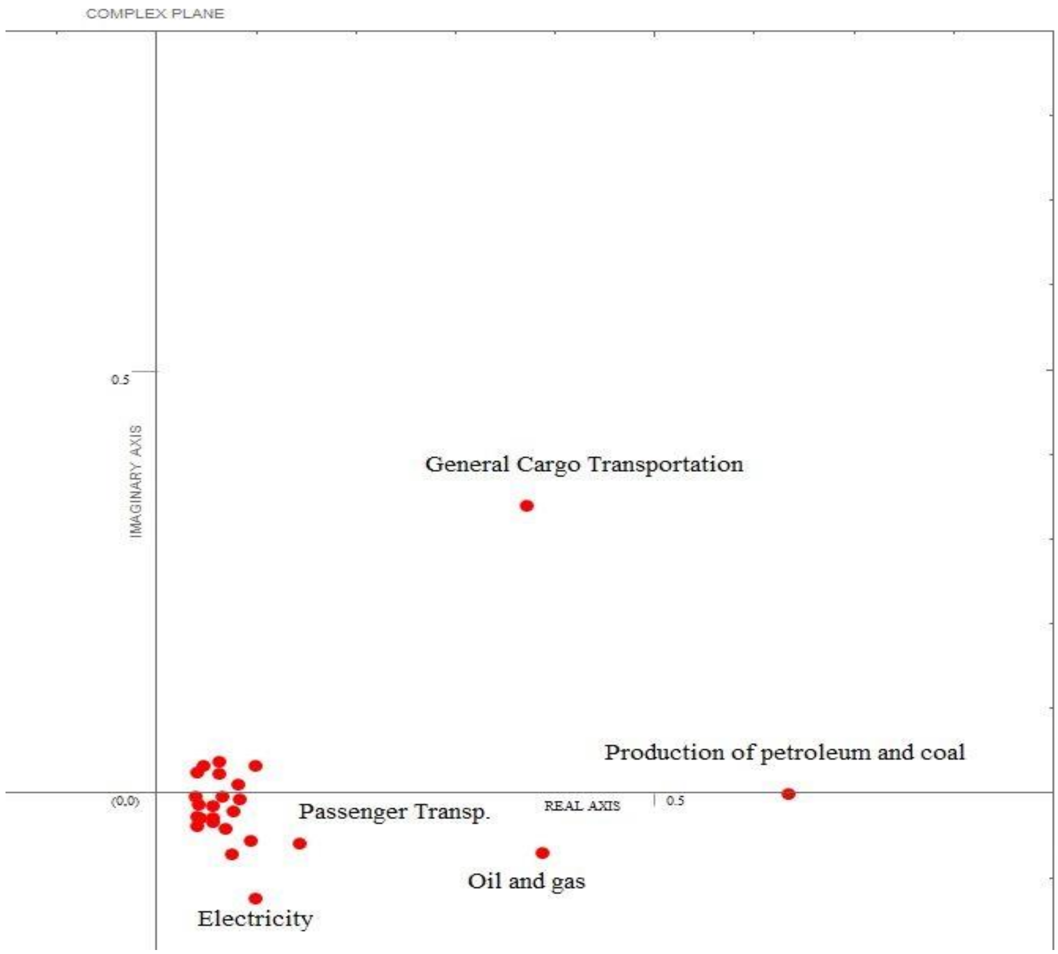


**Figure 6. Layout of second leading Eigenvector on a Complex Plane**




The third and fourth subspaces, shown in table 5, revolve around the activities of the manufactures of parts for motor vehicles. Analysis of the subspaces generated in the rotation of the Hermitian matrix is analogous to that performed for the first two subspaces, namely, that the third subspace shows a better articulation than the fourth subspace; however, the correction of the latter, reveals a more accurate linkage to their suppliers and clients. When inspecting phases, shown in Figures 7 and 8, a wide but smooth variation is observed in the fourth phase.

**Table 5. Third and Fourth Eigenvectors**

| | | | |
|---|---|---|---|
| Manufacture of parts for motor vehicles | 0.5462 | Manufacture of parts for motor vehicles | 0.6851 |
| Production of automobiles and trucks | 0.5238 | Production of automobiles and trucks | 0.6764 |
| Wholesale Grocery and food | 0.4138 | Wholesale Grocery and food | 0.0902 |
| Employment services | 0.1914 | General cargo transportations | 0.0847 |
| Manufacture of plastic products | 0.1793 | Employment services | 0.0721 |
| Lessors of real estate | 0.1191 | Manufacture of plastic products | 0.0689 |
| Manufacture of electronic components | 0.1151 | Manufacture of electronic components | 0.0674 |
| Oil and gas | 0.1135 | Production of rubber products | 0.0617 |
| General cargo transportations | 0.1047 | Foreign collective passenger transport of fixed routes | 0.0616 |
| Generation, transmission and distribution of electricity | 0.0919 | Manufacture of internal combustion engines , turbines and transmissions | 0.0581 |
| Residential building | 0.0906 | Manufacture of communication equipment | 0.0560 |
| Manufacture of iron and steel products | 0.0891 | Manufacture of other metal products | 0.0537 |
| Manufacture of internal combustion engines , turbines and transmissions | 0.0853 | Basic industries of iron and steel | 0.0508 |
| Slaughter, meat packing and processing of livestock, poultry and other eatable animals | 0.0777 | Manufacture of audio and video equipment | 0.0470 |
| Manufacture of communication equipment | 0.0732 | Basic aluminum industry | 0.0448 |
| Manufacture of petroleum and coal products | 0.0706 | Manufacture of other electrical equipment and accessories | 0.0423 |
| Production of rubber products | 0.0705 | Lessors of real estate | 0.0393 |
| Manufacture of other electrical equipment and accessories | 0.0699 | Manufacture of resins and synthetic rubbers and chemical fibers | 0.0390 |
| Basic industries of iron and steel | 0.0698 | Repair and maintenance of cars and trucks | 0.0363 |
| Manufacture of audio and video equipment | 0.0687 | Manufacture of iron and steel products | 0.0361 |
| Elaboration of bakery products and tortillas | 0.0687 | Oil and gas | 0.0357 |
| Manufacture of resins and synthetic rubbers and chemical fibers | 0.0655 | Production of cloths | 0.0310 |
| Nonresidential building | 0.0592 | Production of equipment for generation and distribution of electricity | 0.0271 |
| Manufacture of paper and paperboard | 0.0586 | Manufacture of machinery and equipment for manufacturing industries, except metal working | 0.0252 |
| Manufacture of other metal products | 0.0578 | Residential building | 0.0250 |
| Business administration services | 0.0488 | Cast molding of metal parts | 0.0228 |
| Production of equipment for generation and distribution of electricity | 0.0478 | Tanning and dressing of leather and fur | 0.0224 |
| TransportForeign collective passenger transport of fixed routes | 0.0456 | Coatings and metal finished products | 0.0219 |
| Accounting, audit and related services | 0.0453 | Manufacture of forged and stamped metal products | 0.0209 |
| Production of mensuration instruments, control, and medical equipment | 0.0447 | Brokerage services for freight | 0.0200 |

**Figure 7 Third Phase**

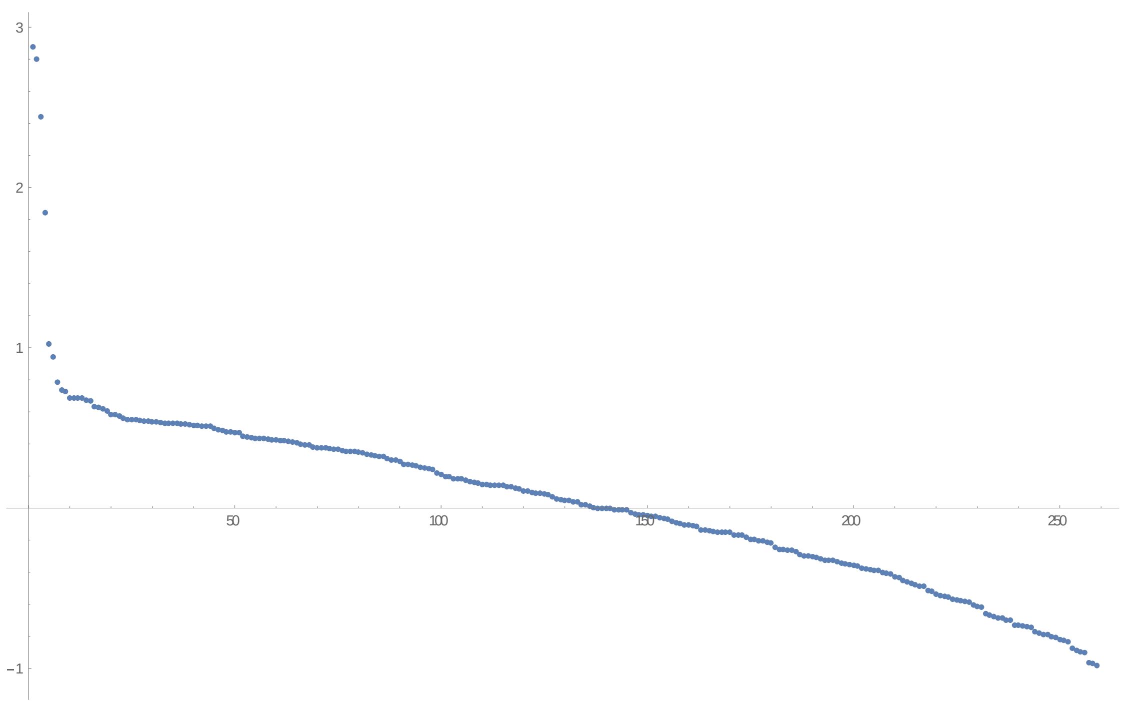
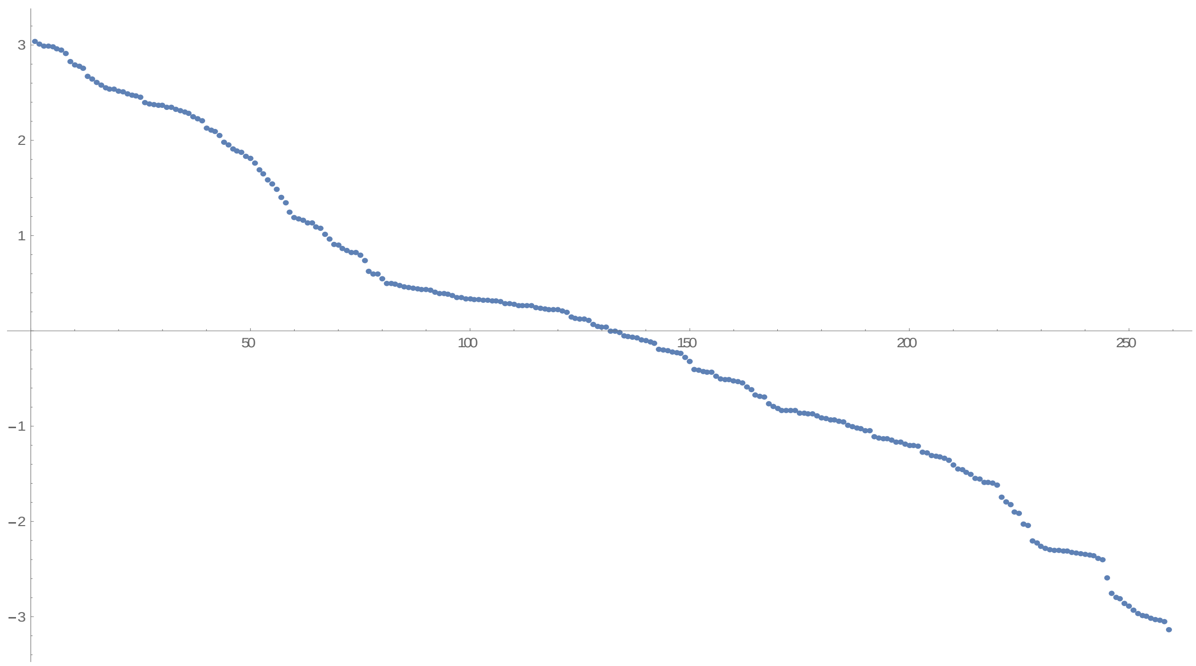


**Figure 8 Fourth Phase**

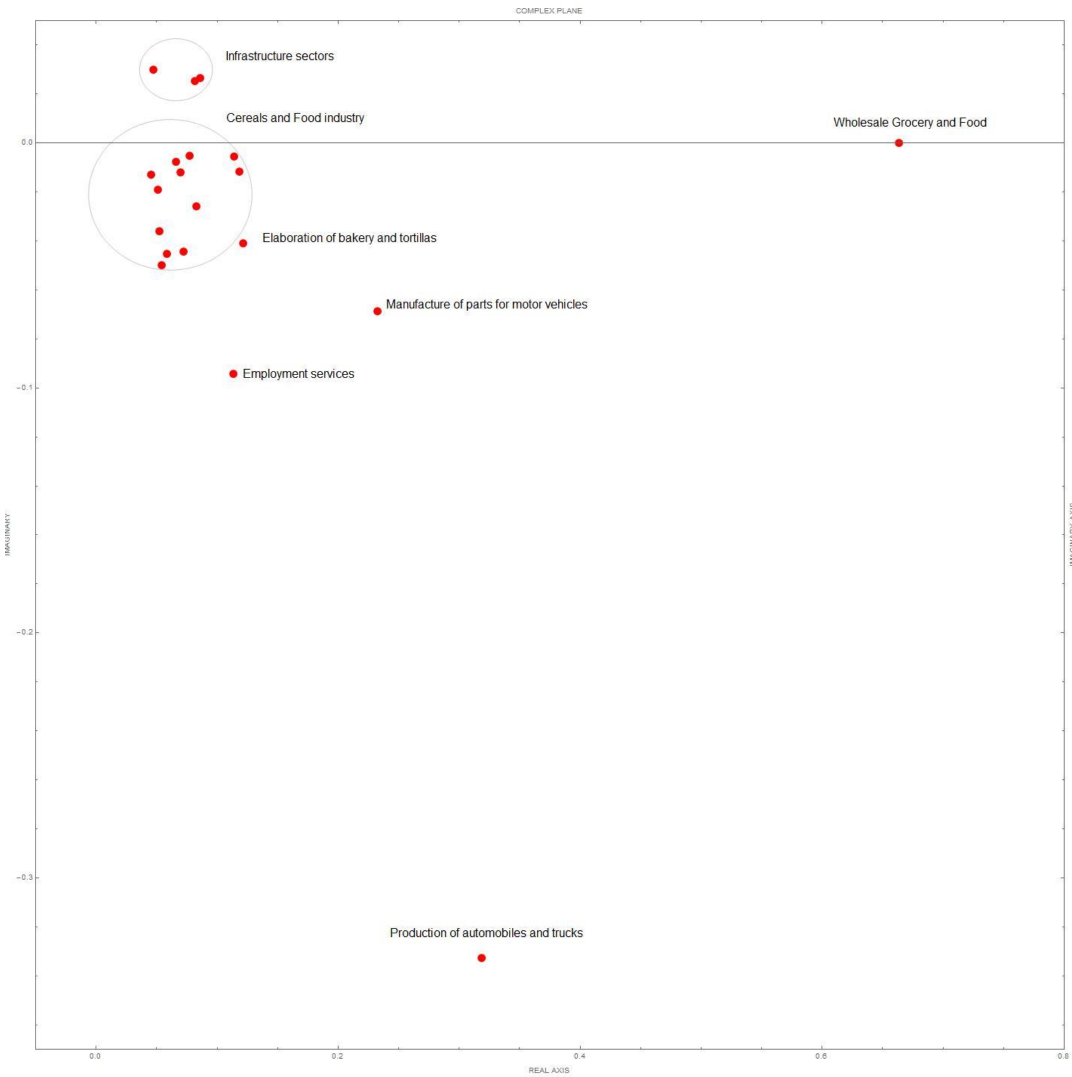



The sub-spaces fifth and sixth, show the associated sectors with the sector Wholesale Grocery and Food, whose core activity is strongly linked with the industrial foods and agricultural production. It also are related to construction, transport facilities and employment services. In this case we just report in the table 6,the sixth eigenvector, showing its 25 most prominent activities bonded with this trading sector; we also display its associated graph in Figure 9. It is interesting that the production of automobiles and trucks have a higher ranking in the real part of the eigenvector, but its phase shows its place at a considerable distance, so this activity is important indirectly.

**Table 6. Sixth Eigenvector: Wholesale of Groceries and Food**

| | | |
|---|---|---|
| Wholesale Grocery and food | 0.39176 | 0.0000 |
| Production of automobiles and trucks | 0.332086 | -2.1554 |
| Manufacture of parts for motor vehicles | 0.31589 | -2.9701 |
| Employment services | 0.256877 | -0.7172 |
| Elaboration of bakery products and tortillas | 0.235103 | 0.6891 |
| Slaughter, meat packing and processing of livestock, poultry and other eata | 0.230128 | 0.6816 |
| Cultivation of oleaginous, leguminous seeds and cereals | 0.22578 | -0.2862 |
| Lessors of real estate | 0.202655 | -0.6894 |
| Grinding grains and obtaining oils and fats | 0.199325 | 0.2753 |
| Production of electronic components | 0.182456 | 2.8045 |
| Production of equipment for generation and distribution of electricity | 0.169254 | -0.2480 |
| Cattle farm | 0.145539 | 0.1325 |
| Residential building | 0.140388 | 0.8597 |
| Poultry farm | 0.137756 | 0.2744 |
| Manufacture of audio and video equipment | 0.134415 | -2.7480 |
| Elaboration of foods for animals | 0.112994 | 0.3341 |
| Manufacture of basic chemicals | 0.112138 | -0.1652 |
| Manufacture of communication equipment | 0.103834 | -2.9241 |
| Non-residential building | 0.097385 | 0.8700 |
| Manufacture of paper and paperboard | 0.086837 | -0.2834 |
| Other construction work | 0.086305 | 0.0885 |
| Business Management Services | 0.083459 | -0.5620 |
| Beverage industry | 0.080023 | 0.6854 |
| Basic industries of iron and steel | 0.078193 | 0.8158 |
| Accounting, audit and related services | 0.076345 | -0.7501 |

**Figure 9. Sixth Complex Eigenvector: Wholesale of Groceries and Food**




## 3.4 Anchors and Clusters

### Anchors

A complex Hermitian matrix can be represented, due to its complete orthonormal eigenvector system, in a spectral (Fourier) representation in which each eigenvector form a subgroup that can be viewed as independent with respect to the economic behavior of the other relevant members of each subgroup. The eigenvector components are interpreted as an economic ordering induced by technological and market behavior of each subgroup member. In addition, each member has for each subgroup structure/eigenvector in the spectral representation, a different order rank. This ordering status depends on his relation to the respective anchor of the subgroup. In other words, each sector appear, in our case, 259 times (one for each eigenvector) but in different position.

In our application to the Mexican economy, we obtained two sets of 259 anchors, i.e. for domestic and total transactions (including imports). Their contrast reveals the impact of foreign transactions in the entire economy. A full analysis of these estimations is out of the scope of this paper, and their inclusion in Table 7, below, is for demonstration purposes only.

**Table 7. Anchors for Domestic and Total Interindustrial Mexican Tables**

| DOMESTIC TRANSACTIONS MATRIX | TOTAL TRANSACTIONS MATRIX |
|---|---|
| | |
| Production of petroleum and coal | Production of petroleum and coal |
| Production of petroleum and coal | Production of petroleum and coal |
| Whole sale of groceries and food | Manufacture of parts for motor vehicles |
| Whole sale of groceries and food | Manufacture of parts for motor vehicles |
| Slaughter, meat packing and processing of livestock, poultry and other eatable animals | Whole sale of groceries and food |
| Slaughter, meat packing and processing of livestock, poultry and other eatable animals | Whole sale of groceries and food |
| Mining of metallic minerals | Manufacture of electronic components |
| Mining of metallic minerals | Cultivation of oleaginous, leguminous seeds and cereals |
| Other construction works | Slaughter, meat packing and processing of livestock, poultry and other eatable animals |
| Other construction works | Manufacture of electronic components |
| Elaboration of bakery products and tortillas | Slaughter, meat packing and processing of livestock, poultry and other eatable animals |
| Manufacture of parts for motor vehicles | Manufacture of plastic products |
| Cultivation of oleaginous, leguminous seeds and cereals | Residential building |
| Insurance and bonding institutions | Manufacture of resins and synthetic rubbers and chemical fibers |
| Manufacture of basic chemicals | Mining of metallic minerals |
| Manufacture of basic chemicals | Manufacture of resins and synthetic rubbers and chemical fibers |
| other cultivations | Industries of non ferrous metals, except aluminum |
| other cultivations | Industries of non ferrous metals, except aluminum |
| Insurance and bonding institutions | Manufacture of plastic products |
| Manufacture of basic chemicals | Generation, transmission and distribution of electricity |
| Manufacture of iron and steel products | Manufacture of iron and steel products |
| Manufacture of plastic products | Insurance and bonding institutions |
| Elaboration of sugars, chocolates, sweet and similar | other cultivations |
| Beverages industries | other cultivations |
| Commercial banks | Insurance and bonding institutions |

The anchors obtained for the matrix of domestic transactions are more traditional activities such as mining, agriculture, production of foods, production of breads, cookies and tortillas, drinks and some products of light manufactures as the plastic products, and works related with the industry of the construction. In the case of the matrix of total transactions it stands out immediately the production of parts for automobiles, manufactures electronic components, residential construction, and production of resins, synthetic fibers and chemical products, that is to say, a more modern industry associated to the global chains of value.

**Clusters**.

As already pointed out, the eigenvectors matrix associated with positive characteristic values of the Hermitian matrix may be used to generate a set of a clusters, with the additional advantage of being automatically estimated[20]. It is based on the same Eigensystem of the adjacent Hermitian complex we have been working with. The number of relevant clusters is determined automatically. Nodes are assigned to clusters using the inner product matrix $S_{nxn}$ calculated from a matrix $R_{nxl}$ of the l eigenvectors as column vectors which correspond to the positive eigenvalues of H. It can be shown that assigning the vertices of the network to clusters such that a node i belongs to cluster Cl if $Re(S_{i,Cl})=max_j Re(S_{i,j})$ a good partitioning can be found.

Applying this methodology, we obtained three main of clusters leaded by: a) oil and gas production; b) manufactures of parts for motor vehicles, and c) wholesale of groceries and food.

[20] See Hoser and Schroder op. cit pp 439-444. Methodology extended by Solis V and Garcia-P op. cit 2009

These clusters can be split up in finer partitions. In our work, we found up to 12 clusters which incorporate different technological and market structural bonds.

The extensive treatment of this clustering procedure also escapes to the purposes of this article; but the outline provided give you the fundamental principles of the clusters estimation for matrices of input output, matrices of bilateral trade, matrices of national and social accounting, and related tables.

## 4. Final Remarks

The method presented in this paper has proven, empirically, to be a robust approach to economic analysis. The application of its tools to the Mexican economy are consistent and corroborate the knowledge we have about this country, and also provided us with new insights. During the seven years since we produce the first paper with this methodology (Solis Garcia, 2009), we have explored literally dozens of input output tables, bilateral trade tables and social accounting matrices, confirming the soundness of this approach.

From an analytical point of view, the methodology is presented with all the theoretical study elements that enable new applications and extensions of the tools for input-output analysis. The methods developed in this paper can be extended to introduce perturbations within technologies and market structures, association with multiple matrices, and a tighter link to graph and multivariable analysis.

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