Eigenvalue distribution, matrix size and the linearity of wage-profit curves

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Abstract

The linearity of wage curves is important to answer questions such as the existence of re-switching of techniques and reverse capital deepening. Schefold (2011) attributes more complex forms of the wage curve to non-vanishing subdominant eigenvalues and even though the existence of those complex curves is accepted in theory they have not been proven to exist in real economy.

While experimenting with equilibrium computations using randomly generated matrices, Bródy (1997) noticed that the speed of convergence towards equilibrium increased with matrix size. The relative size of the second eigenvalue with respect to the first determines the convergence speed, so Bródy conjectured that this relative size tended to fall as a random matrix becomes larger. While this does not appear to hold for observed direct requirements input-output matrices (Mariolis and Tsoulfidis, 2012, Table 1, p. 6), Bidard and Schatterman (2001) proved that in a random matrix with independently and identically distributed entries the speed of convergence increases with the size of the matrix because the relative size of all subdominant eigenvalues tends to zero as the matrix size approaches infinity. Schefold (2011) then showed that zero subdominant eigenvalues imply linear wage-profit curves for any given numeraire.

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Our concern is with actual input-output tables. We successively aggregate the BEA US 2002 make and use tables using the NAICS codes, in order to build 176 squared industry by industry direct requirements tables ranging in size from 403 to 15 industries. This allows us to assess the effects of the size of the matrix on the distribution of the moduli of eigenvalues and on their arithmetic and geometric means. We find that the distribution of the moduli does indeed shift downward as the size of the matrix increases, so that the average size of the moduli of subdominant eigenvalues falls by either measure. In this particular year (2002), as the matrix size increases the arithmetic mean seems to stabilize around 0.05 and the geometric mean around 0.02. This stabilization seems to contradict the random matrix hypothesis, but its implications are unclear: on the one hand, input-output matrices are not random; on the other hand, the random matrix hypothesis only applies in the limit as matrix size approaches infinity. In any case, the fact that the subdominant moduli do fall to some small level has major implications for the analysis of relative prices and choice of technique.

Key-words: capital controversy, Bródy conjecture, wage-profit curve linearity.
Topic: Mathematical Treatments of Input-Output Relationships

Contents

1. Introduction

The Cambridge Capital Controversies widely discussed on the 1960's addressed a coherent and logical critique to the neoclassical economic theory. Even though Sraffa's 1960 book was well accepted as a consistent and logical theory, the capital controversy was not a strong enough argument to fulfill the revolution in the mainstream economic theories. Joan Robinson and Samuelson argued that *reswitching* and *reverse capital deepening* were not necessarily significant in reality.

Since then, the debate has deepened the study of the empirical relevance of those phenomena (*reswitching* and *reverse capital deepening*) as so to give strength to the critique. A growing body of evidence that wage profit curves were linear and even individual price curves were linear appeared giving strength to Joan Robinson's and Samuelson's sides. The works of Baldoni (1984), Salvadori and Steedman (1988), Shaikh (1998) and Shaikh (2012), only to cite a few, make important contributions to the evidence of the linearity of the wage-profit curve. Schefold was responsible for a wide range of work on the mathematical field and apparently changed side on this discussion after getting acquainted with a paper by Bródy on 1997. Schefold and Hanh (2006) while working with random tables claim that *reswitching* and *reverse capital deepening* happen only very rarely, making them insubstantial counterexamples to criticize the use of the surrogate production function.

Bródy (1997) initiated a series of works on the behavior of eigenvalues on direct requirements input-output tables and its relation with the number of industries depicted on those tables. While experimenting with equilibrium computations using randomly generated matrices, Bródy (1997) noticed that the speed of convergence towards equilibrium of a system depicted by a matrix increased with it's size. Since the relative size of the second eigenvalue with respect to the first determines the convergence speed, Bródy claims that this relative size tends to fall as a random matrix becomes larger. This claim got to be know as the Bródy conjecture and a range of works both testing its validity and following up on its economic consequences succeeded Bródy's paper. According to Bródy, the economic consequence of his finding was that the larger an economic system, the faster it would converge to equilibrium[1].

Schefold (2013) derived one of the most interesting consequences of Bródy's conjecture by observing that if all but the dominant eigenvalues would go to zero with size, then in larger system the wage-profit curve would be linear.

So there are two economic conclusions based on experiments on random tables in the literature, one regarding convergence of the economic system towards equilibrium and one regarding linearity of the wage-profit curve which have important consequences to the capital controversies. Nonetheless, the mathematical observation on which those two conclusions are based arises from experiments from random tables and as we intend to show on the present work does not seem to hold on real tables.

The present paper discusses the validity of the mathematical arguments on which Bródy and Schefold bases their findings.

Even though it is quite accepted that this is right for a certain type of non-negative random matrices, this does not appear to hold for observed input-output matrices (Mariolis and Tsoulfidis 2012). Bidard and Schatterman (2001) proved that in a random matrix with independently and identically distributed entries the speed of convergence increases with the size of the matrix because the relative size of all subdominant eigenvalues tends to zero as the matrix size approaches infinity. Schefold (2011) also departs from a work with random tables to show that zero subdominant eigenvalues imply linear wage curves. Ochoa (1984), Bienenfeld (1988) and Shaikh (1998, 2012) find empirical evidence of the linearity of this curve. Nonetheless the reason of this linearity does not necessarily is the one argued by Schefold (2011).

To draw a conclusion between Bródy's findings and real economies, one have to ask the question of how liable is the assumption that real input-output tables are

[1] Bródy is preoccupied with the equilibrium in Leontief and Neumann's systems.

random. Experiments from Mariolis and Tsoufildis (2010, 2012) with real input-output tables goes against Bródy's findings leaving us with the question of what is the structure inside real input-output tables that bring about these different results. This paper generates 176 different aggregations for the US 2002 input output table to compute a series of eigenvalues and fill out the curve of subdominant eigenvalues. Previous empirical works on the subject (Mariolis and Tsoufildis (2010, 2012)) consists on only a few observations per year and per country. The methodology here developed for filling the series between matrix sizes is a contribution that allows us to make a series of tests to try to find out what are the relationship between the size, distribution of coefficients and eigenvalues on real tables.

The next section intends to analyze Bródy conjecture and compare it to real tables. The third section displays the empirical results on the US 2002 tables regarding the distribution of eigenvalues according to the size of tables. Finally, concluding remarks are presented. Furthermore methodological aspects of this work can be found on the appendix.

2. *On Bródy's Conjecture*

Bródy (1997) is preoccupied with the speed of convergence to equilibrium of the economic system. This speed would be given by the relative second eigenvalue of the direct requirements A table.

A decomposition of $\mathbf{v}$ in a basis made of the eigenvectors of A shows that the speed of convergence depends on the ratio of the first two eigenvalues in terms of maximum modulus. According to Bródy's (1997) experiments with random matrices, this ratio tends statistically to zero when the size of the matrix tends to infinity. The consequence of this is that the speed of convergence of the sequence {v} increases with the size of the matrix.

Let A_{nxn} be the direct requirements table of an economy with N sectors. Any vector v in R^n can be written as a linear combination of the eigenvectors of table A_{nxn}[2].

$$v = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \qquad (1.1)$$

Where x_i is the ith eigenvector of the A table. There is a particular $v*$ with all a coefficients in (1.1) equal to one.

$$v^* = x_1 + x_2 + \cdots + x_n \qquad (1.2)$$

For v^* and A we have the following property:

$$A^m v^* = \lambda_1^m x_1 + \lambda_2^m x_2 + \cdots + \lambda_n^m x_n \qquad (1.3)$$

Where m is the number of iterations and λ_i is the ith eigenvalue of the A table. The Perron-Frobenius [3] theorem presented in Frobenius (1912) asserts that a irreducible non-negative matrix have one dominant eigenvalue λ_0 with multiplicity one. All other eigenvalues λ satisfy $|\lambda| \leq \lambda_0$.

Since $\lambda_1 > \lambda_i \; \forall \; i \neq 1$, the iterations make $\lambda_1^m x_1$ be dominant over $\lambda_2^m x_2 + \cdots + \lambda_n^m x_n$, $\frac{\lambda_2^m x_2 + \cdots + \lambda_n^m x_n}{\lambda_1^m x_1} \, change\ this \xrightarrow[m\to\infty]{} 0$. The speed of this convergence depends on the number of iterations necessary for all λ_i^m to fade away. The second eigenvalue is by definition the higher one within those and consequently the one that takes longer to fade away, therefore the speed of convergence depends on the subdominant relative eigenvalue ${\lambda_2}/{\lambda_1}$.

Bródy goes on to show that there is a certain random way of defining A for which ${\lambda_2}/{\lambda_1}$ goes to zero when the size of the matrix increases in a series of random

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[2] Anx_n is a non-negative full rank table since the non-basic sectors have been excluded and sectors with same technologies have been aggregated. In our further empirical work the 430 sectors in the US 2002 table are reduced to 403 (details on the aggregation can be found on the appendix).

[3] There are two versions of the Frobenius theorem; one presented in Froebenius (1908) and the other in Froebenius (1912), the first one refers to positive matrices. Bródy(1996) refers to this version, but since real economy A tables have coefficients that are equal to zero the version for non-negative tables is more adequate. The conclusion regarding the existence of one dominant eigenvalue that is associated with the only positive real eigenvector is valid in both versions. More on Frobenius theorem see Hawks(2013- chapter 17).

matrices, defined as having all its elements independent and uniformly distributed between (0,1). The expected value of each a_{ij}, $E(a_{ij})=1/2$.

The observed value of those statistics can be inferred from real tables. Departing from the 2002 US use and make tables, made available by the Bureau of Economic Analysis (BEA), we construct 176 direct requirements industry-by-industry A tables[4]. Those 176 tables are the database for the empirical work here presented.

For the following comparison with statistics from real tables, let us use the normalization of the random A tables presented in Bialas & Gurgul (1998) is used., where A*= $(2/n)$A, then Bródy's random matrices expected value (E), standard deviation (SD) and coefficient of variation (CV) as a function of the number of sectors in the table would be respectively:

$$E\left(a_{ij}^{*}\right) = {}^{1}\!/_{n} \qquad (1.4)$$

$$SD\left(a_{ij}^{*}\right) = {}^{1}\!/_{n\sqrt{3}} \qquad (1.5)$$

$$CV\left(a_{ij}^{*}\right) = {}^{E\left(a_{ij}^{*}\right)}\!/_{SD\left(a_{ij}^{*}\right)} = \sqrt{3} \qquad (1.6)$$

for each i,j= 1,…n.

The statistics for the 2002 US A tables are derived as following:

$$E\left(a_{ij}\right) = \sum_{j=1}^{n} \sum_{i=1}^{n} {}^{a_{ij}}\!/_{n^2} \qquad (1.7)$$

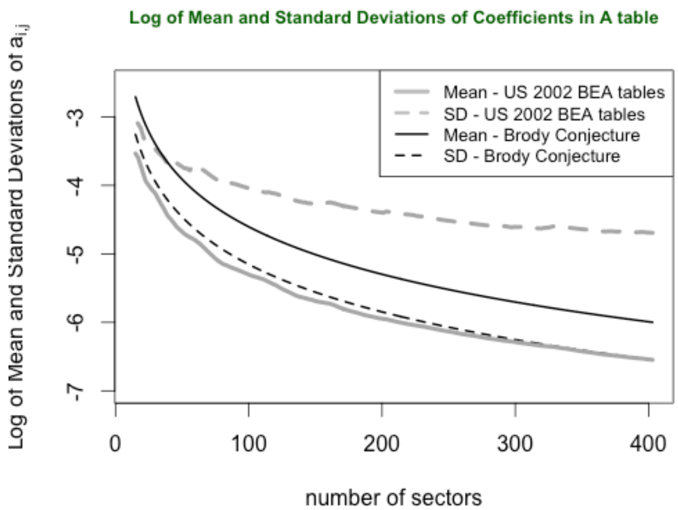
$$SD\left(a_{ij}\right) = \sum_{j=1}^{n} \sum_{i=1}^{n} {}^{\left(a_{ij} - E\left(a_{ij}\right)\right)}\!/_{n^2} \qquad (1.8)$$

$$CV\left(a_{ij}\right) = {}^{E\left(a_{ij}\right)}\!/_{SD\left(a_{ij}\right)} \qquad (1.9)$$

[4] Details on the construction method and on the aggregation correspondence can be found on the methodological Appendix. Further details such as the description of industries in each aggregation can be made available via e-mail.

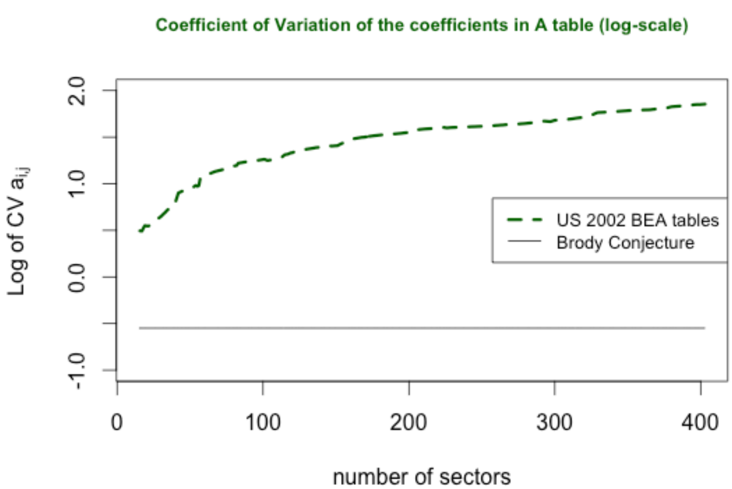
The mean, standard deviation and coefficient of variation of Bródy's random matrices represented by equations 1.4 to 1.6 are plotted together with the values calculated from the US 2002 A tables. Figure 1 and 2 represent respectively the mean and standard deviations on *A* table's coefficients by size.

Figure 1



Standard deviations are higher on real tables than Bródy expected while the means are smaller. Also the increasing distance between those two curves shows that the coefficient of variation is increasing on size, as shown in figure 2 bellow.

Figure 2



These figures seem to show that Bródy's hypothesis of uniform distributions of the coefficients was not precise in describing how coefficients are distributed on real tables. There is no intent here to attack the possibility of constructing random matrices neither on the results that Bródy infers on analyzing them. Nonetheless, the possibility of applying Bródy's findings on random matrices to real Input-Output data lacks empirical support. The problem with Bródy's hypothesis seems to be with the uniform assumption and with the assumption that the distribution of the coefficients does not change with size.

Molnar and Simonovits (1998) expand this result for stochastic matrices to prove that the convergence is also faster for larger matrices. Nonetheless it is still not quite clear if the randomness that is being imposed on those matrices is leaving aside some important feature of real input-output tables. Given the results until now found from real tables this appears to be the case.

Gurgul and Wójtowicz (2015) argue in this direction, that Bródy's results apply to random matrices but that it doesn't bring light into the discussion about real tables. According to them, the speed of convergence of the same system should be invariant to size. So they argue that Bródy's conjecture do not hold in terms of real economy if the ratio of eigenvalues is a function of the number of industries.

In this sense the economic meaning of Bródy's finding is more tangible in Schefold's work, where he discusses what happens to eigenvalues when size increases and what are the consequences of this increase regarding technical choices and linearity of profit wage curve.

Gurgul and Wójtowicz (2015) experiment with random matrices derived from a qui-squared distribution and find a different result than the one that emerges from the distribution used by Bródy. This proves that the ratio $\frac{\lambda_2}{\lambda_1}$ depends on the way randomness is defined and on the kind of distribution restriction that is imposed on the tables. Also they show that if the flow is the same, i.e., if we are to disaggregate a table of a unique economy fixing the total output, it follows that for a random i.i.d., the ratio ${}^{\lambda_2}/_{\lambda_1}$ grows because it will be positively dependent on the size.

This yields to the fact that even though Bródy's conjecture is quite interesting mathematically, one should first look at real tables and see what kind of structure they have in terms of distributions before running experiments. Bródy brought light to the fact that looking at eigenvalues on I-O tables might be interesting in understanding the economy, but the structure should come from real data and not from a decision on how randomness can be imposed in this data.

3. Empirical results on Eigenvalues and distributions

The empirical contributions of this work are of two orders. Firstly we want to address Bródy's conjecture regarding the ratio ${}^{\lambda_2}/_{\lambda_1}$ across tables and discuss the economic meaning of this conjecture while looking at real tables. Secondly we will address the empirical evidence on distribution of eigenvalues within one table and the limiting shape that this distribution approach with size.

Only a few empirical works look at the values of ${}^{\lambda_2}/_{\lambda_1}$ ratio. Mariolis and Tsoufildis (2010) present results for Japan over the years for tables at size 21 and 100 industries. On the first two years analyzed (1980 and 1985) the value of the ratio is smaller on the 100 industries table, while for all the other years (1990, 1995, 2000 and 2005) the ratio increases when one moves from 21 industries to 100 industries. Mariolis and Tsoufildis (2012) addresses Bródy's conjecture by looking at the 1997 and 2002 US tables. The results found by them are presented on table one bellow. According to them, "although λ_{H1} for all aggregations are near each other (…), ρ_{J2} increases with the size of the matrices casting doubt on Bródy's conjecture" (Mariolis and Tsoulfildis 2012, pg. 6). – explain their notation

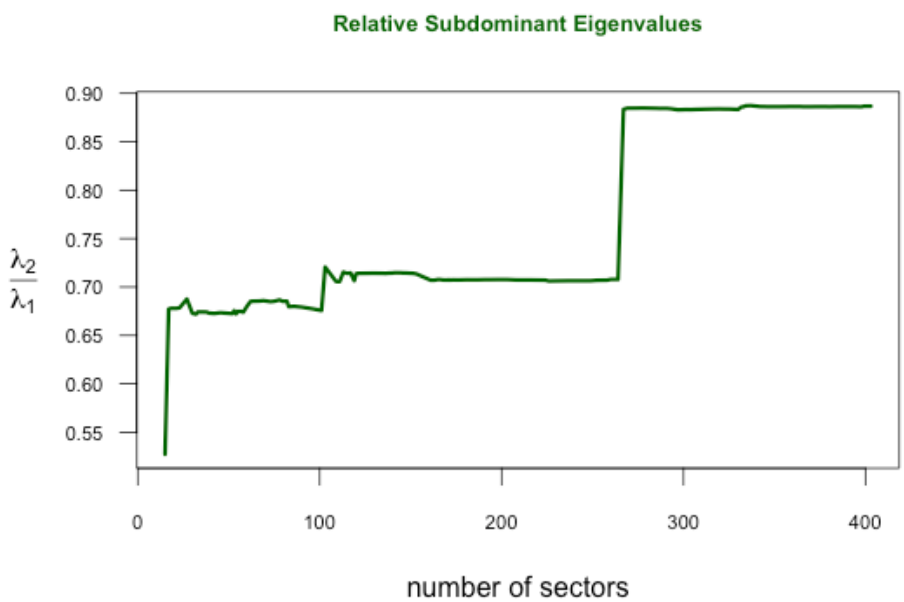
Table 1 The moduli of the second and third eigenvalues, and the average mean of the moduli of the non-dominant eigenvalues

	1997			2002		
n	12	129	488	15	133	426
λ_{H1}	0.97	0.96	1.06	0.92	0.92	0.92
ρ_{J2}	0.25	0.68	0.83	0.36	0.58	0.80
ρ_{J3}	0.25	0.56	0.51	0.25	0.58	0.56
AM	0.08	0.08	0.05	0.08	0.08	0.05

Source: Mariolis and Tsoulfildis (2012) – Table 1.

Our own results regarding λ_2/λ_1 ratio is presented on figure 3 bellow. This results shows how the ratio behaves with the increase on the number of industries present on the A table.

Figure 3



The graph above shows that Brody's conjecture, even though consistent with random tables, does not hold for the US economy in 2002. Bródy's failure to design randomness allowing for a raise in the coefficient of variation of the coefficients present in the tables seems to be on the heart of the different result. When Bródy tries to explain why the random tables constructed by him would have a falling λ_2/λ_1 ratio he recurs to an approximation of this ratio as a function of the expected values of coefficients and standard deviation. In his own words:

"Thus, the relation of the two largest eigenvalues will be close to (and may be perhaps slightly overestimated by) Sigma $n^{1/2}$ Mu=Sigma/(Mu $n^{1/2}$) (…). This tends towards zero as n increases."

In our own notation, Bródy argues that:

$$\frac{\lambda_1{}^*}{\lambda_2} \approx \frac{SD\left(a_{ij}^*\right)\sqrt{n}}{nE\left(a_{ij}^*\right)} = \frac{SD\left(a_{ij}^*\right)}{E\left(a_{ij}^*\right)\sqrt{n}} = \frac{1}{CV\left(a_{ij}^*\right)\sqrt{n}} \qquad (2.1)$$

As shown on equation 1.6 and figures 1 and 2, the ratio between the standard deviation and the expected value, and consequently the coefficient of variation, of the a_{ij}^* in Bródy are constant. Substituting 1.6 into the equation above it is easy to see why the ratio is falling on his approximation.

$$\lim_{n\to\infty} \frac{1}{\sqrt{3n}} = 0$$

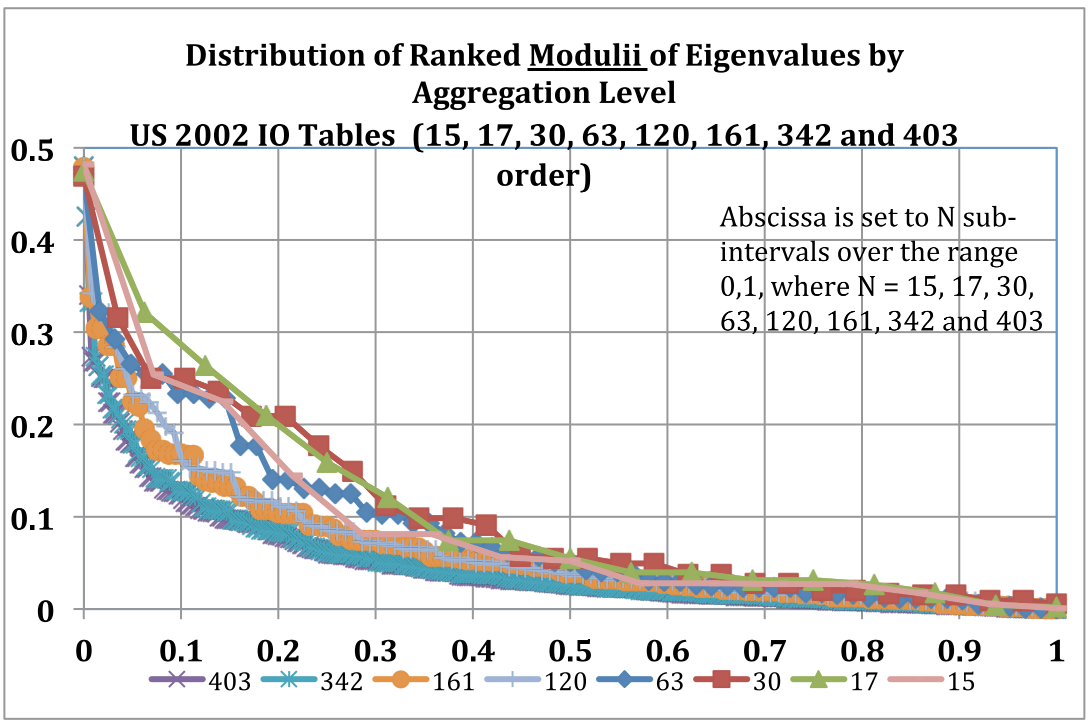
One can argue that the fall of this ratio in Bródy is imposed by the way he defines the distribution of coefficients on his random tables. Equation (2.1) does not represent a good approximation of the ratio in the real table since the distribution is changing with size. It is consequently hard to make a direct causal relation between the coefficient of variation and λ_2/λ_1 value depicted on figures 2 and 3. Even though it is easy to see that Bródy conjecture can not be confirmed on those tables, the reason behind the increasing λ_2/λ_1 ratio still needs a more profound study.

Schefold's work follow a different direction and discusses the theoretical implications of all subdominant eigenvalues going to zero with increase in size. Nonetheless, Schefold keeps on the track of Bródy in the sense of experimenting with random tables. He proposes extensions to Bródy's conjecture given that it holds. Again, the question we address to his work is if those conclusions also apply to real tables.

Schefold argues that if all eigenvalues go to zero but the first then the wage profit curve is linear. More precisely, he uses Bródy's experiments and his own with random matrices to argue that $z \underset{n\to\infty}{\longrightarrow} 0$, where $z = \lambda_2 + \lambda_3 + \cdots + \lambda_n$. He goes on to show that if this sum is zero, then the wage-profit curve is linear.

The theoretical implication of this goes back to Samuelson (1962) surrogate production function and the capital controversy that followed to criticize the possibility of adopting the curve. If the envelope curve of techniques is a convex curve, than it is possible to define a production function y=f(k). Nonetheless, this convexity of the envelope depends on the linearity of the wage –profit curve. "If and only if the individual wage curves are linear (...) the paradoxes of capital theory will then be absent" (Schefold (2009)).

Figure 4



Schefold argues in terms of the shape of curves of eigenvalues. He says that the bigger the table, the closer to an exponential distribution the eigenvalues curve would be.

Even though the economic meaning of the distribution to which those tables approach is not clear, there seem to be a limiting shape. All curves seem to be stacking to the 403 industries one and already the 342 industries curve is quite similar to the 403 industries curve. Even though almost all eigenvalues fall in size noticeably, there are still a few of them besides the first that are of substantial size.

Figure 5

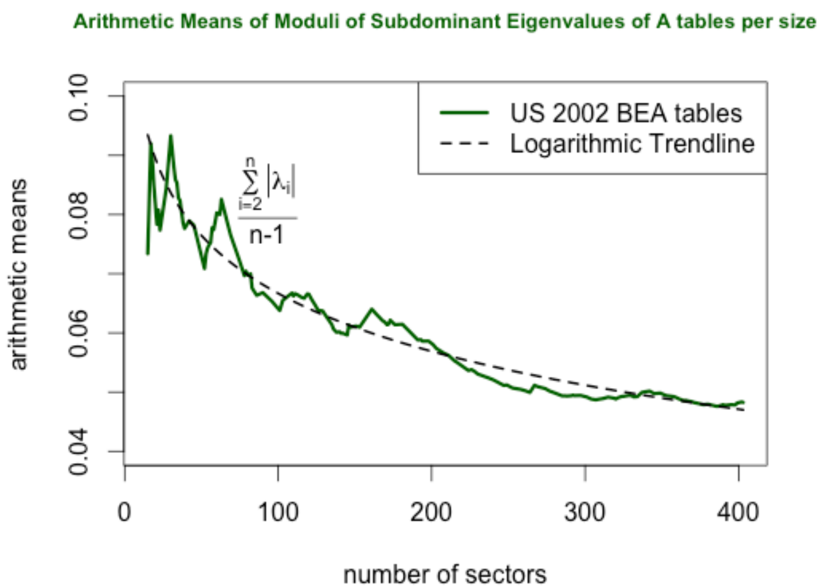
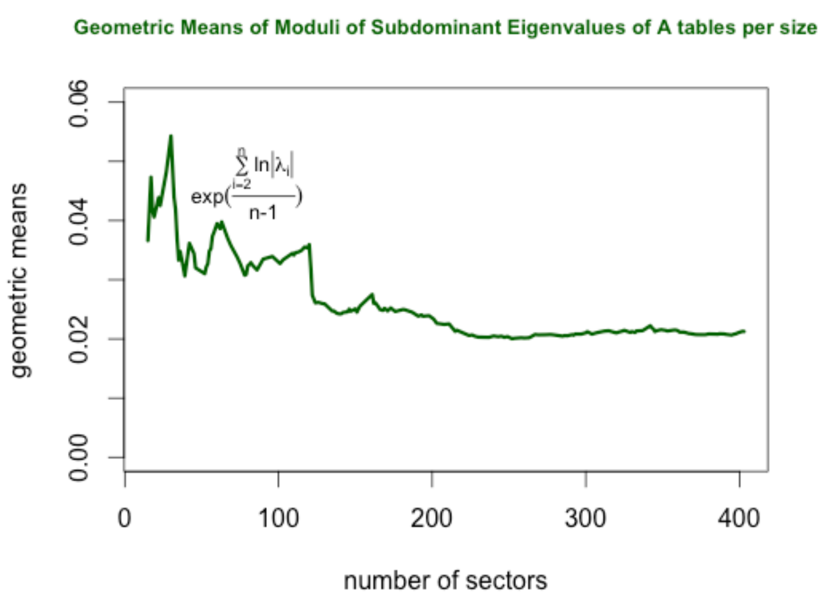


Figure 6



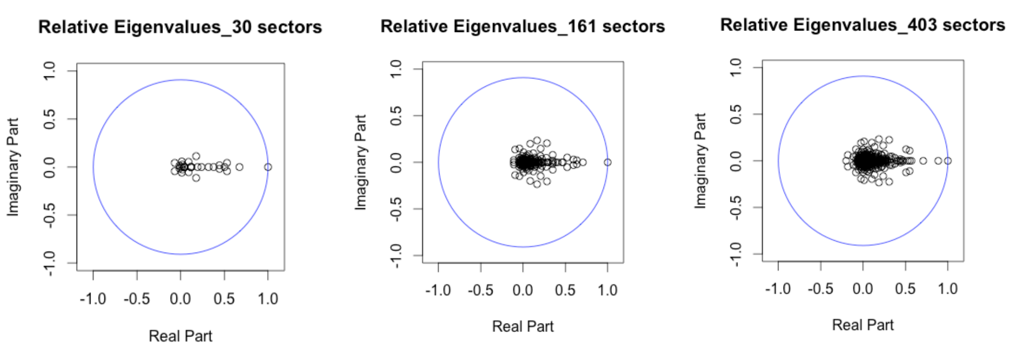
The geometric and arithmetic averages depicted on figure 5 and 6 respectively add evidence to the hypothesis that the subdominant eigenvalues cease to fall after a certain number of industries. Together with figures 4 and 7, these averages reinforce the idea of a few eigenvalues becoming larger with size while most of them become very small but not zero.

Figure 7 shows how the eigenvalues at each size (30 sectors, 161 sectors and 403 sectors) distribute themselves in the unit circle. The Y-axis represents the imaginary parts while the X-axis represents the real part. The symmetric distribution along the X-axis is due to the fact that the complex eigenvalues are always given in pairs where the imaginary parts cancel each other. But it is easy to see that there is no symmetry along the vertical axis. The first eigenvalue seems like an attractor for at least a few of the subdominants one with size and the sum of subdominant eigenvalue don't appear to be approaching zero with size as Schefold defends.

In effect, this sum is increasing with size, since the average is almost constant.

$\lambda_2 + \lambda_3 + \cdots + \lambda_{403} \sim 14{,}40$ and $\frac{\lambda_2+\lambda_3+\cdots+\lambda_{403}}{\lambda_1} \sim 30$ for the $A_{403x403}$ table.

Figure 7



The hypothesis that this sum approaches zero is an important aspect of the argument of linearity of the wage-profit curve within Schefold. Our findings actually give strength to the theory proposed by Sraffa (1960) that these curves have more complex forms. If the wage-profit curve is not linear, it is impossible to represent production as a function of capital labor ratio. What Sraffa (1960) showed us is that a fall in the price of one factor does not necessarily mean that reducing the relative use of this factor to the other will make the cost of production fall because this also depends on the technique applied in the production of capital goods relative to the consumption goods[5].

4. Conclusions

It seems that starting from mathematical properties from random tables to imply properties and then try to find economical meanings to these properties have been misleading the discussion of what we can infer about an economy by studying eigenvalues on input-output tables since Bródy's work on 1997. We decided to take the opposite road and look at eigenvalues at real tables to see if there seem to be any mathematical properties that arise from the data.

It was possible to see that the US 2002 tables contradicts Brody's conjecture and for those tables the ratio λ_2/λ_1 increases with size. Furthermore, the values observed on real tables show that the average value of subdominant eigenvalues does not go to zero and they stop falling after a certain increase of the table, also contradicting Schefold's conjecture that the profit-wage curve would be linear for big input-output tables. As he showed, non-vanishing subdominant eigenvalues are responsible for more complex wage-profit curves as predicted theoretically by Sraffa (1962). Our findings do not back up Schefold's explanation for the empirical support of wage-profit linearity found in the works of Ochoa (1984), Bienenfeld (1988) and Shaikh (1998, 2012).

This leaves us with the question of what do these mathematical properties of real input output tables show in terms of economic theory. One of the conclusions we can draw from the tables here presented is that the more disaggregated we look into the industries in the US 2002 economy, the more unalike the columns of the A table are, which indicates that the techniques are more heterogeneous the more specific the industries we are looking at.

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[5] Serrano(2005)

The economic intuition behind this would be that once we are looking to the production of commodities very closely, differentiating as much as we can one industry from another, the more different would be the kind and proportions of inputs used in its production. This is a quite easy idea to accept, at least far more logical than supporting that one can produce bread and iron using exactly the same inputs.

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6. *Methodological Appendix*

The correspondence of aggregated sectors

This appendix aims to describe the steps that where followed to construct the 176 A tables in different sizes. The whole description of the industries in each table can be requested through e-mail to Luiza Nassif Pires to pirel110@newschool.edu.

We depart from the detailed level US 2002 make and use tables available at http://www.bea.gov/industry/io_benchmark.htm#2002data.

We then treat those make and use tables to exclude non-basic sectors and aggregate linearly dependent sectors together. After this treatment of the 430 industries table we have a 403 industries by 430 commodities make and use table. These 2 tables are the ones that are aggregated iteratively.

The next step was to construct a rule to aggregate those tables. We have a description for how to go from the 403 industries level to seven other basic levels of classification compatible with the North American Industry Classification System (NAICS). Those basic levels are: 15, 17, 30, 63, 120, 161 and 342.

To go from the 403 industries to the 342 industries, we add industries in the make and use table respecting the NAICS nomination in each. Every time each few industries are put together, we calculate a new A table from the make and use. After going through the whole table we are left with the 342 industries table. We then depart from the 342 industries table to aggregate again iteratively constructing A table at each step until we end up with the 161 industries table. And so on until the 15 industries tables are constructed.

The 15, 63, 120, 342 and 403 industries tables were constructed respecting the NAICS codes in which the tables are available at BEA. The benchmark tables are available every 5 years (1997, 2002 and 2007 being the most recent ones) and they are available at the following levels:

Aggregations from benchmark tables:

15 industries– sector level

133 industries - summary level – became 120

430 industries – detailed level – became 403

The annual tables are made available in different levels than the benchmark tables.

Aggregations from annual tables:

15 industries – sector level

69 industries – summary level – became 63

388 industries – detailed level – became 342

The 161 industries tables were constructed respecting the nomination used by the Bureau of Labor Statistics on the following file:

http://www.bls.gov/emp/classifications-crosswalks/sect300.xls

170 industries became 161 in this case.

The 30 industries level consists in the 2 digits NAICS industries and the a few 3 digits NAICS - the 32 and 33 industries are open in 3 digits.

The 17 industries level consists on the BEA sector level with manufacturing open in 3 sectors: 1. food and leather, 2.wood, paper and 3.petroleum and metals

Construction of A tables

The A tables are built from the make and the use table

Input output tables are constructed from the use and the make tables. The use table displays the use of each commodity and elements of value added, in the row, by each industry and components of final demand, in the column, as represented in figure 1. The direct requirements table is built from the intermediary consumption table:

$$X_{m\times n} = \left[x_{i,j}\right]$$

Where n is the number of commodities, m is the number of industries and each $x_{i,j}$ represents the commodity j used by industry i in their total production and the industry's output vector Y_i (in gray on figure 1). The X table is normalized by the total output to produce a direct requirements table industry by commodities, B table, $B_{m \times n} = \left[b_{i,j}\right]$, where each element $\left[b_{i,j}\right] = \frac{x_{i,j}}{Y_i}$ is the value of commodity j that is used in the production of one monetary unit of industry i.

Figure 8

<table>
<tr><th colspan="4">The Use Table</th></tr>
<tr><td></td><td>Industries</td><td>Government, households, rest of the world</td><td>Total</td></tr>
<tr><td>Commodities</td><td>Intermediary consumption</td><td>Final demand</td><td>Total final demand + total intermediary comsumption</td></tr>
<tr><td>Profits, wages and taxes</td><td>Vallue added</td><td rowspan="2"></td><td>Total Vallue added</td></tr>
<tr><td></td><td>Industry output</td><td>Total output</td></tr>
</table>

The data presented on the use table is always displayed in terms of commodities used by industries. To transform the B table into a symmetric industry by industry one, it is necessary to adopt one of the following assumptions, Industry Technology Assumption (ITA) or Commodity Technology Assumption (CTA). According to the ITA, each industry has the same production function for all the commodities it produces. This means that all monetary unities produced by a given industry is assumed to be homogenous in terms of the inputs that constitutes it. According to the CTA, each commodity has a unique production function that is independent of which industry produces it. In this case, each commodity has a unique input structure no matter the industry producing it. As is more usually assumed, in this paper the Industry Technology assumption is adopted[6].

To build the symmetric direct requirements table industry by industry, the make table industry by commodity is used to produce the market share table, $D_{n \times m} = [d_{j,i}]$ that express the share of each industry *j* in the production of commodity *i*, where $d_{j,i}$ is the production of commodity *i* by industry *j* divided by the total output of the commodity. By pre-multiplying the direct requirements table B industry by commodities by the market share table D, we get the A table, direct requirements industry by industry.

$A_{n \times n} = D_{n \times m} \times B_{m \times n} = [a_{i,j}]$,

Where each $a_{i,j}$ is the share of the product of industry *i* that is used as input in the production of one dollar of output of industry *j*.

The sum of the coefficients in one column expresses the ratio of inputs in the total output of the given industry. If we assume that every industry has a strictly positive value added, then $\sum_{i=1}^{n} a_{i,j} < 1$. Furthermore, it is possible to see that this sum express how each monetary unit of output of each industry is shared between inputs and value added, therefore, $1 - \sum_{i=1}^{n} a_{i,j} = \frac{VA_j}{Y_j}$.

How to disaggregate to create a consistent series of random matrices- a critique of Bródy's method

[6] More details on this matter can be found in https://bea.gov/papers/pdf/alttechassump.pdf, according to which one of the advantages of assuming ITA is the fact that it only produces non-negative symmetric coefficients table, unlike CTA that can produce negative coefficients. CTA can only be used in squared tables where the number of industries is equal to the number of commodittis $n = m$, while for rectangular use tables $n \neq m$, only ITA can be used.

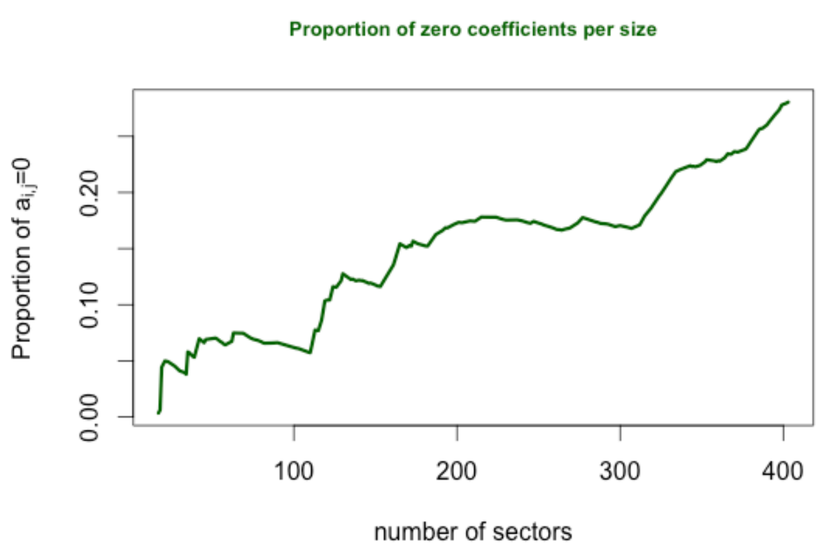
This appendix try to shed some light on the consequences of how the aggregation or disaggregation of tables can influence their structure as so to understand why the random hypothesis present in Bródy might not be accurate when working with real tables.

To go from a small table to a big table by disaggregating it, it would be necessary to do it so by adding columns on the B table and adding rows on the D table.

Regarding the B table, if we disaggregate one column into two by dividing each value by 2 (which is basically the rule that Bródy applies) with size all the divergences will fade out. If on the other hand we fix the total flow of the table, i.e., define that the new bigger table still represented the same economy with the same total output, then the rule to disaggregate would be different.

Bródy supposes that the distribution of the coefficients is the same for each size. He is not allowing for a change in the distribution of coefficients when tables become bigger. Nonetheless, in real I-O tables the size and the distribution of its coefficients are related. Also, he supposes that values are distributed uniformly around the mean value (1/2) and that there are no coefficients that are zero. Nonetheless, in real tables the zeros play an important role.

Figure 9



On figure 8 it is possible to see that the number of zeros on the A table increases. In figure 9, it is clarified that this rise in zeros on the A table are due to increases in zeros at the B table rather then the D table. A thought experiment of how adding zeros to the B table influences the distribution of values within the coefficients can bring some light to why it is possible to argue that the increasing proportion of zeros in the A table has the effect of increasing the disparities between columns. This is consistent with the idea of subdominant eigenvalues not going to zero since.

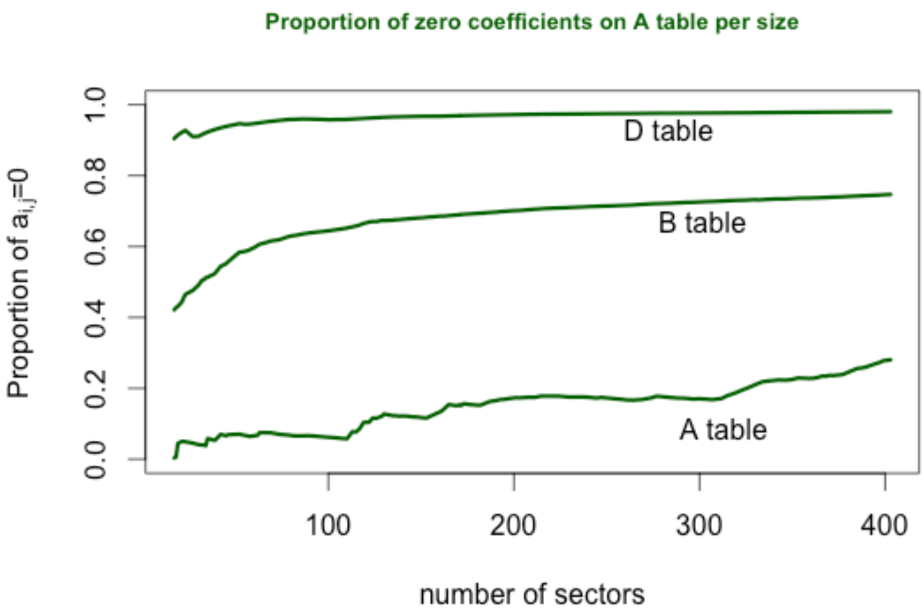
On real input-output tables, while splitting two columns and distributing the values on the original one within two new columns everytime a zero is added in one of the new columns, the other sticks with a higher value then the original.

This goes in the same line of argument as Gurgul and Wójtowicz (2015) about the fact that when disaggregating columns from the same economy for the same year, the flow is fixed and the only way of making this division increasing the uniformity of values would be by allowing the total value of production represented by the table to change.

To disaggregate the A table we need to disaggregate the use and the make table. The make table will be used to build the market share table (D) and the use to build the direct coefficients industries by commodities table (B). D*B=A. We can say that the D table preserves its structure with size and is quite uniform and the B table is the one that is responsible for making it divergent. It would be quite nice to try to test eigenvalues on those tables but unfortunately they are not square matrices.

Take one column v on a B_{nxm} table. Let's split v in two to create a new table that will have (n+1)x(m) cells). Let's say that the original D table have 20% of zeros and that the v column have the same proportion of zeros. Let's also follow the rule that was observed on figure 9 and say that the new table (n+1)x(n+1) have a higher percentage of zeros. Let's also say that the value of production of the sector V will be split in two equally.

Figure 10



First step: when disaggregating v into v_1 and v_2 besides replicating on the 2 new columns the original zeros, a few zeros will have to be placed in both new columns.

Second step: The proportions need to be preserved; this means that if we want to split the values equally between both vectors, we need to replicate the original values (and not divide it by 2 as Bródy does- those values will be divided by two with the adding of new rows on the D table, but column-wise it is not so).

Third step: In the correspondent row of v_2, where a new zero was placed in v'1, the new value will have to be the double of the first since the new quotient of the new columns are half the value of the original column.

We set up a simple example.

V	V_1	V_2
0.08	**0.00**	0.15
0.00	0.00	0.00

0.05	0.05	0.05
0.00	0.00	0.00
0.07	0.07	0.07
0.05	0.10	**0.00**
0.06	0.06	0.06
0.05	0.05	0.05
0.00	0.00	0.00
0.05	0.05	0.05
0.05	0.05	0.05

By this simple example it is easy to see why the rise in zeros are making the new and bigger table less "uniform". Of course if we are also adding a row the values will be divided by 2 or zeros added will mean just replicating the value since the sum needs to be preserved in the whole table. But the disaggregation by column is a factor of rising inequality and according to figure 10 is the one also responsible for the rise in the number of zeros. This is in accordance with both the coefficient of variation and the rise in the subdominant relative eigenvalue seen on figures 2 and 3.