Decomposition of Average Propagation Length

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Abstracts

Dietzenbacher proposed the concept of Average Propagation Length (APL) (Dietzenbacher et al (2005) and Dietzenbacher et al (2007)). APL has been used as length of production process or length of supply chain. On the other hand, international division of labor in the production process is getting focused in face to globalization. The phenomenon of increasing cross-border transaction of intermediate inputs is called the fragmentation of production process. APL is used as index of fragmentation of production process (Romero et al (2009), Escaith et al (2013)).

However, APL includes both propagation in domestic transaction and that in cross-border transaction. To capture fragmentation, two propagations should be separated. In this paper, new concept of cross-border APL (APLxB) and the method of APL decomposition in general are proposed. Finally, we show increase of APLxB.

1. Introduction

Dietzenbacher et al (2005) proposed Average Propagation Length (APL). Power weighted sum of input coefficient produce measure how many transactions are repeated in the propagation process. APL has been used as length of production process or length of supply chain.

On the other hand, in face to globalization, international division of labor in the production process becomes increasingly focused. Production process does not complete within one country and cross border transaction of intermediate input commodities occupies larger share international trade. This phenomenon is called the *fragmentation of production process*[1].

Sometimes, APL is used as index of fragmentation of production process. But, APL includes domestic transaction. Fragmentation is really captured when APL excludes domestic transaction.

We discuss how APL should be extended to treat fragmentation and name new index as cross border APL (APLxB). Next, we show APLxB can be generalized and APL can be decomposed by those of submatrices covering whole coefficient matrix. Finally, we compare APL and decomposed APLs empirically.

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1 The concept of “trade in value added” is used in the same context.

2. Average Propagation Length and cross border transaction

To start with, we describe how original APL is. In the input-output framework, unit of production in sector j requires a_{ij} $(a_{ij} \geq 0)$ unit of product in sector i as first round. To produce a_{ij} unit of product i, $a_{hi}a_{ij}$ unit of product has second round, and so on. Summing up, we get total effect of final demand increase.

$$p_{ij} = a_{ij} + \sum_h a_{ih}a_{hj} + \sum_h \sum_g a_{ig}a_{gh}a_{hj} + \cdots$$

$$P = A + A^2 + A^3 \cdots = (I - A)^{-1} - I = B - I \quad (1)$$

Where B is Leontief inverse. Number of rounds reflect distance between origin and destination of demand. Therefore, average length can be obtained by ratio of sum of (1) weighted by number of rounds to (1).

Weighted sum is

$$s_{ij} = a_{ij} + 2\sum_h a_{ih}a_{hj} + 3\sum_h \sum_g a_{ig}a_{gh}a_{hj} + \cdots$$

or

$$S = A + 2A^2 + 3A^3 \cdots \quad (2)$$

Taking account that $S - AS = A + A^2 + A^3 \cdots = B - I$, Matrix S can be rewritten as:

$$S = (I - A)^{-1}(B - I) = B(B - I) \quad (2)'$$

Length of propagation of sector j to product i is given by the ratio of (i, j) factor of (2) to that of (1).

$$APL_{ij} = s_{ij} / p_{ij}$$

APL_{ij} shows average number of transactions from demand sector j to production sector i. The larger the APL, the longer path of propagation.

APL can be applied to international input-output table. Taking a_{ij}^{rs} as sector j country s input coefficient of sector i country r, input coefficient matrix is

$$A = \begin{pmatrix} A^{11} & A^{12} & \cdots & A^{1R} \\ A^{21} & A^{22} & \cdots & A^{2R} \\ \vdots & \vdots & \ddots & \vdots \\ A^{R1} & A^{R2} & \cdots & A^{RR} \end{pmatrix}, \quad A^{rs} = \begin{pmatrix} a_{11}^{rs} & a_{12}^{rs} & \cdots & a_{1N}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} & \cdots & a_{2N}^{rs} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1}^{rs} & a_{N2}^{rs} & \cdots & a_{NN}^{rs} \end{pmatrix} \quad (3)$$

Where R is number of countries.

For simplicity, we consider two country case where each country produces one commodity as shown in Figure 1. a_1 and a_2 stand for domestic input coefficients and m_1 and m_2 stand for import coefficient. The sum of formula shown outside rectangle is numerator of APL. Numerator of APL (s_{ij}) captures all transaction including domestic transaction. Numerator of APL may increase by a_1 or a_2 which means domestic production process is strengthened. On the other hand, increase of m1 or m2 with reduction of a1 or a2 may increase APL. In the latter case, fragmentation and APL increase is consistent.

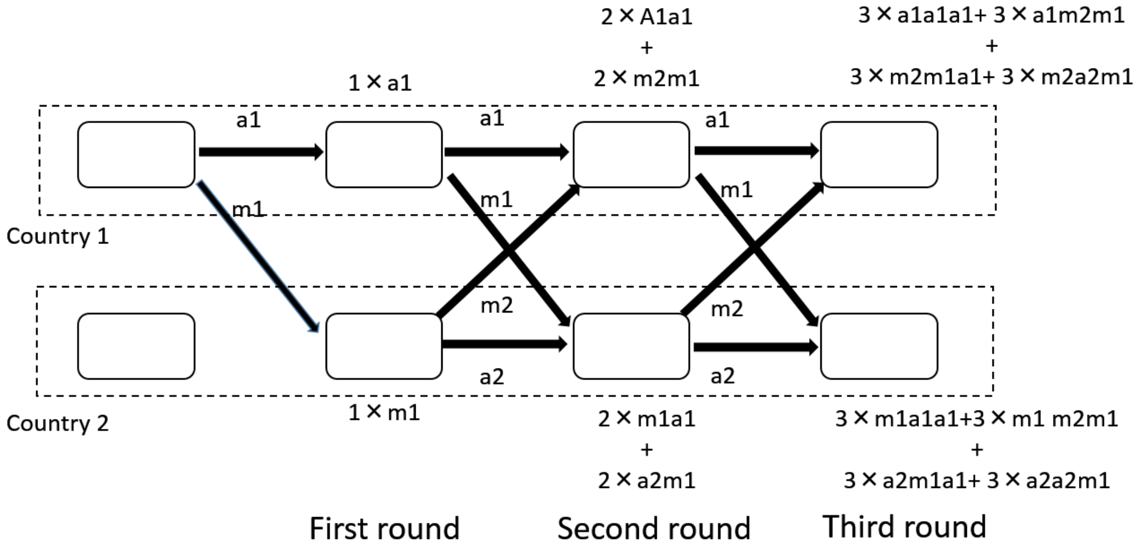

Figure 1 APL in two country case

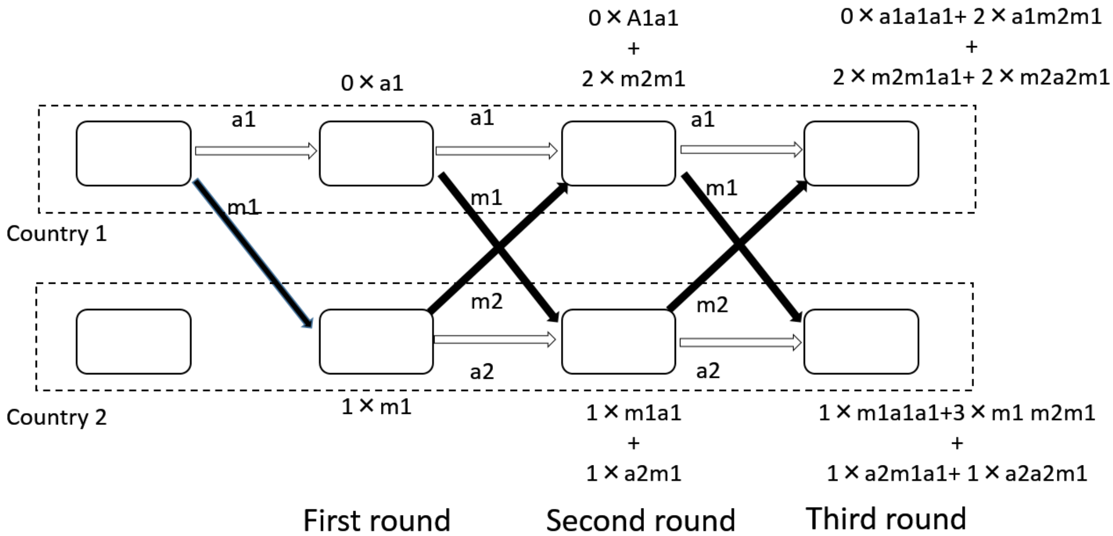

Figure 2

3 Cross border APL

In order to evaluate fragmentation using APL, we should seek another APL, which reflects cross border transaction. In figure 2, black arrows reflect cross border transaction and white arrows reflect domestic transaction. We can extract cross border transaction using weighting of order of m1 and m2. In the propagation of round k, product of a1, a2 m1and m2 in order of k. The original APL evaluates these products equally. On the other hand, APL reflecting cross border transaction should ignore the domestic path, ie., a1 and a2. For example, the second round in country 1 caused by final demand in country 1 is $0 \times a_1^{\,2} + 2 \times m_2 m_1$ instead of $2 \times a_1^{\,2} + 2 \times m_2 m_1$. Numerator reflecting cross border transaction is the sum of these value. Using new numerator and denominator of original APL, new APL is defined. We call this APL reflecting cross border transaction as cross border APL (APLxB)

To formulate APLxB in general, we divide international input coefficient matrix A into domestic coefficient matrix (Ad) and import coefficient matrix (Af) as (**).

$$A_d = \begin{pmatrix} A^{11} & 0 & \cdots & 0 \\ 0 & A^{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & & \cdots & A^{RR} \end{pmatrix}, A_f = \begin{pmatrix} 0 & A^{12} & \cdots & A^{1R} \\ A^{21} & 0 & \cdots & A^{2R} \\ \vdots & \vdots & \ddots & \vdots \\ A^{R1} & A^{R2} & \cdots & 0 \end{pmatrix} \qquad (4)$$

Here, identity $A = A_d + A_f$ holds. A to the power of k, A^k, equals to the sum of all combination of product of A_d and A_f at order k.

$$\begin{aligned} A^k &= (A_d + A_f)^k \\ &= A_d^{\,k} + A_d^{\,k-1} A_f + A_d^{\,k-2} A_f A_d + \ldots + A_d A_f^{\,k-1} + A_f^{\,k} \end{aligned}$$

To evaluate cross border transaction, each product is weighted by order of Af.

$$0 \times A_d^{\,k} + 1 \times A_d^{\,k-1} A_f + 1 \times A_d^{\,k-2} A_f A_d + \ldots + (k-1) \times A_d A_f^{\,k-1} + k \times A_f^{\,k}$$

Weighted powers $2A^2$,$3A^3$…in equation (2) become following T_1, T_2, T_3…, where T_k is weighted propagation effect at round k.

$$\begin{aligned} T_1 &= 1 \times A_f + 0 \times A_d \\ T_2 &= 2 \times A_f^{\,2} + 1 \times (A_f A_d + A_d A_f) + 0 \times A_d^{\,2} \\ T_3 &= 3 \times A_f^{\,3} + 2 \times (A_f^{\,2} A_d + A_f A_d A_f + A_d A_f^{\,2}) + 1 \times (A_f A_d^{\,2} + A_d A_f A_d + A_d^{\,2} A_f) + 0 \times A_d^{\,3} \\ &\vdots \end{aligned} \qquad (5)$$

Eliminating A_d using $A_d = A - A_f$, we have

$$
\begin{aligned}
&T_1 = A_f \\
&T_2 = A_f A + A A_f = A_f A + A T_1 \\
&T_3 = A_f A^2 + A A_f A + A^2 A_f = A_f A^2 + A T_2 \\
&\quad\vdots
\end{aligned}
\tag{6}
$$

We expect following relation holds in general.

$$
T_k = \begin{cases} A_f & for\, k = 1 \\ AT_{k-1} + A_f A^{k-1} & for\, k = 2,3,4,... \end{cases}
\tag{7}
$$

Proposition1: Weighted sum of round k by power of A_f, T_k, is identical to equation (7).

As a preliminary definition, we introduce matrix poly-nominal function $\Phi(G, H, k, l)$. Let *G* and *H* be square matrices and consider the expansion of $(G + H)^k$. Here, we define matrix poly-nominal function, $\Phi(G, H, k, l)$, as the sum of poly-nominal including order l product of H from $(G + H)^k$, where $0 \le l \le k$.

For example, in case of k=3,

$$(G + H)^3 = G^3 + (G^2 H + GHG + HG^2) + (GH^2 + HGH + H^2 G) + H^3.$$

And $\Phi(G, H, 3, l), l = 0,1,2,3$ are

$$\Phi(G, H, 3, 0) = G^3$$

$$\Phi(G, H, 3, 1) = G^2 H + GHG + HG^2$$

$$\Phi(G, H, 3, 2) = GH^2 + HGH + H^2 G$$

$$\Phi(G, H, 3, 3) = H^3$$

Expanding $\Phi(G, H, 3, l)$ left side G or H, we get

$$\Phi(G, H, 3, 0) = GG^2 = G\Phi(G, H, 2, 0)$$

$$\Phi(G, H, 3, 1) = G(GH + HG) + HG^2 = G\Phi(G, H, 2, 1) + H\Phi(G, H, 2, 0)$$

$$\Phi(G, H, 3, 2) = GH^2 + H(GH + HG) = G\Phi(G, H, 2, 2) + H\Phi(G, H, 2, 1)$$

$$\Phi(G,H,3,3)=HH^2=H\Phi(G,H,2,2)$$

In general, function $\Phi(G,H,k,l)$ has following properties:

$$\Phi(G,H,0,0)=0 \tag{8}$$

$$\Phi(G,H,k,0)=G^k \tag{9}$$

$$\Phi(G,H,k,k)=H^k \tag{10}$$

$$\Phi(G,H,k,l)=\begin{cases} G\Phi(G,H,k-1,0) & if\ l=0 \\ G\Phi(G,H,k-1,l)+H\Phi(G,H,k-1,l-1) & if\ 0<l<k \\ H\Phi(G,H,k-1,k-1) & if\ l=k \end{cases} \tag{11}$$

From definition of function $\Phi(G,H,k,l)$, following equality holds

$$\sum_{l=0}^{k}\Phi(G,H,k,l)=(G+H)^k \tag{12}$$

Using function $\Phi(G,H,k,l)$, propagation at k-th stage

$$A^k=\sum_{l=0}^{k}\Phi(A_d,A_f,k,l)=\sum_{l=0}^{k}\Phi(A-A_f,A_f,k,l)$$

Definition of T_k is the sum of propagation ($A^k=(A_d+A_f)^k$) at round k weighted by the order of A_f

(5). T_k can be written as:

$$T_k=\sum_{l=0}^{k}l\Phi(A-A_f,A_f,k,l) \tag{13}$$

[Proof of Proposition 1]

Equation (13) equals to (7) for for k=1:

$$T_1=\sum_{l=0}^{1}l\Phi(A-A_f,A_f,1,l)=0\times(A-A_f)+1\times A_f=A_f \tag{14}$$

If following two equalities,

$$\sum_{l=0}^{k}l\Phi(A-A_f,A_f,k,l)=\Phi(A,A_f,k,1)\quad for\, k=2,3,4,... \tag{15}$$

$$\Phi(A,A_f,k,1)=A\Phi(A,A_f,k-1,1)+A_fA^{k-1}\quad for\, k=2,3,4,... \tag{16}$$

Then (7) is proved, because

$$T_k = \sum_{l=0}^{k} l\Phi\left(A-A_f, A_f, k, l\right) = \Phi\left(A, A_f, k, 1\right) = A\Phi\left(A, A_f, k-1, 1\right) + A_f A^{k-1} = T_{k-1} + A_f A^{k-1} \tag{17}$$

Since the latter equality (16) is obvious from (11) in the case of l=k. We concentrate on (15).

In case of k=1, (11) holds as follows:

$$T_1 = 0 \times \left(A - A_f\right) + 1 \times A_f = \Phi\left(A, A_f, 1, 1\right)$$

Next, we assume (15) holds in case of (k-1) as the assumption of mathematical induction, that is,

$$T_{k-1} = \sum_{l=0}^{k-1} l\Phi\left(A-A_f, A_f, k-1, l\right) = \Phi\left(A, A_f, k-1, 1\right) \tag{18}$$

We will show (15) holds in case of k. Let's expand Left hand side of (15) for left term ($A - A_f$

and A_f). From (11) in the case of (l=0),

$$T_k = \sum_{l=0}^{k} l\Phi\left(A-A_f, A_f, k, l\right) = \sum_{l=0}^{k} l\left[\left(A-A_f\right)\Phi\left(A-A_f, A_f, k-1, l\right) + A_f\Phi\left(A-A_f, A_f, k-1, l-1\right)\right]$$

Then,

$$T_k = \left(A-A_f\right)\sum_{l=0}^{k-1} l\Phi\left(A-A_f, A_f, k-1, l\right) + A_f\sum_{l=1}^{k} l\Phi\left(A-A_f, A_f, k-1, l-1\right) \tag{19}$$

The first term of RHS of (19) is sum of $l = 0,1,\ldots k-1$. It does not include the case of l=k in (11). Similarly, the second term of RHS of (19) is sum of $l = 1,2,\ldots k$. It does not contain the case of l=0.

Furthermore, from the assumption of mathematical induction (18), the first term of RHS of (19) is

$$\left(A-A_f\right)\sum_{l=0}^{k-1} l\Phi\left(A-A_f, A_f, k-1, l\right) = \left(A-A_f\right)\Phi\left(A, A_f, k-1, 1\right) \tag{20}$$

The second term of RHS of (19) can be rewritten as

$$\begin{aligned} A_f\sum_{l=1}^{k} l\Phi\left(A-A_f, A_f, k-1, l-1\right) &= A_f\sum_{l=0}^{k-1}\left(l+1\right)\Phi\left(A-A_f, A_f, k-1, l\right) \\ &= A_f\sum_{l=0}^{k-1} l\Phi\left(A-A_f, A_f, k-1, l\right) + A_f\sum_{l=0}^{k-1}\Phi\left(A-A_f, A_f, k-1, l\right) \end{aligned} \tag{21}$$

We rewrite the first term of RHS of (21) from assumption of mathematical induction (18), and the last tern of RHS of (21) from (12). Then, we get

$$A_f \sum_{l=1}^{k} l\Phi\left(A - A_f, A_f, k-1, l-1\right) = A_f \Phi\left(A, A_f, k-1, 1\right) + A_f \left[\left(A - A_f\right) + A_f\right]^{k-1} \\ = A_f \Phi\left(A, A_f, k-1, 1\right) + A_f A^{k-1} \quad (22)$$

From (20) and (22), we can rewrite (19) as:

$$T_k = \left[\left(A - A_f\right)\Phi\left(A, A_f, k-1, 1\right)\right] + \left[A_f \Phi\left(A, A_f, k-1, 1\right) + A_f A^{k-1}\right] \\ = A\Phi\left(A, A_f, k-1, 1\right) + A_f A^{k-1} = \Phi\left(A, A_f, k, 1\right)$$

Now, (15) holds.

Since (15) and (16) hold, (17) holds. Therefore proposition 1 is proved

[End of proof]

Let T be the sum of T_k

$$T = \sum_{k=1}^{\infty} T_k \quad (23)$$

Average number of cross border transaction is given by the ratio of t_{ij}^{rs} to p_{ij}^{rs} in (1)

$$APLxB_{ij}^{rs} = t_{ij}^{rs} / p_{ij}^{rs} \quad (24)$$

We call (9) as Average Length of Propagation cross border (APLxB). Using APLxB, we can measure the degree of fragmentation.

4 Decomposition of APL

Af, discussed in previous section, is an off diagonal sub matrix. In the proof of Theorem, the character of Af is not used. Therefore, Theorem applies to any sub matrix of A. Suppose that there are several sub matrices A_qs of which sum equals to overall matrix A.

$$A = \sum_q A_q$$

We call APLq corresponding to submatrix Aq in the same way as APLxB corresponds to Af. In this section, we show the sum of APLq equals to APL. It means APL can be decomposed to sum of APLq.

Weighted propagation effects at k-th stage Tqk is expressed as

$$T_{qk} = \begin{cases} A_q & if\ k = 1 \\ A_q A^{k-1} + AT_{qk-1} & if\ k > 1 \end{cases} \quad (25)$$

Proposition 2: Sum of Tqk equals to

$$\sum_q T_{qk} = kA^k$$

[Proof]

If k=1, then the sum of Tq1 equals to A

$$\sum_q T_{q1} = \sum_q A_q = A$$

We suppose (**) $\sum_q T_{qk-1} = (k-1)A^{k-1}$ holds for k>1 as mathematical induction assumption.

$$\sum_q T_{qk} = \sum_q \left(A_q A^{k-1} + AT_{qk-1}\right) = \left(\sum_q A_q\right) A^{k-1} + A\sum_q T_{qk-1} = A^k + (k-1)AA^{k-1} = kA^k$$

Then, we confirm the sum of Tqk equals to Sk in APL.

$$\sum_q T_{qk} = kA^k = S_k \quad for\, k = 1,2,...$$

[end of proof]

Summing above equations for all round k, we have

$$\sum_{k=1}^{\infty}\sum_q T_{qk} = \sum_{k=1}^{\infty} S_k$$

Since numerator of APLq and APL are the sum of Tqk and Sk, sum of APLq equals to APL

APLq corresponding to submatrix Aq is the quotient of (i,j,r,s) factor of sum of (25), $T_q = \sum_{k=1}^{\infty} T_{qk}$, to (i,j,r,s) factor of(1).

$$APLxB_{qij}^{rs} = t_{qij}^{rs} \big/ p_{ij}^{rs} \tag{26}$$

APLq shows how many times pass through submatrix Aq in the propagation process of country r sector i caused by final demand in country s sector j.

Original APL equals to the sum of APLq if submatrix Aq covers A.

$$APL_{ij}^{rs} = \sum_q APLxB_{qij}^{rs} \tag{27}$$

5 Numerical results of APLxB and decomposition of APL

In this section, we show brief result of new indicators APLxB and APLq. We use World Input Output Database (WIOD) which contains 41 regions (40 countries and Rest of the World), 35 sectors and 17 years (1995-2011). We calculate various APL as effect of country s sector j on country r

$$APL_j^{rs} = \sum_i APL_{ij}^{rs}$$

APL, APLq and APLxB are 41 by 1435 matrix, which show average length of origin (j,s) to final destination (r).

We choose each country's domestic input coefficient matrix as $A_q, (q = 1,...R)$ and other parts of whole coefficient matrix as A_f [2]. To illustrate this, consider five country case. As is depicted in Fig 3, the sum of Aq and Af covers A. Therefore, original APL equals to the sum of Aq and APLxB.

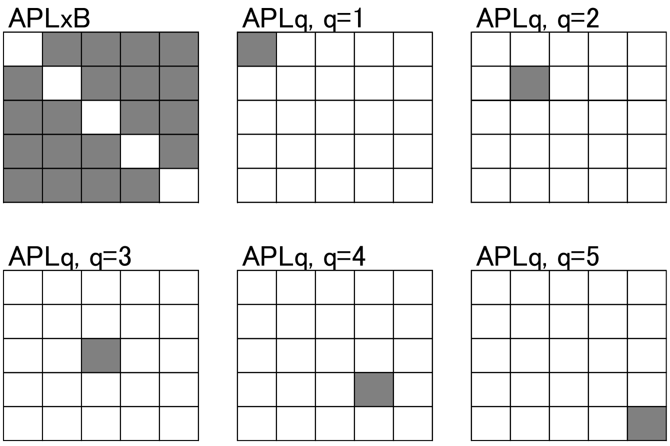

Figure 3 example of division of A

First we show descriptive statistics of APLs. APLq is divided into three categories, APLqj: demand origin (q=j), APLqi: demand destination (q=i), and APLqs: third countries (sum of q except i and j). We distinguish effect on foreign countries ($i \neq j$, "Abroad" in the table) and effect on own

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[2] The smallest submatrix Aq is one entry a_{ij}^{rs} case

$$A_q = \begin{cases} a_{i^*j^*}^{r^*s^*} & if\ i = i^* \ \&\ j = j^* \ \&\ r = r^* \ \&\ s = s^* \\ 0 & else \end{cases}$$

$APLq_{ij}^{rs}$ calculated from such Aq reflects how many times pass through (i^*,j^*,r^*,s^*) during propagation process from (j, s) to (i,r).

country　($i = j$, "Domestic" in the table).　In the domestic case, main body of APL is own country's APLqj and third country APLq and cross border APL is small.　On the other hand, in the abroad case, cross border APL is near to half of original APL.

Table 1 Descriptive statistics of APLs[3]

		All	Abroad	Domestic
No.of Sample		971,782	948,080	23,702
	APL	3.2319	3.2724	1.6104
	APLqj	0.9601	0.9451	1.5624
Mean	APLqi	0.5659	0.5410	
	APLqs	0.2655	0.2719	0.0101
	APLxB	1.4785	1.5145	0.0379

Next, we show scatter diagrams of abroad case and domestic case.　In the abroad case, APL positively correlates to APLqi, APLqj, APLqs and APLxB.　APLqs positively correlates to APLxB. In the domestic case, APL positively correlates to APLqj, APLqs and APLxB.　APLqs positively crrelates to APLxB.　APLqj negatively correlate to APLqs and APLxB.

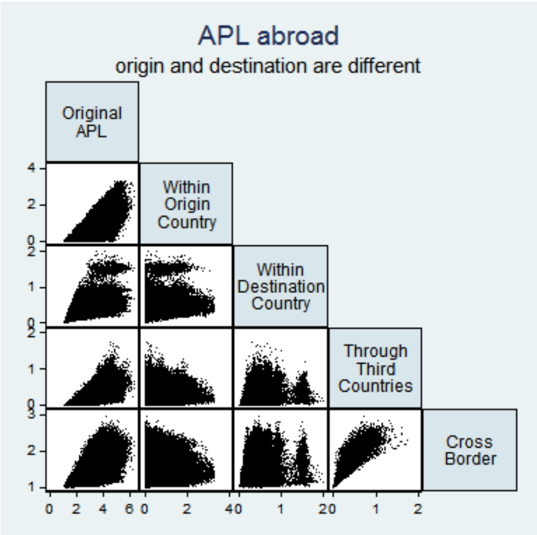

Figure 4

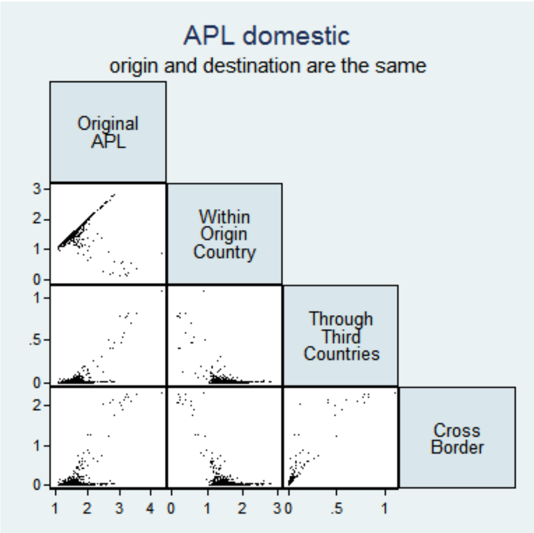

Figure 5

3 Origin of demand (41x35), destination of demand(41)　sample period (1995-2011) have 41x35x41x17= 1,000,195 combinations. Some of them have zero denominator of APL.　These cases are omitted from sample.

Lastly, we estimated time trend of various APLs as shown in Table 2. In two cases, same trends are observed. First, APL has upward trend. We criticized increase in APL does not automatically mean fragmentation, because domestic transaction may increase instead of cross brder transaction increase. But APLqi and APLqj have downward trend which mean APL within origin and destination of demand is decreasing, while APLqs and APLxB have upward trend.

Table 2 Estimation result of time trend

	All cases		Abroad		Domestic	
	Coef	z value	Coef	z value	Coef	z value
APL	0.005533	81.80	0.00564	81.43	0.00145	8.66
APLqj	-0.002968	-68.73	-0.00303	-68.70	-0.00033	-2.84
APLqi	-0.000487	-34.28	-0.00049	-34.41		
APLqs	0.002517	140.08	0.00257	140.08	0.00030	5.54
APLxB	0.006463	209.86	0.00659	210.36	0.00148	9.53

Estimation method: Fixed Effect model

6 Conclusion

Original APL reflects both domestic transaction and cross border transaction. We defined cross border APL (APLxB) as expansion of APL. Empirically, we found APL and APLxB have upward trend. We also decomposed APL into APLxB, APL within origin of demand, APL within destination od demand and APL within third countries. We found APL within origin and destination are decreasing and APL within third countries is increasing. Now, we confirm fragmentation in production process is going on.

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