The Hadamard-multiplicative GLS-based Model for Matrix Updating with a Solution Space of Reducible Dimension

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The paper presents a new Hadamard-multiplicative GLS-based model for updating economic tables within homothetic and angular paradigm for structural similarity of rectangular matrices. It is illustrated that all matrices from homothetic family of initial matrix have an excellent structural similarity. The ensuing notions of homothetic and angular measures for matrix similarity are studied. It is shown that a minimization of angular measure can be implemented via the minimization of homothetic measure. Analytical solution of uniparametrical constrained minimization problem for homothetic measure as an objective function is derived in matrix notation. Special attention is paid to sensitivity of optimal solution for homothetic measure to small changes in row and column totals of the target matrix. The proposed model is quite applicable for updating the economic matrices and tables with some negative entries. Several illustrative numerical examples are given.

Keywords: matrix updating methods, RAS multiplicative pattern, matrix homothety, angular and homothetic measures of matrix similarity, constrained minimization, Lagrange multipliers

JEL Classification: C61; C67

1. An introduction to the problems of matrix updating

The subject of this study is a general problem of updating rectangular (or square) matrices, which can be formulated as follows. Let $\mathbf{A}$ be an initial matrix of dimension $N\times M$ with row and column marginal totals $\mathbf{u_A} = \mathbf{A}\mathbf{e}_M$, $\mathbf{v}'_\mathbf{A} = \mathbf{e}'_N\mathbf{A}$ where $\mathbf{e}_N$ and $\mathbf{e}_M$ are $N\times 1$ and $M\times 1$ summation column vectors with unit elements. Further, let $\mathbf{u} \neq \mathbf{u_A}$ and $\mathbf{v} \neq \mathbf{v_A}$ be exogenous column vectors of dimension $N\times 1$ and $M\times 1$, respectively. The problem is to estimate a target matrix $\mathbf{X}$ of dimension $N\times M$ at the highest possible level of its structural similarity (or closeness, etc.) to initial matrix $\mathbf{A}$ subject to $N+M$ equality constraints

$$\mathbf{X}\mathbf{e}_M = \mathbf{u}, \qquad \mathbf{e}'_N\mathbf{X} = \mathbf{v}' \tag{1}$$

and under the consistency condition

$$\mathbf{e}'_N\mathbf{u} = \mathbf{e}'_M\mathbf{v}. \tag{2}$$

It is assumed that initial matrix $\mathbf{A}$ does not include any zero rows or zero columns, does not have less than $N+M$ nonzero elements, does not include any rows or columns with a unique nonzero element, and does not contain any pairs of rows and columns with four nonzero elements in the intersections. Otherwise, it is advisable to clean matrix $\mathbf{A}$ from those undesirable features before applying any matrix updating method in practice.

Clearly, the system of equations (1) is dependent at consistency condition (2) which provides an existence of target matrix **X**. However, it is easy to show that any $N+M-1$ among $N+M$ constraints (1) are mutually independent. Furthermore, it is evident that any feasible solution of matrix updating problem **X** can be simply transformed into another one by letting, e.g.,

$$x_{nm}^{\text{new}} = x_{nm} + \varepsilon, \qquad x_{nj}^{\text{new}} = x_{nj} - \varepsilon; \qquad x_{im}^{\text{new}} = x_{im} - \varepsilon, \qquad x_{ij}^{\text{new}} = x_{ij} + \varepsilon$$

where ε is an arbitrary scalar, or

$$x_{nm}^{\text{new}} = x_{nm} + \varepsilon, \quad x_{nj}^{\text{new}} = x_{nj} - \varepsilon/2, \quad x_{nk}^{\text{new}} = x_{nk} - \varepsilon/2; \qquad x_{im}^{\text{new}} = x_{im} - \varepsilon, \quad x_{ij}^{\text{new}} = x_{ij} + \varepsilon/2, \quad x_{ik}^{\text{new}} = x_{ik} + \varepsilon/2,$$

and so on.

Thus, general problem of matrix updating significantly depends on a definition of the measure for structural similarity between initial and target matrices. Various definitions of this measure generate a great manifold of different methods and techniques for matrix updating. As Temurshoev et al. (2011, p. 92) rightly noted, “it is impossible to consider *all* updating methods, because theoretically their number is infinite”.

A notion of structural similarity between initial and target matrices has a vague framework that can be slightly refined in an axiomatic manner. In this context let us consider a particular case of strict proportionality between row and column marginal totals $\mathbf{u} = k\mathbf{u}_{\mathbf{A}}$ and $\mathbf{v} = k\mathbf{v}_{\mathbf{A}}$ for target and initial matrices with the same scalar multiplier k. Here the main question arises: can we accept the matrix $\mathbf{X} = k\mathbf{A}$ as optimal solution for proportionality case of a general matrix updating problem? At first sight this solution can be appreciated as rather logical and, moreover, it allows preserving in X the same location of zeros as in the initial matrix. However, it is to be emphasized that the above question indeed seems neither simple nor evident, and its proposition cannot be proved formally.

Nevertheless, in most practical situations an affirmative answer to this question is almost obvious. In particular, the well-known and widely used RAS and Kuroda’s methods for matrix updating serve as an additional instrumental confirmation to such an answer.

2. The proportionality case from the viewpoint of RAS method

The key idea of the RAS method is triple factorization of target matrix

$$\mathbf{X} = \mathbf{RAS} = \langle \mathbf{r} \rangle \mathbf{A} \langle \mathbf{s} \rangle = \hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}} \tag{3}$$

where **r** and **s** are unknown $N\times 1$ and $M\times 1$ column vectors. Here angled bracketing around a vector’s symbol or putting a “hat” over it denotes a diagonal matrix, with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697).

Putting (3) into (1), we have the system of nonlinear equations

$$\hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}}\mathbf{e}_M = \hat{\mathbf{r}}\mathbf{A}\mathbf{s} = \langle \mathbf{A}\mathbf{s} \rangle \mathbf{r} = \mathbf{u}\,, \qquad \mathbf{e}'_N \hat{\mathbf{r}}\mathbf{A}\hat{\mathbf{s}} = \mathbf{r}'\mathbf{A}\hat{\mathbf{s}} = \mathbf{s}'\langle \mathbf{A}'\mathbf{r} \rangle = \mathbf{v}'\,.$$

Proper transformations of this system lead to following pair of iterative processes:

$$\mathbf{r}_{(i)} = \left\langle \mathbf{A}\left\langle \mathbf{A}'\mathbf{r}_{(i-1)} \right\rangle^{-1} \mathbf{v} \right\rangle^{-1} \mathbf{u}\,, \quad i = 1 \div I; \qquad \mathbf{s}_{(I)} = \left\langle \mathbf{A}'\mathbf{r}_{(I)} \right\rangle^{-1} \mathbf{v}\,; \tag{4}$$

$$\mathbf{s}_{(j)} = \left\langle \mathbf{A}'\left\langle \mathbf{A}\mathbf{s}_{(j-1)} \right\rangle^{-1} \mathbf{u} \right\rangle^{-1} \mathbf{v}\,, \quad j = 1 \div J; \qquad \mathbf{r}_{(J)} = \left\langle \mathbf{A}\mathbf{s}_{(J)} \right\rangle^{-1} \mathbf{u} \tag{5}$$

where i and j are iteration numbers, and the character " $\div$ " between the lower and upper bounds of index's changing range means that the index sequentially runs all integer values in the specified range.

As concerning a case of strict proportionality between row and column marginal totals $\mathbf{u} = k\mathbf{u}_{\mathbf{A}}$ and $\mathbf{v} = k\mathbf{v}_{\mathbf{A}}$, it can be easily shown that under starting condition $\mathbf{r}_{(0)} = \mathbf{e}_N$ or $\mathbf{s}_{(0)} = \mathbf{e}_M$ the RAS method iterative process (4) or (5) demonstrates one-step convergence to pair of vectors $\mathbf{r} = \mathbf{e}_N$, $\mathbf{s} = k\mathbf{e}_M$ or to $\mathbf{r} = k\mathbf{e}_N$, $\mathbf{s} = \mathbf{e}_M$, respectively. Hence, RAS algorithm's implementation gives $r_n s_m = k$ for any n and m, $n = 1 \div N$, $m = 1 \div M$, from which $\mathbf{X} = k\mathbf{A}$. Besides, it is easy to see that replacing the initial matrix $\mathbf{A}$ with its homothety $k\mathbf{A}$ leaves the RAS method iterations (4) and (5) invariant.

3. The proportionality case from the viewpoint of Kuroda's method

Kuroda (1988) proposed an original method for matrix updating that reduces to constrained minimization of the twofold-weighted quadratic objective function

$$\mathrm{f}_{\mathrm{K}}\left(\mathbf{x}_{\mathbf{u}}, \mathbf{x}_{\mathbf{v}}\right) = \frac{1}{2}\mathbf{x}'_{\mathbf{u}}\mathbf{W}_1\mathbf{x}_{\mathbf{u}} + \frac{1}{2}\mathbf{x}'_{\mathbf{v}}\mathbf{W}_2\mathbf{x}_{\mathbf{v}} \tag{6}$$

where $\mathbf{W}_1$ and $\mathbf{W}_2$ are the nonsingular diagonal matrices of order NM with the relative reliability or relative confidence factors (weights), $\mathbf{x}_{\mathbf{u}}$ and $\mathbf{x}_{\mathbf{v}}$ are NM-dimensional column vectors that are defined through applying the vectorization operator "vec", which transforms a matrix into a vector by stacking the columns of the matrix one underneath the other (see Magnus and Neudecker, 2007), as follows:

$$\mathbf{x}_{\mathbf{u}} = \mathrm{vec}\left(\hat{\mathbf{u}}^{-1}\mathbf{X} - \hat{\mathbf{u}}_{\mathbf{A}}^{-1}\mathbf{A}\right), \qquad \mathbf{x}_{\mathbf{v}} = \mathrm{vec}\left(\mathbf{X}\hat{\mathbf{v}}^{-1} - \mathbf{A}\hat{\mathbf{v}}_{\mathbf{A}}^{-1}\right).$$

By the way, the vectorization operator, if it applies to matrix $\mathbf{B}$ with dimensions $N \times M$ for obtaining NM-element column vector $\mathbf{b}$, can be represented as the column vector expansion

$$\mathbf{b} = \mathrm{vec}\,\mathbf{B} = \sum_{m=1}^{M} \left(\mathbf{e}_{M,m} \otimes \mathbf{E}_N\right)\mathbf{B}\mathbf{e}_{M,m} \tag{7}$$

where $\mathbf{E}_N$ is an identity matrix of order N, the character " $\otimes$ " denotes the Kronecker product for two matrices, and $\mathbf{e}_{M,m}$ is mth column vector from the natural basis for M-dimensional vector space, with mth element equal to 1 and zeros elsewhere. It can be shown that the corresponding inverse transformation is determined by matrix expansion

$$\mathbf{B} = \operatorname{vec}^{-1}\mathbf{b} = \sum_{m=1}^{M}\left(\mathbf{e}'_{M,m} \otimes \mathbf{E}_N\right)\mathbf{b}\,\mathbf{e}'_{M,m}.$$

To continue with the Kuroda's method, in the proportionality case the row and column marginal totals for target matrix are $\mathbf{u} = k\mathbf{u}_{\mathbf{A}}$ and $\mathbf{v} = k\mathbf{v}_{\mathbf{A}}$, hence

$$\hat{\mathbf{u}}^{-1}\mathbf{X} - \hat{\mathbf{u}}_{\mathbf{A}}^{-1}\mathbf{A} = \hat{\mathbf{u}}_{\mathbf{A}}^{-1}\left(\frac{1}{k}\mathbf{X} - \mathbf{A}\right), \qquad \mathbf{X}\hat{\mathbf{v}}^{-1} - \mathbf{A}\hat{\mathbf{v}}_{\mathbf{A}}^{-1} = \left(\frac{1}{k}\mathbf{X} - \mathbf{A}\right)\hat{\mathbf{v}}_{\mathbf{A}}^{-1}.$$

Thus, at $\mathbf{X} = k\mathbf{A}$ the vectors $\mathbf{x}_{\mathbf{u}}$ and $\mathbf{x}_{\mathbf{v}}$ vanish, and the quadratic function (6) reaches its absolute minimum value equal to zero. It means that from viewpoint of Kuroda's method, as well as RAS method, the matrix $\mathbf{X} = k\mathbf{A}$ provides the optimal solution for proportionality case of a general matrix updating problem.

4. The proportionality case once more: Kullback – Leibler divergence

The RAS method is fairly associated with a more general notion of conditional minimizing non-negative function called the Kullback – Leibler divergence that can be used for comparing "true" and "test" probability distributions (see Kullback and Leibler, 1951). Letting $a = \mathbf{e}'_N\mathbf{A}\mathbf{e}_M$ and $x = \mathbf{e}'_N\mathbf{X}\mathbf{e}_M$, we have the first distribution as $\mathbf{A}/a$, and the second one – as $\mathbf{X}/x$, or possibly vice versa, but with much more vague interpretation. So all elements of $\mathbf{A}$ and $\mathbf{X}$ are implied to be non-negative.

In these denotations the Kullback – Leibler divergence (sometimes called "information gain") has genuine representation as

$$\mathrm{f}_{\mathrm{KL}}(\mathbf{A}/a;\mathbf{X}/x) = \sum_{n=1}^{N}\sum_{m=1}^{M}\frac{a_{nm}}{a}\ln\left(\frac{x}{a}\frac{a_{nm}}{x_{nm}}\right) = \frac{1}{a}\mathrm{f}_{\mathrm{KL}}(\mathbf{A};\mathbf{X}) + \ln\frac{x}{a} \tag{8}$$

and inverse representation, with an opposite order of its arguments, as

$$\mathrm{f}_{\mathrm{KL}}(\mathbf{X}/x;\mathbf{A}/a) = \sum_{n=1}^{N}\sum_{m=1}^{M}\frac{x_{nm}}{x}\ln\left(\frac{a}{x}\frac{x_{nm}}{a_{nm}}\right) = \frac{1}{x}\mathrm{f}_{\mathrm{KL}}(\mathbf{X};\mathbf{A}) + \ln\frac{a}{x} \tag{9}$$

where $\mathrm{f}_{\mathrm{KL}}(\mathbf{A};\mathbf{X})$ and $\mathrm{f}_{\mathrm{KL}}(\mathbf{X};\mathbf{A})$ are corresponding Kullback – Leibler functions for non-normalized data.

Thus, the approach based on the Kullback – Leibler divergence comes to minimization of objective function (8) or (9) subject to linear constraints (1) under the consistency condition (2). It is easy to see that in the proportionality case with $\mathbf{X} = k\mathbf{A}$ the non-negative functions (8) and

(9) reach their absolute minimum (zero) values since

$$\mathrm{f}_{\mathrm{KL}}(\mathbf{A}/a\,;k\mathbf{A}/ka)=\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{a_{nm}}{a}\ln\left(\frac{ka}{a}\frac{a_{nm}}{ka_{nm}}\right)=\frac{a}{a}\ln(1)=0\,,$$

$$\mathrm{f}_{\mathrm{KL}}(k\mathbf{A}/ka\,;\mathbf{A}/a)=\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{ka_{nm}}{ka}\ln\left(\frac{a}{ka}\frac{ka_{nm}}{a_{nm}}\right)=\frac{ka}{ka}\ln(1)=0\,.$$

It means that from viewpoint of the Kullback – Leibler divergence approach, as well as RAS and Kuroda's methods, the matrix homothety $\mathbf{X}=k\mathbf{A}$ can be considered as optimal solution for proportionality case of a general matrix updating problem.

Notice, finally, that the function $\mathrm{f}_{\mathrm{KL}}(\mathbf{X};\mathbf{A})$ for non-normalized data from the inverse representation of Kullback – Leibler divergence (9) despite a certain shortcoming in its interpretation serves as an objective function in mathematical programming formulation of the RAS method, e.g., in Appendix 7.1 "RAS as a Solution to the Constrained Minimum Information Distance Problem" to Miller and Blair (2009). Moreover, it is to be emphasized that the Kullback – Leibler divergence is not a distance function really because the symmetry and triangle inequality conditions do not hold for it.

5. On measurement of structural similarity between initial and target matrices

Acceptance of the matrix $\mathbf{X}=k\mathbf{A}$ as optimal solution for proportionality case leads to establishing the fact that the matrices from homothetic family $k\mathbf{A}$ demonstrate an excellent structural similarity between each other. This conclusion can be helpful for refining a collection of matrix updating methods based on constrained minimization of the distance functions.

A quite common approach to define a measure for the structural similarity between initial and target matrices is to use some matrix norm for the difference $\|\mathbf{X}-\mathbf{A}\|$ to be minimized subject to linear constraints (1) under the consistency condition (one can find the proper reviews, e.g., in Miller and Blair, 2009 and Temurshoev et al., 2011), so that the optimal solution is $\mathbf{X}^*=\arg\min_{\mathbf{X}}\|\mathbf{X}-\mathbf{A}\|$. However, now we can set a goal to dispose the target matrix as close as possible not to initial matrix **A**, but to its homothetic family k**A**. As a result, the optimal solution becomes $(\mathbf{X}^*,k^*)=\arg\min_{\mathbf{X},k}\|\mathbf{X}-k\mathbf{A}\|$, and, clearly, it cannot be "worse" (in terms of the certain matrix norm chosen) than the original one.

The problem to minimize the distance between target matrix **X** and uniparametrical family k**A** is presented above in preliminary formulation. The further handling of this problem becomes more operational with a vectorization of matrices **A** and **X** by applying a vector expansion (7) for transforming them into NM-element column vectors, respectively, $\mathbf{a}=\mathrm{vec}\mathbf{A}$ and $\mathbf{x}=\mathrm{vec}\mathbf{X}$. It is fruitful to express the latter vector in form of multiplicative pattern $\mathbf{x} = \hat{\mathbf{a}}\mathbf{q}$ where $\mathbf{q}$ is *NM*-dimensional column vector of unknown relative coefficients. Note that if the initial matrix contains at least one zero element then diagonal matrix $\hat{\mathbf{a}}$ is singular.

In vector notation the transition from $\|\mathbf{x} - \mathbf{a}\|$ to $\|\mathbf{x} - k\mathbf{a}\|$ leads to an idea of orthogonal projecting an unknown target vector $\mathbf{x}$ onto the homothetic ray $k\mathbf{a}$ in *NM*-dimensional vector space with scalar product operation. To make this vector norm minimization problem independent on scale of initial data, it is expedient to introduce into consideration the relative distance function $\|\mathbf{q} - k\mathbf{e}_{NM}\|$ instead of $\|\mathbf{x} - k\mathbf{a}\|$, i.e., to consider the orthogonal projection of an unknown target vector $\mathbf{q}$ onto the relative homothetic ray $k\mathbf{e}_{NM}$.

6. Reducing a dimension of the target vector with zero elements

The multiplicative vector pattern $\mathbf{x} = \hat{\mathbf{a}}\mathbf{q}$ can be written in matrix notation as

$$\mathbf{X} = \mathbf{Q} \circ \mathbf{A} \tag{10}$$

where $\mathbf{Q}$ is $N \times M$ matrix of unknown coefficients q_{nm}, and the character "$\circ$" denotes the Hadamard's product for two matrices of the same dimensions. For example, in the RAS method $\mathbf{Q} = \mathbf{r}\mathbf{s}'$, as shown above. It is important to note here that model (10) provides in $\mathbf{X}$ (and in $\mathbf{x}$ too) the same location of zeros as in the initial matrix (vector).

It is easy to see that Hadamard-multiplicative model (10) is not strictly identifiable if the initial and target matrices do contain one or more zero elements. Indeed, if the matrices $\mathbf{A}$ and $\mathbf{X}$ are known both and $a_{nm} = x_{nm} = 0$ for some n and m then one cannot determine the coefficient q_{nm} unambiguously. That is why it is advisable to exclude all zero elements from the initial and target vectors in advance.

In this context, to formulate properly the problem of matrix updating on the criterion base presented above, one needs to convert left-hand sides of the constraints (1) into vector notation. It is easy to see that the $N \times NM$ matrix $\mathbf{G} = \mathbf{e}'_M \otimes \mathbf{E}_N$, which consists of M identity matrix $\mathbf{E}_N$ located horizontally, and the $M \times NM$ matrix $\mathbf{H} = \mathbf{E}_M \otimes \mathbf{e}'_{\mathrm{N}}$ – N-fold successive replication of each column from identity matrix $\mathbf{E}_M$ – are the proper substitutes of summation vectors $\mathbf{e}_M$ and $\mathbf{e}'_N$ in (1) respectively. Thus, the system of equations (1) and original *NM*-dimensional multiplicative pattern $\mathbf{x} = \hat{\mathbf{a}}\mathbf{q}$ can be combined as follows:

$$\mathbf{X}\mathbf{e}_M = \mathbf{G}\mathbf{x} = \mathbf{G}\hat{\mathbf{a}}\mathbf{q} = \mathbf{u}, \qquad \mathbf{X}'\mathbf{e}_N = \mathbf{H}\mathbf{x} = \mathbf{H}\hat{\mathbf{a}}\mathbf{q} = \mathbf{v}. \tag{11}$$

The procedure of eliminating all zero elements from the initial and target vectors can be implemented during simultaneous premultiplication of vectors $\mathbf{a}$, $\mathbf{x}$ and $\mathbf{q}$ by identity matrix of order NM which is transformed by deleting those rows that correspond to zero elements of **a**. Let $\underline{\mathbf{E}}_{NM}$ denote this rectangular matrix with dimensions $J \times NM$ where $J \leq NM$ is the number of nonzero elements in the initial matrix (and vector). Then the matrices **G** and **H** in (11) must be replaced by $N \times J$ matrix $\mathbf{G}\underline{\mathbf{E}}'_{NM}$ and $M \times J$ matrix $\mathbf{H}\underline{\mathbf{E}}'_{NM}$ respectively. For convenience, we will keep on the previous denotations **a**, **x**, **q**, **G** and **H** throughout the text below.

Thus, the operation of excluding zero elements allows reducing the dimension of solution space for the matrix updating problem in more or less degree. It is to be emphasized that in practice the macroeconomic matrices of high dimensions often appear to be very sparse, up to more than 90% of zero elements. In such cases, the efficiency of ensuing computations increases rather significantly.

Recall that under consistency condition (2) any $N+M-1$ among $N+M$ constraints (11) are mutually independent. Note also that each column of **G** and **H** includes exactly one nonzero (unit) element such that $\mathbf{e}'_N\mathbf{G} = \mathbf{e}'_M\mathbf{H} = \mathbf{e}'_J$.

7. Vector formulations of the Hadamard-multiplicative model

Harthoorn and van Dalen (1987) applied Hadamard-multiplicative model (10) in the problem of minimizing the quadratic function

$$\mathrm{f}_{\mathrm{HvD}}(\mathbf{q}) = (\mathbf{q} - \mathbf{e}_{NM})' \hat{\mathbf{a}} \hat{\mathbf{w}}^{-1} \hat{\mathbf{a}} (\mathbf{q} - \mathbf{e}_{NM}) \tag{12}$$

subject to linear constraints (11). Here $\hat{\mathbf{w}}$ is the known diagonal $NM \times NM$ matrix of relative confidence factors for the elements of initial vector $\hat{\mathbf{a}}$. Objective function (12) is being minimized and so determines the shortest (weighted) path from the point **q** to the point $\mathbf{e}_{NM}$.

With a goal to dispose the target vector **q** as close to the relative homothetic ray $k\mathbf{e}_J$ as possible, we can formulate the generalized problem of matrix updating in the solution space of reduced dimension as follows: to minimize the objective distance function $\mathrm{f}(\mathbf{q};k) = \|\mathbf{q} - k\mathbf{e}_J\|$ with a scalar parameter k subject to mutually dependent linear constraints (11). In practical case the generalized problem can be instantiated by means of certain choice of a vector norm, e.g., one from well-known family of Hölder norms with parameter p, etc.

At the choice of Euclidean norm (p = 2) which corresponds to Frobenius matrix norm (as in Harthoorn and van Dalen’s method), the uniparametrical objective function of matrix updating problem in the Euclidean solution space of reduced dimension in accordance with general least squares (GLS) principles becomes

$$\mathrm{f}(\mathbf{q};k) = (\mathbf{q} - k\mathbf{e}_J)' \mathbf{W} (\mathbf{q} - k\mathbf{e}_J) \tag{13}$$

where k is a scalar parameter unknown a priori, and $\mathbf{W} = \hat{\mathbf{w}}$ is a nonsingular diagonal matrix of order J with the relative reliability (relative confidence) factors for elements of vector $\mathbf{q}$. In terms of GLS vector $k\mathbf{e}_J$ can be interpreted as a mean of random vector $\mathbf{q}$, and $\mathbf{W}$ – as a inverse covariance matrix for $\mathbf{q}$ in case of zero autocorrelations. Usually vector $\mathbf{w}$ is assumed to be normalized by multiplying it on a proper factor, i.e., $\mathbf{e}'_J\mathbf{w} = 1$.

Objective function (13) is to be minimized subject to mutually dependent linear constraints (11); it determines the shortest weighted path from the solution point $\mathbf{q}$ to the relative homothetic ray $k\mathbf{e}_J$. This function is rather similar to (12) proposed by Harthoorn and van Dalen (1987). Nevertheless, there are at least two significant distinctions between them. First, Harthoorn and van Dalen do have used metric (not relative) measure based on vector $\mathbf{x} - \mathbf{a}$, and secondly, they have not used the operation of orthogonal projecting onto a homothetic ray.

8. Homothetic and angular measures for matrix similarity

Let $(\mathbf{y}, \mathbf{z}) = \mathbf{y}'\mathbf{W}\mathbf{z} = \mathbf{z}'\mathbf{W}\mathbf{y}$ be an scalar product of vectors $\mathbf{y}$ and $\mathbf{z}$ in J-dimensional weighted Euclidean space. Orthogonal projection of $\mathbf{q}$ on the ray $k\mathbf{e}_J$ is determined by coefficient $k^{\perp} = \mathbf{e}'_J\mathbf{W}\mathbf{q}/\mathbf{e}'_J\mathbf{W}\mathbf{e}_J$ from evident condition $\mathbf{e}'_J\mathbf{W}(\mathbf{q} - k^{\perp}\mathbf{e}_J) = 0$ and equals vector $k^{\perp}\mathbf{e}_J$. Hence, the shortest path from the point $\mathbf{q}$ to the ray $k\mathbf{e}_J$ is lying along the vector

$$\boldsymbol{\delta} = \mathbf{q} - k^{\perp}\mathbf{e}_J = \left(\mathbf{E}_J - \frac{\mathbf{e}_J\mathbf{e}'_J\mathbf{W}}{\mathbf{e}'_J\mathbf{W}\mathbf{e}_J}\right)\mathbf{q}\,. \tag{14}$$

Note that the scalar product of vectors $\boldsymbol{\delta}$ and $\mathbf{e}_J$ equals $\mathbf{e}'_J\mathbf{W}\boldsymbol{\delta} = \mathbf{e}'_J\mathbf{W}\mathbf{q} - \mathbf{e}'_J\mathbf{W}\mathbf{q} = 0$ so $\boldsymbol{\delta}$ is indeed orthogonal to homothetic ray $k\mathbf{e}_J$ and besides has zero weighted sum of all elements. The shortest path from the point $\mathbf{q}$ to the ray $k\mathbf{e}_J$ can serve as a measure for deviation of relative target vector $\mathbf{q}$ from homothetic ray $k\mathbf{e}_J$ that is further called homothetic measure of structural similarity between initial and target matrices.

It can be shown that symmetric idempotent matrix in parentheses in (14) has zero eigenvalue with unit multiplicity and corresponding eigenvector $\mathbf{e}_J$, and also has unit eigenvalue with multiplicity J–1 and corresponding eigenvector $\mathbf{z}$ from the hyperplane $\mathbf{e}'_J\mathbf{W}\mathbf{z} = 0$, which is orthogonal to homothetic ray. Therefore, this singular matrix has rank J–1.

Nevertheless, the most natural measure for similarity between a vector and a ray can be defined as a value of the angle $\beta_{\mathbf{qe}}$ between $\mathbf{q}$ and $k\mathbf{e}_J$ at $k \geq 0$, which is assumed to be acute. It is easy to detect a linkage between angular and homothetic measures for matrix similarity because a solution of the right triangle with the sides $\mathbf{q}, k^{\perp}\mathbf{e}_J$ and $\boldsymbol{\delta}$ gives

$$\sin^2 \beta_{\mathbf{qe}} = \frac{\boldsymbol{\delta}'\mathbf{W}\boldsymbol{\delta}}{\mathbf{q}'\mathbf{W}\mathbf{q}}. \tag{15}$$

From geometrical viewpoint, one can establish that angular measure (15) and homothetic measure based on (14) are consistent only for any pair of relative target vectors **q** and **p** satisfying orthogonality condition $\mathbf{e}'_J\mathbf{W}(\mathbf{q}-\mathbf{p})=0$, i.e., all testing target vectors must have the same orthogonal projection onto homothetic ray.

As a conclusion, an angle between target vector **q** and homothetic ray $k\mathbf{e}_J$ at $k \geq 0$ can be considered as a universal measure of structural similarity between target and initial matrices. Main "technical" disadvantage of angular measure appears to be its nonlinearity along with arising difficulties of using (15) to construct competing (in particular, with Harthoorn and van Dalen's method) algorithms of matrix updating. Based on orthogonal projecting operation and associated with length of vector (14), homothetic measure is the simplified version of an angular measure with some shortcomings. Nevertheless, homothetic measure demonstrates a row of helpful properties and may become operational in various algorithmic schemes.

9. Minimization of angular measure via homothetic measure

The expressions (14) and (15) in conjunction with monotonicity of function $\sin^2 x$ at acute angles x generates the following nonlinear programming problem: to minimize the fractional quadratic objective function, or, as it is sometimes called, Rayleigh quotient

$$\mathrm{F}(\mathbf{q}) = \frac{1}{\mathbf{q}'\mathbf{W}\mathbf{q}}\mathbf{q}'\left(\mathbf{E}_J - \frac{\mathbf{e}_J\mathbf{e}'_J\mathbf{W}}{\mathbf{e}'_J\mathbf{W}\mathbf{e}_J}\right)\mathbf{q} = \frac{\mathrm{f}(\mathbf{q})}{\mathbf{q}'\mathbf{W}\mathbf{q}} \tag{16}$$

subject to mutually dependent linear constraints (11). Note that angular measure (16) has the same value F(**q**) along a straight line $k\mathbf{q}$ at any $k \neq 0$. Recall that symmetric idempotent matrix in parentheses has rank J–1. Singularity of this matrix serves as an obvious technical obstacle for the analytical solving of constrained minimization problems (16), (11), but this obstacle can be bypassed in a special way proposed below.

It can be shown that nonlinear programming problem (16), (11) with auxiliary constraint $\mathbf{e}'_J\mathbf{W}\mathbf{q} = k \cdot \mathbf{e}'_J\mathbf{W}\mathbf{e}_J$ (where k is assumed to be an arbitrary constant) is equivalent to quadratic optimization problem that prescribes to minimize uniparametrical objective function (13) subject to constraints (11) and the orthogonality condition

$$\mathbf{e}'_J\mathbf{W}(\mathbf{q} - k\mathbf{e}_J) = 0, \tag{17}$$

in which k is playing the role of an instrumental variable. Clearly, the solution point **q** for this quadratic optimization problem is lying on the hyperplane (17), which is orthogonal to homothetic ray and crosses it at the point $k\mathbf{e}_J$. As established earlier, angular measure (15) and homothetic measure based on (14) are consistent on this orthogonal hyperplane.

Thus, the solution of nonlinear programming problem (16), (11) can be obtained in two stages. At first stage one needs to solve quadratic optimization problem (13), (11), (17) for every k and to find uniparametrical vector family $\mathbf{q}_*(k)$ that provides a local constrained minimum of homothetic measure

$$\mathrm{f}_*(k) = \min_{\mathbf{q}} \left\{ \mathrm{f}\,(\mathbf{q}; k) \;\middle|\; \mathbf{G\hat{a}q} = \mathbf{u},\;\; \mathbf{H\hat{a}q} = \mathbf{v},\;\; \mathbf{e}'_J\mathbf{W}(\mathbf{q} - k\mathbf{e}_J) = 0 \right\} \tag{18}$$

on each hyperplane (17). As a result, we obtain a geometric place of feasible points located at a minimal distance from points $k\mathbf{e}_J$ on homothetic ray at various values of parameter k.

At second stage the unconstrained minimum

$$\mathrm{F}^* = \min_k \left\{ \mathrm{F}(k) = \frac{\mathrm{f}_*(k)}{\mathbf{q}'_*(k)\,\mathbf{W}\mathbf{q}_*(k)} \right\} \tag{19}$$

is to be found together with corresponding vector $\mathbf{q}^*$ as the optimal solution of angular measure minimization problem (16), (11). Besides, the other unconstrained minimum

$$\mathrm{f}_* = \min_k \left\{ \mathrm{f}_*(k) \right\} \tag{20}$$

corresponds to global minimization of homothetic measure along homothetic ray.

10. Uniparametrical constrained minimization of homothetic measure

The Lagrangean function for problem to minimize quadratic objective function (13) with unknown scalar parameter k subject to linear constraints (11) and (17) is

$$\mathrm{L_f}\,(\mathbf{q}; k; \boldsymbol{\lambda}, \boldsymbol{\mu}, \gamma) = (\mathbf{q} - k\mathbf{e}_J)'\mathbf{W}(\mathbf{q} - k\mathbf{e}_J) - \boldsymbol{\lambda}'(\mathbf{G\hat{a}q} - \mathbf{u}) - \boldsymbol{\mu}'(\mathbf{H\hat{a}q} - \mathbf{v}) - \gamma(\mathbf{e}'_J\mathbf{Wq} - k\mathbf{e}'_J\mathbf{We}_J) \tag{21}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are vectors of Lagrange multipliers with dimensions N×1 and M×1, and γ is a scalar Lagrange multiplier. By setting the partial derivatives of (21) with respect to $\mathbf{q}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, γ equal to zero, we obtain the system of J+N+M+1 linear equations

$$2\mathbf{W}(\mathbf{q} - k\mathbf{e}_J) - \mathbf{\hat{a}G}'\boldsymbol{\lambda} - \mathbf{\hat{a}H}'\boldsymbol{\mu} - \gamma\mathbf{We}_J = \mathbf{0}_J\,, \quad \mathbf{G\hat{a}q} - \mathbf{u} = \mathbf{0}_N\,, \quad \mathbf{H\hat{a}q} - \mathbf{v} = \mathbf{0}_M\,, \quad \mathbf{e}'_J\mathbf{Wq} - k\mathbf{e}'_J\mathbf{We}_J = 0\,.$$

While $\mathbf{W}$ is nonsingular matrix, the first equation can be resolved with respect to $\mathbf{q}$ as

$$\mathbf{q} = k\mathbf{e}_J + \frac{1}{2}\mathbf{W}^{-1}\mathbf{\hat{a}}\left(\mathbf{G}'\boldsymbol{\lambda} + \mathbf{H}'\boldsymbol{\mu}\right) + \frac{1}{2}\gamma\mathbf{e}_J.$$

Putting this expression into fourth equation gives

$$\gamma\mathbf{e}_J = -\frac{\mathbf{e}_J\mathbf{a}'}{\mathbf{e}'_J\mathbf{We}_J}\left(\mathbf{G}'\boldsymbol{\lambda} + \mathbf{H}'\boldsymbol{\mu}\right),$$

and after backward substitution we obtain

$$\mathbf{q} = k\mathbf{e}_J + \frac{1}{2}\left(\mathbf{W}^{-1}\mathbf{\hat{a}} - \frac{\mathbf{e}_J\mathbf{a}'}{\mathbf{e}'_J\mathbf{We}_J}\right)\left(\mathbf{G}'\boldsymbol{\lambda} + \mathbf{H}'\boldsymbol{\mu}\right). \tag{22}$$

The second and the third equations from system above and (22) can be combined into *N*+*M* equations with Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ as unknown variables:

$$\left(\mathbf{G}\hat{\mathbf{a}}\mathbf{W}^{-1}\hat{\mathbf{a}}\mathbf{G}' - \frac{\mathbf{G}\mathbf{a}\mathbf{a}'\mathbf{G}'}{\mathbf{e}_J'\mathbf{W}\mathbf{e}_J}\right)\boldsymbol{\lambda} + \left(\mathbf{G}\hat{\mathbf{a}}\mathbf{W}^{-1}\hat{\mathbf{a}}\mathbf{H}' - \frac{\mathbf{G}\mathbf{a}\mathbf{a}'\mathbf{H}'}{\mathbf{e}_J'\mathbf{W}\mathbf{e}_J}\right)\boldsymbol{\mu} = \boldsymbol{\pi}_{11}\boldsymbol{\lambda} + \boldsymbol{\pi}_{12}\boldsymbol{\mu} = 2(\mathbf{u} - k\mathbf{G}\mathbf{a}), \quad (23)$$

$$\left(\mathbf{H}\hat{\mathbf{a}}\mathbf{W}^{-1}\hat{\mathbf{a}}\mathbf{G}' - \frac{\mathbf{H}\mathbf{a}\mathbf{a}'\mathbf{G}'}{\mathbf{e}_J'\mathbf{W}\mathbf{e}_J}\right)\boldsymbol{\lambda} + \left(\mathbf{H}\hat{\mathbf{a}}\mathbf{W}^{-1}\hat{\mathbf{a}}\mathbf{H}' - \frac{\mathbf{H}\mathbf{a}\mathbf{a}'\mathbf{H}'}{\mathbf{e}_J'\mathbf{W}\mathbf{e}_J}\right)\boldsymbol{\mu} = \boldsymbol{\pi}_{21}\boldsymbol{\lambda} + \boldsymbol{\pi}_{22}\boldsymbol{\mu} = 2(\mathbf{v} - k\mathbf{H}\mathbf{a}). \quad (24)$$

It can be shown that $\boldsymbol{\pi}_{11}\mathbf{e}_N - \boldsymbol{\pi}_{12}\mathbf{e}_M = \mathbf{0}_N$, $\boldsymbol{\pi}_{21}\mathbf{e}_N - \boldsymbol{\pi}_{22}\mathbf{e}_M = \mathbf{0}_M$, i.e., the columns of symmetric matrix $\boldsymbol{\pi}$, which is formed by blocks $\boldsymbol{\pi}_{11}, \boldsymbol{\pi}_{12}, \boldsymbol{\pi}_{21}, \boldsymbol{\pi}_{22}$, are linearly dependent, so matrix $\boldsymbol{\pi}$ is singular. Thus, the general solution to corresponding homogeneous system (23), (24) is $\boldsymbol{\lambda}^{(0)} = c\mathbf{e}_N$, $\boldsymbol{\mu}^{(0)} = -c\mathbf{e}_M$ with the same scalar constant *c*.

Since general solution to nonhomogeneous linear system equals the sum of general solution to corresponding homogeneous system and any particular solution to nonhomogeneous system, let $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(0)} + \boldsymbol{\lambda}^{(1)}$ and $\boldsymbol{\mu} = \boldsymbol{\mu}^{(0)} + \boldsymbol{\mu}^{(1)}$, where $\boldsymbol{\lambda}^{(1)}$, $\boldsymbol{\mu}^{(1)}$ is particular solution to system (23), (24). Recall that $\mathbf{e}_N'\mathbf{G} = \mathbf{e}_M'\mathbf{H} = \mathbf{e}_J'$, so putting these formulas into round-bracketed vector expression in the right-hand side of (22) gives

$$\mathbf{G}'\boldsymbol{\lambda} + \mathbf{H}'\boldsymbol{\mu} = \mathbf{G}'\left(c\mathbf{e}_N + \boldsymbol{\lambda}^{(1)}\right) + \mathbf{H}'\left(-c\mathbf{e}_M + \boldsymbol{\mu}^{(1)}\right) = c\mathbf{e}_J - c\mathbf{e}_J + \mathbf{G}'\boldsymbol{\lambda}^{(1)} + \mathbf{H}'\boldsymbol{\mu}^{(1)} = \mathbf{G}'\boldsymbol{\lambda}^{(1)} + \mathbf{H}'\boldsymbol{\mu}^{(1)}.$$

Therefore, to find any particular solution of system (23), (24) means to solve uniparametrical constrained minimization problem with quadratic objective function (13) subject to linear constraints (11) and (17). Notice that *N*+*M*–1 among *N*+*M* equations of system (23), (24) – as well as constraints of system (11) – are mutually independent under consistency condition (2).

11. Analytical solutions for Lagrange multipliers

The Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ can be found from system (23), (24) in two ways.

Since any *N*+*M*–1 among *N*+*M* equations (23), (24) are mutually independent, without loss of generality any one of them can be eliminated from the system together with the corresponding unknown Lagrange multiplier. The reduced system can be solved in standard manner using well-known formulas for the inverse of a partitioned matrix (for details, see Miller and Blair, 2009, Appendix A).

Another way is based on the “easy-to-check” fact that, although matrix $\boldsymbol{\pi}$ is singular, but its square blocks $\boldsymbol{\pi}_{11}$ and $\boldsymbol{\pi}_{22}$ are not if initial matrix **A** does not have any zero rows and columns. So one can construct a pair of iterative processes resembling (4) and (5) and then solve the system (23), (24) by numerical methods.

Anyway, after huge algebraic transformations we get

$$\boldsymbol{\lambda}(k) = 2\mathbf{z}_{\boldsymbol{\lambda}} + 2k\mathbf{y}_{\boldsymbol{\lambda}}\,, \qquad \boldsymbol{\mu}(k) = 2\mathbf{z}_{\boldsymbol{\mu}} + 2k\mathbf{y}_{\boldsymbol{\mu}}\,,$$

from which

$$\mathbf{G}'\boldsymbol{\lambda}(k) + \mathbf{H}'\boldsymbol{\mu}(k) = 2\left(\mathbf{G}'\mathbf{z}_{\boldsymbol{\lambda}} + \mathbf{H}'\mathbf{z}_{\boldsymbol{\mu}}\right) + 2k\left(\mathbf{G}'\mathbf{y}_{\boldsymbol{\lambda}} + \mathbf{H}'\mathbf{y}_{\boldsymbol{\mu}}\right) = 2\mathbf{z} + 2k\mathbf{y} \tag{25}$$

where $\mathbf{z}_{\boldsymbol{\lambda}}, \mathbf{y}_{\boldsymbol{\lambda}}$ and $\mathbf{z}_{\boldsymbol{\mu}}, \mathbf{y}_{\boldsymbol{\mu}}$ are column vectors computable from (23), (24) with dimensions $N\times1$ and $M\times1$, respectively. It is important to note that J-dimensional column vector $\mathbf{y}$ does not depend on $\mathbf{u}$ and $\mathbf{v}$ (marginal totals for target matrix) in contrast to vector $\mathbf{z}$ of dimensions $J\times1$.

Now a general solution of constrained minimization problem for homothetic measure (13), (11), (17) can be derived by putting (25) into (22). Thus, we have the uniparametrical vector family

$$\mathbf{q}_*(k) = k\mathbf{e}_J + \left(\mathbf{W}^{-1}\hat{\mathbf{a}} - \frac{\mathbf{e}_J\mathbf{a}'}{\mathbf{e}_J'\mathbf{W}\mathbf{e}_J}\right)(\mathbf{z} + k\mathbf{y}) = k\mathbf{e}_J + \mathbf{D}(\mathbf{z} + k\mathbf{y}) \tag{26}$$

which is obtained in accordance with a requirement formulated in (18). Note that $\mathbf{D}$ is a square matrix of order J. Along with (13) and (26) the scalar function (18) becomes polynomial of second order as

$$\mathrm{f}_*(k) = (\mathbf{z} + k\mathbf{y})'\mathbf{D}'\mathbf{W}\mathbf{D}(\mathbf{z} + k\mathbf{y}) = \mathbf{z}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z} + 2k\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z} + k^2\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y}\,. \tag{27}$$

These analytical results complete a first stage of solving process for nonlinear programming problem (16), (11).

12. The global minimum of homothetic measure

Formula (26) describes a geometric place of feasible points $\mathbf{q}_*(k)$ located at a minimal distance from points $k\mathbf{e}_J$ on homothetic ray at various values of parameter k. It is important to emphasize that vector $\mathbf{q}_*(k) - k\mathbf{e}_J = \mathbf{D}(\mathbf{z} + k\mathbf{y})$ is orthogonal to homothetic ray at any k according to model construction because of the orthogonality condition (17) among constraints. Indeed,

$$\mathbf{e}_J'\mathbf{W}\mathbf{D} = \mathbf{e}_J'\mathbf{W}\left(\mathbf{W}^{-1}\hat{\mathbf{a}} - \frac{\mathbf{e}_J\mathbf{a}'}{\mathbf{e}_J'\mathbf{W}\mathbf{e}_J}\right) = \mathbf{a}' - \mathbf{a}' = \mathbf{0}_J' \tag{28}$$

so that $\mathbf{e}_J'\mathbf{W}\left[\mathbf{q}_*(k) - k\mathbf{e}_J\right] = \mathbf{e}_J'\mathbf{W}\mathbf{D}(\mathbf{z} + k\mathbf{y}) = 0$ at any k.

The first and the second derivatives of quadratic function (27) with respect to k are determined as

$$\frac{\mathrm{d}\mathrm{f}_*(k)}{\mathrm{d}k} = 2\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z} + 2k\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y}\,, \qquad \frac{\mathrm{d}^2\mathrm{f}_*(k)}{\mathrm{d}k^2} = 2\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y} \geq 0\,.$$

Clearly, this convex function of k has a unique minimum at the parameter value that equals $k_* = -\mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{z} / \mathbf{y}'\mathbf{D}'\mathbf{W}\mathbf{D}\mathbf{y}$ (here the first derivative vanishes).

As a result, the global minimum of homothetic measure along homothetic ray

$$\mathrm{f}_* = \min_k \min_{\mathbf{q}} \left\{ \mathrm{f}\,(\mathbf{q}; k) \;\middle|\; \mathbf{G}\hat{\mathbf{a}}\mathbf{q} = \mathbf{u},\;\; \mathbf{H}\hat{\mathbf{a}}\mathbf{q} = \mathbf{v},\;\; \mathbf{e}'_J \mathbf{W}(\mathbf{q} - k\mathbf{e}_J) = 0 \right\},$$

which corresponds to a requirement formulated in (20), is reached according to (26) at the point

$$\mathbf{q}_* = \mathbf{Dz} + k_*(\mathbf{e}_J + \mathbf{Dy}) = \mathbf{Dz} - \frac{\mathbf{y}'\mathbf{D}'\mathbf{WDz}}{\mathbf{y}'\mathbf{D}'\mathbf{WDy}}(\mathbf{e}_J + \mathbf{Dy}) \tag{29}$$

with the objective function value

$$\mathrm{f}_* = \mathbf{z}'\mathbf{D}'\mathbf{WDz} - 2\frac{(\mathbf{y}'\mathbf{D}'\mathbf{WDz})^2}{\mathbf{y}'\mathbf{D}'\mathbf{WDy}} + \frac{(\mathbf{y}'\mathbf{D}'\mathbf{WDz})^2}{\mathbf{y}'\mathbf{D}'\mathbf{WDy}} = \mathbf{z}'\mathbf{D}'\mathbf{WDz} - \frac{(\mathbf{y}'\mathbf{D}'\mathbf{WDz})^2}{\mathbf{y}'\mathbf{D}'\mathbf{WDy}} \tag{30}$$

where **z** and **y** are *J*-dimensional column vectors defined in (25). It can be shown that formulas (29) and (30) also define the solution of a quadratic programming problem, in which it is required to minimize the objective function (13) subject to linear constraints (11), and unknown parameter k is determined from a statement that vector $\mathbf{q}_*(k) - k\mathbf{e}_J$ is to be orthogonal to homothetic ray.

12. The solution of unconstrained minimization problem for angular measure

At second stage of solving process for nonlinear programming problem (16), (11) we have another objective function

$$\mathrm{F}(k) = \frac{\mathrm{f}_*(k)}{\mathbf{q}'_*(k)\mathbf{W}\,\mathbf{q}_*(k)} = \frac{(\mathbf{z} + k\mathbf{y})'\mathbf{D}'\mathbf{WD}(\mathbf{z} + k\mathbf{y})}{[k\mathbf{e}_J + \mathbf{D}(\mathbf{z} + k\mathbf{y})]'\mathbf{W}[k\mathbf{e}_J + \mathbf{D}(\mathbf{z} + k\mathbf{y})]} \tag{31}$$

that is obtained by substituting (27) and (26) into the function in braces from (19). In accordance with a requirement formulated in (19) it is to be minimized. However, this function of k is fractional quadratic and, hence, can have more than one minimum.

Putting (28) into the denominator of (31) gives

$$[k\mathbf{e}_J + \mathbf{D}(\mathbf{z} + k\mathbf{y})]'\mathbf{W}[k\mathbf{e}_J + \mathbf{D}(\mathbf{z} + k\mathbf{y})] = k^2 \cdot \mathbf{e}'_J\mathbf{W}\mathbf{e}_J + (\mathbf{z} + k\mathbf{y})'\mathbf{D}'\mathbf{WD}(\mathbf{z} + k\mathbf{y})$$

so $\mathrm{F}(k) \le 1$ at any values of k. Note that the maximum of $\mathrm{F}(k)$ equals 1 and is achieved at zero value of k.

It can be shown that first derivative of the fractional quadratic function (31) with respect to k is determined as

$$\frac{\mathrm{dF}(k)}{\mathrm{d}k} = \frac{-2k \cdot \mathbf{e}'_J\mathbf{W}\mathbf{e}_J \cdot (\mathbf{z}'\mathbf{D}'\mathbf{WDz} + k \cdot \mathbf{z}'\mathbf{D}'\mathbf{WDy})}{\left[k^2 \cdot \mathbf{e}'_J\mathbf{W}\mathbf{e}_J + (\mathbf{z} + k\mathbf{y})'\mathbf{D}'\mathbf{WD}(\mathbf{z} + k\mathbf{y})\right]^2}.$$

It seems clear that function (31) has a unique maximum at $k = 0$ and a unique minimum at $k^* = -\mathbf{z}'\mathbf{D}'\mathbf{WDz}/\mathbf{z}'\mathbf{D}'\mathbf{WDy} = -\mathbf{z}'\mathbf{D}'\mathbf{WDz}/\mathbf{y}'\mathbf{D}'\mathbf{WDz}$. The first case concerns an orthogonality of the relative target vector to homothetic ray whereas the second one is associated with a minimal angle between the relative target vector and homothetic ray.

As a result, the global minimum of angular measure $\mathrm{F}^* = \min_k \{ \mathrm{F}(k) \}$, which corresponds to a requirement formulated in (19), is reached at the point

$$\mathbf{q}^* = \mathbf{Dz} + k^*(\mathbf{e}_J + \mathbf{Dy}) = \mathbf{Dz} - \frac{\mathbf{z}'\mathbf{D}'\mathbf{WDz}}{\mathbf{y}'\mathbf{D}'\mathbf{WDz}}(\mathbf{e}_J + \mathbf{Dy}) \tag{32}$$

with the objective function value

$$\mathrm{F}^* = \frac{\left(\mathbf{z}' - \frac{\mathbf{z}'\mathbf{D}'\mathbf{WDz}}{\mathbf{y}'\mathbf{D}'\mathbf{WDz}}\mathbf{y}'\right)\mathbf{D}'\mathbf{WD}\left(\mathbf{z} - \frac{\mathbf{z}'\mathbf{D}'\mathbf{WDz}}{\mathbf{y}'\mathbf{D}'\mathbf{WDz}}\mathbf{y}\right)}{\left(\frac{\mathbf{z}'\mathbf{D}'\mathbf{WDz}}{\mathbf{y}'\mathbf{D}'\mathbf{WDz}}\right)^2 \mathbf{e}'_J\mathbf{W}\mathbf{e}_J + \left(\mathbf{z}' - \frac{\mathbf{z}'\mathbf{D}'\mathbf{WDz}}{\mathbf{y}'\mathbf{D}'\mathbf{WDz}}\mathbf{y}'\right)\mathbf{D}'\mathbf{WD}\left(\mathbf{z} - \frac{\mathbf{z}'\mathbf{D}'\mathbf{WDz}}{\mathbf{y}'\mathbf{D}'\mathbf{WDz}}\mathbf{y}\right)} \tag{33}$$

where **z** and **y** are *J*-dimensional column vectors defined in (25). Thus, formulae (32) and (33) define the solution of a nonlinear programming problem in which it is required to minimize the objective function (16) subject to linear constraints (11).

13. Notes on sensitivity analysis in minimization problem for homothetic measure

Applying a technique of partial derivatives for sensitivity analysis in minimization problem for homothetic measure represents a quite complicated task because of mutually dependent linear constraints (11). In such cases the Lagrange multipliers can not be identified unambiguously. Any disturbance of marginal total vector **u** through the frame of consistency condition (2) generates some compensating changes in the elements of **v**, and vice versa. Clearly, some disturbances lead to an increasing of constrained minimum for objective function (13), while others contribute to decrease it.

Along with vector-valued linear function $\mathbf{G}'\boldsymbol{\lambda} + \mathbf{H}'\boldsymbol{\mu}$, $N\times M$-dimensional matrix

$$\mathbf{L} = \boldsymbol{\lambda}\mathbf{e}'_M + \mathbf{e}_N\boldsymbol{\mu}' = \boldsymbol{\lambda}\mathbf{e}'_M + \mathbf{e}_N\boldsymbol{\mu}' + k\mathbf{e}_N\mathbf{e}'_M - k\mathbf{e}_N\mathbf{e}'_M = (\boldsymbol{\lambda} + k\mathbf{e}_N)\mathbf{e}'_M + \mathbf{e}_N(\boldsymbol{\mu}' - k\mathbf{e}'_M)$$

is invariant under any change of parameter *k*. It is easy to see that its element l_{nm} can be considered as a coefficient of the constrained minimum's sensitivity under impact of the simultaneous increasing u_n and v_m by the same small value ε. Thus, to decrease the minimum $\mathrm{f}_*(\mathbf{u},\mathbf{v}) = \mathrm{f}_*(k)$, a small scalar ε is to be chosen with the sign reversed from the sign of l_{nm}.

In this context the larger absolute values of matrix **L**'s elements are of great interest. Let l_{nm} be an element with the largest absolute value of any one in matrix **L**. Then the best strategy for a local enhancing of constrained minimum is to disturb u_n and v_m by the same small value $-\varepsilon \cdot \mathrm{sgn}(l_{nm})$ where $\varepsilon > 0$ and $\mathrm{sgn}(\cdot)$ is a signum function.

Further, let $l_+ > 0$ and $l_- < 0$ be a maximal and a minimal elements of **L** respectively. Then the best two-component strategy for a local enhancing of constrained minimum at the fixed grand total $x=\mathbf{e}_N'\mathbf{u}=\mathbf{e}_M'\mathbf{v}$ is to decrease the elements of **u** and **v** corresponding to l_+ by –ε and to increase the elements of **u** and **v** corresponding to l_- by ε simultaneously.

In general, total sensitivity effect is formulated as

$$\Delta_{\mathrm{f}}(\boldsymbol{\Delta}_{\mathbf{u}},\boldsymbol{\Delta}_{\mathbf{v}})=\mathrm{f}_*(\mathbf{u}+\boldsymbol{\Delta}_{\mathbf{u}},\mathbf{v}+\boldsymbol{\Delta}_{\mathbf{v}})-\mathrm{f}_*(\mathbf{u},\mathbf{v})=\boldsymbol{\Delta}_{\mathbf{u}}'\boldsymbol{\lambda}+\boldsymbol{\mu}'\boldsymbol{\Delta}_{\mathbf{v}}\,, \tag{34}$$

where vectors $\boldsymbol{\Delta}_{\mathbf{u}}$ and $\boldsymbol{\Delta}_{\mathbf{v}}$ are exogenous disturbances for **u** and **v** respectively satisfying the consistency condition $\mathbf{e}_N'\boldsymbol{\Delta}_{\mathbf{u}}=\mathbf{e}_M'\boldsymbol{\Delta}_{\mathbf{v}}=x_{\boldsymbol{\Delta}}$. To express the right-hand side of (43) in matrix **L** terms it is necessary to consider two cases, namely, $x_{\boldsymbol{\Delta}}=0$ and $x_{\boldsymbol{\Delta}}\neq 0$.

The disturbances $\boldsymbol{\Delta}_{\mathbf{u}}$ and $\boldsymbol{\Delta}_{\mathbf{v}}$ with zero sums $\mathbf{e}_N'\boldsymbol{\Delta}_{\mathbf{u}}=\mathbf{e}_M'\boldsymbol{\Delta}_{\mathbf{v}}=x_{\boldsymbol{\Delta}}=0$ play an important role in statistical practice. They entail the redistributions of **u**’s and **v**’s components while the grand total $x+x_{\boldsymbol{\Delta}}$ is being fixed. It is easy to see from (34) that the total redistribution effect depends on the marginal totals of matrix **L** and is estimated by

$$\Delta_{\mathrm{f}}(\boldsymbol{\Delta}_{\mathbf{u}},\boldsymbol{\Delta}_{\mathbf{v}}|\,x_{\boldsymbol{\Delta}}=0)=\boldsymbol{\Delta}_{\mathbf{u}}'(\boldsymbol{\lambda}+\bar{\mu}\mathbf{e}_N)+(\boldsymbol{\mu}+\bar{\lambda}\mathbf{e}_M)'\boldsymbol{\Delta}_{\mathbf{v}}=\frac{1}{M}\boldsymbol{\Delta}_{\mathbf{u}}'\mathbf{L}\mathbf{e}_M+\frac{1}{N}\mathbf{e}_N'\mathbf{L}\boldsymbol{\Delta}_{\mathbf{v}} \tag{35}$$

where $\bar{\lambda}$ and $\bar{\mu}$ are average values of the Lagrange multipliers. Here the first summand implies that in the total effect calculation an each value $(\boldsymbol{\Delta}_{\mathbf{u}})_n$ is uniformly distributed among *M* components of $\boldsymbol{\Delta}_{\mathbf{v}}$ and generates *M* elementary effects, sum of which is proportional to a row marginal total *n* for **L** divided by *M*. By analogy, the second summand in (35) implies that an each value $(\boldsymbol{\Delta}_{\mathbf{v}})_m$ is uniformly distributed among *N* components of $\boldsymbol{\Delta}_{\mathbf{u}}$ and generates *N* simple effects, sum of which is proportional to a column marginal total *m* for **L** divided by *N*.

On the other hand, the bilinear function of disturbances $\boldsymbol{\Delta}_{\mathbf{u}}'\mathbf{L}\boldsymbol{\Delta}_{\mathbf{v}}$ can be transformed as follows:

$$\boldsymbol{\Delta}_{\mathbf{u}}'\mathbf{L}\boldsymbol{\Delta}_{\mathbf{v}}=\boldsymbol{\Delta}_{\mathbf{u}}'(\boldsymbol{\lambda}\mathbf{e}_M'+\mathbf{e}_N\boldsymbol{\mu}')\boldsymbol{\Delta}_{\mathbf{v}}=\boldsymbol{\Delta}_{\mathbf{u}}'\boldsymbol{\lambda}(\mathbf{e}_M'\boldsymbol{\Delta}_{\mathbf{v}})+(\boldsymbol{\Delta}_{\mathbf{u}}'\mathbf{e}_N)\boldsymbol{\mu}'\boldsymbol{\Delta}_{\mathbf{v}}=x_{\boldsymbol{\Delta}}(\boldsymbol{\Delta}_{\mathbf{u}}'\boldsymbol{\lambda}+\boldsymbol{\mu}'\boldsymbol{\Delta}_{\mathbf{v}})\,.$$

Hence, the total sensitivity effect can be represented as

$$\Delta_{\mathrm{f}}(\boldsymbol{\Delta}_{\mathbf{u}},\boldsymbol{\Delta}_{\mathbf{v}}|\,x_{\boldsymbol{\Delta}}\neq 0)=\boldsymbol{\Delta}_{\mathbf{u}}'\boldsymbol{\lambda}+\boldsymbol{\mu}'\boldsymbol{\Delta}_{\mathbf{v}}=\frac{1}{x_{\boldsymbol{\Delta}}}\boldsymbol{\Delta}_{\mathbf{u}}'\mathbf{L}\boldsymbol{\Delta}_{\mathbf{v}} \tag{36}$$

where the disturbance grand total $x_{\boldsymbol{\Delta}}$ is assumed to be nonzero. Recall, that in contrast to (36) formula (35) is well defined only for the redistribution case $x_{\boldsymbol{\Delta}}=0$.

13. Numerical examples and concluding remarks

Consider the Eurostat input–output data set given in “Box 14.2: RAS procedure” (see Eurostat, 2008, p. 452) for compiling several numerical examples. The 3×4-dimensional initial matrix **A** combines the entries in intersections of the columns “Agriculture”, “Industry”, “Services”, “Final d.” with the rows “Agriculture”, “Industry”, “Services” in “Table 1: Input-output data for year 0”. Note that all the elements of this matrix are nonzero. The row marginal total vector $\mathbf{u}$ of dimension 3×1 is the proper part of the column “Output” in “Table 2: Input-output data for year 1”, and the column marginal total vector $\mathbf{v}'$ of dimension 1×4 involves the proper entries of the row “Total” in the near-mentioned data source.

Initial matrix $\mathbf{A}$ and marginal totals $\mathbf{u}$, $\mathbf{v}'$ are presented in the left half of Table 1. The first numerical example is to handle the data set available by RAS method with iterative processes (4) or (5) and by methods (29), (30) and (32), (33) proposed to solve the constrained minimization problem for homothetic and angular measures (13), (11) and (16), (11) – briefly, by HOM and ANG methods respectively. The computation results at $\mathbf{W} = \mathbf{E}_{NM} / \mathbf{e}'_{NM}\mathbf{E}_{NM}\mathbf{e}_{NM}$ are grouped in the right half of Table 1 for RAS method and in Table 1a for HOM and ANG methods; they seem to be very similar among themselves.

Table 1. Initial matrix **A** with nonzero elements and RAS results for its updating

	A				$\mathbf{u_A}$	**u**
	20.00	34.00	10.00	36.00	100.00	94.78
	20.00	152.00	40.00	188.00	400.00	412.86
	10.00	72.00	20.00	98.00	200.00	212.68
$\mathbf{v}'_\mathbf{A}$	50.00	258.00	70.00	322.00	700.00	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

RAS	**X**				$\mathbf{u_X}$	**u**
	17.94	32.77	9.76	34.31	94.78	94.78
	19.36	158.08	42.12	193.30	412.86	412.86
	9.98	77.17	21.70	103.84	212.68	212.68
$\mathbf{v}'_\mathbf{X}$	47.28	268.02	73.58	331.44	720.32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

Table 1a. HOM and ANG results for updating of data set from Table 1

HOM	**X**				$\mathbf{u_X}$	**u**
	18.35	32.41	10.03	33.99	94.78	94.78
	19.07	158.82	42.60	192.37	412.86	412.86
	9.86	76.79	20.95	105.08	212.68	212.68
$\mathbf{v}'_\mathbf{X}$	47.28	268.02	73.58	331.44	720.32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

ANG	**X**				$\mathbf{u_X}$	**u**
	18.33	32.41	10.04	34.00	94.78	94.78
	19.08	158.81	42.58	192.40	412.86	412.86
	9.87	76.80	20.96	105.04	212.68	212.68
$\mathbf{v}'_\mathbf{X}$	47.28	268.02	73.58	331.44	720.32	
$\mathbf{v}'$	47.28	268.02	73.58	331.44		720.32

Nevertheless, HOM and ANG methods demonstrate the stable 5-percentage advantage over RAS method both in homothetic measure of matrix similarity based on (14) and in angular measure (15) as follows:

$$|\boldsymbol{\delta}^{\mathrm{RAS}}| = 0.0549, \qquad |\boldsymbol{\delta}^{\mathrm{HOM}}| = 0.0522, \qquad |\boldsymbol{\delta}^{\mathrm{ANG}}| = 0.0522, \qquad |\boldsymbol{\delta}^{\mathrm{HOM}}| / |\boldsymbol{\delta}^{\mathrm{RAS}}| = 95.10\%;$$

$$\beta_{\mathbf{qe}}^{\mathrm{RAS}} = 3.1161°, \qquad \beta_{\mathbf{qe}}^{\mathrm{HOM}} = 2.9677°, \qquad \beta_{\mathbf{qe}}^{\mathrm{ANG}} = 2.9675°, \qquad \beta_{\mathbf{qe}}^{\mathrm{ANG}} / \beta_{\mathbf{qe}}^{\mathrm{RAS}} = 95.23\%.$$

The next numerical example is assigned to test the methods’ response to zero elements in the initial matrix. So let us disturb one element of our data set, say (3, 1), by putting it equal to zero for years 0 and 1. After recalculation of the marginal totals we get the data set in the left half of Table 2.

The results of computations are collected in the right half of Table 2 for RAS method and in Table 2a for HOM and ANG methods; as earlier, they seem to be very similar among themselves.

Table 2. Initial matrix **A** with zero element and RAS results for its updating

	A				$\mathbf{u}_\mathbf{A}$	**u**	**RAS**	**X**				$\mathbf{u}_\mathbf{X}$	**u**
	20.00	34.00	10.00	36.00	100.00	94.78		18.02	32.74	9.75	34.27	94.78	94.78
	20.00	152.00	40.00	188.00	400.00	412.86		19.46	158.05	42.11	193.25	412.86	412.86
	0.00	72.00	20.00	98.00	190.00	202.88		**0.00**	77.23	21.72	103.92	202.88	202.88
$\mathbf{v}'_\mathbf{A}$	40.00	258.00	70.00	322.00	690.00		$\mathbf{v}'_\mathbf{X}$	37.48	268.02	73.58	331.44	710.52	
$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52	$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52

Table 2a. HOM and ANG results for updating of data set from Table 2

HOM	**X**				$\mathbf{u}_\mathbf{X}$	**u**	**ANG**	**X**				$\mathbf{u}_\mathbf{X}$	**u**
	18.36	32.40	10.04	33.98	94.78	94.78		18.35	32.40	10.05	33.98	94.78	94.78
	19.12	158.80	42.58	192.37	412.86	412.86		19.13	158.78	42.55	192.39	412.86	412.86
	0.00	76.82	20.96	105.10	202.88	202.88		**0.00**	76.84	20.98	105.07	202.88	202.88
$\mathbf{v}'_\mathbf{X}$	37.48	268.02	73.58	331.44	710.52		$\mathbf{v}'_\mathbf{X}$	37.48	268.02	73.58	331.44	710.52	
$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52	$\mathbf{v}'$	37.48	268.02	73.58	331.44		710.52

Again, HOM and ANG methods still keep on the 5-percentage advantage over RAS method both in homothetic and angular measures as follows:

$$\left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 0.0543, \qquad \left|\boldsymbol{\delta}^{\mathrm{HOM}}\right| = 0.0516, \qquad \left|\boldsymbol{\delta}^{\mathrm{ANG}}\right| = 0.0516, \qquad \left|\boldsymbol{\delta}^{\mathrm{HOM}}\right| \Big/ \left|\boldsymbol{\delta}^{\mathrm{RAS}}\right| = 95.04\%;$$

$$\beta_{\mathbf{qe}}^{\mathrm{RAS}} = 3.0805°, \qquad \beta_{\mathbf{qe}}^{\mathrm{HOM}} = 2.9291°, \qquad \beta_{\mathbf{qe}}^{\mathrm{ANG}} = 2.9286°, \qquad \beta_{\mathbf{qe}}^{\mathrm{ANG}} \Big/ \beta_{\mathbf{qe}}^{\mathrm{RAS}} = 95.07\%.$$

An advantage of HOM and ANG methods observed here is not so impressive because of small number of “free” variables $NM-(N+M)$ and $NM-(N+M)-1$ in our numerical examples. However, if the dimensions of updating matrix tend to grow, then this advantage rapidly increases. At the dimensions more than 3×7 (7×3) and 4×5 (5×4) a total amount of free variables starts to exceed total number of RAS variables, so flexibility of HOM and ANG methods substantially grows. Computational experiments with 15×20-dimensional matrices indicates that HOM and ANG methods seem to be almost twice more effective than RAS in the sense of homothetic measure based on (14) and angular measure (15).

As it is well-known, “… RAS can only handle non-negative matrices, which limits its application to SUTs that often contain negative entries…” – see Temurshoev et al. (2011, p. 92). Therefore, the final numerical example is assigned to test the methods’ response to negative elements in the initial matrix with Generalized RAS (GRAS) method proposed by Junius and Oosterhaven (2003) and redeveloped by Lenzen et al. (2007) instead of RAS. Let us disturb three elements of our data set, say (1, 3), (3, 1) and (3, 3), by reversing their sign for years 0 and 1.

After proper recalculation of the marginal totals we obtain the data set in the left half of Table 3.

The results of computations are grouped in the right half of Table 3 for GRAS method and in Table 3a for HOM and ANG methods; now they demonstrate wide differences in the elements of three target matrices calculated, especially in x_{13}, x_{23}, x_{24} and x_{33} .

Table 3. Initial matrix **A** with negative elements and RAS results for its updating

	A				$\mathbf{u}_\mathbf{A}$	**u**	**GRAS**	**X**				$\mathbf{u}_\mathbf{X}$	**u**
	20.00	34.00	**-10.00**	36.00	80.00	74.50		18.13	32.85	**-10.87**	34.40	74.50	74.50
	20.00	152.00	40.00	188.00	400.00	412.86		19.62	158.95	39.84	194.44	412.86	412.86
	-10.00	72.00	**-20.00**	98.00	140.00	148.92		**-10.07**	76.22	**-19.84**	102.60	148.92	148.92
$\mathbf{v}'_\mathbf{A}$	30.00	258.00	10.00	322.00	620.00		$\mathbf{v}'_\mathbf{X}$	27.68	268.02	9.14	331.44	636.28	
$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28	$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28

Table 3a. HOM and ANG results for updating of data set from Table 3

HOM	**X**				$\mathbf{u}_\mathbf{X}$	**u**	**ANG**	**X**				$\mathbf{u}_\mathbf{X}$	**u**
	18.55	32.30	**-10.21**	33.87	74.50	74.50		18.56	32.31	**-10.26**	33.89	74.50	74.50
	19.27	159.99	39.34	194.26	412.86	412.86		19.30	159.91	39.47	194.18	412.86	412.86
	-10.13	75.73	**-19.99**	103.31	148.92	148.92		**-10.18**	75.80	**-20.07**	103.37	148.92	148.92
$\mathbf{v}'_\mathbf{X}$	27.68	268.02	9.14	331.44	636.28		$\mathbf{v}'_\mathbf{X}$	27.68	268.02	9.14	331.44	636.28	
$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28	$\mathbf{v}'$	27.68	268.02	9.14	331.44		636.28

An advantage of HOM and ANG methods in this case seems to be significant too. Indeed, the received estimates of homothetic and angular measures are

$$\left|\boldsymbol{\delta}^{\mathrm{GRAS}}\right| = 0.0486, \qquad \left|\boldsymbol{\delta}^{\mathrm{HOM}}\right| = 0.0438, \qquad \left|\boldsymbol{\delta}^{\mathrm{ANG}}\right| = 0.0438, \qquad \left|\boldsymbol{\delta}^{\mathrm{HOM}}\right| \Big/ \left|\boldsymbol{\delta}^{\mathrm{GRAS}}\right| = 90.19\%;$$

$$\beta_{\mathbf{qe}}^{\mathrm{GRAS}} = 2.7657°, \qquad \beta_{\mathbf{qe}}^{\mathrm{HOM}} = 2.5102°, \qquad \beta_{\mathbf{qe}}^{\mathrm{ANG}} = 2.5081°, \qquad \beta_{\mathbf{qe}}^{\mathrm{ANG}} \big/ \beta_{\mathbf{qe}}^{\mathrm{GRAS}} = 90.69\%.$$

The HOM and ANG methods appear to be especially effective under the complicated circumstances because of its immanent flexibility. They are quite applicable for updating the economic matrices and tables with some negative entries. In practice the proposed GLS-based methods provides generating much more compact distributions of the Hadamard-multiplicative model's factors in comparison with other methods.

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