# The Linear Matrix-Valued Cost and Production Functions in the Rectangular and Square Input–Output Models

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A general linear problem of input–output analysis is considered in this study as a system of equations written in terms of free variables for any rectangular supply and use table given. This system spans the regular linear equations for material and financial balances, a batch of predetermined values for exogenous variables and an additional set of linkage equations that provides the exact identifiability for all unknown variables.

The study is concerned with some operational opportunities for constructing a set of the identifying linear equations in the cases of evaluating the response of the economy to exogenous changes in final demand and value added. To this end, two matrix-valued linear cost functions with product and industry outputs as their arguments (based on Leontief technical coefficients and Ghosh allocation coefficients to be fixed) are involved. The main types of economy's response to exogenous changes is found out, namely, in terms of quantity changing, price changing, and combined price and quantity changes. The latter types of economy's response seem to be implausible artifacts that are out of economic sense. In particular, there are some certain doubts about plausibility of underlying background for an industry technology assumption and a fixed product sales structure assumption, which are used for transforming supply and use tables to symmetric input-output tables.

It is shown that in a square case (all matrices are square) the cost function with product outputs as its arguments forms an underlying algebraic framework of Leontief demand-driven model, whereas the cost function with industry outputs as its arguments provides an algebraic foundation of Ghosh supply-driven model. For a symmetric (production matrix is diagonal) square case, the equivalence of Ghosh supply-driven model and Leontief price model as well as the equivalence of Leontief demand-driven model and Ghosh quantity model are proved.

Besides, two matrix-valued linear production functions with industry and product intermediate consumption as their arguments are involved into consideration (they based on "quasi-reciprocal" technical and allocation coefficients to be fixed). It is shown that the models with the matrix-valued production functions and the models with the matrix-valued cost functions are pairwise equivalent that can be appreciated as a demonstration of general equilibrium in the theory of input–output analysis.. Thus, technical and allocation coefficients should be regarded as helpful ways of economic interpretation rather than as basic framework or operational tools for modeling.

*Keywords*: rectangular supply and use table, matrix-valued production and cost functions, exogenous changes in final demand and value added, demand-driven and supply-driven models, quantity and price models

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## 1. A general linear problem of input–output analysis

Any supply and use table in an economy with $N$ of products (commodities) and $M$ industries (sectors) for the certain time period (say, period 0) is defined by a pair of rectangular matrices, namely supply (production) matrix $\mathbf{X}_0$ and use for intermediates (intermediate consumption) matrix $\mathbf{Z}_0$ of the same dimension $N\times M$ both (see Eurostat, 2008). In mathematical notation supply and use table's data satisfies to the vector equation for material balance of products' intermediate and final uses

$$\mathbf{X}_0\mathbf{e}_M = \mathbf{Z}_0\mathbf{e}_M + \mathbf{y}_0 \tag{1}$$

and to the vector equation for financial balance of industries' intermediate and primary (combined into value added) inputs

$$\mathbf{e}'_N \mathbf{X}_0 = \mathbf{e}'_N \mathbf{Z}_0 + \mathbf{v}'_0 \tag{2}$$

where $\mathbf{e}_N$ and $\mathbf{e}_M$ are $N\times1$ and $M\times1$ summation column vectors with unit elements, $\mathbf{y}_0$ is a column vector of net final demand with dimensions $N\times1$, and $\mathbf{v}_0$ is a column vector of value added with dimensions $M\times1$. Here putting a prime after vector's (matrix's) symbol denotes a transpose of this vector (matrix).

"One of the major uses of the information in an input–output model is to assess the effect on an economy of changes in elements that are exogenous to the model of that economy" (Miller and Blair, 2009, p. 243). For analytical purposes, system of balance equations (1), (2) must be rewritten in terms of free variables. Let $\mathbf{x}_{\downarrow} = \mathbf{X}\mathbf{e}_M$ be $N$-dimensional column vector of product outputs, and let $\mathbf{x}'_{\rightarrow} = \mathbf{e}'_N \mathbf{X}$ be a row vector of industry outputs with dimensions $1\times M$. Also, let $\mathbf{z}_{\downarrow} = \mathbf{Z}\mathbf{e}_M$ be a column vector of product amounts in intermediate use with dimensions $N\times1$, and let $\mathbf{z}'_{\rightarrow} = \mathbf{e}'_N \mathbf{Z}$ be $M$-dimensional row vector of industry expenditures for intermediate consumption. The vectors $\mathbf{x}_{\downarrow}, \mathbf{x}_{\rightarrow}, \mathbf{z}_{\downarrow}, \mathbf{z}_{\rightarrow}$ are sometimes called the product and industry marginal totals for the production matrix **X** and the intermediate consumption matrix **Z**.

The system of $N$+$M$ scalar equations (1), (2) can be written in free vector variables as follows:

$$\mathbf{x}_{\downarrow} = \mathbf{z}_{\downarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\rightarrow} = \mathbf{z}'_{\rightarrow} + \mathbf{v}'. \tag{3}$$

As noted above, the main aim of constructing similar balance models is to assess an impact of the exogenous changes (either absolute or relative) in net final demand and, by virtue of symmetry of the balance equations under consideration, the exogenous changes in gross value added on simultaneous behavior of the economy. Balance models do not usually reflect the true causes of the certain changes in final demand or value added, so the response of the economy to any exogenous disturbance is evaluated in the mode of getting answers to questions like "what would happen if ...? ".

The balance model (3) contains $N$+$M$ linear equations with 3($N$+$M$) scalar variables. Assume that exogenous disturbance is expressed in terms of $k$ exogenous variables. To provide exact (or strict) identifiability of the model it is required to incorporate into the model 2($N$+$M$) – $k$ auxiliary independent equations as a certain set of linkages between the variables. In particular, $N$+2$M$ independent equations are needed at $k = N$, and 2$N$+$M$ equations are needed at $k = M$. The structure of initial supply and use table serves as an informational framework for constructing the auxiliary linkage equations.

In this context, a general linear problem of input–output analysis is considered as the system of equations (3) together with a chosen specification of exogenous disturbance and a corresponding set of linear linkages between the variables, which provides the strict identification of all unknown variables. Note that the general problem of input–output analysis becomes nonlinear if at least one of its equations is nonlinear.

## 2. On constructing the linear linkages between the model's variables

Consider some operational opportunities in constructing a set of identifying linear equations for balance model (3) in the cases of evaluating the response of the economy to exogenous changes in the net final demand vector $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ with dimensions $N\times 1$ or in the gross value added vector $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ with dimensions $M\times 1$. As noted earlier, the unique informational source for constructing the auxiliary linkages between the model variables is the initial supply and use table. Thus, the linkages should be formulated in terms of production matrix $\mathbf{X}_0$ and intermediate consumption matrix $\mathbf{Z}_0$ given.

The simplest formal way of using this available data for input–output modeling comes to factorization of unknown production matrix $\mathbf{X}$ and unknown intermediate consumption matrix $\mathbf{Z}$ into a pair of matrix factors each. There are two following main techniques for pair factorizing any ($N\times M$)-dimensional matrix:

(i) as a product of a matrix with dimensions $N\times M$ and a square matrix of order $M$;

(ii) as a product of a square matrix of order $N$ and a matrix with dimensions $N\times M$.

The rectangular matrices in (i) and (ii) are obviously either $\mathbf{X}_0$ or $\mathbf{Z}_0$. Further, the square matrix factors $\mathrm{f}_{M\times M}(\mathbf{F},\mathbf{F}_0)$ and $\mathrm{f}_{N\times N}(\mathbf{F},\mathbf{F}_0)$ must have the evident properties $\mathrm{f}_{M\times M}(\mathbf{F}_0,\mathbf{F}_0) = \mathbf{E}_M$ and $\mathrm{f}_{N\times N}(\mathbf{F}_0,\mathbf{F}_0) = \mathbf{E}_N$, where $\mathbf{E}_M$ and $\mathbf{E}_N$ are the identity matrices of order $M$ and $N$ respectively, and an auxiliary matrix $\mathbf{F}$ with dimensions $N\times M$ is equal to either $\mathbf{X}$ or $\mathbf{Z}$ in turn. Hence, the postmultiplying and premultiplying square matrix factors can be represented by the simplest patterns in terms of relative changes in auxiliary matrix's product and industry marginal totals as follows:

$$\mathrm{f}_{M\times M}(\mathbf{F},\mathbf{F}_0) = \langle \mathbf{e}'_N \mathbf{F} \rangle \langle \mathbf{e}'_N \mathbf{F}_0 \rangle^{-1}, \qquad \mathrm{f}_{N\times N}(\mathbf{F},\mathbf{F}_0) = \langle \mathbf{F}\mathbf{e}_M \rangle \langle \mathbf{F}_0 \mathbf{e}_M \rangle^{-1} \tag{4}$$

where angled bracketing around a vector's symbol (or putting a “hat” over it) denotes a diagonal matrix with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697).

Because of applying the multiplicative patterns (4), we get four factorizations for unknown intermediate consumption matrix **Z**, namely

$$\mathbf{Z} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X} \rangle \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}, \qquad \mathbf{Z} = \langle \mathbf{X}\mathbf{e}_M \rangle \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{Z}_0, \tag{5}$$

$$\mathbf{Z} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{Z} \rangle \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1}, \qquad \mathbf{Z} = \langle \mathbf{Z}\mathbf{e}_M \rangle \langle \mathbf{Z}_0 \mathbf{e}_M \rangle^{-1} \mathbf{Z}_0, \tag{6}$$

and the other four factorizations for unknown production matrix **X** as follows:

$$\mathbf{X} = \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{Z} \rangle \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1}, \qquad \mathbf{X} = \langle \mathbf{Z}\mathbf{e}_M \rangle \langle \mathbf{Z}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}_0, \tag{7}$$

$$\mathbf{X} = \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{X} \rangle \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}, \qquad \mathbf{X} = \langle \mathbf{X}\mathbf{e}_M \rangle \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}_0 . \tag{8}$$

The obtained formulas (5) – (8) establish the operational base for constructing the various sets of identifying linear equations in addition to the balance model (3).

**3. An incorporation of the matrix-valued linear cost functions into the model**

The linkage equations (5) reflect the dependencies of intermediate consumption matrix **Z** on the vector of industry outputs $\mathbf{x}'_{\rightarrow} = \mathbf{e}'_N \mathbf{X}$ and the vector of product outputs $\mathbf{x}_{\downarrow} = \mathbf{X}\mathbf{e}_M$. These dependencies are linear, since

$$\mathbf{Z} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \langle \mathbf{e}'_N \mathbf{X} \rangle = \mathbf{A}\hat{\mathbf{x}}_{\rightarrow}, \tag{9}$$

$$\mathbf{Z} = \langle \mathbf{X}\mathbf{e}_M \rangle \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{Z}_0 = \hat{\mathbf{x}}_{\downarrow}\mathbf{B} \tag{10}$$

where **A** and **B** are the computable ($N$×$M$)-dimensional matrices of relative coefficients; here the obvious commutativity property of diagonal matrices is used. It is easy to see that formula (9) at $\mathbf{x}'_{\rightarrow} = \mathbf{e}'_N \mathbf{X}_0$ and formula (9) at $\mathbf{x}_{\downarrow} = \mathbf{X}_0 \mathbf{e}_M$ determine the initial intermediate consumption matrix $\mathbf{Z}_0$ given.

One can classify (9) and (10) as the matrix-valued linear cost functions because of their production arguments. Matrices $\mathbf{A} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}$ and $\mathbf{B} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{Z}_0$ are known in special literature as (Leontief) technical coefficients matrix and (Ghosh) allocation coefficients matrix respectively (see, e.g., Miller and Blair, 2009).

From the expression for cost function (9), it follows that

$$\mathbf{z}_{\downarrow} = \mathbf{Z}\mathbf{e}_M = \mathbf{A}\hat{\mathbf{x}}_{\rightarrow}\mathbf{e}_M = \mathbf{A}\mathbf{x}_{\rightarrow}, \qquad \mathbf{z}'_{\rightarrow} = \mathbf{e}'_N \mathbf{Z} = \mathbf{e}'_N \mathbf{A}\hat{\mathbf{x}}_{\rightarrow} = \mathbf{x}'_{\rightarrow} \langle \mathbf{e}'_N \mathbf{A} \rangle$$

since $\mathbf{c}'\hat{\mathbf{d}} = \mathbf{d}'\hat{\mathbf{c}}$ for any pair of vectors **c** and **d** with the same dimensions. After current result's substitution, the system of linear equations (3) becomes

$$\mathbf{x}_{\downarrow} = \mathbf{A}\mathbf{x}_{\rightarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\rightarrow} = \mathbf{x}'_{\rightarrow} \langle \mathbf{e}'_N \mathbf{A} \rangle + \mathbf{v}'. \tag{11}$$

Next, the expression for cost function (10) gives

$$\mathbf{z}_{\downarrow} = \mathbf{Z}\mathbf{e}_M = \hat{\mathbf{x}}_{\downarrow}\mathbf{B}\mathbf{e}_M = \langle \mathbf{B}\mathbf{e}_M \rangle \mathbf{x}_{\downarrow}, \qquad \mathbf{z}'_{\rightarrow} = \mathbf{e}'_N \mathbf{Z} = \mathbf{e}'_N \hat{\mathbf{x}}_{\downarrow}\mathbf{B} = \mathbf{x}'_{\downarrow}\mathbf{B},$$

and the system of linear equations (3) is transformed to

$$\mathbf{x}_{\downarrow} = \langle \mathbf{B}\mathbf{e}_M \rangle \mathbf{x}_{\downarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\rightarrow} = \mathbf{x}'_{\downarrow}\mathbf{B} + \mathbf{v}'. \tag{12}$$

Thus, we get two models, namely (11) and (12), each of which comprises *N*+*M* linear equations with 2(*N*+*M*) scalar variables $\mathbf{x}_{\downarrow}, \mathbf{x}_{\rightarrow}, \mathbf{y}, \mathbf{v}$. To provide exact (strict) identifiability of all variables in the models with exogenous disturbances, it is expedient to supplement the models (11) and (12) with the linear equations that link the vector of product outputs $\mathbf{x}_{\downarrow}$ and the vector of industry outputs $\mathbf{x}_{\rightarrow}$.

To this end, the factorizations (8) do seem to be helpful. By introducing new matrix denotations, we have

$$\mathbf{X} = \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \langle \mathbf{e}'_N \mathbf{X} \rangle = \mathbf{G}\hat{\mathbf{x}}_{\rightarrow}, \tag{13}$$

$$\mathbf{X} = \langle \mathbf{X}\mathbf{e}_M \rangle \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}_0 = \hat{\mathbf{x}}_{\downarrow}\mathbf{H} \tag{14}$$

where **G** and **H** are the computable (*N*×*M*)-dimensional matrices of relative coefficients. Obviously, formula (13) at $\mathbf{x}'_{\rightarrow} = \mathbf{e}'_N \mathbf{X}_0$ and formula (14) at $\mathbf{x}_{\downarrow} = \mathbf{X}_0 \mathbf{e}_M$ determine the initial production matrix $\mathbf{X}_0$ given. Matrices $\mathbf{G} = \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}$ and $\mathbf{H} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}_0$ are known in special literature as product-mix matrix (with shares of each product in output of an industry in a column) and market shares matrix (with contributions of each industry to the output of a product in a row) respectively (see Eurostat, 2008).

The equation (13) implies that $\mathbf{x}_{\downarrow} = \mathbf{X}\mathbf{e}_M = \mathbf{G}\hat{\mathbf{x}}_{\rightarrow}\mathbf{e}_M = \mathbf{G}\mathbf{x}_{\rightarrow}$; it allows eliminating the variable $\mathbf{x}_{\downarrow}$ in turn from the models (11) and (12), which become

$$\mathbf{G}\mathbf{x}_{\rightarrow} = \mathbf{A}\mathbf{x}_{\rightarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\rightarrow} = \mathbf{x}'_{\rightarrow} \langle \mathbf{e}'_N \mathbf{A} \rangle + \mathbf{v}'. \tag{15}$$

$$\mathbf{G}\mathbf{x}_{\rightarrow} = \langle \mathbf{B}\mathbf{e}_M \rangle \mathbf{G}\mathbf{x}_{\rightarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\rightarrow} = \mathbf{x}'_{\rightarrow}\mathbf{G}'\mathbf{B} + \mathbf{v}'. \tag{16}$$

The equation (14) gives $\mathbf{x}'_{\rightarrow} = \mathbf{e}'_N \mathbf{X} = \mathbf{e}'_N \hat{\mathbf{x}}_{\downarrow}\mathbf{H} = \mathbf{x}'_{\downarrow}\mathbf{H}$. After its substitution to the models (11) and (12) for eliminating the variable $\mathbf{x}_{\rightarrow}$, we obtain

$$\mathbf{x}_{\downarrow} = \mathbf{A}\mathbf{H}'\mathbf{x}_{\downarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\downarrow}\mathbf{H} = \mathbf{x}'_{\downarrow}\mathbf{H} \langle \mathbf{e}'_N \mathbf{A} \rangle + \mathbf{v}'. \tag{17}$$

$$\mathbf{x}_{\downarrow} = \langle \mathbf{B}\mathbf{e}_M \rangle \mathbf{x}_{\downarrow} + \mathbf{y}, \qquad \mathbf{x}'_{\downarrow}\mathbf{H} = \mathbf{x}'_{\downarrow}\mathbf{B} + \mathbf{v}'. \tag{18}$$

Note that each model (15) – (18) consists of *N*+*M* linear equations with different numbers of unknown scalar variables, namely *N*+2*M* scalar variables $\mathbf{x}_{\rightarrow}, \mathbf{y}, \mathbf{v}$ as in (15) or (16), and

$2N+M$ scalar variables $\mathbf{x}_{\downarrow}$, $\mathbf{y}$, $\mathbf{v}$ as in (17) or (18). Therefore, supplementing the exogenous value added condition $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ provides a just identifying closure of the models (15) and (16), whereas an exact identifiability of variables in the models (17) and (18) can be achieved by direct incorporation of the exogenous final demand condition $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$.

However, if the values of $N$ and $M$ coincide, alternative choice of exogenous condition appears to be also feasible, namely one can use the condition $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ for closure of the models (15) and (16) as well as the condition $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ for closure of the models (17) and (18). Thus, if all the matrices in (15) – (18) are square, each model has a supplementary solution corresponding to alternative exogenous condition. All possible situations for models (15) – (18) are grouped in Table 1.

Table 1. The various exact identifiable specifications of linear input–output analysis problem with the matrix-valued linear cost functions and exogenous variables

| Model | Model code | Matrices fixed | Vector variables | Number of variables | Exogenous variable | Alternative exogenous variable at $N = M$ |
|---|---|---|---|---|---|---|
| (15) | **AG** | $\mathbf{A}, \mathbf{G}$ | $\mathbf{x}_{\rightarrow}, \mathbf{y}, \mathbf{v}$ | $N+2M$ | $\mathbf{v} = \mathbf{v}_*$ | $\mathbf{y} = \mathbf{y}_*$ |
| (16) | **BG** | $\mathbf{B}, \mathbf{G}$ | $\mathbf{x}_{\rightarrow}, \mathbf{y}, \mathbf{v}$ | $N+2M$ | $\mathbf{v} = \mathbf{v}_*$ | $\mathbf{y} = \mathbf{y}_*$ |
| (17) | **AH** | $\mathbf{A}, \mathbf{H}$ | $\mathbf{x}_{\downarrow}, \mathbf{y}, \mathbf{v}$ | $2N+M$ | $\mathbf{y} = \mathbf{y}_*$ | $\mathbf{v} = \mathbf{v}_*$ |
| (18) | **BH** | $\mathbf{B}, \mathbf{H}$ | $\mathbf{x}_{\downarrow}, \mathbf{y}, \mathbf{v}$ | $2N+M$ | $\mathbf{y} = \mathbf{y}_*$ | $\mathbf{v} = \mathbf{v}_*$ |

So, in terms of exogenous final demand and value added variables, each specification of the linear input–output analysis problem (15) – (18) has two solutions – a regular one and a supplementary one with an alternative exogenous vector at $N = M = K$.

## 4. The solutions of linear input–output analysis problem: algebraic properties

The regular and supplementary solutions of linear input–output analysis problem have some important algebraic properties, which can be explored without direct solving the linear equation systems (15) – (18) under various specifications of an exogenous disturbance. In particular, from (13) and (9) one can conclude that in the model **AG** (15) an unknown production matrix **X** and an unknown intermediate consumption matrix **Z** are associated with the initial matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ by postmultiplying the latters on the same diagonal matrix both as follows:

$$\mathbf{X} = \mathbf{G}\hat{\mathbf{x}}_{\rightarrow} = \mathbf{X}_0 \left\langle \mathbf{e}'_N \mathbf{X}_0 \right\rangle^{-1} \hat{\mathbf{x}}_{\rightarrow} = \mathbf{X}_0 \hat{\mathbf{q}}, \qquad \mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}_{\rightarrow} = \mathbf{Z}_0 \left\langle \mathbf{e}'_N \mathbf{X}_0 \right\rangle^{-1} \hat{\mathbf{x}}_{\rightarrow} = \mathbf{Z}_0 \hat{\mathbf{q}} \quad (19)$$

where $\mathbf{q}$ is a column vector of dimension $M\times 1$ strictly identifiable together with the vector of industry outputs $\mathbf{x}_{\rightarrow}$ at any feasible exogenous condition. However, it is not so difficult to make sure that other models (16) – (18) have the different algebraic properties.

Indeed, formulas (14) and (10) imply that in the model **BH** (18), in full contrast to model **AG**, unknown matrices **X** and **Z** are associated with the initial matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ by premultiplying the latters on the same diagonal matrix both as follows:

$$\mathbf{X} = \hat{\mathbf{x}}_{\downarrow}\mathbf{H} = \hat{\mathbf{x}}_{\downarrow}\left\langle \mathbf{X}_0\mathbf{e}_M \right\rangle^{-1}\mathbf{X}_0 = \hat{\mathbf{p}}\mathbf{X}_0, \qquad \mathbf{Z} = \hat{\mathbf{x}}_{\downarrow}\mathbf{B} = \hat{\mathbf{x}}_{\downarrow}\left\langle \mathbf{X}_0\mathbf{e}_M \right\rangle^{-1}\mathbf{Z}_0 = \hat{\mathbf{p}}\mathbf{Z}_0 \qquad (20)$$

where $\mathbf{p}$ is a column vector of dimension $N\times 1$ exactly identifiable together with the vector of product outputs $\mathbf{x}_{\downarrow}$ at any feasible exogenous condition.

Further, for the models **BG** (16) and **AH** (17) one needs to combine the formulas from (19) and (20) pairwise in a criss-cross manner, namely

$$\mathbf{X} = \mathbf{G}\hat{\mathbf{x}}_{\rightarrow} = \mathbf{X}_0\left\langle \mathbf{q}_{\mathrm{BG}} \right\rangle, \qquad \mathbf{Z} = \hat{\mathbf{x}}_{\downarrow}\mathbf{B} = \left\langle \mathbf{p}_{\mathrm{BG}} \right\rangle\mathbf{Z}_0 \qquad (21)$$

and

$$\mathbf{X} = \hat{\mathbf{x}}_{\downarrow}\mathbf{H} = \left\langle \mathbf{p}_{\mathrm{AH}} \right\rangle\mathbf{X}_0, \qquad \mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}_{\rightarrow} = \mathbf{Z}_0\left\langle \mathbf{q}_{\mathrm{AH}} \right\rangle \qquad (22)$$

respectively. Here $\mathbf{q}_{\bullet}$ is a column vector of dimension $M\times 1$ unambiguously computable from the models **BG** or **AH** together with the vector of industry outputs $\mathbf{x}_{\rightarrow}$ at any feasible exogenous condition, whereas $\mathbf{p}_{\bullet}$ is a column vector of dimension $N\times 1$ unambiguously computable from the models **BG** or **AH** together with the vector of product outputs $\mathbf{x}_{\downarrow}$ at any feasible exogenous condition.

## 5. Economic interpretation of the solutions' properties

The vectors **p**'s with dimensions $N\times 1$ in (19) – (22) have a natural interpretation as the relative price indices for products (goods and services), and the vectors **q**'s with dimensions $M\times 1$ should be considered as the relative volume (quantity) indices for industries (sectors) of the economy. Thus, the expressions (19) – (22) allow recognizing four different types of the economy response to exogenous changes in net final demand and gross value added in accordance with the models (15) – (18).

The model **AG** (15) and its generated disturbances in production and intermediate consumption matrices (19) describe an impact of exogenous changes in final demand or value added exclusively in terms of the production quantity changing at constant prices for the products. Meanwhile, the model **BH** (18) and its response (20) characterize this impact only in terms of price changing at constant level of production in the industries. At the same time, the models **BG** (16) and **AH** (17) and their responses (21) and (22) should be classified as mixed – because they combine price and quantity changes – and are needed some special attention.

In the model **BG** (16) according to (21) we have $\mathbf{q}_{\mathrm{BG}} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{x}_{\rightarrow}$, $\mathbf{p}_{\mathrm{BG}} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{x}_{\downarrow}$ and $\mathbf{x}_{\downarrow} = \mathbf{G}\mathbf{x}_{\rightarrow}$. Putting the latter expression into the second one gives

$$\mathbf{p}_{\mathrm{BG}} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{G}\mathbf{x}_{\rightarrow} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{x}_{\rightarrow} = \mathbf{H} \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{x}_{\rightarrow} = \mathbf{H}\mathbf{q}_{\mathrm{BG}} . \quad (23)$$

As it was shown by (21), in model **BG** the production matrix under estimation undergoes exclusively quantity changes, whereas the matrix of intermediate consumption – only price changes. This result is highly difficult to interpret from economic point of view. Here the total equilibration of resulting supply and use table is provided by "forced" matching a pair of vector indices $\mathbf{p}_{\mathrm{BG}}$ and $\mathbf{q}_{\mathrm{BG}}$, which are functionally linked through matrix **H**. By means of the equation (23), the latter statement can be clearly illustrated using a simple transformation for the initial vector of product outputs as follows:

$$\langle \mathbf{p}_{\mathrm{BG}} \rangle \mathbf{X}_0 \mathbf{e}_M = \langle \mathbf{X}_0 \mathbf{e}_M \rangle \mathbf{H}\mathbf{q}_{\mathrm{BG}} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}_0 \mathbf{q}_{\mathrm{BG}} = \mathbf{X}_0 \mathbf{q}_{\mathrm{BG}} = \mathbf{X}\mathbf{e}_M . \quad (24)$$

So from (21) we have $\mathbf{X} = \mathbf{X}_0 \langle \mathbf{q}_{\mathrm{BG}} \rangle$, but (24) implies that $\left( \mathbf{X} - \langle \mathbf{p}_{\mathrm{BG}} \rangle \mathbf{X}_0 \right) \mathbf{e}_M = \mathbf{0}_N$ where $\mathbf{0}_N$ is a zero column vector of dimension $N \times 1$. These are the reasons why the model **BG** (16) seems to be implausible artifact that is out of economic sense.

Further, in the model **AH** (17) according to formulas (22) we have $\mathbf{p}_{\mathrm{AH}} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{x}_{\downarrow}$, $\mathbf{q}_{\mathrm{AH}} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{x}_{\rightarrow}$ and $\mathbf{x}_{\rightarrow} = \mathbf{H}' \mathbf{x}_{\downarrow}$. Putting the latter expression into the second one gives

$$\mathbf{q}_{\mathrm{AH}} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{H}' \mathbf{x}_{\downarrow} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{x}_{\downarrow} = \mathbf{G}' \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{x}_{\downarrow} = \mathbf{G}' \mathbf{p}_{\mathrm{AH}} . \quad (25)$$

From the formulas (22) it follows that in model **AH** the production matrix undergoes only price changes, whereas the matrix of intermediate consumption – exclusively quantity changes. As earlier with model **BG**, this fact is highly difficult to interpret from economic point of view. Again, the total equilibration of resulting supply and use table is provided by "forced" matching a pair of vector indices $\mathbf{q}_{\mathrm{AH}}$ and $\mathbf{p}_{\mathrm{AH}}$, which are functionally linked through transpose matrix **G**. By means of the equation (25), the latter proposition can be confirmed using a simple transformation for the initial vector of industry outputs as follows:

$$\mathbf{e}'_N \mathbf{X}_0 \langle \mathbf{q}_{\mathrm{AH}} \rangle = \mathbf{p}'_{\mathrm{AH}} \mathbf{G} \langle \mathbf{e}'_N \mathbf{X}_0 \rangle = \mathbf{p}'_{\mathrm{AH}} \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \langle \mathbf{e}'_N \mathbf{X}_0 \rangle = \mathbf{p}'_{\mathrm{AH}} \mathbf{X}_0 = \mathbf{e}'_N \mathbf{X} . \quad (26)$$

So from (22) we have $\mathbf{X} = \langle \mathbf{p}_{\mathrm{AH}} \rangle \mathbf{X}_0$, but (26) gives $\mathbf{e}'_N \left( \mathbf{X} - \mathbf{X}_0 \langle \mathbf{q}_{\mathrm{AH}} \rangle \right) = \mathbf{0}'_M$ where $\mathbf{0}'_M$ is a zero row vector of dimension $1 \times M$. Thus, the model **AH** (17) as well as **BG** (16) seem to be implausible artifact that is out of economic sense.

Nevertheless, the model **AH** is still widely used in current practice of input–output analysis for transformation of supply and use tables to symmetric input–output tables in the product-by-product format on the base of industry technology assumption (model B as in Eurostat, 2008) and in the industry-by-industry format on the base of fixed product sales structure assumption (model D as in Eurostat, 2008)

As a result, only the models **AG** and **BH** are of great theoretical and practical interest among all the specifications of linear input–output analysis problem with matrix-valued cost functions listed above in Table 1. So it is advisable to go to the direct solution of linear equation systems (15) and (18).

**6. Regular and supplementary solutions for the models AG and BH**

Under the exogenous gross value added condition $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$, second equation of system (15) in a very general case $\langle \mathbf{e}'_N \mathbf{A} \rangle \neq \mathbf{E}_M$ can be resolved with respect to the vector of industry outputs, namely

$$\mathbf{x}_{\rightarrow} = \left(\mathbf{E}_M - \langle \mathbf{e}'_N \mathbf{A} \rangle\right)^{-1} \mathbf{v}_* . \tag{27}$$

From (19) it follows that $\hat{\mathbf{q}} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \hat{\mathbf{x}}_{\rightarrow}$, and by substituting (27) we get the regular solution for model **AG** (15) at the exogenous changes in value added:

$$\mathbf{q} = \hat{\mathbf{q}}\mathbf{e}_M = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \left(\mathbf{E}_M - \langle \mathbf{e}'_N \mathbf{A} \rangle\right)^{-1} \mathbf{v}_* = \left(\langle \mathbf{e}'_N \mathbf{X}_0 \rangle - \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle\right)^{-1} \mathbf{v}_* = \hat{\mathbf{v}}_0^{-1} \mathbf{v}_* . \tag{28}$$

Analogously, under the exogenous net final demand condition $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$, first equation of system (18) in a very general case $\langle \mathbf{B}\mathbf{e}_M \rangle \neq \mathbf{E}_N$ can be resolved with respect to the vector of product outputs as

$$\mathbf{x}_{\downarrow} = \left(\mathbf{E}_N - \langle \mathbf{B}\mathbf{e}_M \rangle\right)^{-1} \mathbf{y}_* . \tag{29}$$

From (20) it follows that $\hat{\mathbf{p}} = \hat{\mathbf{x}}_{\downarrow} \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1}$, so by substituting (29) we obtain the regular solution for model **BH** (18) at the exogenous changes in final demand:

$$\mathbf{p}' = \mathbf{e}'_N \hat{\mathbf{p}} = \mathbf{y}'_* \left(\mathbf{E}_N - \langle \mathbf{B}\mathbf{e}_M \rangle\right)^{-1} \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} = \mathbf{y}'_* \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \langle \mathbf{Z}_0 \mathbf{e}_M \rangle\right)^{-1} = \mathbf{y}'_* \hat{\mathbf{y}}_0^{-1} . \tag{30}$$

It should be noted that the solutions (28) and (30) are valid at any numbers of products and industries in the economy. Nevertheless, both these regular solutions are trivial because a response of model **AG** to the disturbance $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ comes to the alternate multiplying the columns of production and intermediate consumption matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ on the growth indices of value added through all industries at constant prices for the products, and a response of model **BH** to the disturbance $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ can be calculated by the alternate multiplying the rows of matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ on the value indices of final demand through all products at constant level of production in the industries.

As noted in Section 3, a choice of alternative exogenous condition at $N = M = K$ is also feasible for finding a supplementary solution of the linear input–output analysis problem. Under the exogenous final demand condition $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$, first equation of system (15) can be resolved with respect to the vector of industry outputs as

$$\mathbf{x}_{\rightarrow} = (\mathbf{G} - \mathbf{A})^{-1}\mathbf{y}_*, \tag{31}$$

of course, if an inverse of square (at $N = M = K$) matrix $\mathbf{G} - \mathbf{A}$ exists as it is expected to be. Note that matrix $\mathbf{G}$ (together with initial production matrix $\mathbf{X}_0$) usually has the dominant main diagonal.

From (19) it follows that $\hat{\mathbf{q}} = \langle \mathbf{e}'_K \mathbf{X}_0 \rangle^{-1} \hat{\mathbf{x}}_{\rightarrow}$, and by substituting (31) we get the supplementary solution for model **AG** (15) at the exogenous changes in final demand:

$$\mathbf{q} = \hat{\mathbf{q}}\mathbf{e}_K = \langle \mathbf{e}'_K \mathbf{X}_0 \rangle^{-1} (\mathbf{G} - \mathbf{A})^{-1} \mathbf{y}_* = \left[ (\mathbf{G} - \mathbf{A}) \langle \mathbf{e}'_K \mathbf{X}_0 \rangle \right]^{-1} \mathbf{y}_* = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}_*. \tag{32}$$

Similarly, under the exogenous value added condition $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$, second equation of system (18) can be resolved with respect to the vector of product outputs, namely

$$\mathbf{x}_{\downarrow} = (\mathbf{H}' - \mathbf{B}')^{-1} \mathbf{v}_*, \tag{33}$$

if an inverse of square (at $N = M = K$) matrix $\mathbf{H} - \mathbf{B}$ exists as it is expected to be because matrix $\mathbf{H}$ (together with $\mathbf{G}$ and initial production matrix $\mathbf{X}_0$) usually has the dominant main diagonal.

From (20) it follows that $\hat{\mathbf{p}} = \hat{\mathbf{x}}_{\downarrow} \langle \mathbf{X}_0 \mathbf{e}_K \rangle^{-1}$, and by substituting (33) we obtain the supplementary solution for model **BH** (18) at the exogenous changes in value added:

$$\mathbf{p} = \hat{\mathbf{p}}\mathbf{e}_K = \langle \mathbf{X}_0 \mathbf{e}_K \rangle^{-1} (\mathbf{H}' - \mathbf{B}')^{-1} \mathbf{v}_* = \left[ (\mathbf{H}' - \mathbf{B}') \langle \mathbf{X}_0 \mathbf{e}_K \rangle \right]^{-1} \mathbf{v}_* = (\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1} \mathbf{v}_*. \tag{34}$$

The supplementary solutions (32) and (34) are valid only if the values of $N$ and $M$ coincide, but in contrast to regular solutions (28) and (30) they are not trivial. It is interesting here to pay attention to the fact that models **AG** (15) and **BH** (18) do demonstrate a remarkable set of duality properties in pairwise comparison of the regular solutions (28) and (30) as well as the supplementary solutions (32) and (34) at $N = M = K$.

## 7. The Leontief and Ghosh quantity and price models

The model **AG** (15), its regular solutions (28) together with supplementary solution (32) and the resulting disturbances in production and intermediate consumption matrices (19) describe an impact of exogenous changes in final demand in terms of the production quantity changing at constant prices for the products. The model **BH** (18), its regular solutions (30) together with supplementary solution (34) and the resulting disturbances in production and intermediate consumption matrices (20) characterize an impact of exogenous changes in value added in terms of price changing at constant level of production in the industries.

Model **AG** (15) at *N* = *M* = *K* is well-known as a Leontief demand-driven model (see Miller and Blair, 2009, Section 2.2.2). It serves to assess an impact of exogenous (absolute or relative) changes in final demand on the economy at constant prices. Indeed, as it follows from (19), the main model's statements are $\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}$ and $\mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}$ where

$$\mathbf{q} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}_* = [\langle\mathbf{X}_0\mathbf{e}_K\rangle(\mathbf{H} - \mathbf{B})]^{-1}\mathbf{y}_* = (\mathbf{H} - \mathbf{B})^{-1}\langle\mathbf{X}_0\mathbf{e}_K\rangle^{-1}\mathbf{y}_* \tag{35}$$

according to (32). Total requirements matrix, which links the vector of product outputs with the final demand vector, can be derived as follows:

$$\mathbf{x}_{\downarrow} = \mathbf{X}\mathbf{e}_K = \mathbf{X}_0\mathbf{q} = \mathbf{X}_0(\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}_* = \left[(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{X}_0^{-1}\right]^{-1}\mathbf{y}_* = \left(\mathbf{E}_K - \mathbf{Z}_0\mathbf{X}_0^{-1}\right)^{-1}\mathbf{y}_*\,.$$

Model **BH** (18) at *N* = *M* = *K* is known as a Ghosh supply-driven model (see Miller and Blair, 2009, Section 12.1). It helps to evaluate an impact of exogenous (absolute or relative) changes in value added on the economy at fixed production scales. As it follows from (20), the main model's statements are $\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0$ and $\mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0$ where

$$\mathbf{p} = (\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1}\mathbf{v}_* = [\langle\mathbf{e}'_K\mathbf{X}_0\rangle(\mathbf{G}' - \mathbf{A}')]^{-1}\mathbf{v}_* = (\mathbf{G}' - \mathbf{A}')^{-1}\langle\mathbf{e}'_K\mathbf{X}_0\rangle^{-1}\mathbf{v}_* \tag{36}$$

in accordance with (34). Ghosh analogue of total requirements matrix, which links the vector of industry outputs with the value added vector, can be derived as follows:

$$\mathbf{x}_{\rightarrow} = \mathbf{X}'\mathbf{e}_K = \mathbf{X}'_0\mathbf{p} = \mathbf{X}'_0(\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1}\mathbf{v}_* = \left[(\mathbf{X}'_0 - \mathbf{Z}'_0)(\mathbf{X}'_0)^{-1}\right]^{-1}\mathbf{v}_* = \left[\mathbf{E}_K - \mathbf{Z}'_0(\mathbf{X}'_0)^{-1}\right]^{-1}\mathbf{v}_*\,.$$

Here it is worth to mention the duality properties of models **AG** (15) and **BH** (18) again, because a response of model **AG** to the disturbance of the final demand coefficients $\langle\mathbf{X}_0\mathbf{e}_M\rangle^{-1}\mathbf{y}_*$ is described by matrices **H** and **B**, whereas a response of model **BH** to the disturbance of the value added coefficients $\langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{v}_*$ – by matrices **G** and **A**.

Vectors **q** and **p** are defined above using an assumption that all the matrices in (35) and (36) are square (at *N* = *M* = *K*). In addition, let the initial production matrix $\mathbf{X}_0$ be a diagonal one as in a symmetric input–output table. Then the Leontief and Ghosh models under consideration can be easily led to a "classical" view.

For diagonal matrix $\mathbf{X}_0$ of order *K*,

$$\mathbf{X}_0 = \mathbf{X}'_0 = \langle\mathbf{e}'_K\mathbf{X}_0\rangle = \langle\mathbf{X}_0\mathbf{e}_K\rangle\,. \tag{37}$$

One can obtain the most famous Leontief formula, using (35), (37) and some algebraic properties of diagonal matrices along the sequential transformations of the product output marginal totals $\mathbf{x}_{\downarrow}$ as follows:

$$\mathbf{x}_{\downarrow} = \mathbf{X}\mathbf{e}_K = \mathbf{X}_0\mathbf{q} = \mathbf{X}_0(\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}_* = \left[(\mathbf{X}_0 - \mathbf{Z}_0)\langle \mathbf{e}'_K\mathbf{X}_0\rangle^{-1}\right]^{-1}\mathbf{y}_* = (\mathbf{E}_K - \mathbf{A})^{-1}\mathbf{y}_* .$$

Its analogue for Ghosh supply-driven model can be easily derived in the similar manner, using (36) and then (37) along the sequential transformations of the industry output marginal totals $\mathbf{x}_{\rightarrow}$ as follows:

$$\mathbf{x}_{\rightarrow} = \mathbf{X}'\mathbf{e}_K = \mathbf{X}'_0\mathbf{p} = \mathbf{X}'_0(\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1}\mathbf{v}_* = \left[(\mathbf{X}'_0 - \mathbf{Z}'_0)\langle \mathbf{X}'_0\mathbf{e}_K\rangle^{-1}\right]^{-1}\mathbf{v}_* = (\mathbf{E}_K - \mathbf{B}')^{-1}\mathbf{v}_* .$$

Further, direct putting (37) into Ghosh model (36) gives well-known formula

$$\mathbf{p} = (\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1}\mathbf{v}_* = \left(\langle \mathbf{e}'_K\mathbf{X}_0\rangle - \mathbf{Z}'_0\right)^{-1}\mathbf{v}_* = (\mathbf{E}_K - \mathbf{A}')^{-1}\langle \mathbf{e}'_K\mathbf{X}_0\rangle^{-1}\mathbf{v}_*$$

for so-called Leontief price model (see Miller and Blair, 2009, p. 44). Thus, in the case of a symmetric input-output table the Ghosh supply-driven model coincides with the Leontief price model (see Dietzenbacher, 1997).

It can be shown in similar manner that the Leontief demand-driven model serves as the Ghosh quantity model. Indeed, direct substituting (37) into Leontief model (35) gives

$$\mathbf{q} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}_* = \left(\langle \mathbf{X}_0\mathbf{e}_K\rangle - \mathbf{Z}_0\right)^{-1}\mathbf{y}_* = (\mathbf{E}_K - \mathbf{B})^{-1}\langle \mathbf{X}_0\mathbf{e}_K\rangle^{-1}\mathbf{y}_* .$$

It is appropriate to mention here that all formulas obtained above in this and previous sections demonstrate a remarkable set of duality properties.

## 8. An incorporation of the matrix-valued linear production functions into the model (3)

Up until now in this paper, the study has focused on applying two factorizations of unknown intermediate consumption matrix (5) and two factorizations of unknown production matrix (8) for constructing the various sets of identifying linear equations in addition to the general balance model (3). Now it is time to explore the other two opportunities provided by two factorizations (7) together with two factorization (6).

The linkage equations (7) reflect the dependencies of production matrix $\mathbf{X}$ on the vector of industry expenditures for intermediate consumption $\mathbf{z}'_{\rightarrow} = \mathbf{e}'_N\mathbf{Z}$ and the vector of product amounts in intermediate use $\mathbf{z}_{\downarrow} = \mathbf{Z}\mathbf{e}_M$. These dependencies are linear, since

$$\mathbf{X} = \mathbf{X}_0\langle \mathbf{e}'_N\mathbf{Z}_0\rangle^{-1}\langle \mathbf{e}'_N\mathbf{Z}\rangle = \mathbb{A}\hat{\mathbf{z}}_{\rightarrow} , \tag{38}$$

$$\mathbf{X} = \langle \mathbf{Z}\mathbf{e}_M\rangle\langle \mathbf{Z}_0\mathbf{e}_M\rangle^{-1}\mathbf{X}_0 = \hat{\mathbf{z}}_{\downarrow}\mathbb{B} \tag{39}$$

where $\mathbb{A}$ and $\mathbb{B}$ are the computable ($N\times M$)-dimensional matrices of relative coefficients. It is easy to check that formula (38) at $\mathbf{z}'_{\rightarrow} = \mathbf{e}'_N\mathbf{Z}_0$ and formula (39) at $\mathbf{z}_{\downarrow} = \mathbf{Z}_0\mathbf{e}_M$ determine the initial production matrix $\mathbf{X}_0$ given.

Matrix-valued dependencies (38) and (39) can be classified as the linear production functions because of their cost arguments. Matrices $\mathbb{A} = \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1}$ and $\mathbb{B} = \langle \mathbf{Z}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}_0$ are apparently not known in special literature in contrast to the Leontief technical coefficients matrix $\mathbf{A} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}$ and the Ghosh allocation coefficients matrix $\mathbf{B} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{Z}_0$. It is easy to see that matrices $\mathbb{A}$ and $\mathbf{A}$ as well as $\mathbb{B}$ and $\mathbf{B}$ are in certain “quasi-reciprocal” relations.

From the expression for production function (38), it follows that

$$\mathbf{x}_\downarrow = \mathbf{X}\mathbf{e}_M = \mathbb{A}\hat{\mathbf{z}}_\rightarrow \mathbf{e}_M = \mathbb{A}\mathbf{z}_\rightarrow , \qquad \mathbf{x}'_\rightarrow = \mathbf{e}'_N \mathbf{X} = \mathbf{e}'_N \mathbb{A} \hat{\mathbf{z}}_\rightarrow = \mathbf{z}'_\rightarrow \langle \mathbf{e}'_N \mathbb{A} \rangle .$$

After current result’s substitution, the system of linear equations (3) becomes

$$\mathbb{A}\mathbf{z}_\rightarrow = \mathbf{z}_\downarrow + \mathbf{y} , \qquad \mathbf{z}'_\rightarrow \langle \mathbf{e}'_N \mathbb{A} \rangle = \mathbf{z}'_\rightarrow + \mathbf{v}' . \tag{40}$$

Further, the expression for production function (39) gives

$$\mathbf{x}_\downarrow = \mathbf{X}\mathbf{e}_M = \hat{\mathbf{z}}_\downarrow \mathbb{B} \mathbf{e}_M = \langle \mathbb{B}\mathbf{e}_M \rangle \mathbf{z}_\downarrow , \qquad \mathbf{x}'_\rightarrow = \mathbf{e}'_N \mathbf{X} = \mathbf{e}'_N \hat{\mathbf{z}}_\downarrow \mathbb{B} = \mathbf{z}'_\downarrow \mathbb{B} ,$$

and the system of linear equations (3) is transformed to

$$\langle \mathbb{B}\mathbf{e}_M \rangle \mathbf{z}_\downarrow = \mathbf{z}_\downarrow + \mathbf{y} , \qquad \mathbf{z}'_\downarrow \mathbb{B} = \mathbf{z}'_\rightarrow + \mathbf{v}' . \tag{41}$$

Thus, we get two models (40) and (41), each of which contains $N+M$ linear equations with $2(N+M)$ scalar variables $\mathbf{z}_\downarrow, \mathbf{z}_\rightarrow, \mathbf{y}, \mathbf{v}$. To provide exact identifiability of all variables in the models with exogenous disturbances, let us supplement them with the linear equations that link the vector of product amounts in intermediate use $\mathbf{z}_\downarrow$ and the vector of industry expenditures for intermediate consumption $\mathbf{z}_\rightarrow$.

To this aim, the factorizations (6) do seem to be useful. By introducing new matrix denotations, we have

$$\mathbf{Z} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1} \langle \mathbf{e}'_N \mathbf{Z} \rangle = \mathbb{G} \hat{\mathbf{z}}_\rightarrow , \tag{42}$$

$$\mathbf{Z} = \langle \mathbf{Z}\mathbf{e}_M \rangle \langle \mathbf{Z}_0 \mathbf{e}_M \rangle^{-1} \mathbf{Z}_0 = \hat{\mathbf{z}}_\downarrow \mathbb{H} \tag{43}$$

where $\mathbb{G}$ and $\mathbb{H}$ are the computable $(N\times M)$-dimensional matrices of relative coefficients. Obviously, formula (42) at $\mathbf{z}'_\rightarrow = \mathbf{e}'_N \mathbf{Z}_0$ and formula (43) at $\mathbf{z}_\downarrow = \mathbf{Z}_0 \mathbf{e}_M$ determine the initial intermediate consumption matrix $\mathbf{Z}_0$ given. Matrices $\mathbb{G} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1}$ and $\mathbb{H} = \langle \mathbf{Z}_0 \mathbf{e}_M \rangle^{-1} \mathbf{Z}_0$ are apparently not known in special literature in contrast to their mirror-images, namely the product-mix matrix $\mathbf{G} = \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}$ and the market shares matrix $\mathbf{H} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}_0$.

The equation (42) gives $\mathbf{z}_\downarrow = \mathbf{Z}\mathbf{e}_M = \mathbb{G}\hat{\mathbf{z}}_\rightarrow \mathbf{e}_M = \mathbb{G}\mathbf{z}_\rightarrow$. After its substitution to the models (40) and (41) for eliminating the variable $\mathbf{z}_\downarrow$, we obtain

$$\mathbb{A}\mathbf{z}_{\rightarrow} = \mathbb{G}\mathbf{z}_{\rightarrow} + \mathbf{y}, \qquad \mathbf{z}'_{\rightarrow}\langle \mathbf{e}'_N \mathbb{A}\rangle = \mathbf{z}'_{\rightarrow} + \mathbf{v}'. \tag{44}$$

$$\langle \mathbb{B}\mathbf{e}_M \rangle \mathbb{G}\mathbf{z}_{\rightarrow} = \mathbb{G}\mathbf{z}_{\rightarrow} + \mathbf{y}, \qquad \mathbf{z}'_{\rightarrow}\mathbb{G}'\mathbb{B} = \mathbf{z}'_{\rightarrow} + \mathbf{v}'. \tag{45}$$

The equation (43) implies that $\mathbf{z}'_{\rightarrow} = \mathbf{e}'_N \mathbf{Z} = \mathbf{e}'_N \hat{\mathbf{z}}_{\downarrow} \mathbb{H} = \mathbf{z}'_{\downarrow}\mathbb{H}$; it allows eliminating the variable $\mathbf{z}_{\rightarrow}$ in turn from the models (40) and (41), which become

$$\mathbb{A}\mathbb{H}'\mathbf{z}_{\downarrow} = \mathbf{z}_{\downarrow} + \mathbf{y}, \qquad \mathbf{z}'_{\downarrow}\mathbb{H}\langle \mathbf{e}'_N \mathbb{A}\rangle = \mathbf{z}'_{\downarrow}\mathbb{H} + \mathbf{v}'. \tag{46}$$

$$\langle \mathbb{B}\mathbf{e}_M \rangle \mathbf{z}_{\downarrow} = \mathbf{z}_{\downarrow} + \mathbf{y}, \qquad \mathbf{z}'_{\downarrow}\mathbb{B} = \mathbf{z}'_{\downarrow}\mathbb{H} + \mathbf{v}'. \tag{47}$$

Each model (44) – (47) consists of $N$+$M$ linear equations with different numbers of unknown scalar variables, namely $N$+2$M$ scalar variables $\mathbf{z}_{\rightarrow}, \mathbf{y}, \mathbf{v}$ as in (44) or (45), and 2$N$+$M$ scalar variables $\mathbf{z}_{\downarrow}, \mathbf{y}, \mathbf{v}$ as in (46) or (47). Therefore, supplementing the exogenous value added condition $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ provides a just identifying closure of the models (44) or (45), whereas an exact identifiability of variables in the models (46) or (47) can be achieved by direct incorporation of the exogenous final demand condition $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$.

If all the matrices in (44) – (47) are square, each model also has a supplementary solution corresponding to alternative exogenous condition (see Section 3). All possible situations for models (44) – (47) are listed in Table 2.

Table 2. The various exact identifiable specifications of linear input–output analysis problem with the matrix-valued linear production functions and exogenous variables

| Model | Model code | Matrices fixed | Vector variables | Number of variables | Exogenous variable | Alternative exogenous variable at $N$ = $M$ |
|---|---|---|---|---|---|---|
| (44) | $\mathbb{A}\mathbb{G}$ | $\mathbb{A}, \mathbb{G}$ | $\mathbf{z}_{\rightarrow}, \mathbf{y}, \mathbf{v}$ | $N$+2$M$ | $\mathbf{v} = \mathbf{v}_*$ | $\mathbf{y} = \mathbf{y}_*$ |
| (45) | $\mathbb{B}\mathbb{G}$ | $\mathbb{B}, \mathbb{G}$ | $\mathbf{z}_{\rightarrow}, \mathbf{y}, \mathbf{v}$ | $N$+2$M$ | $\mathbf{v} = \mathbf{v}_*$ | $\mathbf{y} = \mathbf{y}_*$ |
| (46) | $\mathbb{A}\mathbb{H}$ | $\mathbb{A}, \mathbb{H}$ | $\mathbf{z}_{\downarrow}, \mathbf{y}, \mathbf{v}$ | 2$N$+$M$ | $\mathbf{y} = \mathbf{y}_*$ | $\mathbf{v} = \mathbf{v}_*$ |
| (47) | $\mathbb{B}\mathbb{H}$ | $\mathbb{B}, \mathbb{H}$ | $\mathbf{z}_{\downarrow}, \mathbf{y}, \mathbf{v}$ | 2$N$+$M$ | $\mathbf{y} = \mathbf{y}_*$ | $\mathbf{v} = \mathbf{v}_*$ |

It can be shown that the regular and supplementary solutions of the linear equation systems (44) – (47) under various specifications of an exogenous disturbance have similar algebraic properties as the solutions for models (15) – (18) in Section 3.

Indeed, unknown matrices $\mathbf{X}$ and $\mathbf{Z}$ in the model $\mathbb{A}\mathbb{G}$ (44) according to (38) and (42) are associated with the initial matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ by postmultiplying the latters on the same diagonal matrix both as follows:

$$\mathbf{X} = \mathbb{A}\hat{\mathbf{z}}_{\rightarrow} = \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1} \hat{\mathbf{z}}_{\rightarrow} = \mathbf{X}_0 \hat{\mathbf{q}}, \qquad \mathbf{Z} = \mathbb{G}\hat{\mathbf{z}}_{\rightarrow} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{Z}_0 \rangle^{-1} \hat{\mathbf{z}}_{\rightarrow} = \mathbf{Z}_0 \hat{\mathbf{q}} \tag{48}$$

where $\mathbf{q}$ is a column vector of dimension $M\times1$ strictly identifiable together with the vector of industry expenditures for intermediate consumption $\mathbf{z}_{\rightarrow}$ at any feasible exogenous condition. Hence, the model $\mathbb{AG}$ and its response (48) describe an impact of exogenous changes in final demand or value added just in the terms of the production quantity changing at constant prices for the products.

Further, formulas (39) and (43) imply that in the model $\mathbb{BH}$ (47), in full contrast to model $\mathbb{AG}$, matrices $\mathbf{X}$ and $\mathbf{Z}$ are associated with the initial matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ by premultiplying the latters on the same diagonal matrix both as follows:

$$\mathbf{X} = \hat{\mathbf{z}}_{\downarrow}\mathbb{B} = \hat{\mathbf{z}}_{\downarrow}\langle \mathbf{Z}_0\mathbf{e}_M \rangle^{-1}\mathbf{X}_0 = \hat{\mathbf{p}}\mathbf{X}_0\,, \qquad \mathbf{Z} = \hat{\mathbf{z}}_{\downarrow}\mathbb{H} = \hat{\mathbf{z}}_{\downarrow}\langle \mathbf{Z}_0\mathbf{e}_M \rangle^{-1}\mathbf{Z}_0 = \hat{\mathbf{p}}\mathbf{Z}_0 \tag{49}$$

where $\mathbf{p}$ is a column vector of dimension $N\times1$ unambiguously identifiable together with the vector of product amounts in intermediate use $\mathbf{z}_{\downarrow}$ at any feasible exogenous condition. Therefore, the model $\mathbb{BH}$ and its response (49) characterize an impact of exogenous changes in final demand or value added only in the terms of price changing at constant level of production in the industries.

Finally, for the cases of models $\mathbb{BG}$ (45) and $\mathbb{AH}$ (46) one needs to combine appropriate formulas from (48) and (49) pairwise in a criss-cross manner. So, the models $\mathbb{BG}$ (45) and $\mathbb{AH}$ (46) seem to be implausible artifacts that are out of economic sense, as well as the models **BG** (16) and **AH** (17) in Section 4.

## 9. Regular and supplementary solutions for the models $\mathbb{AG}$ and $\mathbb{BH}$

Under the exogenous value added condition $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$, second equation of system (44) in a very general case $\langle \mathbf{e}'_N\mathbb{A}\rangle \neq \mathbf{E}_M$ can be resolved with respect to the vector of industry expenditures for intermediate consumption, namely

$$\mathbf{z}_{\rightarrow} = \left(\langle \mathbf{e}'_N\mathbb{A}\rangle - \mathbf{E}_M\right)^{-1}\mathbf{v}_*\,. \tag{50}$$

From (48) it follows that $\hat{\mathbf{q}} = \langle \mathbf{e}'_N\mathbf{Z}_0\rangle^{-1}\hat{\mathbf{z}}_{\rightarrow}$, and by substituting (50) we obtain the regular solution for model $\mathbb{AG}$ (44) at the exogenous changes in value added:

$$\mathbf{q} = \hat{\mathbf{q}}\mathbf{e}_M = \langle \mathbf{e}'_N\mathbf{Z}_0\rangle^{-1}\left(\langle \mathbf{e}'_N\mathbb{A}\rangle - \mathbf{E}_M\right)^{-1}\mathbf{v}_* = \left(\langle \mathbf{e}'_N\mathbf{X}_0\rangle - \langle \mathbf{e}'_N\mathbf{Z}_0\rangle\right)^{-1}\mathbf{v}_* = \hat{\mathbf{v}}_0^{-1}\mathbf{v}_*\,. \tag{51}$$

Similarly, under the exogenous net final demand condition $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$, first equation of system (47) in a very general case $\langle \mathbb{B}\mathbf{e}_M\rangle \neq \mathbf{E}_N$ can be resolved with respect to the vector of product amounts in intermediate use as

$$\mathbf{z}_{\downarrow} = \left(\langle \mathbb{B}\mathbf{e}_M \rangle - \mathbf{E}_N\right)^{-1} \mathbf{y}_* . \tag{52}$$

From (49) it follows that $\hat{\mathbf{p}} = \hat{\mathbf{z}}_{\downarrow} \langle \mathbf{Z}_0 \mathbf{e}_M \rangle^{-1}$, so by substituting (52) we get the regular solution for model $\mathbb{BH}$ (47) at the exogenous changes in final demand:

$$\mathbf{p}' = \mathbf{e}'_N \hat{\mathbf{p}} = \mathbf{y}'_* \left(\langle \mathbb{B}\mathbf{e}_M \rangle - \mathbf{E}_N\right)^{-1} \langle \mathbf{Z}_0 \mathbf{e}_M \rangle^{-1} = \mathbf{y}'_* \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \langle \mathbf{Z}_0 \mathbf{e}_M \rangle\right)^{-1} = \mathbf{y}'_* \hat{\mathbf{y}}_0^{-1} . \tag{53}$$

It is easy to see that both these regular solutions are trivial. However, as noted above, at $N = M = K$ each model also has a supplementary solution corresponding to alternative exogenous condition. Under the exogenous final demand condition $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$, first equation of system (44) can be resolved with respect to the vector of industry expenditures for intermediate consumption as

$$\mathbf{z}_{\rightarrow} = \left(\mathbb{A} - \mathbb{G}\right)^{-1} \mathbf{y}_* , \tag{54}$$

of course, if an inverse of square (at $N = M = K$) matrix $\mathbb{A} - \mathbb{G}$ exists as it is expected to be. Note that matrix $\mathbb{A}$ (together with initial production matrix $\mathbf{X}_0$) usually has the dominant main diagonal.

From (48) it follows that $\hat{\mathbf{q}} = \langle \mathbf{e}'_K \mathbf{Z}_0 \rangle^{-1} \hat{\mathbf{z}}_{\rightarrow}$, and by substituting (54) we obtain the supplementary solution for model $\mathbb{AG}$ (44) at the exogenous changes in final demand:

$$\mathbf{q} = \hat{\mathbf{q}} \mathbf{e}_K = \langle \mathbf{e}'_K \mathbf{Z}_0 \rangle^{-1} \left(\mathbb{A} - \mathbb{G}\right)^{-1} \mathbf{y}_* = \left[\left(\mathbb{A} - \mathbb{G}\right) \langle \mathbf{e}'_K \mathbf{Z}_0 \rangle\right]^{-1} \mathbf{y}_* = \left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1} \mathbf{y}_* . \tag{55}$$

Analogously, under the exogenous value added condition $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$, second equation of system (47) can be resolved with respect to the vector of product amounts in intermediate use, namely

$$\mathbf{z}_{\downarrow} = \left(\mathbb{B}' - \mathbb{H}'\right)^{-1} \mathbf{v}_* , \tag{56}$$

if an inverse of square (at $N = M = K$) matrix $\mathbb{B} - \mathbb{H}$ exists as it is expected to be because matrix $\mathbb{B}$ (together with $\mathbb{A}$ and initial production matrix $\mathbf{X}_0$) usually has the dominant main diagonal.

From (49) it follows that $\hat{\mathbf{p}} = \hat{\mathbf{z}}_{\downarrow} \langle \mathbf{Z}_0 \mathbf{e}_K \rangle^{-1}$, and by substituting (56) we get the supplementary solution for model $\mathbb{BH}$ (47) at the exogenous changes in value added:

$$\mathbf{p} = \hat{\mathbf{p}} \mathbf{e}_K = \langle \mathbf{Z}_0 \mathbf{e}_K \rangle^{-1} \left(\mathbb{B}' - \mathbb{H}'\right)^{-1} \mathbf{v}_* = \left[\left(\mathbb{B}' - \mathbb{H}'\right) \langle \mathbf{Z}_0 \mathbf{e}_K \rangle\right]^{-1} \mathbf{v}_* = \left(\mathbf{X}'_0 - \mathbf{Z}'_0\right)^{-1} \mathbf{v}_* . \tag{57}$$

The supplementary solutions (55) and (57) are valid only if the values of $N$ and $M$ are equal to each other, but in contrast to regular solutions (51) and (53), they are not trivial. Besides, the regular and supplementary solutions (51) and (55) exactly coincide with the regular and supplementary solution (28) and (32), i.e., model **AG** (15) with the matrix-valued linear *cost* function (9) and model $\mathbb{AG}$ (44) with the matrix-valued linear *production* function (38) are equivalent. Moreover, the regular and supplementary solutions (53) and (57) also exactly coincide with the regular and supplementary solution (30) and (34), i.e., model **BH** (18) with the matrix-valued linear *cost* function (10) and model $\mathbb{BH}$ (47) with the matrix-valued linear *production* function (39) are equivalent, in their turn.

The latter facts can be appreciated as an ostensive demonstration of general equilibrium in the theory of input–output analysis.

## 10. Discussion of results and concluding remarks

It is advisable to point out four main results developed in this paper.

First, a general linear problem of input–output analysis is formulated as the system of balance equations together with a chosen specification of exogenous disturbance and a corresponding set of linear linkages between the variables, which provides the exact identification of all unknown variables. The factorization of unknown production and intermediate consumption matrices are proposed as a common formal method of using data available for input–output modeling. Thereby the operational base for constructing the various sets of identifying equations in addition to the general balance model is provided. It is to be emphasized that the factorizations (5) – (8) do not exhaust all the options for developing the linkage equations in this way.

Secondly, applying the matrix-valued linear cost functions (9) and (10) with product and industry outputs as their arguments do allow recognizing four different types of the economy response to exogenous changes in net final demand and gross value added in accordance with the models (15) – (18): in terms of the production quantity changing at constant prices for the products, in terms of price changing at constant level of production in the industries, and in terms of combined quantity'n'price and price'n'quantity changes. Two latter types of economy's response (and two corresponding models **BG** and **AH**) seem to be implausible artifacts that are out of economic sense. It is important to note that model **AH** is widely used in practice of input–output analysis for transformation of supply and use tables to symmetric input–output tables in the product-by-product format on the base of industry technology assumption and in the industry-by-industry format on the base of fixed product sales structure assumption.

Thirdly, the models **AG** and **BH** together with its regular and supplementary solutions constitute a general algebraic approach to mutual analysis of Leontief and Ghosh quantity and price models. In particular, for a symmetric input–output table, the equivalence of Ghosh supply-driven model and Leontief price model as well as the equivalence of Leontief demand-driven model and Ghosh quantity model are proved.

Fourthly, involving the matrix-valued linear production functions (38) of industry expenditures for intermediate consumption and (39) of product amounts in intermediate use (based on “quasi-reciprocal” technical and allocation coefficients to be fixed) does not lead to analysis expansion. It is shown that model $\mathbb{AG}$ with production function (38) and model **AG** with cost function (9) as well as model $\mathbb{BH}$ with production function (39) and model **BH** with cost function (10) are pairwise equivalent. These equivalencies can be considered as a testimony to presence of general equilibrium in input–output tables.

The latter facts also mean that certain choice of the technical and allocation coefficients' patterns does not have an influence on results of modeling, contrary to a widely accepted point of view that “the center-piece of input–output analysis is a matrix... of technical coefficients” (ten Raa, 1994, p.4). Therefore, technical and allocation coefficients should be regarded as helpful ways of economic interpretation rather than as basic framework or operational tools for modeling.

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