Regularities in Prices of Production and the Concentration of Compositions of Capitals

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Abstract

Recent developments in price of production models have proposed a hypothesis on the structure of the input coefficient matrices to explain the empirical near-linearity and monotonicity found in prices as a function of income distribution —the tendency towards zero of subdominant eigenvalues. The objective of this paper is twofold: First, based on the behavior of observed eigenvalues, the paper shows that they cannot explain by their own the regularities found in prices of production. Second, it is shown theoretically and empirically the existence and relevance of an additional force acting on the input matrix and the labor coefficient vector: the concentration of industries' vertically integrated compositions of capital around their average. It is argued that the combined effect of these two factors produces the empirical regularities in relative prices. The tendency of the vertically integrated labor to means of production proportions to cluster around their average reveals the existence of an economic force acting on the structure of technology of observable economies and calls for an explanation. The paper relies on the US 1987-2007 Input-Output accounts, at the highest disaggregation level (between 370-466 sectors), for the empirical evidence in this paper.

Keywords: Prices of production, Structure of technology, Concentration of capital compositions, Spectral decomposition, Eigenvalues, Eigenlabors, US economy

JEL Class: B12, B14, B24, B51, C67, D24, D33, D46, D57, J23, L16, L23, 051, P16

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Contents

1 Introduction

One of the main objectives of Sraffa's *Production of Commodities by Means of Commodities* is to study the behavior of prices of production and the profit rate as the effect of changes in the wage rate. In his model of prices of production with single product industries and pure circulating capital, relative prices as a function of income distribution depend on the structure of technology of the whole economy. The components of technology are the input coefficient matrix $\mathbf{A}$ and the labor coefficient vector $\mathbf{l}$, whereas their *individual* characteristics and their *relation* define its structure.

Sraffa, and many after him, consider that relative prices in general do not have a simple behavior as income distribution changes (Sraffa, 1960, pp. 12, 15; Pasinetti, 1977, p. 82; Bidard and Krause, 1996, p. 51). It is argued that as the wage rate moves from its maximum to zero, relative prices might alternate in rising and falling, experiencing *complex* patterns (Sraffa, 1960, p. 15); so that the *normal* case is one with complicated curvatures (Schefold, 1976, p. 26-7). However, it is equally possible that this model generates simple price patterns, like constant or linear functions of income distribution.

The reason for this coexistence in degrees of complexities in price trajectories is the absence of relevant constraints on the structure of technology of the model. Movements in relative prices depend in the last instance on "the inequality of the proportions in which labour and means of production are employed in the various industrie" (Sraffa, 1960, p. 12). The interindustry input-output network contained in the input matrix $\mathbf{A}$ links the proportions of all the industries. Hence, the structure of technology is captured by the labor to means of production proportions, or compositions of capital, of all the industries directly and indirectly intervening in the production of every commodity. However, the only constraints on $\mathbf{A}$ and $\mathbf{l}$ that are usually assumed in the price models are that of non-negativity, indecomposability, primitivity, and productivity of $\mathbf{A}$ and the positivity of $\mathbf{l}$ —all of them consistent with "simple" and "complicated" relative price patterns.

In an effort to characterize the technological structure of observed economies, Schefold (1976) proposes the *Regular Sraffian System*, consisting of an input coefficient matrix with non-zero semi-simple eigenvalues $\lambda_t \neq 0$ and an input matrix-labor vector relation such that $\mathbf{l}$ cannot be orthogonal to any right eigenvector $\mathbf{q}_t^{\mathrm{R}}$ of $\mathbf{A}$ —that is, $\mathbf{l}\mathbf{q}_t^{\mathrm{R}} \neq 0$. Schefold proposes these features on the grounds that *irregular* systems are only possible by a fluke (p. 27). However, these non-equality ($\neq$) constraints on the elements of $\mathbf{A}$ and on its relation with $\mathbf{l}$ only prevents a limited set of outcomes, which actually corresponds to the cases of simple behavior, like constant or linear price functions. Therefore, this regular system maintains a highly unconstrained technological structure, which allows

Schefold to conclude that "prices of production follow a twisted curve in function of the rate of profit" (1976, p. 46).

In contrast with the complex outcomes expected from the price of production model, empirical estimations of these models have consistently produced results in the opposite direction.[1] First, observed standard prices of production (prices measured in terms of Sraffa's standard commodity) are in most cases monotonic functions of the profit rate. Linear and quadratic functions approximate well actual price curves. The same characteristics can be found for industries' output-capital ratios. Second, the wage rate as a function of the profit rate (the wage-profit curve) is also a nearly linear curve.[2] Finally, several distance measures show that prices of production, direct prices (prices proportional to total labor time) and market (observable) prices are close to each other, with small overall deviations.

The reason of this contrast between expected and observed price behavior is the existence of forces acting on the technology of actual economies in such a way that regulates $\mathbf{A}$ and $\mathbf{l}$ (and therefore the labor to means of production proportions) by imposing constraints on them. The observed technological structure, represented by the interindustry economic statistics, is a snapshot of the ongoing process of technical change. Hence, the explanation of the empirical regularities in price of production models must consider the technology structure and the forces that ultimately produce the interindustry configuration of labor and means of production relations. This discussion has been generally absent in the theoretical and empirical literature on prices of production until recently.

In order to explain the empirical regularities in price of production models, recent developments in this literature have studied systematically the structure of $\mathbf{A}$, specifically its eigenvalues. Under the hypothesis that all the eigenvalues, with the exception of the maximum, are zero, Schefold derives linear prices as a function of the profit rate and argues that the observed near-linearity in these prices can be explained by small subdominant eigenvalues (2013a, p. 1177). This hypothesis is also known as the random matrix hypothesis. Following this new perspective,[3] the literature has studied the economic and algebraic properties of the spectral representation of the model, i.e. the expression of

[1] A comprehensive review of this empirical literature can be found in Mariolis and Tsoulfidis (2016b) and Shaikh (2016, chapter 9).

[2] The wage-profit curve as well as any commodity aggregate derived from the price of production model (like the aggregated capital-output ratio) depends, in addition to the technology structure, on the output proportions. In order to present our contribution to the literature as simply as possible, this paper concentrates on individual prices of production and on the structure of technology. However, Section 5 briefly considers the regularities found in wage-profit curve estimations in the light of our results.

[3] See Iliadi et al. (2014), Mariolis and Tsoulfidis (2011, 2014, 2015, 2016a), Nassif and Shaikh (2015), Schefold (2013a,b, 2014, 2016) and Shaikh (2016, ch. 9 and appendix 10).

prices in terms of the eigenvalues and eigenvectors of $\mathbf{A}$, and explored the behavior of eigenvalues in actual economies. They have identified a structural constraint on observed $\lambda_{\mathbf{A},t}$: the vast majority of eigenvalues are clustered around zero. These authors consider that the relevant constraint needed to explain the observed behavior of relative prices acts solely on the eigenvalues, implicitly summarizing the structure of labor to means of production proportions in the information contained in $\mathbf{A}$, with no reference to $\mathbf{l}$.

The purpose of this paper is twofold. First, it provides an assessment of the random matrix hypothesis as a hypothesis to explain the empirical regularities in price of production models. It will be argued on theoretical and empirical grounds that these regularities cannot be explained by assuming a tendency to zero of the subdominant eigenvalues. The main problem with this hypothesis is that empirical evidence shows that observed input matrices, at any level of aggregation, have an important *number* of eigenvalues with considerable magnitude, and this number and the *magnitude* itself increases with the size of the matrix. On the other hand, the spectral representations of the price models by the supporters of this hypothesis omit the effects that the relationship between $\mathbf{A}$ and $\mathbf{l}$ might have on relative prices —i.e. relative prices depend not only on λ_t, on the structure of $\mathbf{A}$, but also on the *eigenlabors*, $l_t^* = \mathbf{l}\mathbf{q}_t^{\mathrm{R}}$, the labor vector representation on the coordinate space given by the left eigenvectors of $\mathbf{A}$. Eigenvalues and eigenlabors jointly determine the labor to means of production proportions.

Second, this paper provides an alternative and empirically relevant structural constraint that is consistent with the regularities in prices of production and the eigenvalues. This technological constraint involves $\mathbf{A}$ and $\mathbf{l}$: the concentration of industries' vertically integrated compositions of capital around their average, that is, the tendency for *total* (direct and indirect) labor to means of production proportions of each industry to cluster around the average value.

When estimating standard prices of production, eigenvalues, and eigenlabors for the US economy (1987-2007) at the highest level possible (between 370-466 industries), it is found the following results: 1) in general, standard prices are monotonic functions of the profit rate; 2) eigenlabors cluster around zero and at a rate higher than the one experienced by the eigenvalues; 3) the deviations between the labor vectors and the Perron left eigenvector cluster around a mean value with a fairly symmetric dispersion; 4) the same can be said about the vertically integrated compositions of capital; and 5) for capital-output ratios, these two evaluated at both direct prices and market prices. Hence, the empirical evidence of eigenvalues and eigenlabors suggests a tendency of observed economies towards *Irregular Systems*, i.e. it indicates that the structure of technology of observed economies is constrained in such a way that their industries have similar means of production proportions and a tendency for their vertically integrated compositions of capital to cluster around their average.

The rest of the paper is organized as follows. Section 2 presents the price of production model and the estimations of standard prices for the US economy. Section 3 presents the assessment of the random matrix hypothesis as the explanation of the empirical regularities in standard price, whereas Section 4 presents our alternative hypothesis. Section 5 discusses the precedents and implications of the results.

2 The Sraffian price of production model

This paper considers the Sraffian price of production model for n single product industries and circulating capital only. Prices of production $\underset{n\times 1}{\mathbf{p}} > \mathbf{0}$, or "prices" from now onwards, are commodities' exchange values that yield profits proportional to the value of the capital invested in each industry.[4] These prices depend on two exogenously given economic factors. On the one hand, they depend on income distribution, that is, the functional distribution between wages and profits. On the other hand, prices depend on technology, i.e. on the uses of labor and means of production. Sraffa constructed this model to study the behavior of prices p_j and the rate of profit r as an effect of changes in the wage rate w, holding output and the technology constant. It is assumed that wages are not part of the value of the capital advanced. Labor is assumed to be homogeneous and each unit is paid the same w, whereas means of production are commodities produced within the system and valuated at prices. The mathematical representation of the technology is given by the non-negative input coefficients matrix $\underset{n\times n}{\mathbf{A}} = \{a_{ij}\}$, $a_{ij} \geq 0$ and the strictly positive vector of direct labor coefficients $\underset{n\times 1}{\mathbf{l}} > \mathbf{0}$, which represent the means of production and labor requirements per unit of gross output ($\underset{1\times n}{\mathbf{x}} > \mathbf{0}$), respectively. The price and the output system are then:

$$\mathbf{p} = (1+r)\,\mathbf{pA} + w\mathbf{l} \tag{1}$$

$$\mathbf{x} = \mathbf{Ax} + \mathbf{y}, \tag{2}$$

[4]The present research will not consider the process of equalization of profitability conditions and competition. Recent evidence of the equalization process can be found in Shaikh (2016, ch. 7, 8) and Scharfenaker and Semieniuk (2016).

where $\underset{1\times n}{\mathbf{y}} \geq \mathbf{0}$ is the exogenously given net output vector.[5] This condition assumes that the system is *productive* in the sense that gross output in each industry at least covers its requirements as means of production $\mathbf{Ax} \leq \mathbf{x}$. Finally, it is assumed that matrix $\mathbf{A}$ is *indecomposable* or that all industries produce basic commodities —i.e, every commodity enters directly or indirectly into the production of all the commodities.

Equation (1) shows that $p_j(r)$ depend on the structure of technology, that is, on the *individual* characteristics of $\mathbf{A}$ and on the *relationship* maintained with $\mathbf{l}$. The technology can be represented in terms of the composition of capital or the proportions of labor to means of production. Hence, the structure of technology generates a structure in the composition of capital. Given that all the industries are interconnected due to its direct or indirect participation in the production of the means of production of every industry, the compositions of capital are also interconnected and its structure involves the whole economy. Hence, it is convenient to re-express the price and output system (1)-(2) in terms of total (direct plus indirect) or vertically integrated quantities of labor, means of production, and gross output required to produce one unit of net output:

$$\mathbf{p} = r\mathbf{pH} + w\mathbf{v} \tag{3}$$

$$\mathbf{x} = [\mathbf{I} - \mathbf{A}]^{-1}\mathbf{y}, \tag{4}$$

where $[\mathbf{I} - \mathbf{A}]^{-1}$ is Leontief's inverse, $\mathbf{H} = \mathbf{A}[\mathbf{I} - \mathbf{A}]^{-1}$ is the total inputs coefficient matrix, and $\mathbf{v} = \mathbf{l}[\mathbf{I} - \mathbf{A}]^{-1}$ is the total labor requirements vector or the labor value vector.

The price systems (1) and (3) have $n + 2$ unknowns (n prices and 2 distributive variables) and n equations. Either r or w is assumed to be given. We will consider the $\mathbf{p}$ and w to be functions of r. By selecting a numeráire, which can be an individual or a composite commodity, and giving values to the rate of profit for the range $0 \geq r \geq R$, where R is the maximal rate of profit, *relative prices* are obtained as a function of the profit rate. The solution of the price system (3) in terms of the numeráire $\underset{1\times n}{\mathbf{z}} \geq \mathbf{0}$ is then:

$$1 = \mathbf{pz} \tag{5}$$

$$\mathbf{p}(s) = w\mathbf{v}[\mathbf{I} - s\mathbf{C}]^{-1} \text{ for } 0 < s < 1 \tag{6}$$

$$w = \left[\mathbf{v}[\mathbf{I} - s\mathbf{C}]^{-1}\mathbf{z}\right]^{-1}, \tag{7}$$

[5] This paper will not consider how the net output is divided between consumption and investment. However, see the remarks in Section 5.

where $s = \frac{r}{R}$ is the relative profit rate and $\mathbf{C} = R\mathbf{H}$. When $s = 0$ relative prices are proportional to their total quantities of embodied labor: $\mathbf{p}(0) = w\mathbf{v}$. When $s = 1$ (or $w = 0$) prices are proportional to the left Perron eigenvector (the eigenvector associated to the maximal eigenvalue): $\mathbf{p}(1) = \mathbf{pC} \propto \mathbf{q}_1^{\mathrm{L}}$.[6]

2.1 Standard prices and their changes as an effect of changes in distribution

The *study* of the behavior of prices of production as an effect of changes in income distribution based on Equations (5) and (6) can be complicated: It is not possible to know if the trajectory of relative prices[7] is caused by the influence of the numerator, the denominator, or both (Sraffa, 1960, p. 18). However, Sraffa (1960, ch. IV,V) shows that if we adopt the standard commodity as the standard of measure of prices, then we can be sure that their movement is in fact caused by the prices we are studying and not by the numeráire. The standard commodity $\mathbf{y}_{\mathrm{S}}$ is a composite commodity the value of which is independent of distribution because it is proportional to the *right* Perron eigenvector $\mathbf{y}_{\mathrm{S}} \propto \mathbf{q}_1^{\mathrm{R}}$. By normalizing $\mathbf{y}_{\mathrm{S}}$ and $\mathbf{p}$ such that $\mathbf{v}\bar{\mathbf{y}}_{\mathrm{S}} = 1$ and $\bar{\mathbf{p}}\bar{\mathbf{y}}_{\mathrm{S}} = 1$,[8] we obtain the standard wage $\bar{w}$ and standard prices $\bar{\mathbf{p}}$:

$$\bar{w} = 1 - \frac{r}{R} = 1 - s \tag{8}$$

$$\bar{\mathbf{p}} = r\bar{\mathbf{p}}\mathbf{H} + \bar{w}\mathbf{v} = \mathbf{v} - r\left(\bar{\mathbf{p}}\mathbf{H} - \frac{1}{R}\mathbf{v}\right) \tag{9}$$

$$= (1 - s)\,\mathbf{v}\left[\mathbf{I} - s\mathbf{C}\right]^{-1}. \tag{10}$$

Sraffa (1960, ch. III) shows that the behavior of relative prices, say $\frac{p_f}{p_g}$, as an effect of changes in distribution depends on the compositions of capital of industries f and g and on those of the industries indirectly participating in the production of their means of production. In an economy producing only basic commodities, relative price movements depends then on the compositions of capital of all the industries. If we take the j-th equation of Equation (9) and rearrange it we can see that price movements depend on the

[6]Matrices $\mathbf{A}$, $\mathbf{H}$, and $\mathbf{C}$ have the same eigenvectors but different eigenvalues. See Section 3.1.

[7]Be they in terms of the numeráire $p_j = \frac{\mathbf{v}[\mathbf{I}-s\mathbf{C}]_j^{-1}}{\mathbf{v}[\mathbf{I}-s\mathbf{C}]^{-1}\mathbf{z}}$ or any other commodity $\frac{p_f}{p_g} = \frac{\mathbf{v}[\mathbf{I}-s\mathbf{C}]_f^{-1}}{\mathbf{v}[\mathbf{I}-s\mathbf{C}]_g^{-1}}$, where $[\mathbf{I} - s\mathbf{C}]_j^{-1}$ is the j-th column of matrix $[\mathbf{I} - s\mathbf{C}]^{-1}$.

[8]That is, by normalizing $\mathbf{y}_{\mathrm{S}}$ and $\mathbf{p}$ in such a way that had the economy would produce under the proportions given by $\mathbf{y}_{\mathrm{S}}$, the total quantity of labor and the value of the net output of this system would be the same and equal to 1. See Pasinetti (1977, p. 93-103) for a full discussion on this.

relationship between the vertically integrated compositions of capital of the j-th industry $\frac{\bar{\mathbf{p}}\mathbf{H}_j}{v_j}$ and that of the industry acting as numeráire: the standard *system* $\frac{1}{R}$ (Bienenfeld, 1988, p. 253-4, Eq. A4):[9]

$$\frac{\bar{p}_j}{v_j} = 1 - r\left(\frac{\bar{\mathbf{p}}\mathbf{H}_j}{v_j} - \frac{1}{R}\right). \tag{11}$$

2.2 Empirical evidence for the US economy 1987-2007

This section estimates standard prices in order to assess the conjecture made by Sraffa (1960) and others that the behavior of relative prices as an effect of changes in income distribution generates complex patterns. For this, Equation (10) is estimated at the highest disaggregation ever employed (between 360-466 industries) and evaluate three aspects of these functions: monotonicity, price-labor value reversal, and visual inspection of their shapes. If a function is monotonic, then it is always either non-increasing (decreasing or constant) or non-decreasing (increasing or constant) —it cannot change direction. Price-labor value reversal is a particular case of non-monotonic behavior: the change in the price and labor value relation $p_j \gtrless v_j$ as the relative profit rate changes. Finally, by giving a visual inspection of standard prices for $0 \geq s \geq 1$ allows us to assess the degree of complexities that price movements might have.

The empirical estimation of standard prices requires one to have empirical estimates of the input coefficient matrix $\mathbf{A}$ and the labor coefficient vector $\mathbf{l}$. We take from Torres and Yang (2016) the input coefficient matrices for the US economy for years 1987, 1992, 1997, 2002, and 2007. Full details of the construction can be found in their appendix. However, these are the main aspects of it: The matrices were constructed from the Use and the Make tables after redefinitions from BEA's Benchmark Input-Output Accounts for the mentioned years under the industry-based technology assumption.[10] A number of manipulations were needed to obtain a non-negative and irreducible (only basic commodities) $\mathbf{A}$, with positive industrial value-added and non-fictitious industries, which involved the elimination of a small number of industries and commodities (between 6 and 4 percent of the industries were discarded).

As for $\mathbf{l}$, Shaikh's (2012, p. 98) approach is followed and $\mathbf{l}$ is estimated as a skilled-adjusted labor vector, under the assumption that differences in skills are represented by wage differentials. Hence, each industry's wage bill (obtained from the Use table referred

[9] The standard system is the collection of all the industries producing basic commodities with technology $(\mathbf{A}, \mathbf{l})$, under output proportions given by the standard commodity, and with the total quantity of labor equal to that of the actual systems.

[10] This accounts are based on a commodity-by-industry distinction and deals with industries' secondary production. See Miller and Blair (2009, ch. 5) for a detailed treatment of this topic.

Concept	1987	1992	1997	2002	2007
Number of Basic Industries	456	466	464	411	370
Number of monotonic prices	331	399	207	320	308
Number of non-monotonic prices	125	67	257	91	62
% of non-monotonic prices	27.4	14.4	55.4	22.1	16.8
Prices with 2 changes in direction	0	2	0	3	2
Number of price-labor value reversals	62	37	245	39	33
% with price-labor value reversal over total	13.6	7.9	52.8	9.5	8.9
% with price-labor value reversals over non-monotonic	49.6	55.2	95.3	42.9	53.2
Maximal number of price-labor value reversals	1	1	1	1	1

Table 1: Monotonicity evaluation of standard prices as a function of income distribution: US economy, 1987-2007. Standard prices are calculated at 21 equally spaced relative profit rates $s = \frac{r}{R} \in [0, 0.99]$. Most industries have monotonic standard prices. Industries that experience non-monotonicity have overwhelmingly only one change in direction. Instances of price-labor value reversal were even less frequent. There were no more than 2 changes in direction and only for 7 cases.

above and for the industries considered in the construction of matrix $\mathbf{A}$) is divided by the economy-wide average wage rate, i.e. by the ratio of total employee compensation to total (full and part-time) employment.

Table 1 shows the evaluation of standard prices calculated for 21 equally-spaced relative profit rates s in the range $0 \leq s \leq 1$: $s = \{0, 0.05, 0.10, ..., 0.95, 0.99\}$. Compared with the previous empirical exercises of the literature on standard prices, the present results are of interest for two reasons. First, due to the level of detail, it is between 5 and 10 times higher than previous exercises. Second, because of its results: Except for the year 1997, the results are consistent with those using highly aggregated information, showing that the empirical regularities found in the literature are structural and that aggregation does not generates the regularities.

The first 4 rows shows that most of the prices are monotonic —only between one fifth and one forth of the prices are non-monotonic. Moreover, except for 7 cases, non-monotonic prices experience only *one* change in direction. The referred 7 cases experienced 2 changes in direction, which were the highest for the five years. The remaining rows explore the price-labor value reversal. About half of the non-monotonic prices experienced price-labor value reversal, which means that the latter cases were even less frequent, with no more that one reverse in direction.

In order to complement the analysis of monotonicity and price-labor value reversal and to have a notion of the actual shapes of standard price, Figure 1 presents a random sample of them, two for each year and for each type of behavior: monotonic, non-monotonic with 1 change in direction, price-labor value reversal, and non-monotonic with 2 changes in direction (for this last one, we present all the 7 cases). Following Shaikh

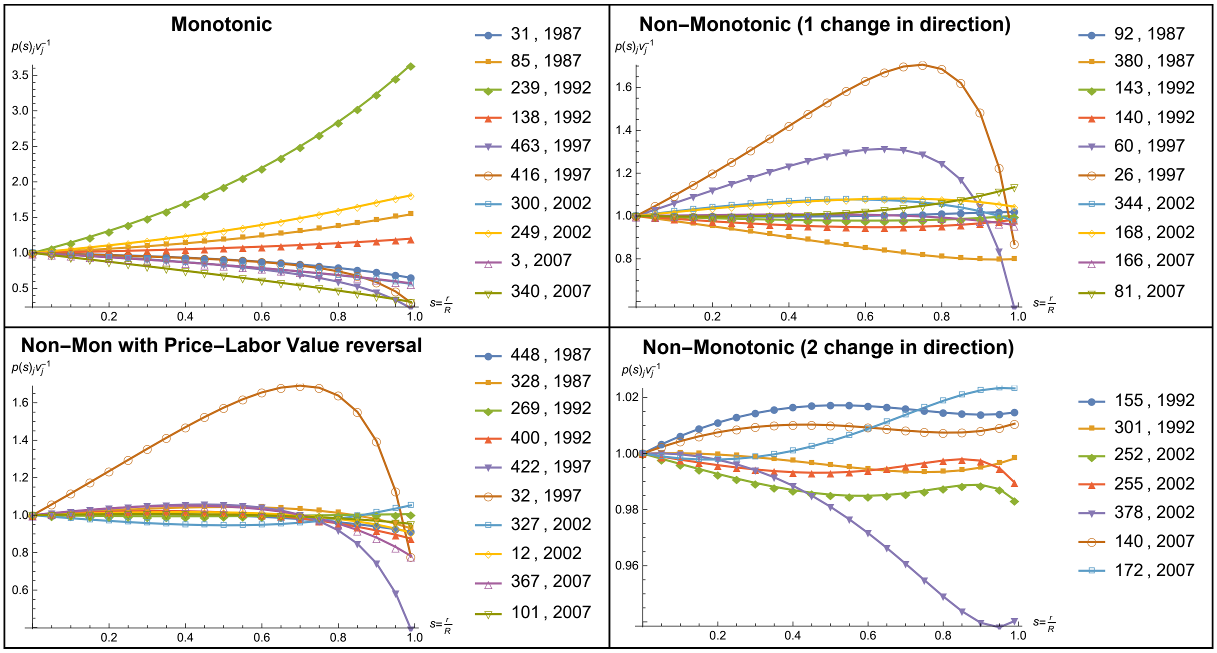

Figure 1: Sample of standard prices, normalized by their labor values, as a function of income distribution by type of behavior: US economy, 1987-2007. The series description indicates the number of industry and the year. Except for Non-Monotonic prices with 2 changes in direction, the rest of the prices are randomly selected (2 for each year). The graphics gives a visual representation of the types of standard prices found in Table 1.

(1998), standard prices are normalized by their labor value (or its standard price at a zero profit rate, $p\left(0\right)_j = v_j$), so they all start at 1. We want to call to attention of two aspects from this figure. First, the randomly selected normalized standard prices give a visual representation of each type of behavior detected in Table 1. Being constrained to normally have between 0 and 1 changes in direction, the degree of curvature of the functions is rather limited. Therefore, the average standard price does not seem to have the complex pattern from Sraffa's (1960) conjecture. Second, if we discard standard prices from 1997, there is a pattern emerging from the scale, partially noticed by Shaikh (1998, p. 230): non-monotonic standard prices present considerable less absolute variability than monotonic prices. Non-monotonic prices are closer to their labor values than monotonic prices. Even more, the standard prices that experience price-labor value reversal and 2 changes in direction (row 2 of Figure 1) have the smallest variability. Of course, this visual appreciation is highly susceptible to the scale of the y axis, that is why the appreciation is in relative terms, comparing monotonic and non-monotonic prices.

3 An evaluation of the random matrix hypothesis to explain the empirical regularities in standard prices

In order to explain the contrast between the expected and observed behavior in standard prices, recent developments in this literature have studied the features of the structure of technology that might explain this paradox.[11] The authors have advanced a hypothesis that constrains the technology and therefore the possible movements of standard prices. The constraint operates only on the input matrix $\mathbf{A}$ and acts particularly on the eigenvalues. By providing an alternative representation of the price model in terms of the eigenvalues and eigenvectors of $\mathbf{A}$, called the *spectral representation*, the authors show how the movements of prices is affected by different eigenvalue configurations. Now is is presented the mathematical results on $\mathbf{A}$ used in the spectral representation of the price model in Section 3.1, whereas Section 2.2 and 3.3 respectively reviews the empirical evidence of eigenvalues and assesses the hypothesis advanced by the modern literature.

We call scalar λ an eigenvalue of $\mathbf{A}$ and $\underset{1\times n}{\mathbf{q}^{\mathrm{L}}} \neq \mathbf{0}$ and $\underset{n\times 1}{\mathbf{q}^{\mathrm{R}}} \neq \mathbf{0}$ respectively the left and right eigenvector corresponding to λ, if $\mathbf{q}^{\mathrm{L}}\mathbf{A} = \lambda\mathbf{q}^{\mathrm{L}}$ and $\mathbf{A}\mathbf{q}^{\mathrm{R}} = \lambda\mathbf{q}^{\mathrm{R}}$ (Meyer, 2001, pp. 490).[12] By the Perron-Frobenius theorem, if matrix $\mathbf{A}$ is non-negative and irreducible, then it has a real, positive, and simple maximum eigenvalue λ_1, associated with a unique positive left $\mathbf{q}_1^{\mathrm{L}} > \mathbf{0}$ and right $\mathbf{q}_1^{\mathrm{R}} > \mathbf{0}$ eigenvector (Meyer, 2001, pp. 673). These vectors will be called the Perron eigenvectors. If in addition $\mathbf{A}$ is productive and primitive, then its maximal eigenvalue is bounded by $0 < \lambda_1 < 1$ and $\lambda_1 > |\lambda|_{t\neq 1}$ (Vegara, 1978, pp. 23; Meyer, 2001, pp. 674). Finally, it is assumed that $\mathbf{A}$ is diagonalizable and, for sake of simplicity, that all its eigenvalues are simple. Under these assumptions, the input matrix has the following spectral decomposition:

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}, \tag{12}$$

where $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, ..., \lambda_n\}$, matrix $\mathbf{Q}$ has as columns eigenvectors $\{\mathbf{q}_t^{\mathrm{R}}\}$ whereas matrix $\mathbf{Q}^{-1}$ has as rows eigenvectors $\{\mathbf{q}_t^{\mathrm{L}}\}$ (Meyer, 2001, pp. 517), and the eigenvalues, ordered according to their modulus, have the following ranking: $1 > \lambda_1 > |\lambda_2| \geq ... \geq |\lambda_n| \geq 0$.

[11]See Iliadi et al. (2014), Mariolis and Tsoulfidis (2011, 2014, 2015, 2016a), Nassif and Shaikh (2015), Schefold (2013a,b, 2014, 2016) and Shaikh (2016, ch. 9 and appendix 10)

[12]We will refer to the eigenvalue of $\mathbf{A}$ with the Greek letter λ. Eigenvalues from different matrices will be indicated in λ's subscript.

The left and right eigenvectors are normal to each other, so (Meyer, 2001, pp. 517)

$$\mathbf{Q}^{-1}\mathbf{Q} = \mathbf{I} \tag{13}$$

$$\mathbf{q}_f^{\mathrm{L}}\mathbf{q}_g^{\mathrm{R}} = \begin{cases} 1 \text{ if } f = g \\ 0 \text{ if } f \neq g. \end{cases} \tag{14}$$

Eigenvectors $\mathbf{q}_t^{\mathrm{R}}$ and $\mathbf{q}_t^{\mathrm{L}}$ are determined up to a scale, that is, only their proportions are uniquely determined. However, no matter the normalization used to obtain each eigenvector, say $\bar{\mathbf{q}}_t^{\mathrm{L}} = k\mathbf{q}_t^{\mathrm{L}}$, such that $\left(k\mathbf{q}_t^{\mathrm{L}}\right)\mathbf{A} = \lambda_t\left(k\mathbf{q}_t^{\mathrm{L}}\right)$, for Equation (12) to hold the normalization must fulfills Equation (14): $\left(k\mathbf{q}_t^{\mathrm{L}}\right)\left(k^{-1}\mathbf{q}_t^{\mathrm{R}}\right) = 1$.

Finally, if $f(z)$ is function defined for all λ_t, then (Meyer, 2001, pp. 517):

$$f(\mathbf{A}) = \mathbf{Q}f(\mathbf{\Lambda})\mathbf{Q}^{-1}. \tag{15}$$

That is, if $f(\mathbf{A})$ is for example a polynomial function of $\mathbf{A}$ and λ_t is an eigenvalue of $\mathbf{A}$, then $f(\lambda_i)$ is an eigenvalue of $f(\mathbf{A})$. It is important to highlight that both $\mathbf{A}$ and $f(\mathbf{A})$ have the same eigenvectors. Therefore, matrix functions such as $[\mathbf{I} - r\mathbf{H}]^{-1}$ in Equation (3) can be equivalently be represented as

$$[\mathbf{I} - r\mathbf{H}]^{-1} = \mathbf{Q}[\mathbf{I} - r\mathbf{\Lambda_H}]^{-1}\mathbf{Q}^{-1}, \tag{16}$$

where $\mathbf{\Lambda_H} = \mathbf{\Lambda}[\mathbf{I} - \mathbf{\Lambda}]^{-1}$ and $\lambda_{\mathbf{H},t} = \lambda_t(1 - \lambda_t)^{-1}$.

3.1 The spectral representation and the random matrix hypothesis

The spectral representation of standard prices in Equation (10) consists in expressing them in terms of the eigenvalues $\lambda_{\mathbf{C},t}$ and eigenlabors v_t^*, that is, in expressing $\mathbf{C}$ and $\mathbf{v}$ in the coordinate space given by the eigenvectors $\left\{\mathbf{q}_t^{\mathrm{L}}\right\}$ and $\left\{\mathbf{q}_t^{\mathrm{R}}\right\}$: $\mathbf{\Lambda_C} = \mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}$ and $\mathbf{v}^* = \mathbf{v}\mathbf{Q} = \left\{v_j^*\right\} = \left\{\mathbf{v}\mathbf{q}_t^{\mathrm{R}}\right\}$, where $\mathbf{\Lambda_C} = \lambda_{\mathbf{H},1}^{-1}\mathbf{\Lambda_H}$, $\lambda_{\mathbf{C},t} = \lambda_{\mathbf{H},1}^{-1}\lambda_{\mathbf{H},t}$, $\lambda_{\mathbf{C},1} = 1$, and $\mathbf{v} = \mathbf{v}^*\mathbf{Q}^{-1} = \sum_{t=1}^{n} v_t^*\mathbf{q}_t^{\mathrm{L}}$. By plugging in Equation (16) into (10) we get:

$$\begin{aligned}\bar{\mathbf{p}} &= (1-s)\,\mathbf{v}\mathbf{Q}[\mathbf{I} - s\mathbf{\Lambda_C}]^{-1}\mathbf{Q}^{-1} \\ &= \sum_{t=1}^{n}\frac{(1-s)}{1-s\lambda_{\mathbf{C},t}}v_t^*\mathbf{q}_t^{\mathrm{L}} = v_1^*\mathbf{q}_1^{\mathrm{L}} + \sum_{t=2}^{n}\frac{(1-s)}{1-s\lambda_{\mathbf{C},t}}v_t^*\mathbf{q}_t^{\mathrm{L}},\end{aligned} \tag{17}$$

For a given particular value of s, Equation (17) expresses standard prices as a linear combination of the left eigenvectors having as weights *two* terms, which represent the *two* aspects of the structure of technology: the eigenvalues $\lambda_{\mathbf{H},t}$ (the individual characteristics of $\mathbf{A}$) *and* the eigenlabors v_t^* (the relationship between $\mathbf{A}$ and $\mathbf{v}$).[13]

We can alternatively re-express solution (10), together with its spectral representation, in terms of dated quantities of labor (Sraffa, 1960, ch. VI). Provided the viability assumption, $\lambda_{\mathbf{H},1} = \frac{1}{R} < 1$, the inverse matrix $[\mathbf{I} - r\mathbf{H}]^{-1}$ is the result of the infinite (but convergent) sum $[\mathbf{I} + r\mathbf{H} + r^2\mathbf{H}^2 + ...]$, so that $\bar{\mathbf{p}} = \left(1 - \frac{r}{R}\right)\mathbf{v}\,[\mathbf{I} - r\mathbf{H}]^{-1}$ is expressed as:

$$\begin{aligned}\bar{\mathbf{p}} &= \mathbf{v} + r\left[\mathbf{v}\mathbf{H} - \frac{1}{R}\mathbf{v}\right] + r^2\left[\mathbf{v}\mathbf{H} - \frac{1}{R}\mathbf{v}\right]\mathbf{H} + r^3\left[\mathbf{v}\mathbf{H} - \frac{1}{R}\mathbf{v}\right]\mathbf{H}^2 + ... \qquad (18)\\ &= \mathbf{v} + r\mathbf{v}^*\left[\mathbf{\Lambda_H} - \lambda_{\mathbf{H},\mathbf{1}}\mathbf{I}\right]\mathbf{Q}^{-1} + r^2\mathbf{v}^*\left[\mathbf{\Lambda_H} - \lambda_{\mathbf{H},\mathbf{1}}\mathbf{I}\right]\mathbf{\Lambda_H}\mathbf{Q}^{-1} + ...\end{aligned}$$

$$\begin{aligned}\bar{p}_j &= v_j + r\sum_{t=2}^{n} v_t^*\left(\lambda_{\mathbf{H},t} - \lambda_{\mathbf{H},1}\right)\mathbf{q}_{t,j}^{\mathrm{L}} + r^2\sum_{t=2}^{n} v_t^*\left(\lambda_{\mathbf{H},t} - \lambda_{\mathbf{H},1}\right)\lambda_{\mathbf{H},t}\mathbf{q}_{t,j}^{\mathrm{L}}\\ &\quad + r^3\sum_{t=2}^{n} v_t^*\left(\lambda_{\mathbf{H},t} - \lambda_{\mathbf{H},1}\right)\lambda_{\mathbf{H},t}^2\mathbf{q}_{t,j}^{\mathrm{L}} + ... \qquad (19)\end{aligned}$$

The conclusion reached by the empirical literature that linear and quadratic functions are accurate approximations of full standard prices, has led the authors working on the spectral representation to identify constraints in the technology structure that might produce these particular functions. If standard prices could be parametrized as $\bar{\mathbf{p}} = \mathbf{a} + r\mathbf{b} + r^2\mathbf{c} + r^3\mathbf{d} + ...$, then constant, linear, and quadratic standard prices would be respectively $\bar{\mathbf{p}}_c = \mathbf{a}$, $\bar{\mathbf{p}}_l = \mathbf{a} + \mathbf{b}r$, and $\bar{\mathbf{p}}_q = \mathbf{a} + \mathbf{b}r + \mathbf{c}r^2$. What then are the economic and mathematical constraints required for standard prices to have such functions?

The case of constant standard prices $\bar{\mathbf{p}}_c = \mathbf{a}$ requires a specific constraint on the relationship between the input matrix and the labor vector: the same (direct and vertically integrated) compositions of capital, and equal to that of the standard industry $\frac{\mathbf{v}\mathbf{H}_j}{v_j} = \frac{1}{R}$, in every industry, so that $\left[\mathbf{v}\mathbf{H} - \frac{1}{R}\mathbf{v}\right] = \mathbf{0}$. This implies that $\mathbf{v} \propto \mathbf{q}_1^{\mathrm{L}}$, so that $\mathbf{v} = v_1^*\mathbf{q}_1^{\mathrm{L}}$ or $v_{t\geq 2}^* = 0$. This constraint does not depend on the eigenvalues because $\mathbf{A}$ is primitive (λ_1 cannot be equal to $|\lambda|_{t\neq 1}$). So $v_t^* = 0$ for $t = 2, ..., n$ is necessarily needed. Hence, because at $r = 0$ we have $\bar{\mathbf{p}} = \mathbf{v}$ and at $r = R$ we have $\mathbf{p} \propto \mathbf{q}_1^{\mathrm{L}}$, the equal compositions of capital constraint on technology (or the zero "subdominant" eigenlabors), imply that

[13] Given that $v_t^* = \frac{l_t^*}{1-\lambda_t}$ and $\lambda_{\mathbf{H},t} = \lambda_t\left(1 - \lambda_t\right)^{-1}$, we can express the spectral representation of standard prices without vertical integration as $\bar{\mathbf{p}} = \left(1 - \frac{r}{R}\right)\sum_{i=1}^{n}\frac{l_j^*}{1-(1+r)\lambda_t}\mathbf{q}_t^{\mathrm{L}} = l_1^*\mathbf{q}_1^{\mathrm{L}} + \sum_{i=2}^{n}\frac{1-(1+r)\lambda_1}{1-(1+r)\lambda_t}l_j^*\mathbf{q}_t^{\mathrm{L}}$. So strictly speaking, the (total) eigenlabor v_t^* captures both the individual characteristics of the input matrix and its relationship with the labor vector. However, for any given eigenvalue, v_t^* is determined by l_t^*.

$\bar{\mathbf{p}} = \mathbf{a} = \mathbf{v} = v_1^* \mathbf{q}_1^{\mathrm{L}}$, i.e., standard prices (prices) are equal (proportional) to labor values at the whole range $0 \leq s \leq 1$.

Under the constraint of zero subdominant eigenvalues $\lambda_{t\geq 2} = 0$, Schefold (2013a, p. 1174) has derived linear standard prices and argues that the observed near-linearities found in the empirical calculations of standard prices can be explained by small subdominant eigenvalues. If $\lambda_{\mathbf{H},t\geq 2} = 0$, then all the quadratic and higher order polynomial terms go to zero and $\bar{\mathbf{p}} = \mathbf{a} + \mathbf{b}r$, where $\mathbf{a} = \mathbf{v}$ and $\mathbf{b} = \sum_{t=2}^{n} v_t^* \left(\lambda_{\mathbf{H},t} - \lambda_{\mathbf{H},1}\right) \mathbf{q}_t^{\mathrm{L}} = (\mathbf{vH} - \frac{1}{R}\mathbf{v})$ or $b_j = v_j \left[\frac{\mathbf{vH}}{v_j} - \frac{1}{R}\right]$.[14] Mariolis and Tsoulfidis (2011, 2014, 2015, 2016a) and Shaikh (2016, p. 440-2, 864-6) have adopted this view as well. Bienenfeld (1988) shows that a quadratic function improves substantially the accuracy of the approximation to full standard prices. Iliadi, et al. (2014, p.4-5), in studying the conditions in which $\mathbf{vC}^{\tau} \to \mathbf{q}_1^{\mathrm{L}}$ for $\tau = 1, 2$, argues that the empirical results in the linear and quadratic approximations are caused by small subdominant eigenvalues.

This constraint on the subdominant eigenvalues comes from Brody's Conjecture or the "random matrix hypothesis" (Shaikh, 2016, pp. 441). Based on a random matrix model[15] and an estimation procedure of the dominant λ_1 and the absolute value of the subdominant eigenvalue $|\lambda_2|$, Brody (1997) conjectures that the spectral gap $\frac{|\lambda_2|}{\lambda_1}$ tends to zero as the size of the random matrix n increases.[16] Brody (1997, p. 256) states that actual "market mechanisms" are "efficient" in the sense that actual price and output proportions "converges in just a few steps" to the equilibrium proportions, suggesting that the feature of the actual economy that generates this efficiency is a technology structure which can be represented with an input matrix with close to zero subdominant eigenvalues. We now present empirical evidence on eigenvalues in order to assess if the random matrix hypothesis can explain the empirical regularities in standard prices.[17]

3.2 Empirical evidence of eigenvalues

Parallel to the developments on the spectral representation of the model, the modern literature on prices of production has conducted an extensive study of eigenvalues of ob-

[14] Bienenfeld (1988) and Shaikh (1998) mention that linear standard prices $\bar{p}_j = a_j + b_j r = v_j + v_j \left[\frac{\mathbf{vH}_j}{v_j} - \frac{1}{R}\right] r$ are the equivalent of Marx's transformation of values into prices of production.

[15] Each column of the input matrix consists of a sample of size n of a positive random variable with a given mean and variance.

[16] Bidard and Schatteman (2001), Goldberg et al. (2000), and Goldberg and Neumann (2003) give proofs of Brody's Conjecture and the same hypothesis on the spectral gap for similar models.

[17] This paper does not evaluate the random matrix models and Brody's Conjecture as adequate representations of the information contained in observed input matrices. It only evaluates the latter as an hypothesis to explain the empirical regularities in standard prices.

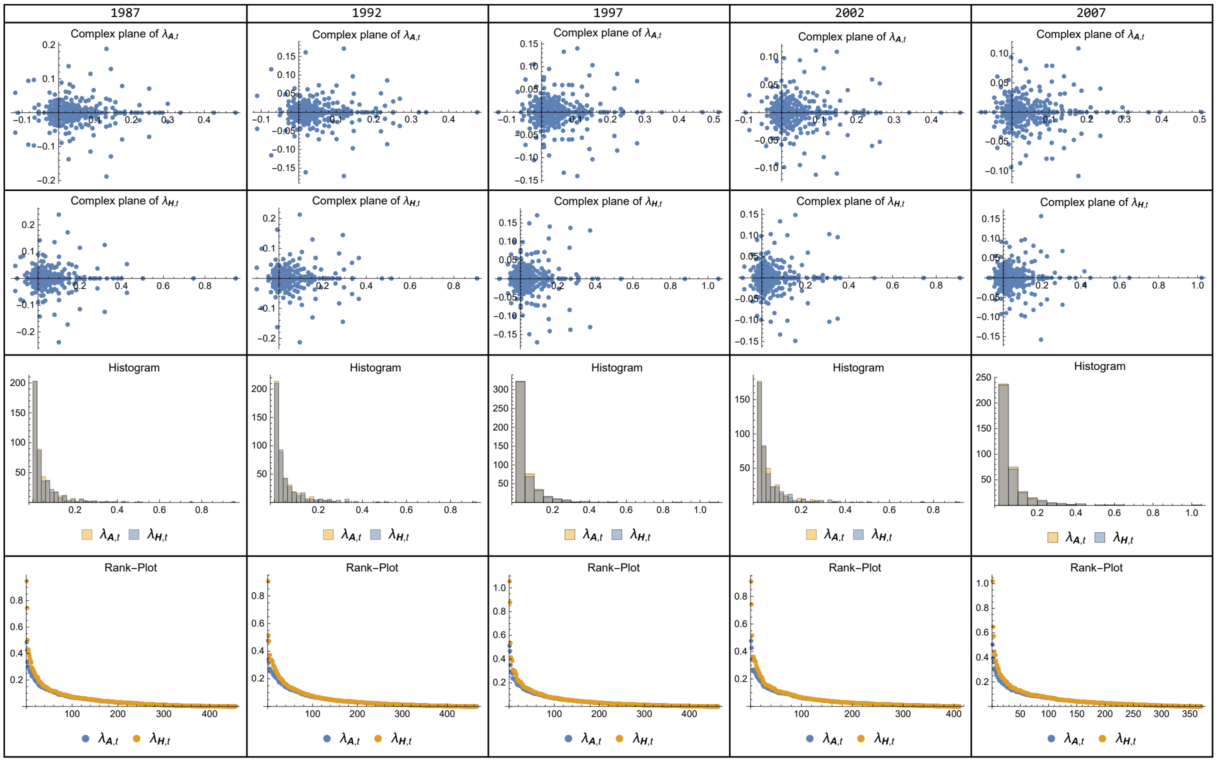

Figure 2: Eigenvalues of the Direct λ_t and the Vertically Integrated $\lambda_{\mathbf{H},t} = \frac{\lambda_t}{1-\lambda_t}$ input coefficient matrices: US economy, 1987-2007. The number of industries is in the range of 370-466. Rows 1 to 4 gives the complex plane, histograms, and rank-plots. There are two main patterns. Although the modulus, the real and the imaginary components of most of the eigenvalues cluster around zero, there is an important *number* of them with considerable magnitude —there is no evidence of zero subdominant eigenvalues.

served input matrices for different countries, years, and aggregation levels.[18] The authors have found the same behavior in every case, indicating the existence of a stylized fact on observable **A**. Our own results displayed in Figure 2 for the US economy for years for 1987-2007, with the number of industries ranging between 370-466, are consistent with their *results*. However, there is a significant difference in our *reading* of these results, which leads to a substantial difference in the assessment of the random matrix hypothesis.

We agree with the literature regarding two statistical characteristics. First, all the graphs show that most of the eigenvalues cluster around zero. That is, relative to the set of all eigenvalues, the real and imaginary components of most of the eigenvalues, as well as their modulus, have small magnitudes. Although this is shared for the eigenvalues of both **A** and **H**, the eigenvalues of the latter cluster strongly around zero. Second, the histogram and rank-plot of eigenvalues' moduli show a fast rate of decline, indicating that

[18]See Iliadi et al. (2014), Mariolis and Tsoulfidis (2011, 2014, 2015, 2016a), Nassif and Shaikh (2015), Schefold (2013a). Gurgul and Wojtowicz (2015) have concentrated on the spectral gap in order to evaluate Brody's Conjecture, with no specific reference to the price of production literature.

only few of them do not cluster around zero. Mariolis and Tsoulfidis (2011) state that an exponential function fits well the rank-plot for all their cases of study.

The difference in the reading of the results relates to the *number* of eigenvalues with considerable magnitude. On the one hand, it has been shown that, contrary to Brody's Conjecture, the spectral gap increases with the level of disaggregation. Mariolis and Tsoulfidis (2014) and Gurgul and Wojtowicz (2015) take two aggregation levels and show respectively for the US and European Union countries that the spectral gap increases considerably. Nassif and Shaikh (2015) takes 176 aggregation levels for the US 2002 input matrix and show that the spectral gap increases in a piecewise fashion from around 0.5 up to 0.85. From the complex plane graphs in Figure 2, we can see that in all cases the fist and second eigenvalue are close together. This evidence suggest that at least we have two significant eigenvalues that, according to Equations (17) and (19), "activates" higher order polynomial terms. However, this is just a particular result from a more general outcome. By plotting the rank of eigenvalues' moduli for eight aggregation levels (15, 17, 30, 63, 120, 161, 342, and 403) Nassif-Pires and Shaikh (2015, figure 4) show that the *number* of eigenvalues with considerable magnitude increases with the matrix size. Even more, it is possible to appreciate that the *magnitude* of the bigger subdominant eigenvalues (not only the second one) increases as well with the size of the matrix. From the rank plot graphs in Figure 2 we can see that there is an important number of eigenvalues which hardly can be considered to be close to zero, compared with the maximum eigenvalue.

3.3 An assessment of the hypothesis

From this section, it is concluded that the empirical regularities in standard prices as a function of income distribution cannot be explained solely by the behavior of observed eigenvalues. Equations (17) and (19) show that the curvature of standard prices is affected by the magnitude of subdominant eigenvalues and that with sufficiently small magnitudes we can approximate nearly linear curves. However, empirical evidence shows that for every aggregation level there is an important number of eigenvalues with considerable magnitude. In addition, this number increases with the size of the matrix. The increase in the number of important eigenvalues is lesser in proportion than the increase in n. This yields to a tendency for most eigenvalues to cluster around zero but cast doubts on the hypothesis that all subdominant eigenvalues tend to zero. So how it is possible to observe nearly linear standard prices and at the same time have an important number of eigenvalues with significant magnitude? The reason is that prices depend not only on income distribution and the eigenvalues, but also on the eigenlabors and eigenvectors.

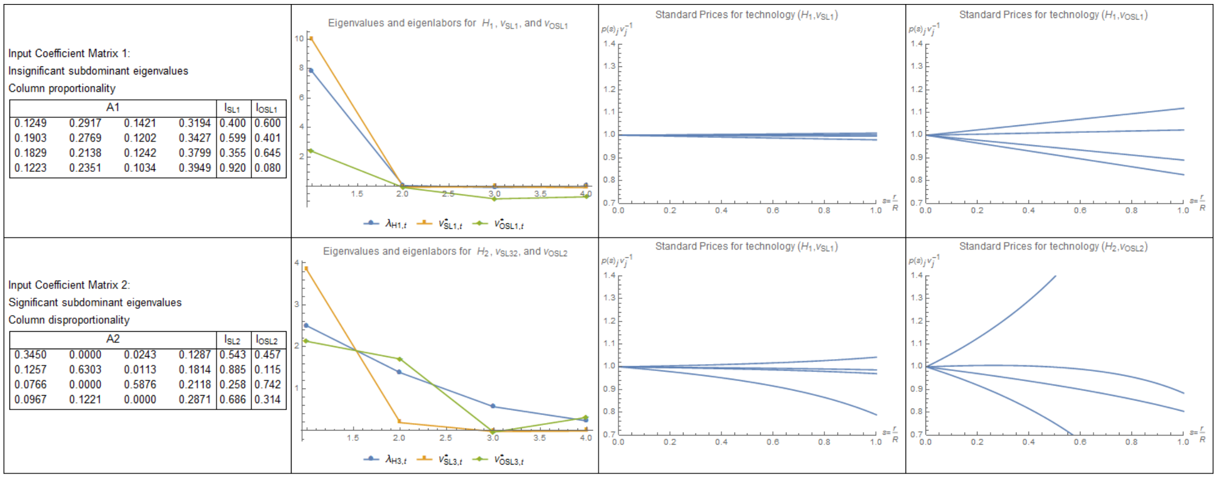

Figure 3: Standard prices $\bar{p}_j(s)$, eigenvalues $(\lambda_{\mathbf{H},t})$, and eigenlabors (v_t^*) for examples of different technology structures. Each row combines a particular input matrix $(\mathbf{A}1, \mathbf{A}2)$ with two opposite total labor vectors $(\mathbf{l})$: nearly proportional $\mathbf{l}_{\text{SL}}$ and disproportional $\mathbf{l}_{\text{OSL}}$ to the Standard Labor proportions of the two matrices. Comparison of columns 3 and 4 shows that the shapes of $\bar{p}_j(s)$ can be altered by varying v_t^*, without any modification of $\lambda_{\mathbf{H},t}$.

The eigenlabors and eigenvectors have not been considered by the authors working on the spectral representation of the model as factors contributing to the regularities in estimated prices. As was mentioned, modern literature has focused its attention solely on the individual characteristics of $\mathbf{A}$ and the eigenvalues, and the constraints on them that would produce the empirical regularities, ignoring the possible effects that the relationship between $\mathbf{A}$ and $\mathbf{l}$ might have on standard prices. An element involved in this omission has been the renormalization procedures of the eigenvectors in the spectral representation of standard prices. By re-scaling the eigenvectors $\bar{\mathbf{q}}_t^{\text{L}} = v_t^* \mathbf{q}_t^{\text{L}}$ in such a way that the new eigenlabors are equal to one, $\bar{\mathbf{v}} = \sum_{t=1}^{n} \bar{\mathbf{q}}_t^{\text{L}}$ so that $\bar{\mathbf{v}}^* = (1, 1, ..., 1)$, the authors have put a veil on the eigenlabors. However, their effect remains intact in the equations.

The eigenlabors seem to have a different effect over standard prices than the eigenvalues in Equation (17): while small eigenvalues tend to generate a constant term out of each term $\frac{(1-s)}{1-s\lambda_{\mathbf{C},t}} v_t^* \mathbf{q}_t^{\text{L}}$, small eigenlabors tend to eliminate the whole term. However, if we look at the alternative representation of standard prices in Equation (17), then it is shown that both eigenvalues and eigenlabors tend to wipe out the coefficients of the k-th order term: $\sum_{t=2}^{n} v_t^* \left(\lambda_{\mathbf{H},t} - \lambda_{\mathbf{H},1}\right) \lambda_{\mathbf{H},t}^{k-1} \mathbf{q}_{t,j}^{\text{L}}$. Figure 3 shows the influence of eigenvalues and eigenlabors on standard prices by constructing 4 examples made out of the combination of two extreme cases of the two aspects of the structure of technology: the individual characteristics of $\mathbf{A}$ and its relationship with $\mathbf{l}$.

As in Figure 1, standard prices $\bar{p}_j$ are divided by their labor value, so all prices start at 1. Columns 1 and 2 present the raw data and the rank-plot graphs for the different eigenvalues and eigenlabors. The first row considers a system with an input matrix with nearly proportional columns, generating small subdominant eigenvalues. The second row is a diagonally dominant matrix with all the eigenvalues with significant magnitude. These two extreme cases of individual characteristics of matrix $\mathbf{A}$ are combined with two different ways in which $\mathbf{A}$ and $\mathbf{l}$ are related. Column 3 shows standard prices when $\mathbf{l}$ is closely proportional to the left Perron eigenvector or what this paper calls the *Standard Labor Proportions* $\mathbf{l} \approx l_1^* \mathbf{q}_1^{\mathrm{L}}$. This closeness generates "subdominant" eigenlabors close to zero $l_{t\geq 2}^* \approx 0$. Column 4 presents standard prices when $\mathbf{l}$ is far from being proportional to the Standard Labor Proportions, so that several eigenlabors have a significant magnitude. It is important to note that the scale of the graphs are the same for all cases.

The comparison of the graphs in column 3 shows that price systems with either high and low subdominant eigenvalues are compatible with nearly linear standard prices, provided that the labor vector is close to the standard labor proportions —i.e. with $v_{t\geq 2}^* \approx 0$. The upper graphs of columns 3 and 4 show that almost linear standard prices are generated by close to zero subdominant eigenvalues, irrespectively of how close or far is the labor vector to the standard labor proportions. Finally, the comparison of column 3 and 4 in row 2 shows how the curvature of standard prices might be affected by the behavior of eigenlabors, even though eigenvalues are significantly different from zero.

4 An additional constraint on technology

From Equations (17) and (19) and the examples in Figure 3, we can see that the movement of standard prices as an effect of changes in distribution is determined not only by the eigenvalues, that is, by the individual characteristics of matrix $\mathbf{A}$, but also by the eigenlabors, or the relationship between $\mathbf{A}$ and $\mathbf{l}$. The empirical regularities of standard prices point to the existence of constraints on the structure of technology. We saw in Section 2.2 how eigenvalues are systematically constrained. Hence, a series of questions arises: Are eigenlabors also constraint? What is the connection between the behavior of eigenlabors and the relationship between the labor vectors and the standard labor proportions. What is the connection of these results with the compositions of capital?

4.1 Eigenlabors and eigenvalues behavior

Figure 4 presents the complex plane, histograms, and rank plots of different aspects of the eigenlabors of the vertically integrated labor requirements v_j^*. There are two results that

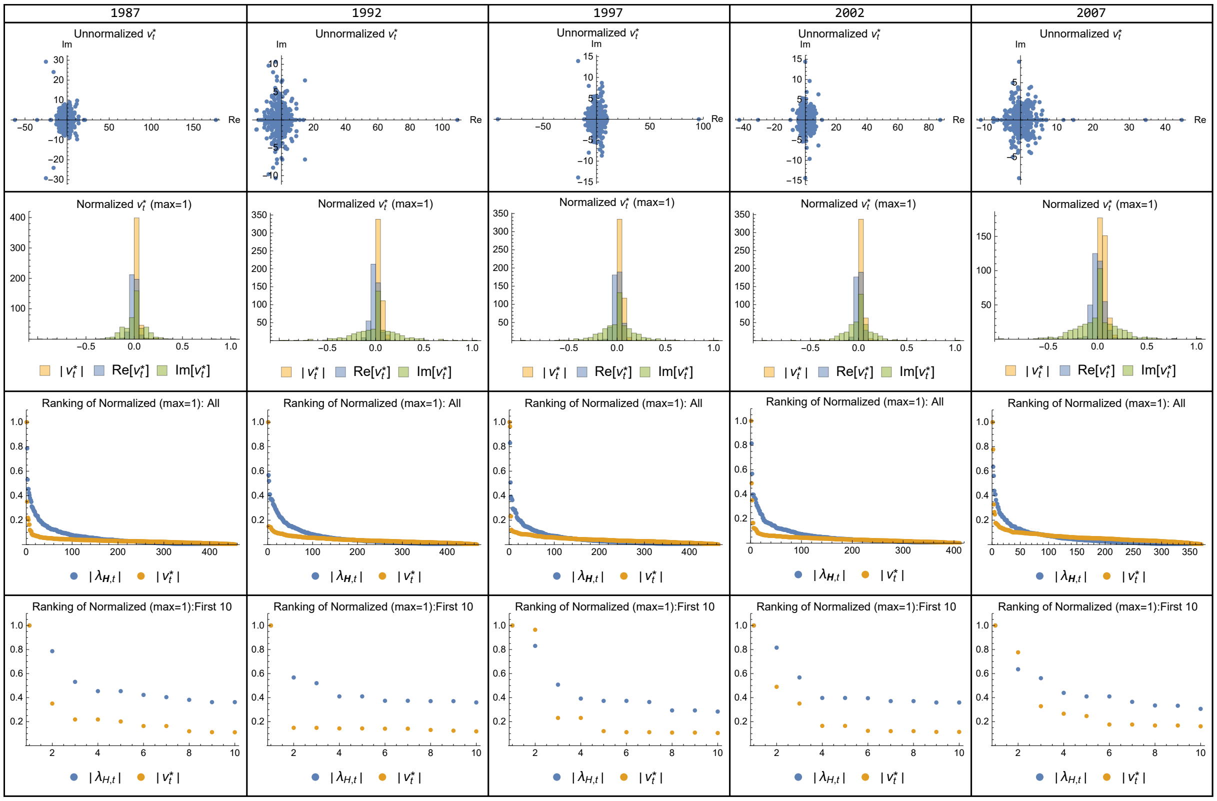

Figure 4: Eigenlabors of the vertically integrated labor coefficients (v_t^*): US economy, 1987-2007. Row 1 and 2 show the complex plane and histograms, whereas row 3 the rank-plots. As the eigenvalues ($\lambda_{\mathbf{H},t}$), the modulus, real, and imaginary components of v_t^* cluster around zero. However, the rate of decay of v_t^* is considerably faster than that for the $\lambda_{\mathbf{H},t}$, yielding a smaller number with considerable magnitude.

are highlighted. First, just as eigenvalues, the eigenlabors tend to cluster around zero and have a fast rate of decay in their magnitudes. Both eigenvalues and eigenlabors belong to the field of complex numbers. Rows 1 and 2 show the real and imaginary components clustering around zero. The empirical frequency distribution of the imaginary part, by the algebraic properties of the eigenvalues, is symmetric, whereas the real parts show a small bias towards negative values. The histogram of the absolute value is positively valuated, and, like the eigenvalues, shows a rapid decay. Second, as shown by the rank-plots in row 3 and 4, compared with the eigenvalues, eigenlabors have a faster rate of decline so there are fewer eigenlabors with significant magnitude. Overall, only one or two eigenlabors have a magnitude that is substantially different from the rest.[19]

[19]There is one aspect that it is briefly mentioned here and in the Appendix and that needs further research. Although scatter plots of the absolute value and the real and imaginary parts of the eigenvalues and the eigenlabors do not show a clear relationship, the eigenlabors with higher magnitude belong to the eigenpairs $\left(\lambda_t, \mathbf{q}_t^{\mathrm{R}}\right)$ containing the eigenvalues with the highest magnitude.

4.2 The relationship between the labor coefficient vectors and the Standard Labor proportions

The conclusion that in general only one or two eigenlabors have a magnitude that is substantially different from the rest, which are clustered around zero, suggests that the labor vector is in some sense close to the standard labor proportions. As it was already shown, the vertically integrated labor vector can be expressed as a linear combination of the n linearly independent left eigenvectors $\mathbf{v} = \sum_{t=1}^{n} v_t^* \mathbf{q}_t^{\mathrm{L}} = \mathbf{v}^* \mathbf{Q}^{-1}$. If $v_{t\geq 2}^* = 0$, then $\mathbf{v} = v_1^* \mathbf{q}_1^{\mathrm{L}}$.[20] Hence, it is of interest to study the relationship between $\mathbf{v}$ and $\mathbf{q}_1^{\mathrm{L}}$.

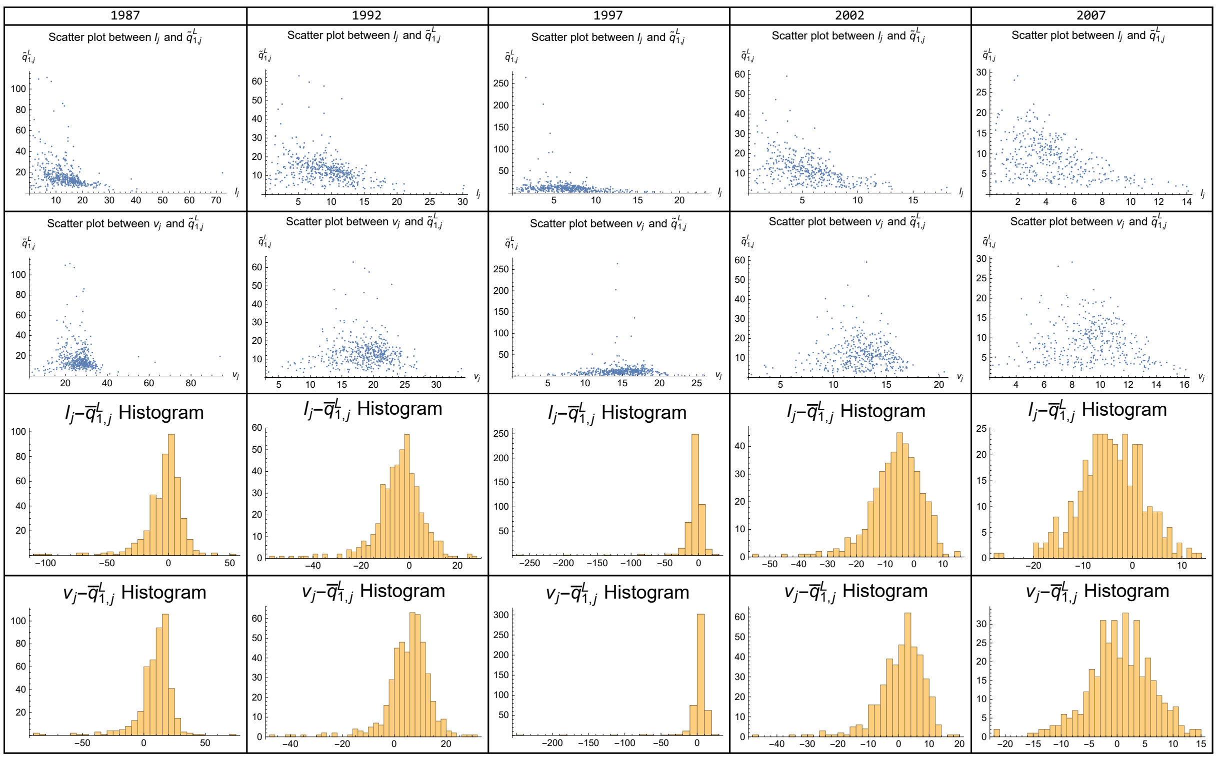

Figure 5: The relationship between the direct ($\mathbf{l}$) and total ($\mathbf{v}$) labor coefficient vectors and the Standard Labor Proportions $\left(\bar{\mathbf{q}}_1^L\right)$ (left Perron eigenvector): US economy, 1987-2007. Rows 1 and 2 show the scatter plots and rows 3 and 4 present the histogram of the their differences. If the scatter plots do not show a clear pattern between the pairs $(l_j, \bar{q}_{1,j}^L)$ and $(v_j, \bar{q}_{1,j}^L)$, the histograms in rows 3 and 4 show a tendency of their deviations to cluster around their average, with a highly symmetric dispersion.

Figure 5 presents different graphics that consider the relationship of both $\mathbf{l}$ and $\mathbf{v}$ with $\bar{\mathbf{q}}_1^{\mathrm{L}}$, the normalized standard labor proportions. The normalization condition is such that we have equal total employment under both proportions: $L = \mathbf{l}\mathbf{x} = \bar{\mathbf{q}}_1^{\mathrm{L}}\mathbf{x}$. Row 1 and 2 present the scatter plots of the labor vectors and the standard labor proportions,

[20] If $\mathbf{0} = \sum_{i=2}^{n} v_t^* \mathbf{q}_t^{\mathrm{R}}$, or as a matter of fact $\mathbf{0} = \sum_{i=2}^{k} v_t^* \mathbf{q}_t^{\mathrm{R}}$ for $2 \leq k \leq n$, then the eigenlabors, the coefficients in that linear combination, have to be zero, because the $\left\{\mathbf{q}_t^{\mathrm{L}}\right\}$ are linearly independent.

whereas rows 3 and 4 show the histograms of their deviations. The scatter plots show no clear sign of relationship between the labor vectors and the standard labor proportions. The relationship between $\mathbf{l}$ and $\bar{\mathbf{q}}_1^{\mathrm{L}}$ seems to suggest a non-linear negative relationship but that of $\mathbf{v}$ with $\bar{\mathbf{q}}_1^{\mathrm{L}}$ is less clear. In spit of these associations, it is interesting that the deviations between $\mathbf{l}$ and $\bar{\mathbf{q}}_1^{\mathrm{L}}$ and between $\mathbf{v}$ and $\bar{\mathbf{q}}_1^{\mathrm{L}}$ gives a smooth, unimodal, and close to symmetric empirical frequency distribution, with few negative outliers coming from some extraordinary high values in the re-scaled standard labor proportions. The same type of distribution is repeated for the whole period, even though 1997 gives a more visual sharpness due to the significant outliers. Hence, the empirical frequency distributions show the existence of a systematic force maintaining the labor vectors and the standard labor proportions close to each other, that is, a force generating a certain relationship between the labor vector and the input matrix.

4.3 The concentration of industries' vertically integrated compositions of capital around the average

We have seen that most of both eigenlabors and eigenvalues cluster around zero and only a few of them have a considerable magnitude. In addition, there is a smooth, unimodal, and close to symmetric empirical frequency distribution in the deviations between the labor vectors and the standard labor proportions. This indicates the existence of relevant constraints acting on the technology and producing the empirical regularities in standard prices according to the spectral decompositions in Equations (17) and (19). However, it is still needed to map the findings and constraints in Figures 4 and 5 in terms of the compositions of capital, the ultimate source of relative price movements as an effect of changes in income distribution. The extreme case of constant prices happens when there are equal compositions of capital, i.e. when the labor vectors are proportional to the standard labor proportions —when $l^*_{t\geq 2} = v^*_{t\geq 2} = 0$. Hence, evidence in Figures 4 and 5 suggests that industries's compositions of capital should *not* be far from each other.

Figure 6 shows the empirical frequency distributions of several variables associated with the compositions of capital —each variable shows a remarkable consistency for the whole period. Row 1 presents the *direct* composition of capital of the different industries valuated at labor values $p(0)_j = v_j$: $k_j(0) = \frac{\mathbf{p}(0)\mathbf{A}_j}{l_j} = \frac{\mathbf{v}\mathbf{A}_j}{l_j}$. There is a smooth empirical frequency distribution where the *value* compositions of capital tend to cluster around a unique mode but with a significant skew towards zero, the lower bound, and with a long right tail, produced by few outliers, specially for 2007. Row 2 shows the histogram of the

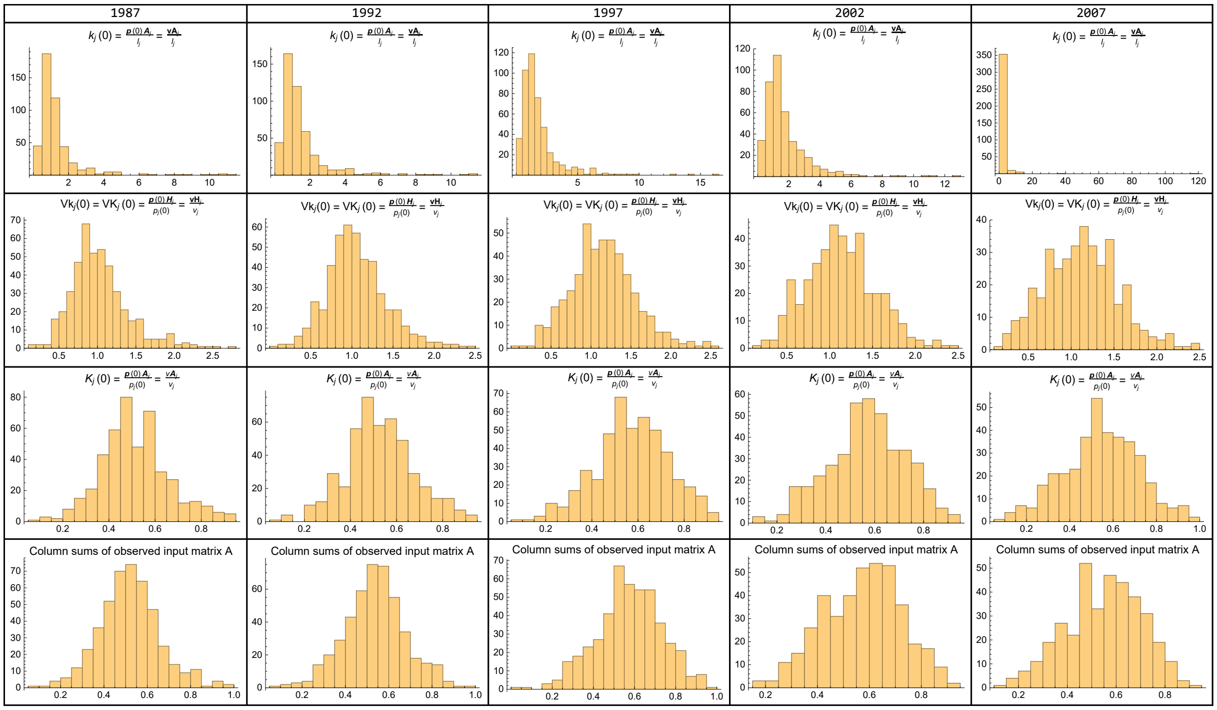

Figure 6: Histogram of industries' composition of capital and capital-output ratios: US economy, 1987-2007. Rows 1 and 2 present industries' direct k_j and vertically integrated Vk_j composition of capital evaluated at direct prices $p_j(0) = v_j$. Rows 3 and 4 show the capital-output ratios at direct prices $K_j(0)$ and at market prices (i.e. column sums of the observed $\mathbf{A}$). All graphs show a smooth and unimodal empirical frequency distributions (EFD). Except for k_j, the EFDs are highly symmetric. The type of EFD in every row is consistent in every year. Overall evidence suggests a tendency for the compositions of capital and capital-output ratios to cluster around their averages.

vertically integrated value compositions of capital $Vk_j\,(0) = \frac{\mathbf{p}(0)\mathbf{H}_j}{v_j} = \frac{\mathbf{vH}_j}{v_j}$. There is also a smooth and unimodal empirical frequency distribution as well, however with a significant reduction in the skewness —we can see that it is almost symmetric. One additional finding is that the outlier values from row 1 seem to be reduced in the vertically integrated case. Hence, the empirical evidence from industries' value composition of capital suggests a tendency for their concentration around an average value, pointing towards a constraint on the dispersion from this central value.

There is a close relationship between compositions of capital and capital-output ratios in the price of production model: Under the assumption of uniform wages and profit rates, a tendency for the equalization of capital-output ratios implies a tendency for the equalization of compositions of capital. The vertically integrated value composition of capital $Vk_j\,(0)$ also corresponds to industries' *vertically integrated* capital-output ratios when prices are proportional to labor values: $VK_j\,(0) = \frac{\mathbf{p}(0)\mathbf{H}_j}{p_j(0)} = \frac{\mathbf{vH}_j}{v_j} = Vk_j\,(0)$. Rows 4 and 5 show the *direct* capital-output ratios valuated at prices proportional to labor values

$K_j(0) = \frac{\mathbf{p}(0)\mathbf{A}_j}{\mathbf{p}(0)} = \frac{\mathbf{v}\mathbf{A}_j}{\mathbf{v}}$ and at market prices. The latter corresponds to the column sums of the constructed input matrix from national accounts. These two variables also show a smooth, unimodal, and highly symmetric empirical frequency distribution, suggesting also a concentration tendency of capital-output ratios around their average.

5 Discussion of the results

Sraffa did not construct his model to explain the price of production dynamics caused by *actual* changes in income distribution. Studying prices of production as a function of actual income distribution while keeping the technology constant would discard one of the most important linkages in capitalist economies. Sraffa designed his model to study "such properties of an economic system as do not depend on changes in the scale of production or in the proportions of 'factors' [compositions of capital]" (Sraffa, 1960, p. v). The objective of this paper is to use his framework to identify the structure of technology of actual economies that, under the assumption of uniform rates of wage and profit, would produce the empirical regularities found in the literature.

One major implication of the empirical behavior of eigenvalues, eigenlabors, labor vectors-standard labor proportion deviations, compositions of capital, and capital-output ratios is the existence of systematic forces shaping the structure of technology in actual economies. Particularly, the empirical evidence shows that the technology is constrained towards an *Irregular* Sraffian System, that is, away from the two conditions (constraints) put forward by Schefold (1976) that would represent the characteristics of observable economies: non-zero semi simple eigenvalues and non-zero eigenlabors (the non-orthogonality of the labor vector and the right eigenvectors $\mathbf{l}\mathbf{q}_t^{\mathrm{R}} = \mathbf{l}_t^* \neq 0$).

In spite of its importance for prices of production, there has not been much discussion about the interindustry structure of technology in the history of economic thought. The vast majority of the constraints on technology and its relationship with income distribution is located at a macroeconomic level. For instance, although Ricardo concludes that the effects of changes in income distribution on relative prices are small and that relative price movements are dominated by relative quantities of labor bestowed in their production, he does not describe the interindustry structure of technology that produces this result. One exception is Marx, who addresses the interindustry structure in both the schemes of reproduction and in the transformation of values into prices of production. In the discussion of the latter, Marx actually states that there is a *tendency towards the equalization* of the organic composition of capital around the social average:

> The capital invested in some spheres of production has a mean, or average, composition, that is, it has the same, or almost the same composition as the average social capital ... *Competition* [emphasis added] so distributes the social capital among the various spheres of production that the prices of production in each sphere take shape according to the model of the prices of production in these spheres of average composition (Marx, 1967, p. 173).

> But it is evident that the balance among spheres of production of different composition must tend to equalize them with the spheres of average composition, be it exactly or only approximately the same as the social average. Between the spheres more or less approximating the average there is again a tendency toward equalization, seeking the ideal average, i.e., an average that does not really exist, i.e., a tendency to take this ideal as a standard (Marx, 1967, p. 173).

Marx considers that competition is the economic force that produces the tendency of compositions of capital towards the social average. However, he does not explain the mechanisms and the process that would produce this result.

Later on, the case of equal compositions of capital became hotly debated when Samuelson (1962) used this assumption in the construction of his surrogate production function. In the recent literature on prices of production, Shaikh (2016, p. 386-8) maintains that industries' vertically integrated profit-wage ratios of observed economies tend to be alike. When comparing the direct and vertically integrated profit-wage ratios, Shaikh finds that the standard deviation of the latter constitutes one third of the former for the US economy in 1998. Under the assumption of uniform wage and profit rates, the tendency towards small deviations in the profit-wage ratios, valuated at market prices, implies small deviations in the market valuated compositions of capital. However, he uses this result to argue that relative market prices are mainly determined by relative unit labor costs and does not consider the behavior of profit-wage ratios to explain the empirical regularities in standard price movements as an effect of changes in income distribution. Finally, Mariolis and Tsoulfidis provide aggregated distance measures between the direct labor vector $\mathbf{l}$ and the standard labor proportions $\mathbf{q}_{\mathrm{I}}^{\mathrm{L}}$ and conclude that they are significant, leading them to state that their studied economies "deviate considerably from the equal value composition of capital case" (2015, p. 10; 2016a, p. 5)

Given the findings in this paper, two important tasks are to be addressed. First, the construction of economic hypotheses that explain the concentration of the compositions of capital and the statistical structure identified in this paper. The technology, represented in different angles by the eigenvalues, eigenlabors, compositions of capital, and capital-output ratios, has a structure and must be explained.

Second, the study the implications that this paper has for other branches in the price of production literature. For instance, the regularities in the close relationship between market prices, prices of production, and direct prices in actual economies, mentioned in the introduction of this paper, could be approached from the structure of technology identified in this paper. Smith, Ricardo, and Marx believed that market prices tend to (turbulently) gravitate around prices of production. The mobility of capital produced by the discrepancies in these prices generates a stable distribution of capital among the different branches, which, combined with the average composition of capital in each sector, determines the social division of labor.[21] However, under this framework, there is no reason to argue that this social division of labor would be such that would produce prices of production, and therefore market prices, proportional to total labor time —and yet these are the results from the empirical literature.[22] The tendency of value compositions of capital to cluster around their average can help explain this behavior.

Another area of study of the implications of our spectral representation and statistical approach is the wage-profit relation, a branch that has captured the most attention in the price of production literature since the Cambridge Capital Controversies. The reason for not considering this variable in this paper is that it involves, as any other composite commodity or macroeconomic aggregate, another dimension in the social coordination of capitalism: output proportions, i.e. the quantity side of the price of production model.

Let us consider the wage-profit curve $w(r)$,[23] taking as numeráire the gross output valuated at prices of production $\mathbf{px} = 1$, and the economy-wide capital-output ratio $\kappa(r)$:

$$w(r) = \left[\mathbf{v}(\mathbf{I} - r\mathbf{H})^{-1}\mathbf{x}\right]^{-1} \tag{20}$$

$$\kappa(r) = \frac{\mathbf{pAx}}{\mathbf{px}} = \frac{\mathbf{v}(\mathbf{I} - r\mathbf{H})^{-1}\mathbf{Ax}}{\mathbf{v}(\mathbf{I} - r\mathbf{H})^{-1}\mathbf{x}}. \tag{21}$$

As in the case of standard prices, the modern literature working on the spectral decomposition of the price of production model has concentrated on the structure of the input matrix and considers that the constraint in having small subdominant eigenvalues explains the near-linearities found in the empirical estimations of wage-profit curves. The

[21] See Rubin (2007, ch. 18) and Foley (2011, 2016) for an exposition.

[22] See the revision of the literature in Mariolis and Tsoulfidis (2016b, ch. 3) and Shaikh (2016, ch. 9).

[23] By substituting Equation (6) into $\mathbf{p}$ in the numeráire and solving for w.

spectral representation of Equations (20) and (21) is:

$$w(r) = \frac{1}{\sum_{t=1}^{n} \frac{v_t^* x_t^*}{1 - r\lambda_t}} \tag{22}$$

$$\kappa(r) = \sum_{t=1}^{n} \delta_t \lambda_t, \tag{23}$$

where $\delta_t = \frac{\frac{v_t^* x_t^*}{1 - r\lambda_{\mathbf{H},t}}}{\sum_{t=1}^{n} \frac{v_t^* x_t^*}{1 - r\lambda_{\mathbf{H},t}}}$ and $\sum_{t=1}^{n} \delta_t = 1$. The authors working under this approach have ignored the effects that not only eigenlabors v_t^* might have, but also the eigenoutputs x_t^*. Hence, given that $\mathbf{x} = [\mathbf{I} - \mathbf{A}]^{-1} \mathbf{y}$ (where the net output $\mathbf{y}$ must be decomposed into the consumption and investment), composite commodities and macroeconomic variables will depend not only on the structure of technology but also on the *structure of demand*.

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Appendices

A Eigenvalues-eigenlabors relationship

The indexation of the t-th eigenlabor $v_t^* = \mathbf{v}\mathbf{q}_t^{\mathrm{L}}$ depends on the eigenpair $\left(\lambda_{\mathbf{H},t}, \mathbf{q}_t^{\mathrm{L}}\right)$ is associated with. The rank-plots from Figure 4 showed a strong decay rate in the magnitude of eigenlabors based on the rank order of the eigenvalues. That is, eigenlabors with the highest magnitude are associated with the eigenvalues with higher magnitude. However, based on the scatter-plot of the modulus, real, and imaginary components of the eigenvalues and eigenlabors in Figure 7, we can see that there is no clear relationship between those variables. However, more work is needed to identify the relationship between them.

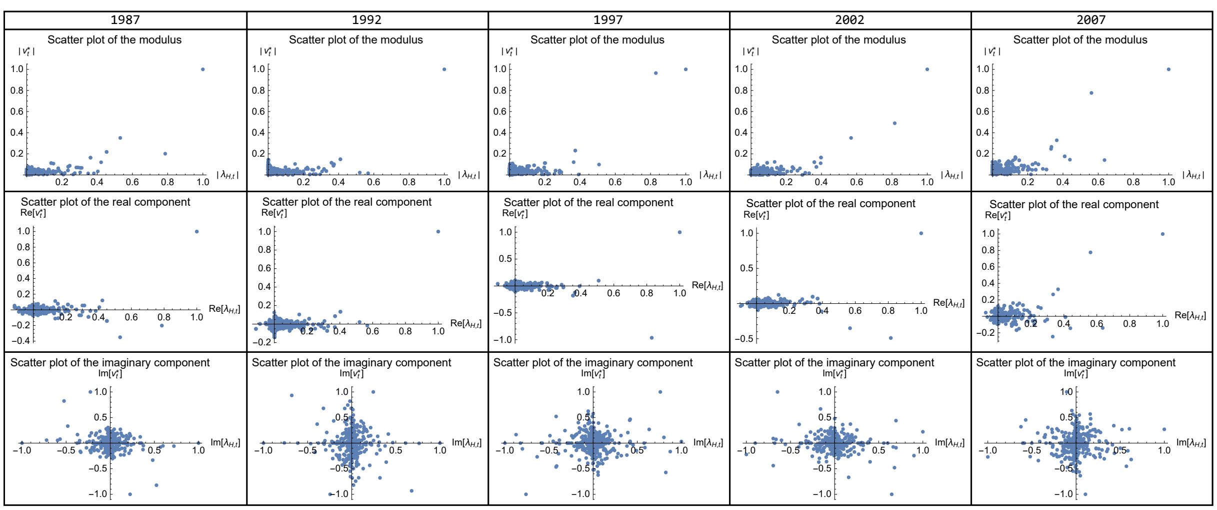

Figure 7: The relationship between the t-th pairs of modulus, real, and imaginary components of the eigenlabors (v_t^*) and the eigenvalues $(\lambda_{\mathbf{H},t})$: US economy, 1987-2007. Each series is normalized according to their yearly maximal value. There seems to be no clear relationship between v_t^* and $\lambda_{\mathbf{H},t}$.