Determinants of country positioning in global value chains*

Kirill Muradov†

May 2017

Abstract

This paper enhances the technique to identify country and industry positions in global value chains. The modified GVC production line position index can be computed at any level of aggregation and has the desired statistical properties. This allows the author to explore whether all countries can simultaneously upgrade their positions in global value chains and how the country GVC positions respond to the global changes in the underlying factors. The relevance of upgrading GVC positions is also critically reviewed. For an empirical application of the proposed analytical indicators, the paper utilises the 2015 edition of the OECD Inter-Country Input-Output (ICIO) tables.

1 Introduction

Available evidence confirms that global value chains (GVCs) can be an important avenue for developing countries to build productive capacity and to integrate in the world economy at lower costs. But the gains from GVC participation are not automatic and require careful policy-making. According to the OECD, WTO and UNCTAD, "For policy makers, a starting point for the incorporation of GVCs in a development strategy is an understanding of where their countries and their industrial structures stand in relation to GVCs" (OECD et al., 2013).

Naturally, countries differ in where they are located in the value chain. Position upstream in the value chain means that production requires mostly primary inputs, and outputs are supplied to intermediate users. This is typical for producers of raw materials or knowledge (e.g. research, design) that are required at the beginning of the production process. Position downstream means that production requires more intermediate inputs, and outputs are supplied to final rather than intermediate users. Producers located downstream often specialise in assembling processed goods and providing customer services. The relative position in the GVC can change over time.

Positioning closer to the beginning of the production process is generally believed to secure higher value added shares and increase technological sophistication (OECD, 2013). Therefore, moving upstream or upgrading country position in GVCs are current policy priorities for many countries.

This paper refines a technique to identify country and industry positions in GVCs and explores how and why these positions evolve over time. Production, trade and consumption

*Paper prepared for the 25th International Input-Output Conference (Atlantic City, New Jersey, USA, June 19-23, 2017).

†Centre for Information and Analysis of Foreign Trade, Moscow, Russia; e-mail: kmuradov@mail.ru.

in the global economy are described in an inter-country input-output framework. Technically, the author isolates global value chain from domestic value chain and measures its length with respect to a country or an industry in two directions, forwards to the destination of outputs and backwards to the origin of inputs. Length corresponds to the average number of production stages a typical product has to undergo along this value chain. The key analytical indicator is a modified GVC position index that relates the average number of production stages that link output to final users to the average number of production stages that link the same output to primary producers through GVCs.

The paper goes further than Wang et al. (2016) who first proposed the GVC production line position index in 2016. First, it modifies and, hopefully, simplifies the calculation of the index, also ensuring its useful statistical properties. Second, it performs a structural decomposition of the modified GVC position index to isolate the impact of such factors as country and industry of origin of imported input requirements, changes in technology and the outsourcing effect, structure and total value of final demand at aggregate country level. Third, it critically reviews the message that upgrading country position in global value chains helps retaining higher value added share.

For an empirical application of the proposed analytical indicators, the paper utilises the 2015 edition of the OECD Inter-Country Input-Output (ICIO) tables. Calculations cover 61 countries plus rest of the world, 34 industries and the years 2000, 2005, 2008 and 2011. Perhaps the most important finding is that global value chains appear to be an equilibrium system where some countries can be positioned upstream only if other countries are positioned downstream. Country positions are not independent from each other, and upgrading the position of one country will most likely cause downgrading the positions of some other countries

The paper proceeds as follows. Section 2 explains the concepts underlying the measurement procedures and reviews the setup of the inter-country input-output framework. Section 3 discusses the results with appropriate visualisations. Section 4 provides a summary and conclusions. For the interested reader, Appendices B-G describe the system of analytical indicators of structure and length of value chains that serves as a foundation to the GVC position index.

2 Concepts and analytical framework

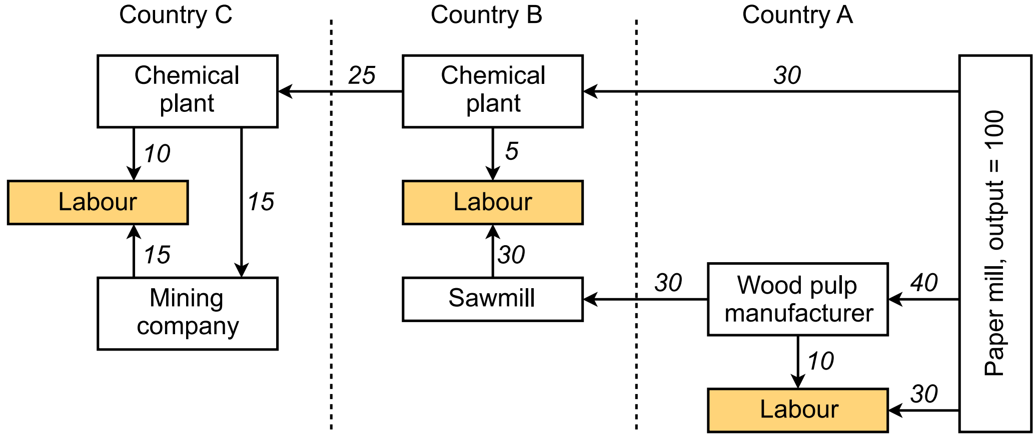
2.1 An illustrative example: where a paper mill is positioned in value chains?

For an illustrative introduction to the ideas explored in this paper, let us consider an example of a paper mill in Fig. 1. Suppose that the paper mill produces paper and cardboard worth 100 units of which 40 are supplied to a packaging manufacturer in the home country A and 60 are sent to a printing facility in country B. There are three value chains linking the producer and consumers (see the lower part of the figure, 1b):

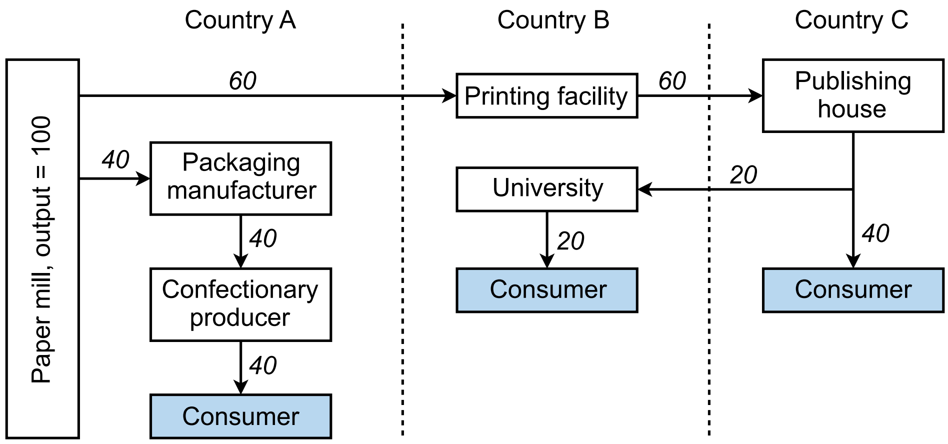
1. *Paper mill in country A → Packaging manufacturer in country A → Confectionary producer in country A → Consumer in country A*;

2. *Paper mill in country A → Printing facility in country B → Publishing house in country C → Consumer in country C*;

3. *Paper mill in country A → Printing facility in country B → Publishing house in country C → University in country B → Consumer in country B.*



(a) Value chain of inputs



(b) Value chain of outputs

Figure 1: An illustrative example: a paper mill relative position in global value chains

The value of the paper and cardboard is carried forward from one production stage to the next until it reaches consumers, and Fig. 1b records these numbers against each arrow that corresponds to the delivery of intermediate or final products. In real economy, we would most likely have providers of trade and transport services in between each pair of producers or producers and consumers. We drop these in our simplified example and suppose that each enterprise is responsible for both production and delivery of its output to the purchaser. A production stage therefore consists of production per se and a sales transaction. Accordingly, we have purely domestic production stages (e.g., the whole value chain No. 1), production stages in partner countries (e.g., from the publishing house to consumer in value chain No. 2), and cross-border production stages (e.g., from the publishing house to the university in value chain No. 3). We may also distinguish intermediate production stages linking two producers (e.g., between the packaging manufacturer and the confectionary producer in value chain No. 1) and final production stages linking producers and consumers (e.g., between the confectionary producer and consumer in value chain No. 1).

Now, we will count the total number of production stages between the paper mill and its indirect consumers. There are three production stages in value chains No. 1 and 2, and four production stages in value chain No. 3. Total is 10 production stages, and the average is $10/3 = 3\frac{1}{3}$. Given that value chains in real economy may be very long or virtually infinite, a more reasonable approach is to weigh the number of transactions from producer to consumer by the share of initial output that reaches this consumer. The results for the three value chains are as follows:

1. *40 units of paper reach consumer in country A as confectionary in 3 stages*: $3 \times \dfrac{40}{100} = 1.2$;
2. *40 units of paper reach consumer in country C as books or magazines in 3 stages*: $3 \times \dfrac{40}{100} = 1.2$;
3. *20 units of paper reach consumer in country B as educational services in 4 stages*: $4 \times \dfrac{20}{100} = 0.8$.

The weighted average number of production stages from the paper mill to all consumers is the sum of the numbers for individual value chains, $1.2 + 1.2 + 0.8 = 3.2$. Note that we may count the number of production stages separately for value chains No. 1 and No. 2-3 using 40 and 60 units as denominators. Then the paper worth 40 units would have to go through $3 \times \frac{40}{40} = 3$ production stages and that worth 60 units through $3 \times \frac{40}{60} + 4 \times \frac{20}{60} = 3\frac{1}{3}$ stages. Taking the weighted average $3 \times \frac{40}{100} + 3\frac{1}{3} \times \frac{60}{100}$ again yields 3.2.

The last stage in each value chain corresponds to the completion of production and delivery of the final product to consumer. Therefore, we may count the weighted average number of final production stages with respect to total output of the paper mill: $1 \times \frac{40}{100} + 1 \times \frac{20}{100} + 1 \times \frac{40}{100} = 1$. This result meets our expectation: however long is the value chain, there is always one completion stage, and this is true for individual industries or the whole economy. The remaining are intermediate production stages, and their weighted average number is: $2 \times \frac{40}{100} + 2 \times \frac{40}{100} + 3 \times \frac{20}{100} = 2.2$.

Employing this logic, we may further differentiate among domestic intermediate production stages at home (in country A), domestic intermediate production stages in partners (countries B and C, though these do not appear in Fig. 1b) and cross-border intermediate production stages (between countries A, B and C). The weighted average number of intermediate production stages that paper and cardboard outputs have to undergo at home is: $2 \times \frac{40}{100} = 0.8$. And the weighted average number of intermediate production stages spanning across borders is: $2 \times \frac{40}{100} + 3 \times \frac{20}{100} = 1.4$. The cross-border value chain of paper products is therefore longer than purely domestic intermediate value chain. Note that these results also reflect the importance of the relevant paths in value chain, because the numbers are weighted by the share of output received by consumer in total output of producer.

Whereas Fig. 1b outlines the paths of outputs of the paper mill, Fig. 1a traces the origin of its inputs. Here the arrows with corresponding numbers may be understood as flows of payments for the inputs required in production. To produce 100 units of paper and cardboard, the paper mill has to purchase wood pulp in the home country A worth 40 units, chemicals in country B worth 30 units and has to pay 30 units to its workers (labour is the only factor of production in our simplified example). There are six value chains linking the producer and workers (labour) who ultimately contribute value to the inputs:

1. *Paper mill in country A → Labour in country A*;
2. *Paper mill in country A → Wood pulp manufacturer in country A → Labour in country A*;
3. *Paper mill in country A → Wood pulp manufacturer in country A → Sawmill in country B → Labour in country B*;
4. *Paper mill in country A → Chemical plant in country B → Labour in country B*;

5. *Paper mill in country A → Chemical plant in country B → Chemical plant in country C → Labour in country C*;

6. *Paper mill in country A → Chemical plant in country B → Chemical plant in country C → Mining company in country C → Labour in country C.*

The payments from the paper mill flow back from one production stage to the previous one until they directly or indirectly reach workers. Value chain No. 1 contains one production stage, value chains No. 2 and 4 two production stages, value chains No. 3 and 5 three production stages, and value chain No. 6 four productions stages. The simple average is 2.5. The weighted number of productions stages for the six value chains in Fig. 1a are:

1. *30 units reach workers in country A in 1 stage as payments for the production of paper*:
   $$1 \times \frac{30}{100} = 0.3;$$

2. *10 units reach workers in country A in 2 stages as payments for the production of wood pulp*:
   $$2 \times \frac{10}{100} = 0.2;$$

3. *30 units reach workers in country B in 3 stages as payments for the supply of wood*:
   $$3 \times \frac{30}{100} = 0.9;$$

4. *5 units reach workers in country B in 2 stages as payments for the production of chemicals*:
   $$2 \times \frac{5}{100} = 0.1;$$

5. *10 units reach workers in country C in 3 stages as payments for the production of chemicals*:
   $$3 \times \frac{10}{100} = 0.3;$$

6. *15 units reach workers in country C in 4 stages as payments for the supply of minerals*:
   $$4 \times \frac{15}{100} = 0.6.$$

The weighted average number of production stages that extend from the paper mill to all workers is: $0.3 + 0.2 + 0.9 + 0.1 + 0.3 + 0.6 = 2.4$. It is natural to label the very first stage in the production process, i.e. that links workers and the respective plants, as primary (in contrast to final production stage). Then the average weighted number of primary production stages is: $1 \times \frac{30}{100} + 1 \times \frac{10}{100} + 1 \times \frac{30}{100} + 1 \times \frac{5}{100} + 1 \times \frac{10}{100} + 1 \times \frac{15}{100} = 1$. The average number of intermediate production stages is: $0 \times \frac{30}{100} + 1 \times \frac{10}{100} + 2 \times \frac{30}{100} + 1 \times \frac{5}{100} + 2 \times \frac{10}{100} + 3 \times \frac{15}{100} = 1.4$. Splitting further the number of intermediate production stages we may obtain $1 \times \frac{10}{100} + 1 \times \frac{30}{100} = 0.4$ domestic intermediate production stages at home (in country A), $1 \times \frac{15}{100} = 0.15$ domestic intermediate production stages in partner countries (in country C), and $1 \times \frac{30}{100} + 1 \times \frac{5}{100} + 2 \times \frac{10}{100} + 2 \times \frac{15}{100} = 0.85$ cross-border intermediate production stages.

Finally, with the above calculations at hand, we may compare the length of value chain of outputs with that of inputs of the paper mill. This will indicate its relative position in value chains: whether it is closer to its consumers who eventually use the outputs or to the workers who initially contribute value to the inputs. This can simply be achieved by calculating the ratio of the weighted average number of all production stages forwards and backwards: $\frac{3.2}{2.4} = 1\frac{1}{3}$. The result exceeding unity means that the outputs of the paper mill need to undergo more production stages to reach their final users than they have undergone since labour was employed in their production. In other words, there is longer value chain after production compared to that before production, and the paper mill is positioned upstream in value chains. The result less than unity would signify the opposite, i.e. that the enterprise is positioned downstream in value chains. The result that equals unity indicates a neutral position in value chains.

We may treat the intermediate production stages within partner countries and across borders as production stages relevant to global value chains, because these production stages occur if domestically produced outputs are involved in the production process beyond domestic economy. This leads to comparing the length of only those parts of value chains that can be classified as global value chains. From the itemised count above: $\frac{0+1.4}{0.15+0.85} = 1.4$. The paper mill is therefore upstream in global value chains, because the global value chain of its outputs is longer than that of it inputs.

The rest of this section explains how these indicators can be calculated with respect to the real economy with myriads of products, industries and dozens of partner countries. The technical solution is not that cumbersome, provided that the data on inter-industry transactions are organized in the form of input-output accounts, and the computations are performed in block matrix environment.

2.2 The input-output framework: notation and setup

Global value chain analysis requires a global input-output table where single-country tables are combined and linked via international trade matrices. Such inter-country or multi-regional input-output tables have been described by Isard (1951), Moses (1955), and Leontief and Strout (1963), among others, but have not been compiled at a global scale until late 2000s. The release of experimental global input-output datasets, including WIOD, Eora, Exiobase, OECD ICIO, GTAP-MRIO[1] and others,[2] has fuelled research into the implications of global value chains on trade, economy and the environment.

If there are K countries and N industries in each country, the key elements of the inter-country input-output system may be described by block matrices and vectors. The KN×KN matrix of intermediate demand $\mathbf{Z}$ is therefore as follows:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & \cdots & \mathbf{Z}_{1K} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} & \cdots & \mathbf{Z}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Z}_{K1} & \mathbf{Z}_{K2} & \cdots & \mathbf{Z}_{KK} \end{bmatrix} \quad \text{where a block element} \quad \mathbf{Z}_{rs} = \begin{bmatrix} z_{rs}^{11} & z_{rs}^{12} & \cdots & z_{rs}^{1N} \\ z_{rs}^{21} & z_{rs}^{22} & \cdots & z_{rs}^{2N} \\ \vdots & \vdots & \ddots & \vdots \\ z_{rs}^{N1} & z_{rs}^{N2} & \cdots & z_{rs}^{NN} \end{bmatrix}$$

The lower index henceforth denotes a country with $r \in \{1, \ldots, K\}$ corresponding to the exporting country and $s \in \{1, \ldots, K\}$ to the partner country. The upper index denotes industry. $\mathbf{Z}_{rs}$ is therefore an N×N matrix where each element z_{rs}^{ij} is the monetary value of the intermediate inputs supplied by the producing industry $i \in \{1, \ldots, N\}$ in country r to the purchasing (using) industry $j \in \{1, \ldots, N\}$ in country s.

Similarly, the KN×K matrix of final demand is:

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_{11} & \mathbf{f}_{12} & \cdots & \mathbf{f}_{1K} \\ \mathbf{f}_{21} & \mathbf{f}_{22} & \cdots & \mathbf{f}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{f}_{K1} & \mathbf{f}_{K2} & \cdots & \mathbf{f}_{KK} \end{bmatrix} \quad \text{where a block element} \quad \mathbf{f}_{rs} = \begin{bmatrix} f_{rs}^{1} \\ f_{rs}^{2} \\ \vdots \\ f_{rs}^{N} \end{bmatrix}$$

[1] Multi-regional versions of GTAP input-output tables were compiled on an *ad hoc* basis in various research projects and were not publicly released.

[2] See the special issue of *Economic Systems Research*, 2013, vol. 25, no. 1 for an overview.

Each block $\mathbf{f}_{rs}$ is an N×1 vector with elements f^i_{rs} representing the value of the output of industry i in country r sold to final users in country s.

Total output of each industry is recorded in the KN×1 column vector $\mathbf{x}$:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_K \end{bmatrix} \quad \text{where a block element} \quad \mathbf{x}_r = \begin{bmatrix} x^1_r \\ x^2_r \\ \vdots \\ x^N_r \end{bmatrix}$$

And the value added by each industry is recorded in the 1×KN row vector $\mathbf{v}'$:

$$\mathbf{v}' = \begin{bmatrix} \mathbf{v}'_1 & \mathbf{v}'_2 & \cdots & \mathbf{v}'_K \end{bmatrix} \quad \text{where a block element} \quad \mathbf{v}'_s = \begin{bmatrix} v^1_s & v^2_s & \cdots & v^N_s \end{bmatrix}$$

$\mathbf{v}'_s$ is a 1×N vector where each element v^j_s describes the value added generated by industry j in country s throughout the production process.

To better reflect the results of production, net of any taxes, subsidies or margins related to sales, the transactions in $\mathbf{Z}$ and $\mathbf{F}$ should be valued at basic prices. Meanwhile, from the producer's perspective, intermediate inputs should enter the calculation at purchasers' prices, inclusive of all costs associated with their purchase. Accordingly, the taxes or margins payable on intermediate inputs should also be accounted for as inputs to production. These are usually recorded as 1×KN row vectors below $\mathbf{Z}$:

$$\mathbf{m}'_{\mathbf{Z}}(g) = \begin{bmatrix} \mathbf{m}'_{\mathbf{Z}}(g)_1 & \mathbf{m}'_{\mathbf{Z}}(g)_2 & \cdots & \mathbf{m}'_{\mathbf{Z}}(g)_K \end{bmatrix}$$

$$\text{where a block element} \quad \mathbf{m}'_{\mathbf{Z}}(g)_s = \begin{bmatrix} m_{\mathbf{Z}}(g)^1_s & m_{\mathbf{Z}}(g)^2_s & \cdots & m_{\mathbf{Z}}(g)^N_s \end{bmatrix}$$

$\mathbf{m}'_{\mathbf{Z}}(g)_s$ is a 1×N row vector of the g^{th} margin where each element $m_{\mathbf{Z}}(g)^j_s$ is the amount of tax paid, subsidy received or trade/transport margin on all intermediate inputs purchased by industry j in country s. $\mathbf{m}'_{\mathbf{Z}}(g)$ is in fact a condensed form of the valuation layer that conforms to the dimension of $\mathbf{Z}$:

$$\mathbf{M}_{\mathbf{Z}}(g) = \begin{bmatrix} \mathbf{M}_{\mathbf{Z}}(g)_{11} & \mathbf{M}_{\mathbf{Z}}(g)_{12} & \cdots & \mathbf{M}_{\mathbf{Z}}(g)_{1K} \\ \mathbf{M}_{\mathbf{Z}}(g)_{21} & \mathbf{M}_{\mathbf{Z}}(g)_{22} & \cdots & \mathbf{M}_{\mathbf{Z}}(g)_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{\mathbf{Z}}(g)_{K1} & \mathbf{M}_{\mathbf{Z}}(g)_{K2} & \cdots & \mathbf{M}_{\mathbf{Z}}(g)_{KK} \end{bmatrix}$$

$$\text{where a block element} \quad \mathbf{M}_{\mathbf{Z}}(g)_{rs} = \begin{bmatrix} m_{\mathbf{Z}}(g)^{11}_{rs} & m_{\mathbf{Z}}(g)^{12}_{rs} & \cdots & m_{\mathbf{Z}}(g)^{1N}_{rs} \\ m_{\mathbf{Z}}(g)^{21}_{rs} & m_{\mathbf{Z}}(g)^{22}_{rs} & \cdots & m_{\mathbf{Z}}(g)^{2N}_{rs} \\ \vdots & \vdots & \ddots & \vdots \\ m_{\mathbf{Z}}(g)^{N1}_{rs} & m_{\mathbf{Z}}(g)^{N2}_{rs} & \cdots & m_{\mathbf{Z}}(g)^{NN}_{rs} \end{bmatrix}$$

In N×N matrices $\mathbf{M}_{\mathbf{Z}}(g)_{rs}$, each element $m_{\mathbf{Z}}(g)^{ij}_{rs}$ depicts the amount of g^{th} margin (tax paid, subsidy received or trade/transport cost) paid on intermediate inputs purchased by industry j in country s from industry i in country r. $\mathbf{M}_{\mathbf{Z}}(g)$ is then a matrix of bilateral margins that changes the valuation of intermediate inputs. If the industry that produces the margins, e.g., domestic trade and transportation services, is modelled endogenous to the inter-industry system (in other words, is inside $\mathbf{Z}$), the summation of $\mathbf{M}_{\mathbf{Z}}(g)$ column-wise will result in a zero vector $\mathbf{m}'_{\mathbf{Z}}(g)$. Taxes and subsidies on products are usually recorded exogenous to the system, so vector $\mathbf{m}'_{\mathbf{Z}}(g)$ contains non-zero values. International transport margins are also modelled as though they were provided from outside the system, which is the result of the "Panama assumption" (see Streicher and Stehrer, 2015 for an extensive discussion).

For a complete account of trade costs, valuation terms should also be compiled with respect to final products – 1×K row vector $\mathbf{m}'_{\mathbf{F}}(g)$ and KN×K matrix $\mathbf{M}_{\mathbf{F}}(g)$.

The fundamental accounting identities in the monetary input-output system imply that total sales for intermediate and final use equal total output, $\mathbf{Zi}_{KN} + \mathbf{Fi}_K = \mathbf{x}$, and the purchases of intermediate and primary inputs at basic prices plus margins and net taxes on intermediate inputs equal total input (outlays) that must also be equal to total output, $\mathbf{i}'_{KN}\mathbf{Z} + \sum_{g=1}^{G} \mathbf{m}'_{\mathbf{Z}}(g) + \mathbf{v}' = \mathbf{x}'$, where $\mathbf{i}_{KN}$ and $\mathbf{i}_K$ are, respectively, KN×1 and K×1 summation vectors, and G is the number of the valuation layers (margins).[3]

The key to the demand-driven input-output analysis is the Leontief inverse, which in the case of the inter-country input-output table is defined as follows:

$$(\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \mathbf{I}_N-\mathbf{A}_{11} & -\mathbf{A}_{12} & \cdots & -\mathbf{A}_{1K} \\ -\mathbf{A}_{21} & \mathbf{I}_N-\mathbf{A}_{22} & \cdots & -\mathbf{A}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbf{A}_{K1} & -\mathbf{A}_{K2} & \cdots & \mathbf{I}_N-\mathbf{A}_{KK} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} & \cdots & \mathbf{L}_{1K} \\ \mathbf{L}_{21} & \mathbf{L}_{22} & \cdots & \mathbf{L}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{L}_{K1} & \mathbf{L}_{K2} & \cdots & \mathbf{L}_{K} \end{bmatrix} = \mathbf{L}$$

$\mathbf{I}$ and $\mathbf{I}_N$ are, respectively, KN×KN and N×N identity matrices. $\mathbf{A}_{rs}$ blocks are N×N technical coefficient matrices where an element $a_{rs}^{ij} = \frac{z_{rs}^{ij}}{x_s^j}$ describes the amount of input by industry i in country r required per unit of output of industry j in country s. In block matrix form, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$. Leontief inverse $\mathbf{L}$ is a KN×KN multiplier matrix that allows total output to be expressed as a function of final demand: $\mathbf{x} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{Fi}_K = \mathbf{LFi}_K$.

The supply-side counterpart to the Leontief inverse, or the matrix of output (demand) multipliers, is the Ghosh inverse, or the matrix of input (supply) multipliers:

$$(\mathbf{I}-\mathbf{B})^{-1} = \begin{bmatrix} \mathbf{I}_N-\mathbf{B}_{11} & -\mathbf{B}_{12} & \cdots & -\mathbf{B}_{1K} \\ -\mathbf{B}_{21} & \mathbf{I}_N-\mathbf{B}_{22} & \cdots & -\mathbf{B}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbf{B}_{K1} & -\mathbf{B}_{K2} & \cdots & \mathbf{I}_N-\mathbf{B}_{KK} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} & \cdots & \mathbf{G}_{1K} \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \cdots & \mathbf{G}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_{K1} & \mathbf{G}_{K2} & \cdots & \mathbf{G}_{KK} \end{bmatrix} = \mathbf{G}$$

where $\mathbf{B}_{rs}$ are N×N allocation coefficient matrices with elements $b_{rs}^{ij} = \frac{z_{rs}^{ij}}{x_r^i}$ that describe the proportion of output of industry i in country r sold as intermediate input to industry j in country s. In block matrix form, $\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{Z}$. Ghosh inverse $\mathbf{G}$ is a KN×KN multiplier matrix that links total output and primary inputs in the following way: $\mathbf{x}' = \left(\sum_{g=1}^{G} \mathbf{m}'_{\mathbf{Z}}(g) + \mathbf{v}'\right)(\mathbf{I}-\mathbf{B})^{-1} = \left(\sum_{g=1}^{G} \mathbf{m}'_{\mathbf{Z}}(g)\right)\mathbf{G} + \mathbf{v}'\mathbf{G}$.

In addition to the usual operations on block matrices, the decompositions in this paper will often require extracting only diagonal or off-diagonal block elements. Accordingly, the "hat" and "check" operators in this paper, unless otherwise specified, apply to blocks and do not apply to the elements within those blocks. For example:

[3] We assume here that the inter-country input-output table does not contain purchases abroad by residents or domestic purchases by non-residents or any statistical discrepancies. The sum of intermediate purchases at basic prices, net taxes, margins on intermediate inputs and value added at basic prices is therefore equal to industry output at basic prices.

$$\widehat{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & 0 & \cdots & 0 \\ 0 & \mathbf{A}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_{KK} \end{bmatrix} \quad \text{and} \quad \breve{\mathbf{F}} = \begin{bmatrix} 0 & \mathbf{f}_{12} & \cdots & \mathbf{f}_{1K} \\ \mathbf{f}_{21} & 0 & \cdots & \mathbf{f}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{f}_{K1} & \mathbf{f}_{K2} & \cdots & 0 \end{bmatrix}$$

2.3 Identifying country and industry positions in global value chains

The measurement of the number of production stages or the length of production chains has attracted the interest of many input-output economists. The idea of simultaneously counting and weighting the number of inter-industry transactions was formalized by Dietzenbacher et al. (2005). Their "average propagation length" (APL) is the average number of steps it takes an exogenous change in one industry to affect the value of production in another industry. It is essentially the APL concept that underlies the count of the number of production stages from the paper mill to its consumers and workers in our simplified example above. The only difference is that Dietzenbacher et al. (2005), and many authors in the follow-up studies, neglect the completion stage. First applications of the APL concept to measure the length of cross-border production chains appear in Dietzenbacher and Romero (2007) and Inomata (2008), though Oosterhaven and Bouwmeester (2013) warn that the APL should only be used to compare pure interindustry linkages and not to compare different economies or different industries.

Fally (2011, 2012) proposes the recursive definitions of two indices that quantify the "average number of embodied production stages" and the "distance to final demand". Miller and Temurshoev (2015), by analogy with Antràs et al. (2012), use the logic of the APL and derive the measures of "output upstreamness" and "input downstreamness" that indicate industry relative position with respect to the final users of outputs and initial producers of inputs. They show that their measures are mathematically equivalent to those of Fally and the well known indicators of, respectively, total forward linkages and total backward linkages. Fally (2012) indicates that the average number of embodied production stages may be split to account for the stages taking place within the domestic economy and abroad. This approach was implemented in OECD (2012), De Backer and Miroudot (2013) and elaborated in Miroudot and Nordström (2015).

Ye et al. (2015) generalize previous length and distance indices and propose a consistent accounting system to measure the distance in production networks between producers and consumers at the country, industry and product levels from different economic perspectives. Their "value added propagation length" may be shown to be equal to Fally's embodied production stages and Miller–Temurshoev's input downstreamness when aggregated across producing industries.

Finally, Wang et al. (2016) develop a technique of additive decomposition of the average production length. Therefore, they are able to break the value chain into various components and measure the length of production along each component. Their production length index system includes indicators of the average number of domestic, cross-border and foreign production stages. They also propose new participation and production line position indices to clearly identify where a country or industry is in global value chains. Importantly, Wang et al. (2016) clearly distinguish between average production length and average propagation length, and between shallow and deep global value chains.

This paper builds on the technique and ideas of Wang et al. (2016) and the derivation of the weighted average number of border crossings by Muradov (2016). Quantification of country and industry positions in global value chains is actually a part of a holistic system of analytical indicators of structure and length of value chains. The core features of this system include: (1) factorization of the Leontief and Ghosh global inverse matrices into domestic and international components, (2) decomposition of outputs and inputs in accordance with their paths along domestic or cross-border value chains, (3) classification of production stages into final, intermediate, primary, domestic and cross-border, (4) explicit count of production stages within and beyond domestic economy with respect to each output or input component. Most results are identical to those in Wang et al. (2016), but some results are thought to provide more information on the structure and length of value chains. A detailed exposition of the system, including step-by-step derivation of all equations, may be found in Appendices B-D. This section only utilises the results that are relevant to identifying country and/or industry positions in global value chains.

In the indexing of variables below, $\mathbf{X}_D$ signifies the direction towards the destination of outputs and $\mathbf{X}_O$ towards the origin of inputs, *ips* abbreviates intermediate production stages, p production stages in partner countries and cb cross-border production stages. We can therefore denote the weighted average number of intermediate production stages in partners that outputs undergo on the way to their eventual destination as $\mathbf{C}_{(\mathbf{X}_D)ips.p}$ and the weighted average number of cross-border intermediate production stages as $\mathbf{C}_{(\mathbf{X}_D)ips.cb}$:

$$\mathbf{C}_{(\mathbf{X}_D)ips.p} = \frac{\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} (\mathbf{H} - \mathbf{I}) \left(\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} - \mathbf{I}\right) \mathbf{HF}}{\mathbf{LF}} \tag{1}$$

$$\mathbf{C}_{(\mathbf{X}_D)ips.cb} = \frac{\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \mathbf{H}(\mathbf{H} - \mathbf{I})\mathbf{F}}{\mathbf{LF}} \tag{2}$$

In equations (1) and (2), $\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}$ is a KN×KN block-diagonal matrix of local Leontief inverses that describes the production chain confined to the domestic economy, $\mathbf{H} = \left(\mathbf{I} - \breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)^{-1}$ is a KN×KN global inverse that is responsible for the production chain beyond national borders that exists because of international trade in intermediates. Importantly, the fraction sign henceforth denotes the element-to-element division to simplify notation.

Both $\mathbf{C}_{(\mathbf{X}_D)ips.p}$ and $\mathbf{C}_{(\mathbf{X}_D)ips.cb}$ are KN×K matrices of dimensionless numbers where each element $\left[\mathbf{C}_{(\mathbf{X}_D)ips.p}\right]^i_{rs}$ and $\left[\mathbf{C}_{(\mathbf{X}_D)ips.cb}\right]^i_{rs}$ quantifies the average weighted number of intermediate production stages, respectively, in partner countries and across borders, that the outputs of industry i in country r have to undergo until they eventually reach final user in country s.

Counting the number of production stages in the reverse direction yields the following indicators:

$$\mathbf{C}_{(\mathbf{X}_O)ips.p} = \frac{\mathbf{VQ}\left(\left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1} - \mathbf{I}\right)(\mathbf{Q} - \mathbf{I})\left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} \tag{3}$$

$$\mathbf{C}_{(\mathbf{X}_O)ips.cb} = \frac{\mathbf{V}(\mathbf{Q} - \mathbf{I})\mathbf{Q}\left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} \tag{4}$$

Equations (3) and (4) mirror equations (1) and (2), but build on a KN×KN block-diagonal matrix of Ghosh local inverses $\left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}$ and another KN×KN global inverse

$\mathbf{Q} = \left(\mathbf{I} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1} \breve{\mathbf{B}}\right)^{-1}$. $\mathbf{Q}$ is similar to $\mathbf{H}$, and the relationship between the two matrices is explored in more detail in Appendix E.2.

$\mathbf{C}_{(\mathbf{X}_O)ips.p}$ and $\mathbf{C}_{(\mathbf{X}_O)ips.cb}$ are K×KN matrices of dimensionless numbers where each element $\left[\mathbf{C}_{(\mathbf{X}_O)ips.p}\right]_{rs}^{j}$ and $\left[\mathbf{C}_{(\mathbf{X}_O)ips.cb}\right]_{rs}^{j}$ counts the average weighted number of intermediate production stages, respectively, in partner countries and across borders, that directly and indirectly link the outputs of industry j in country s to primary inputs in country r.

The sum of $\mathbf{C}_{(\mathbf{X}_D)ips.p}$ and $\mathbf{C}_{(\mathbf{X}_D)ips.cb}$, aggregated to the KN×1 dimension, gives the weighted average number of GVC-related production stages that the outputs of industry i in country r undergo until they end up in final demand in all partners.[4] Given that $\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \mathbf{H} = \mathbf{L}$, this can be simplified to:

$$\mathbf{c}_{(\mathbf{X}_D)ips.p} + \mathbf{c}_{(\mathbf{X}_D)ips.cb} = \frac{\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} (\mathbf{H} - \mathbf{I}) \left(\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} - \mathbf{I}\right) \mathbf{H}\mathbf{F}\mathbf{i}_K}{\mathbf{L}\mathbf{F}\mathbf{i}_K} + \\ + \frac{\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \mathbf{H}(\mathbf{H} - \mathbf{I})\mathbf{F}\mathbf{i}_K}{\mathbf{L}\mathbf{F}\mathbf{i}_K} = \frac{\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right) \mathbf{L}\mathbf{f}}{\mathbf{L}\mathbf{f}} \tag{5}$$

where $\mathbf{i}_K$ is a K×1 summation vector and $\mathbf{f}$ is a column vector of final demand ($\mathbf{f} = \mathbf{F}\mathbf{i}_K$).

Similarly, a 1×KN vector of the weighted average numbers of GVC-related production stages that link the outputs of industry j in country s to primary inputs in all partners may be obtained as follows:

$$\mathbf{c}'_{(\mathbf{X}_O)ips.p} + \mathbf{c}'_{(\mathbf{X}_O)ips.cb} = \frac{\mathbf{i}'_K \mathbf{V}\mathbf{Q} \left(\left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1} - \mathbf{I}\right) (\mathbf{Q} - \mathbf{I}) \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}}{\mathbf{i}'_K \mathbf{V}\mathbf{G}} + \\ + \frac{\mathbf{i}'_K \mathbf{V}(\mathbf{Q} - \mathbf{I})\mathbf{Q} \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}}{\mathbf{i}'_K \mathbf{V}\mathbf{G}} = \frac{\mathbf{v}'\mathbf{G} \left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)}{\mathbf{v}'\mathbf{G}} \tag{6}$$

Equation (6) utilises that $\mathbf{Q} \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1} = \mathbf{G}$.

Finally, taking element-by-element ratio of the weighted average numbers of GVC-related production stages forwards and backwards yields a position index of industry $i = j$ in country $r = s$ in global value chains:

$$\mathbf{p}_{(\mathbf{C},\mathbf{X}_{D/O})GVC} = \frac{\mathbf{c}_{(\mathbf{X}_D)ips.p} + \mathbf{c}_{(\mathbf{X}_D)ips.cb}}{\mathbf{c}_{(\mathbf{X}_O)ips.p} + \mathbf{c}_{(\mathbf{X}_O)ips.cb}} = \\ = \frac{\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right) \mathbf{L}\mathbf{f}}{\mathbf{L}\mathbf{f}} \oslash \left[\frac{\mathbf{v}'\mathbf{G} \left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)}{\mathbf{v}'\mathbf{G}}\right]' = \\ = \frac{\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right) \mathbf{L}\mathbf{f}}{\left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)' \mathbf{G}'\mathbf{v}} \oslash \frac{\mathbf{L}\mathbf{f}}{\mathbf{G}'\mathbf{v}} \tag{7}$$

[4] Note that partner country may coincide with the home country if outputs return home after processing abroad.

where $\oslash$ signifies the element-by-element division, as do fractions. If value added $\mathbf{v}'$ includes net taxes on products, or net taxes on products are zero, then $\mathbf{Lf} = \mathbf{G}'\mathbf{v}$ and the last term in the equation above would equal a vector of ones and might be dropped:

$$\mathbf{p}_{(\mathbf{C},\mathbf{x}_{D/O})GVC} = \frac{\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{Lf}}{\left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)'\mathbf{G}'\mathbf{v}} \tag{8}$$

The GVC position index in equation (7) represents a modified average production line position index proposed by Wang et al. (2016). There are important differences between the two. First, the index of Wang et al. (2016) implicitly calculates the weighted average numbers of domestic productions stages in the home economy, in addition to GVC-related production stages forwards and backwards. On the contrary, the index in equation (7) includes a calculation of the weighted average numbers of only GVC-related production stages. Second, the index of Wang et al. (2016) compares weighted average numbers of production stages forwards and backwards that only GVC-related parts of outputs and inputs[5] have to undergo. Meanwhile, the index in this paper compares the same numbers with respect to all outputs and inputs. The index in equation (7) therefore utilises the indicators of production length in GVCs that are normalised with respect to total outputs. This ensures that the relative size of the GVC-related production activities is properly accounted for, and the index is able to properly handle the cases where the production length is significant but the share of output (value added, final products) that is relevant to this production chain does not really matter for the whole industry or economy.

Equation (8) unveils some useful properties of the modified production line position index, given that $\mathbf{Lf} = \mathbf{G}'\mathbf{v} = \mathbf{x}$. Rewrite the denominator in the fraction on the right side of equation (8) in terms of Leontief local and global inverses:

$$\left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)'\mathbf{G}'\mathbf{v} = \left(\hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}} - \hat{\mathbf{x}}^{-1}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\hat{\mathbf{x}}\right)'\mathbf{x} = \left[\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\hat{\mathbf{x}}\right]'\hat{\mathbf{x}}^{-1}\mathbf{x} =$$
$$= \left[\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\widehat{\mathbf{Lf}}\right]'\mathbf{i}_{KN}$$

Then insert the above result in equation (8) and rewrite the production line position index:

$$\mathbf{p}_{(\mathbf{C},\mathbf{x}_{D/O})GVC} = \frac{\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{Lf}}{\left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)'\mathbf{G}'\mathbf{v}} = \frac{\left[\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\widehat{\mathbf{Lf}}\right]\mathbf{i}_{KN}}{\left[\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\widehat{\mathbf{Lf}}\right]'\mathbf{i}_{KN}} \tag{9}$$

Equation (9) implies that the production line position index is now merely an element-by-element ratio of the row sum to the column sum of a KN×KN matrix that appears in square brackets in both numerator and denominator. Hence two interrelated conclusions.

First, the production line GVC position index at the global level, or the global weighted average of the index, will always equal unity because this requires division of the sum of all

[5] In fact, the subject of measurements in Wang et al. (2016) are value added and final products at industry level. It may be easily shown that the results, however, are invariant to using either variables that involve double counting (outputs, inputs) or those that remove double counting (value added, final demand) at industry level. Further details are in Appendix B.3.7.

elements of a matrix by itself. The values of the index at the country-industry or aggregate country, industry levels are therefore distributed around one.

Second, the production line GVC position index cannot be more than unity or less than unity for all N industries in all K countries or any aggregation thereof because this would require the total row sum exceed the total column sum of the same matrix. This means that whereas some countries or industries are positioned upstream in value chains, other countries or industries need to be positioned downstream.

2.4 Structural decomposition analysis of country positions in global value chains

As country and industry positions in global value chains shift over time, the next research question is to identify the key factors underlying these shifts. For this purpose, this paper employs the structural decomposition analysis that allows the researcher to attribute the changes in a matrix (vector, scalar) that is a product of various components to the changes in those components or determinants.

There may be multiple options to identify the determinants that influence the GVC-related indicators, as some of the recent investigations show (e.g., Nagengast and Stehrer, 2016; Timmer et al., 2016). In choosing the determinants, we should ensure that they are capable of capturing important economics effects and that they are independent of each other: change in one determinant must not entail an automatic change in other determinants (see Dietzenbacher and Los, 2000 for a discussion). This paper explores a multiplicative decomposition of the changes in the GVC production line position index with the following determinants of $\mathbf{L}$, $\widehat{\mathbf{A}}$ and $\mathbf{f}$.

The global Leontief inverse $\mathbf{L}$ may be itemised as $\left(\mathbf{I} - \breve{\mathbf{A}} - \widehat{\mathbf{A}}\right)^{-1}$ where the matrix of technical coefficients for imports $\breve{\mathbf{A}}$ is decomposed as a product of four matrices:

$$\breve{\mathbf{A}} = \overset{Par}{\mathbf{A}} \circ \left[\mathbf{S}'_K \left[\overset{Imp}{\mathbf{A}} \circ \overset{Tec}{\mathbf{A}}\hat{\mathbf{a}}\right]\right] \tag{10}$$

$\mathbf{S}_K$ is a N×KN country-wise aggregation matrix (see Appendix G for the description) and $\circ$ signifies the element-by-element multiplication. $\mathbf{a}'$ is a KN×1 row vector that sums the technical coefficients along the columns of $\mathbf{A}$:

$$\mathbf{a}' = \mathbf{i}'_{KN}\mathbf{A}$$

Vector $\mathbf{a}'$ reports the total industry requirements for intermediate inputs as a share in total industry output. Since $\mathbf{a}' + \mathbf{v}'_c = \mathbf{i}'_{KN}$, where $\mathbf{v}'_c = \mathbf{v}'\hat{\mathbf{x}}^{-1}$ are value added coefficients, $\mathbf{a}'$ is capable of capturing the effect of outsourcing: an increase in $\mathbf{a}'$ with a corresponding decrease in $\mathbf{v}'_c$ (at constant prices) means that the producer decides to substitute its primary inputs with intermediate inputs from other producers, or purchase necessary inputs rather than produce those in-house.

$\overset{Tec}{\mathbf{A}}$ is a N×KN matrix that allocates total industry requirements for intermediate inputs to the industry source of those inputs or, by and large, to individual products. This matrix is thought to describe technology, or the so called production recipe in input-output models. $\overset{Tec}{\mathbf{A}}$ changes if the production technology changes. Note that the industries that supply the intermediate inputs are aggregated across all countries at this stage:

$$\overset{Tec}{\mathbf{A}} = \mathbf{S}_K\mathbf{A}\hat{\mathbf{a}}^{-1} = \left[\sum_{r=1}^{K}[\mathbf{A}]_{r1}[\hat{\mathbf{a}}^{-1}]_1 \quad \sum_{r=1}^{K}[\mathbf{A}]_{r2}[\hat{\mathbf{a}}^{-1}]_2 \quad \cdots \quad \sum_{r=1}^{K}[\mathbf{A}]_{rK}[\hat{\mathbf{a}}^{-1}]_K\right]$$

The apparent dependency of $\overset{Tec}{\mathbf{A}}$ on $\mathbf{a}'$ in the equation above needs explaining. In fact, $\overset{Tec}{\mathbf{A}}$ may be rewritten as solely a function of $\mathbf{A}$ because $\mathbf{a}'$ is a function of $\mathbf{A}$. So both $\overset{Tec}{\mathbf{A}}$ and $\mathbf{a}'$ depend on $\mathbf{A}$ but in different ways. It is also obvious that the dependency between $\overset{Tec}{\mathbf{A}}$ and $\mathbf{a}'$ is not automatic. A change in technology may not necessarily alter the combination of intermediate and primary inputs. And outsourcing may hypothetically involve all intermediate inputs in such a way that technology remains unchanged. Then both $\overset{Tec}{\mathbf{A}}$ and $\mathbf{a}'$ are thought to capture two different effects and are useful for this structural decomposition.

$\overset{Imp}{\mathbf{A}}$ is another N×KN matrix that identifies how much of the total industry requirements for intermediate inputs are imported. So it is capable of capturing the substitution between domestic and imported intermediate inputs:

$$\overset{Imp}{\mathbf{A}} = \left[\mathbf{S}_K\breve{\mathbf{A}}\right]\oslash[\mathbf{S}_K\mathbf{A}] = \left[\sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r1}\oslash\sum_{r=1}^{K}[\mathbf{A}]_{r1} \quad \sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r2}\oslash\sum_{r=1}^{K}[\mathbf{A}]_{r2} \quad \cdots \quad \sum_{r=1}^{K}[\breve{\mathbf{A}}]_{rK}\oslash\sum_{r=1}^{K}[\mathbf{A}]_{rK}\right]$$

Finally, $\overset{Par}{\mathbf{A}}$ is a KN×KN matrix that distributes the imported intermediate inputs across partner countries. A change in $\overset{Par}{\mathbf{A}}$ generally describes changes in import procurement decisions with respect to partner countries:

$$\overset{Par}{\mathbf{A}} = \breve{\mathbf{A}}\oslash\left[\mathbf{S}'_K\mathbf{S}_K\breve{\mathbf{A}}\right] = \begin{bmatrix} 0\oslash\sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r1} & \mathbf{A}_{12}\oslash\sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r2} & \cdots & \mathbf{A}_{13}\oslash\sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r3} \\ \mathbf{A}_{21}\oslash\sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r1} & 0\oslash\sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r2} & \cdots & \mathbf{A}_{13}\oslash\sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r3} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{31}\oslash\sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r1} & \mathbf{A}_{32}\oslash\sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r2} & \cdots & 0\oslash\sum_{r=1}^{K}[\breve{\mathbf{A}}]_{r3} \end{bmatrix}$$

The matrix of technical coefficients for domestic inputs $\widehat{\mathbf{A}}$ is written as a product of $\overset{Imp}{\mathbf{A}}$, $\overset{Tec}{\mathbf{A}}$ and $\mathbf{a}'$:

$$\widehat{\mathbf{A}} = \langle\mathbf{S}'_K\left[(\mathbf{J}_{N\times KN} - \overset{Imp}{\mathbf{A}})\circ\overset{Tec}{\mathbf{A}}\,\hat{\mathbf{a}}\right]\rangle \tag{11}$$

where $\mathbf{J}_{N\times KN}$ is a N×KN matrix of ones and the angle brackets $\langle\ \ \rangle$ signify taking the block-diagonal elements of a block matrix, similar to the "hat" sign.

Now, insert the decomposed $\breve{\mathbf{A}}$ and $\widehat{\mathbf{A}}$ into the global Leontief inverse $\mathbf{L}$:

$$\mathbf{L} = \left(\mathbf{I} - \breve{\mathbf{A}} - \widehat{\mathbf{A}}\right)^{-1} = \left(\mathbf{I} - \overset{Par}{\mathbf{A}}\circ\left[\mathbf{S}'_K\left[\overset{Imp}{\mathbf{A}}\circ\overset{Tec}{\mathbf{A}}\,\hat{\mathbf{a}}\right]\right] - \langle\mathbf{S}'_K\left[(\mathbf{J}_{N\times KN} - \overset{Imp}{\mathbf{A}})\circ\overset{Tec}{\mathbf{A}}\,\hat{\mathbf{a}}\right]\rangle\right)^{-1}$$

The decomposition of the KN×1 vector of final demand $\mathbf{f}$ involves two factors:

$$\mathbf{f} = \mathbf{F}_s\mathbf{f}_K \tag{12}$$

$\mathbf{f}_K$ is a K×1 column vector of aggregate final demand at country level:

$$\mathbf{f}_K = \mathbf{S}'_N\mathbf{f}$$

And the KN×K matrix $\mathbf{F}_s$ identifies the product structure of aggregate final demand with the shares of individual industries in its diagonal blocks:

$$\mathbf{F}_s = \left[\hat{\mathbf{f}}\mathbf{S}_N\right]\hat{\mathbf{f}}_K^{-1} = \begin{bmatrix} \frac{1}{\sum_{i=1}^{N} f_1^i}\mathbf{f}_1 & 0 & \cdots & 0 \\ 0 & \frac{1}{\sum_{i=1}^{N} f_2^i}\mathbf{f}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sum_{i=1}^{N} f_K^i}\mathbf{f}_K \end{bmatrix}$$

Inserting the decomposed $\mathbf{L}$, $\widehat{\mathbf{A}}$ and $\mathbf{f}$ in equation (9) allows us to express the GVC production line position index as a function of six factors: $\mathbf{p}_{(\mathbf{C},\mathbf{X}_{D/O})GVC}\left(\overset{Par}{\mathbf{A}}, \overset{Imp}{\mathbf{A}}, \overset{Tec}{\mathbf{A}}, \mathbf{a}', \mathbf{F}_s, \mathbf{f}_K\right)$. A change of the index from period 0 to period 1 can then be fully attributed to the changes into the underlying factors. Following Dietzenbacher and Los (1998), it is reasonable to construct two polar multiplicative decompositions and to take their geometric average. With the lower indices denoting time periods 0 and 1 and the position index denoted by $\mathbf{p}$ for brevity, the right-to-left decomposition is as follows:

$$\begin{aligned} \frac{\mathbf{p}_1}{\mathbf{p}_0} = & \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,0}\right)} \circ \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,0}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)} \circ \\ & \circ \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)} \circ \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)} \circ \\ & \circ \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)} \circ \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)} \end{aligned} \quad (13)$$

As in the previous equations, the fraction sign signifies the element-by-element division and ∘ the element-by-element multiplication.

The left-to-right decomposition mirrors the above:

$$
\begin{aligned}
\frac{\mathbf{p}_1}{\mathbf{p}_0} = {} & \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)} \circ \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)} \circ \\
& \circ \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)} \circ \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)} \circ \\
& \circ \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,1}\right)} \circ \frac{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,1}\right)}{\mathbf{p}\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)}
\end{aligned}
\tag{14}
$$

The geometric average is calculated pairwise for components in equations (13) and (14) that account for the change in the same variable. For example, the change in the GVC position index for each i-th industry in country r because of changes in the propensity to import intermediate inputs may be expressed as follows:

$$
p^i_{r,1/0}(\overset{Imp}{\mathbf{A}}) = \sqrt{\frac{p^i_r\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)}{p^i_r\left(\overset{Par}{\mathbf{A}}_1, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_0, \mathbf{a}'_0, \mathbf{F}_{s,0}, \mathbf{f}_{K,0}\right)} \circ \frac{p^i_r\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_1, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)}{p^i_r\left(\overset{Par}{\mathbf{A}}_0, \overset{Imp}{\mathbf{A}}_0, \overset{Tec}{\mathbf{A}}_1, \mathbf{a}'_1, \mathbf{F}_{s,1}, \mathbf{f}_{K,1}\right)}}
$$

The averaging of the two polar decompositions results in six factors, each being a KN$\times$1 vector where each r, i-th element describes the growth or decline in the GVC position index of the industry i in country r solely because of the global changes in one of the underlying factors.

3 Data and results

3.1 Data

For an empirical application of the proposed index, this paper utilizes the 2015 edition of the OECD Inter-Country Input-Output (ICIO) tables. The tables cover 62 countries (including the rest of world as a single country) and 34 industries. The years covered are 1995, 2000, 2005 and 2008 to 2011. All values are reported at current prices.

The OECD ICIO tables are consistent with the matrix setup described in section 2. However, there are a number of peculiarities that are worth noting. First, manufacturing industries in China and Mexico are additionally disaggregated into those operating in standard mode and those operating in processing exports mode. Therefore, the tables have heterogeneous classification with 69 industries for China, 50 industries for Mexico and 34 industries for all other countries. Second, taxes less subsidies on products are not explicitly distinguished from value added at basic prices. Therefore, vector $\mathbf{v}$ includes net taxes on products and $\mathbf{v}'\mathbf{G} = \mathbf{x}'$. Third, statistical discrepancy is recorded in a column vector that equals the difference between total industry output and intermediate plus final use. Therefore, $\mathbf{Lf} \neq \mathbf{x}$.

3.2 Country and industry positions in global value chains: results and discussion

Positions in global value chains were identified with respect to each industry in each country (country-industry level), all industries aggregated within countries (country level) and each industry aggregated between countries (industry level, see Appendix G for details on the aggregation options). As the GVC production line position index virtually never equals one but may only marginally differ from one, a third category in addition to "upstream" and "downstream" was introduced – "neutral". Therefore, upstream position corresponds to the index values exceeding 1.01, downstream to those less than 0.99, while the values between 0.99 and 1.01 signify that the industry or country in question are neutrally positioned in global value chains. In other words, they are equally far from the destinations of their GVC-related outputs and origins of their GVC-related inputs. Table 1 reports the summary statistics of these measurements and allows us to immediately make a number of important observations.

First, as explicitly shown in subsection 2.3, the upstream position of some industries/countries balances the downstream position of other industries/countries in global value chains. Industries tend to be positioned downstream: as of 2011, 57% of all industries at the disaggregate country-industry level and 64% at the global aggregate level have the GVC position index less than 0.99. The downstream position also prevailed for countries before 2011, but then the share of countries positioned downstream decreased to 43% at the expense of those positioned neutrally (11%) and upstream (46%).

Second, balance appears to persist with respect to both positions and changes of positions in global value chains, although the latter is not substantiated by a mathematically proof in this paper. Whereas some countries and industries shift upstream, other shift downstream. Through the entire period 2000-2011, average positions of most industries downgraded, i.e. moved closer to final users, while average positions of most countries upgraded, i.e. moved closer to producers at origin of value chains. At the disaggregate country-industry level, none of the two direction prevailed: nearly half of all industries moved upstream while another half downstream. As follows from Table 1, the global financial crisis of 2008 coincided with an apparent shift downstream, observed for 59% of individual industries, 70% of aggregate industries and 64% of countries. In the subsequent period of recovery, most GVC positions upgraded for 62% of individual industries, 72% of countries, but 55% of aggregate industries continued to drift downstream.

Third, as might be reasonably expected, GVC position measurements are relatively stable in time at the aggregate industry level. Out of 33 industries[6] only 4 switched their positions between upstream, downstream and/or neutral at least once in the three periods considered here. For individual industries and countries, fluctuations between involvement in upstream or downstream global value chains was much more common. The relative stability in the aggregate industry GVC positions – also observed in the low mean absolute changes of the index – may be useful to explain country positions. It is natural to suppose that countries where the economy relies on the upstream industries would be positioned upstream and vice versa. As an experiment, a weighted average of the aggregate industry GVC position indices in 2011 was calculated for each country with the shares of the respective industries in country total value added as weights. The results successfully predicted country GVC positions (upstream or downstream) in 40 of 61 cases. Industry "Wholesale and retail trade; repairs" (C50T52) that has high GVC position index and usually accounts for a large share of total value added is likely to be responsible for the overestimation of country positions in

[6] "Private households with employed persons" (C95) industry was dropped because of zero output in the majority of countries.

Table 1: Summary statistics of the GVC production line position index

	2000	2005	2008	2011	2000-2005	2005-2008	2008-2011	2000-2011
Country-industry level								
Number of upstream industries ($\mathbf{p} > 1.01$)	837	840	791	856				
Number of downstream industries ($\mathbf{p} < 0.99$)	1180	1183	1225	1168				
Number of neutral industries ($0.99 < \mathbf{p} < 1.01$)	28	22	24	16				
Number of industries that moved upstream ($\mathbf{p}_1 > \mathbf{p}_0$)					988	841	1270	1044
Number of industries that moved downstream ($\mathbf{p}_1 < \mathbf{p}_0$)					1065	1200	770	1001
Number of industries that switched position between upstream, downstream and/or neutral at least once								455
Number of industries that did not switch position								1590
Mean absolute change in $\mathbf{p}$					0.37	0.32	0.26	0.43
Aggregate country level								
Number of upstream countries ($\mathbf{p} > 1.01$)	26	25	26	28				
Number of downstream countries ($\mathbf{p} < 0.99$)	35	36	32	26				
Number of neutral countries ($0.99 < \mathbf{p} < 1.01$)	0	0	3	7				
Number of countries that moved upstream ($\mathbf{p}_1 > \mathbf{p}_0$)					31	22	44	38
Number of countries that moved downstream ($\mathbf{p}_1 < \mathbf{p}_0$)					30	39	17	23
Number of countries that switched position between upstream, downstream and/or neutral at least once								18
Number of countries that did not switch position								43
Mean absolute change in $\mathbf{p}$					0.16	0.12	0.12	0.20
Aggregate industry level								
Number of upstream industries ($\mathbf{p} > 1.01$)	13	12	12	10				
Number of downstream industries ($\mathbf{p} < 0.99$)	19	20	21	21				
Number of neutral industries ($0.99 < \mathbf{p} < 1.01$)	1	1	0	2				
Number of industries that moved upstream ($\mathbf{p}_1 > \mathbf{p}_0$)					12	10	15	10
Number of industries that moved downstream ($\mathbf{p}_1 < \mathbf{p}_0$)					21	23	18	23
Number of industries that switched position between upstream, downstream and/or neutral at least once								4
Number of industries that did not switch position								29
Mean absolute change in $\mathbf{p}$					0.07	0.08	0.05	0.11

Note: The total number of industries at the country-industry level varies and is less than $KN = 62 \times 34 = 2108$ because the GVC production line position index for some industries in some countries or a change thereof involves division by zero and cannot be defined. The results at the aggregate country level exclude the rest of world. The results at the aggregate industry level exclude "Private households with employed persons" (C95) because of zero output in the majority of countries.
Source: OECD ICIO tables, author's calculations.

many cases.

A visualisation of country and industry positions in global value chains in scatter plots as in Fig. 2 and 3 provides another dimension in terms of the number of GVC-related production stages that underly the computation of the index. Location in the lower right part of the plot signifies upstream position in global value chains and that in the upper left part signifies downstream position. The diagonal line corresponds to neutral position in global value chains.

One may easily see that the values of the position index for industries are more dispersed around unity than those for countries.

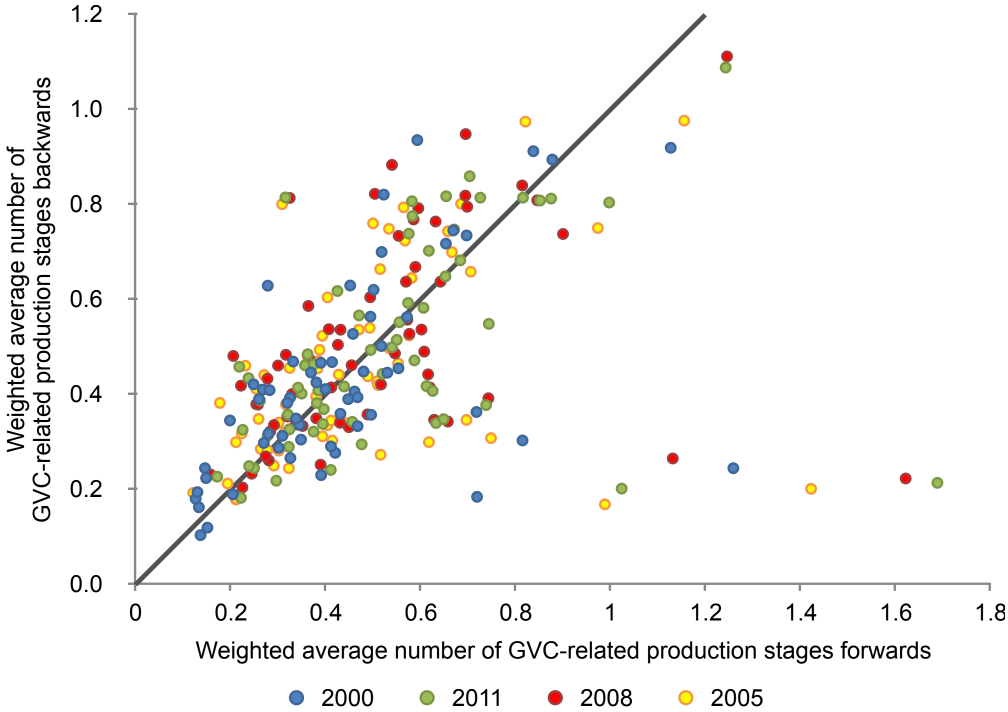


Figure 2: Visualisation of the GVC production line position index, at the aggregate country level

Source: OECD ICIO tables, author's calculations.

The full country and industry rankings based on the GVC production line position index for 2011 may be found in Appendix A. The closest to the origin and the farthest to the destination of global value chains is mining and quarrying. Only a few other good-producing industries are positioned upstream. These industries supply basic metals, chemicals, pulp and paper products. Most manufacturing industries are positioned downstream. Notably, manufacturing of various types of equipment and motor vehicles requires that inputs pass along a longer and more complex production chain than the outputs of these industries. However, the upstream position is more typical for service providers. And only a few service industries are heavily oriented towards backward value chain because their outputs almost immediately end up in final demand: public administration, education and health services.

Accordingly, the most upstream in the global production segment are countries with a significant role of extractive industries – Brunei, Saudi Arabia, Norway, Russia, Chile, Australia, South Africa. Lower part of the list displays a rather diverse group of countries that include some of the world's largest economies (China, India), developing economies in Asia (Viet Nam, Cambodia, Turkey) and some of the smaller EU members (Malta, Croatia, Cyprus). These countries have the lowest average production line position index and are the most downstream in terms of global production. Apparently, the reliance on agriculture and construction pulls large developing economies downstream.

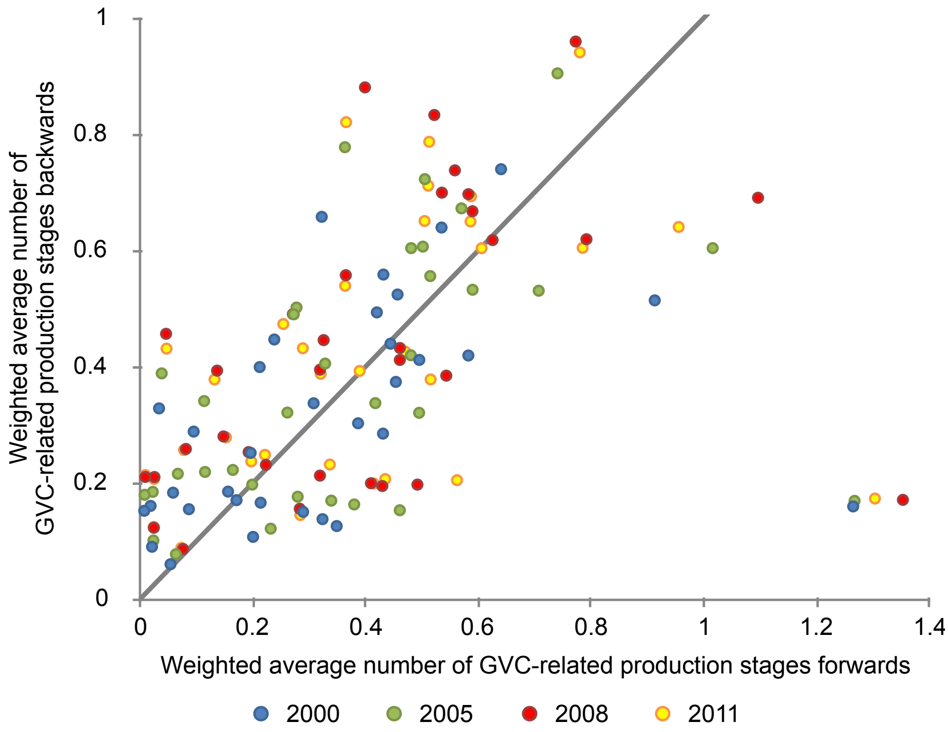


Figure 3: Visualisation of the GVC production line position index, at the aggregate industry level

Source: OECD ICIO tables, author's calculations.

3.3 Results of the structural decomposition

Prior to the discussion of the results of the structural decomposition analysis, two points are worth noting. First, structural decomposition technique usually applies to an expression where a scalar is defined as a product of various matrices and/or vectors. A change in a globally aggregated variable is therefore attributed to the changes in the underlying arrays of data, including data on individual countries (see, for example, Nagengast and Stehrer, 2016; Timmer et al., 2016). This paper adopts an opposite approach. Since the globally aggregated GVC position index always equals unity and does not change, the variable decomposed here is a vector of GVC position indices aggregated to country level. A change in GVC position of an individual country is attributed to the changes in matrices or vectors that describe industrial structure of the global economy.

Second, ideally, the structural decomposition should be run at constant prices to isolate the price changes over time. However, such data are not available for the OECD ICIO tables. Moreover, Timmer et al. (2016) apply structural decomposition to study the changes in global import intensity using the 2016 edition of the World Input-Output Database (WIOD), only available at current prices. To address a possible price effect, they perform the analysis excluding trade in mineral fuels and still find that global import intensities did not rise over the 2011-2014 period. They suggest that the results are not driven by sizable changes in relative prices. The decomposition in this paper was also run at current prices which is thought to provide largely valid results.

Fig. 4 shows the results for selected countries that exemplify different paths in global value chains: Portugal continuously moved upstream, France only moved downstream, China moved upstream in 2000-2008 and then downstream in 2008-2011, and Viet Nam moved downstream in 2000-2008 and then upstream in 2008-2011. The change in the underlying factors is the same, but their contributions vary from one country to another. As regards Portugal and France, it is the distribution of intermediate imports across partner countries

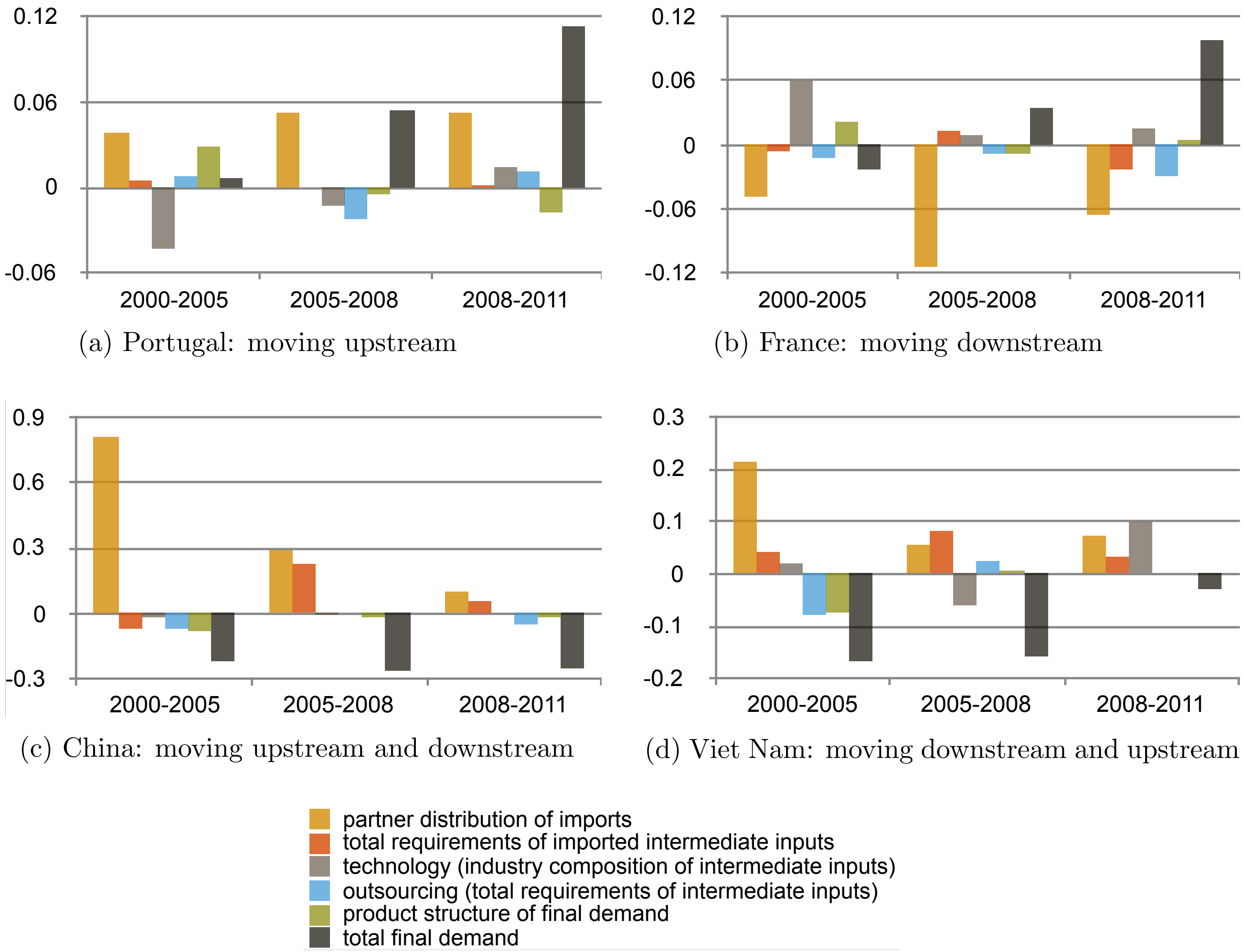


Figure 4: Results of the structural decomposition of the change in GVC positions for selected countries, 2000-2011

Source: OECD ICIO tables, author's calculations.

that is largely responsible for pulling Portugal upstream and France downstream in global value chains. In cases of China and Viet Nam, the change because of individual factors was mostly unidirectional, but, unlike Portugal and France, these countries experience a negative contribution of total final demand.

The change in total final demand was mostly responsible for the shifts in country GVC positions, as Table 2 shows. The second most influential factor was the partner distribution of imports. Surprisingly, factors that record changes in technology and propensity to import intermediate inputs turned out to be less important, and their importance was declining over the entire period. Moreover, there were no countries whose GVC positions were primarily affected by changes in the product structure of final demand. To ensure the relevance of these results for policy purposes, the change in the underlying factors should be split into a change within the domestic economy and a change in the international economy because interested policy makers may apply regulatory measures to the former not the latter. For example, to investigate the drivers of the GVC position of France, the partner distribution of imports may be split into that of France with respect to its partners and that of all other countries. However, this would require a more complex structural decomposition and may be a subject for a future research.

Table 2: Summary statistics of the structural decomposition of the change in the GVC production line position index

	Number of countries for which this factor contributed the maximum absolute change		
Factor	2000-2005	2005-2008	2008-2011
partner distribution of imports	18	20	17
total requirements of imported intermediate inputs	10	6	9
technology (industry composition of intermediate inputs)	9	7	7
outsourcing (total requirements of intermediate inputs)	4	6	0
product structure of final demand	0	0	0
total final demand	20	22	28

Note: The results exclude the rest of world.
Source: OECD ICIO tables, author's calculations.

3.4 To upgrade or not to upgrade position in global value chains?

Here we revert to a prevailing belief that the position upstream in global value chains helps deriving more value added and, generally, more incomes from production. We test it by plotting value added coefficients against GVC production line position index for each industry in each country in Fig. 5 and for each aggregated industry in Fig. 6. The vertical line denotes the values of the index that equal unity, or neutral position in global value chains. Dots to the right of the line correspond to the upstream industries, and those to the left to the downstream industries. The scatter plot additionally distinguishes the industries pertaining to agriculture (marked in green), mining and quarrying (yellow), manufacturing and electricity, gas and water supply (blue) and construction with services (red).

There is no simple pattern to describe the relationship between the two variables considered in Fig. 5 and 6. By and large, position indices of $\mathbf{p} = 1 \pm 0.5$ are typical for manufacturing industries with most of those found downstream ($0.5 < \mathbf{p} < 1$). In the vast majority of cases, the share of output that these industries retain as value added does not exceed 50%. The closer an industry is to final user, the higher the probability that it is a service industry. Services are also prevalent among the industries further from final user ($\mathbf{p} > 1.5$). This is in line with the "smile curve" pattern: production chain starts with service activities such as research, design (high value added), proceeds to manufacturing activities (low value added) and ends up again with services such as marketing, distribution (high value added). Using the "value added propagation length" indicator Ye et al. (2015) confirmed the emergence of the "smile curves" for various value chains.

Looking at the results at the aggregate industry level (Fig. 6), we may establish that the most upstream service industries include renting of machinery and equipment, R&D and related service activities, wholesale and retail trade, financial intermediation. The most downstream service industries are health, public administration, education, hotels and restaurants. All of these service industries ensure value added share more than 50% in 2011, with the exception of hotels and restaurants (48%). These plus mining are the industries that a country should prioritise if it wants to propel itself to "higher value-added activities". However, as is evident from this subsection, this does not necessarily require upgrading position in global value chains. Meanwhile, giving priority to the development of manufacturing will most likely push a country to a neutral position in global value chains and derive less value added from production.

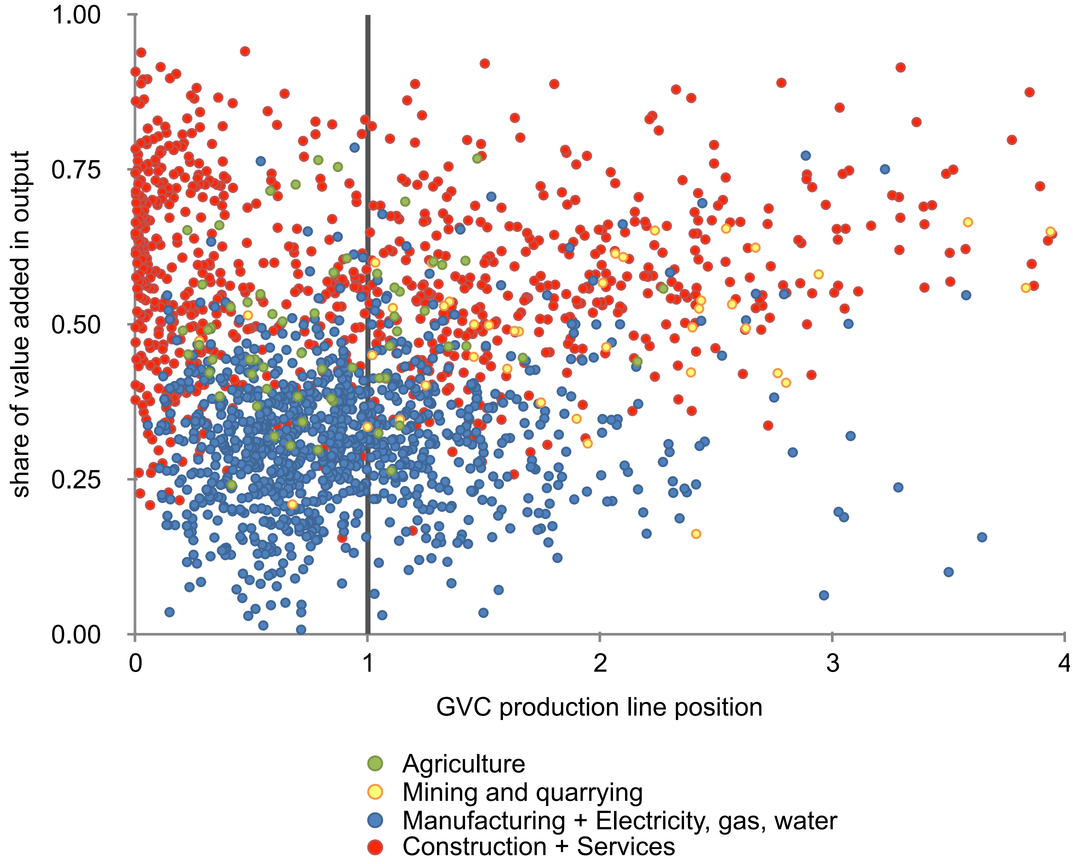


Figure 5: Relationship between value added share and GVC production line position at country-industry level, 2011

Source: OECD ICIO tables, author's calculations.

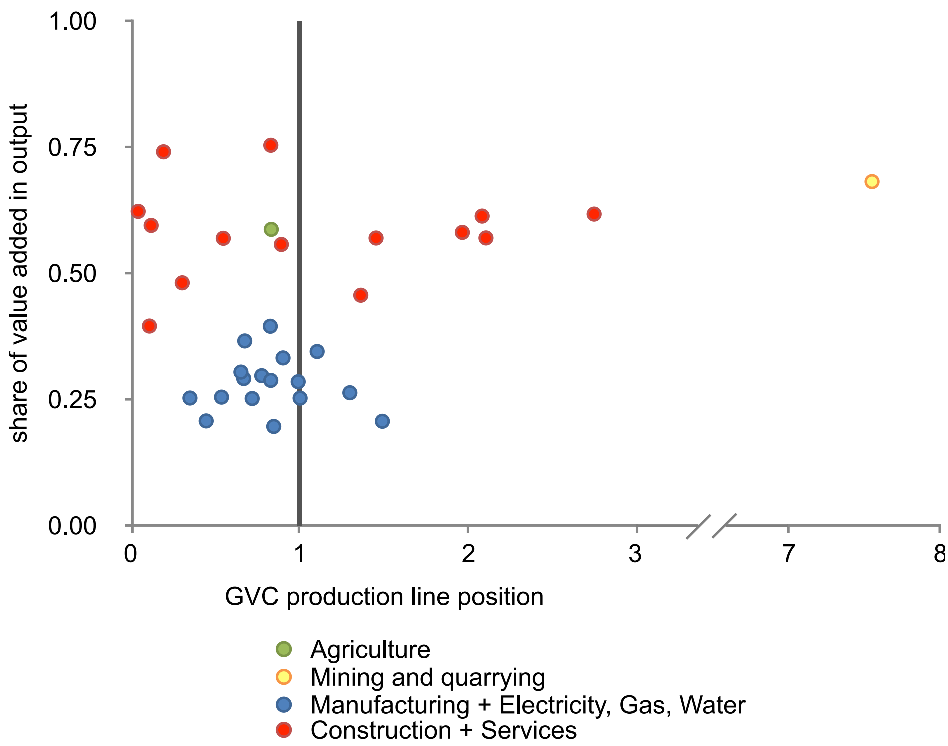


Figure 6: Relationship between value added share and GVC production line position at aggregate industry level, 2011

Source: OECD ICIO tables, author's calculations.

4 Conclusion

The paper has explored the properties and behaviour of the GVC production line position index, first proposed by Wang et al. (2016). First, the domestic component is removed and all length components of the index are normalised with respect to total output. This means that total output serves as a common denominator ensuring that all lengths are entirely comparable. This ensures that the GVC position index at the global level, i.e. after aggregation across all countries and industries, is exactly 1. Second, it is shown that GVCs are an equilibrium system where some countries can be positioned upstream only if other countries are positioned downstream. Country positions in GVCs are not independent from each other, and upgrading the position of one country will most likely cause downgrading the positions of some other countries. Third, a structural decomposition of the modified GVC position index is performed to isolate the impact of such factors as country and industry of origin of imported input requirements, production technology, outsourcing effect, structure and total value of final demand.

Relative positions in GVCs appear rather stable for industries, but fluctuating for countries over time. Global changes in partner distribution of imported inputs and aggregate final demand at country level are found to have the largest impact on country positions in GVCs. Meanwhile, the industry composition of output may well explain cross-country variation in GVC positions.

The results clearly illustrate that while some countries move upstream, other countries move downstream in GVCs. Although there is pronounced growth in the overall complexity of cross-border value chains, it is unlikely that all countries simultaneously upgrade their positions as measured by the proposed index. Moreover, upgrading a position in global value chains does not automatically secure higher value added share of output. For example, shifting from downstream to mostly neutral position, i.e. up the value chain, may require a country to develop more manufacturing activities with lower value added shares. Therefore, there is no simple solution to the problem of upgrading positions in global value chains.

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A GVC production line position index in 2011: country and industry rankings

Table A.1: GVC production line position index: country rankings, 2011

Country (1-30)	GVC position index	Country (31-61)	GVC position index
Brunei Darussalam	7.8730	Lithuania	1.0026
Saudi Arabia	5.0610	Estonia	1.0017
Norway	1.9522	Malaysia	1.0015
Russian Federation	1.8618	United Kingdom	0.9984
Chile	1.8596	Mexico	0.9915
Australia	1.7020	Latvia	0.9678
South Africa	1.6122	Argentina	0.9580
Hong Kong SAR	1.5344	Costa Rica	0.9416
Switzerland	1.4662	Korea	0.8974
Iceland	1.3549	France	0.8967
Colombia	1.3546	Slovak Republic	0.8912
Netherlands	1.3324	Greece	0.8870
Sweden	1.2441	Bulgaria	0.8789
Singapore	1.2398	Italy	0.8699
Brazil	1.2155	Poland	0.8304
Germany	1.1722	Spain	0.8252
Canada	1.1665	Hungary	0.8183
Indonesia	1.1640	Romania	0.8088
Luxembourg	1.1415	Czech Republic	0.8000
New Zealand	1.1128	Thailand	0.7789
Austria	1.0807	Portugal	0.7729
Chinese Taipei	1.0767	United States	0.7577
Israel	1.0730	Malta	0.7514
Denmark	1.0674	Cyprus	0.7474
Ireland	1.0534	Viet Nam	0.7211
Philippines	1.0526	China	0.6924
Belgium	1.0407	Tunisia	0.6878
Japan	1.0186	India	0.6765
Finland	1.0059	Croatia	0.5472
Slovenia	1.0055	Turkey	0.4763
		Cambodia	0.3872

Source: OECD ICIO tables, author's calculations.

Table A.2: GVC production line position index: industry rankings, 2011

Industry	GVC position index
Mining and quarrying	7.5552
Renting of machinery and equipment	2.7516
R&D and other business activities	2.1084
Wholesale and retail trade; repairs	2.0857
Financial intermediation	1.9679
Basic metals	1.4926
Computer and related activities	1.4550
Transport and storage	1.3642
Chemicals and chemical products	1.2993
Pulp, paper, paper products, printing and publishing	1.1047
Rubber and plastics products	1.0034
Wood and products of wood and cork	0.9920
Fabricated metal products	0.9021
Post and telecommunications	0.8919
Coke, refined petroleum products and nuclear fuel	0.8469
Agriculture, hunting, forestry and fishing	0.8321
Real estate activities	0.8294
Computer, Electronic and optical equipment	0.8292
Electricity, gas and water supply	0.8263
Machinery and equipment, nec	0.7755
Electrical machinery and apparatus, nec	0.7185
Manufacturing nec; recycling	0.6744
Other non-metallic mineral products	0.6684
Other transport equipment	0.6517
Other community, social and personal services	0.5467
Textiles, textile products, leather and footwear	0.5363
Motor vehicles, trailers and semi-trailers	0.4451
Food products, beverages and tobacco	0.3486
Hotels and restaurants	0.3027
Education	0.1916
Public admin. and defence; compulsory social security	0.1179
Construction	0.1077
Health and social work	0.0407

Source: OECD ICIO tables, author's calculations.

B Destination of outputs

B.1 Factorization of the Leontief inverse

In the inter-country case, the Leontief inverse links the industry that produces output and country where this output is eventually consumed or invested. We will denote the KN×K matrix of total output reallocated to the final destinations as $\mathbf{X}_D$:

$$\mathbf{X}_D = \mathbf{LF} = \begin{bmatrix} \sum_{t=1}^{K} \mathbf{L}_{1t}\mathbf{f}_{t1} & \sum_{t=1}^{K} \mathbf{L}_{1t}\mathbf{f}_{t2} & \cdots & \sum_{t=1}^{K} \mathbf{L}_{1t}\mathbf{f}_{tK} \\ \sum_{t=1}^{K} \mathbf{L}_{2t}\mathbf{f}_{t1} & \sum_{t=1}^{K} \mathbf{L}_{2t}\mathbf{f}_{t2} & \cdots & \sum_{t=1}^{K} \mathbf{L}_{2t}\mathbf{f}_{tK} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{t=1}^{K} \mathbf{L}_{Kt}\mathbf{f}_{t1} & \sum_{t=1}^{K} \mathbf{L}_{Kt}\mathbf{f}_{t2} & \cdots & \sum_{t=1}^{K} \mathbf{L}_{Kt}\mathbf{f}_{tK} \end{bmatrix}$$

$$\text{where a block element} \quad \mathbf{X}_{D,rs} = \begin{bmatrix} \sum_{j=1}^{N} l_{rs}^{1j} f_{rs}^{j} \\ \sum_{j=1}^{N} l_{rs}^{2j} f_{rs}^{j} \\ \vdots \\ \sum_{j=1}^{N} l_{rs}^{Nj} f_{rs}^{j} \end{bmatrix}$$

A typical entry in $\mathbf{X}_D$ describes the output of industry i in country r required to satisfy final demand in country s directly or via the production chain. The sum of outputs destined for final demand in all countries, or the row sum of $\mathbf{X}_D$, equals total output, $\mathbf{X}_D\mathbf{i}_K = \mathbf{LFi}_K = \mathbf{x}$.

Given that $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ and $\mathbf{A} = \widehat{\mathbf{A}} + \breve{\mathbf{A}}$, the Leontief inverse may be decomposed into a product of two matrices:

$$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \left(\mathbf{I} - \widehat{\mathbf{A}} - \breve{\mathbf{A}}\right)^{-1} = \left(\mathbf{I} - \widehat{\mathbf{A}} - \breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)\right)^{-1} =$$
$$= \left(\left(\mathbf{I} - \breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\left(\mathbf{I} - \widehat{\mathbf{A}}\right)\right)^{-1} = \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\left(\mathbf{I} - \breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)^{-1}$$

The first factor $\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}$ is equal to a block-diagonal matrix of local Leontief inverses:

$$\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} = \begin{bmatrix} \mathbf{I}_N - \mathbf{A}_{11} & 0 & \cdots & 0 \\ 0 & \mathbf{I}_N - \mathbf{A}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{I}_N - \mathbf{A}_{KK} \end{bmatrix}^{-1} =$$
$$= \begin{bmatrix} (\mathbf{I}_N - \mathbf{A}_{11})^{-1} & 0 & \cdots & 0 \\ 0 & (\mathbf{I}_N - \mathbf{A}_{22})^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\mathbf{I}_N - \mathbf{A}_{KK})^{-1} \end{bmatrix}$$

The power series expansion shows that this matrix describes the production chain confined to the domestic economy where domestic industries purchase intermediates from each other:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}=\mathbf{I}+\widehat{\mathbf{A}}+\widehat{\mathbf{A}}\widehat{\mathbf{A}}+\widehat{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{A}}+\dots$$

Each term $\widehat{\mathbf{A}}^t$ corresponds to the domestic transactions among producers at t^{th} round of production. In the absence of international trade in intermediates ($\breve{\mathbf{A}}=0$), the global Leontief inverse $\mathbf{L}$ would be equal to the matrix of local Leontief inverses.

The second factor is responsible for the production chain beyond national borders that exists because of international trade in intermediates. This matrix can also be expanded as a power series:

$$\left(\mathbf{I}-\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^{-1}=\mathbf{I}+\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+$$
$$+\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+\dots$$

In the above expression, each term $\left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^u$ corresponds to a sequential border crossing at u^{th} tier of cross-border supply. Within each term, the local Leontief inverses $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}$ capture the domestic transactions in u^{th} tier partner downstream the value chain.

The first factor therefore corresponds to the value chain at the country of origin while the second term to the value chain across borders and within the partner economies. For brevity, the second factor will henceforth be denoted as $\mathbf{H}$. Then the factorization of the Leontief inverse can be written as:

$$\mathbf{L}=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\mathbf{I}-\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^{-1}=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}\tag{B.1}$$

B.2 Forward decomposition of output: from production to final destination

Using the factorization of the Leontief inverse, we can rewrite the equation of bilateral output reallocated to the final destinations:

$$\mathbf{X}_D=\mathbf{LF}=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{HF}\tag{B.2}$$

Splitting final demand into domestic and international sales of final products $\mathbf{F}=\widehat{\mathbf{F}}+\breve{\mathbf{F}}$, and detaching the first term $\mathbf{I}$ in the power series from both inverse matrices $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}$ and $\mathbf{H}$ results in the following decomposition of equation (B.2):

$$\begin{aligned}\mathbf{X}_D&=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{HF}=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}\widehat{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}\breve{\mathbf{F}}=\\&=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}=\\&=\widehat{\mathbf{F}}+\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\widehat{\mathbf{F}}+(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}+\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}+\\&+\breve{\mathbf{F}}+\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\breve{\mathbf{F}}+(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}+\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}\end{aligned}\tag{B.3}$$

Below is a brief interpretation of the eight terms in equation (B.3).

1. $\widehat{\mathbf{F}}$ – final products sold directly to domestic users. These do not involve intermediate production stages, do not cross borders.[7]

2. $\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\widehat{\mathbf{F}} = \left(\widehat{\mathbf{A}}+\widehat{\mathbf{A}}\widehat{\mathbf{A}}+\ldots\right)\widehat{\mathbf{F}}$ – intermediate products supplied to domestic producers only and eventually embodied in final products for domestic use. These involve at least one intermediate production stage, do not cross borders.

3. $(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}} = \left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+\ldots\right)\widehat{\mathbf{F}}$ – exported intermediate products that are eventually embodied in final products for domestic final use in direct and indirect partner countries. These outputs undergo at least one intermediate production stage and cross at least one border.

4. $\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}} =$
$= \left(\widehat{\mathbf{A}}+\widehat{\mathbf{A}}\widehat{\mathbf{A}}+\ldots\right)\left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+\ldots\right)\widehat{\mathbf{F}}$ – intermediate products supplied to domestic producers, then embodied in intermediate exports and eventually embodied in final products for domestic final use in direct and indirect partner countries. These outputs undergo at least two intermediate production stages, cross at least one border.

5. $\breve{\mathbf{F}}$ – final products sold directly to users in partner countries. These do not involve intermediate production stages, cross one border.

6. $\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\breve{\mathbf{F}} = \left(\widehat{\mathbf{A}}+\widehat{\mathbf{A}}\widehat{\mathbf{A}}+\ldots\right)\breve{\mathbf{F}}$ – intermediate products supplied to domestic producers only and eventually embodied in final exports to direct partner countries. These involve at least one intermediate production stage, cross one border.

7. $(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}} = \left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+\ldots\right)\breve{\mathbf{F}}$ – exported intermediate products that are eventually embodied in final exports to indirect partner countries. These outputs undergo at least one intermediate production stage and cross at least two borders.

8. $\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}} =$
$= \left(\widehat{\mathbf{A}}+\widehat{\mathbf{A}}\widehat{\mathbf{A}}+\ldots\right)\left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}+\ldots\right)\breve{\mathbf{F}}$ – intermediate products supplied to domestic producers, then embodied in intermediate exports and eventually embodied in final exports to indirect partner countries. These outputs undergo at least two intermediate production stages, cross at least two borders.

First, second, fifth and sixth terms are less relevant for value chain analysis as these are only used in domestic inter-industry transactions and, at best, are sold to direct partner country as final products. Meanwhile, third, fourth, seventh and eighth terms are those included in longer and more complex value chain because they undergo intermediate production stages in partner countries and cross-border intermediate production stages. Fig. B.1 provides a simplified outline of the paths of these components in the downstream value chain.

[7] Production stages and border crossings for the purpose of this interpretation are not weighted.

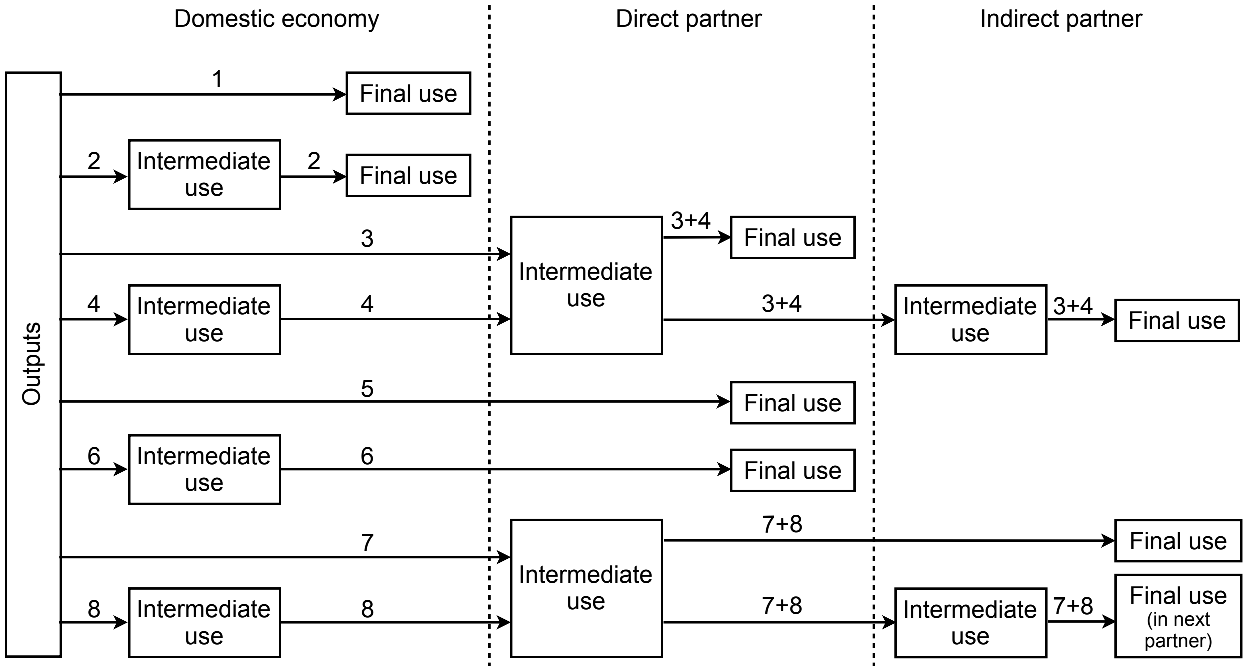

Figure B.1: Forward decomposition of output: a simplified outline of the eight terms in equation (B.3)
Note: indirect partner may coincide with the domestic economy.

The sum of the third, fifth and seventh terms is equal to cumulative exports, described by Muradov (2016):

$$\mathbf{E}_{cum} = (\mathbf{H} - \mathbf{I})\widehat{\mathbf{F}} + \check{\mathbf{F}} + (\mathbf{H} - \mathbf{I})\check{\mathbf{F}} = \mathbf{H}\check{\mathbf{F}} + (\mathbf{H} - \mathbf{I})\widehat{\mathbf{F}} = \mathbf{H}\mathbf{F} - \widehat{\mathbf{F}} \tag{B.4}$$

In $\mathbf{E}_{cum}$, each element describes the amount of product of industry i in country r that is first exported and eventually used for final demand in country s. Cumulative exports therefore include direct bilateral exports to partner countries and indirect exports via third countries. Total cumulative exports to all destinations are equal to total direct gross exports, $\mathbf{E}_{cum}\mathbf{i}_K = \mathbf{E}_{bil}\mathbf{i}_K$.[8]

B.3 Number of production stages: from production to final demand

The sequence of production stages required for industry output to reach final users can be approximated as a power series:

$$\mathbf{X}_D = \mathbf{LF} = \mathbf{F} + \mathbf{AF} + \mathbf{AAF} + \mathbf{AAAF} + \ldots = \left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \ldots\right)\mathbf{F}$$

Each term in the power series corresponds to the number of transactions between producers and final users. A transaction may be understood as production by industry i in country r for industry j or final user in country s. A transaction therefore involves production by industry i in country r, delivery of products to industry j or final user in country s and

[8]Gross bilateral exports in the inter-country input-output system may be obtained by summing the international sales of outputs for intermediate and final use:

$$\mathbf{E}_{bil} = \check{\mathbf{Z}}\mathbf{S}_N + \check{\mathbf{F}}$$

where $\mathbf{S}_N$ is a industry-wise aggregation matrix, see Appendix G.

receipt of payment from them in return for the products supplied.[9] Industry i in country r may produce and deliver a product directly to final user in country s, and this is described by the first term in the power series, **F** (or **I** in the brackets). Or it may produce intermediate products and sell those to industry j in country s for eventual use in the production/delivery of final products, which is described by all other terms in the power series.[10]

Production and delivery of products to final users is always in the end of production chain, so there is one transaction between producer and final user per each term in the power series. The total number of **F** in the power series, or simply total number of terms, is equal to the total unweighted number of these transactions. Meanwhile, the number of **A** in each term indicates how many transactions among producers – that involve production and delivery of intermediate products – precede the production and delivery of final products. For example, **AAAF** signifies that a product of industry i in country r has to undergo three transactions in intermediates and one transaction in final products. The unweighted sum of all **A** leads to an infinite series $1 + 2 + 3 \ldots$

We will henceforth refer to the transactions that involve the production and delivery of intermediate products between producers as "intermediate production stages" and to the transactions that involve the production and delivery of products to final user as "final production stages".

The sum of the number of production stages weighted by the decreasing shares of output at each successive production round yields a measure conceptually similar to the average propagation length (see Dietzenbacher et al., 2005).

First, we define weights (using the fraction sign for the element-to-element division, to simplify notation):

$$\frac{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} = \frac{\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} + \frac{\mathbf{AF}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} + \frac{\mathbf{AAF}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} + \ldots$$

Each term in the expression above describes a decreasing share of industry output that reaches final demand after the respective production stage. The longer the value chain, the smaller share of output reaches final user.

We will apply the weights separately to intermediate and final production stages. To count intermediate production stages, we apply the weights to the sequence of numbers starting from zero that conform to the number of **A** in the power series:

$$\begin{aligned}\mathbf{C}_{(\mathbf{x}_D)ips} &= 0 \times \frac{\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} + 1 \times \frac{\mathbf{AF}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} + 2 \times \frac{\mathbf{AAF}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} + \ldots = \\ &= \frac{(\mathbf{A}+2\mathbf{A}^2+\ldots)\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} = \frac{(\mathbf{I}-\mathbf{A})^{-1}\left((\mathbf{I}-\mathbf{A})^{-1}-\mathbf{I}\right)\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} = \frac{\mathbf{L}(\mathbf{L}-\mathbf{I})\mathbf{F}}{\mathbf{LF}}\end{aligned} \tag{B.5}$$

To count final production stages, we apply the weights to one additional transaction at each round:

[9] The payment may not coincide with the supply of products or may not occur at all. In this case, transaction involves the emergence of claim or obligation to pay.

[10] Recall that, as has been mentioned in subsection 2.1, in the real economy there are trade and transport industries that are usually responsible for deliveries of intermediate products from one industry to another. The interpretation in this and the following sections corresponds to the basic price concept that is customary in the input-output analysis: we treat trade and transport services as intermediate inputs supplied to producers who implicitly use those inputs for the deliveries of their outputs.

$$\mathbf{C}_{(\mathbf{x}_D)fps} = 1 \times \frac{\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} + 1 \times \frac{\mathbf{AF}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} + 1 \times \frac{\mathbf{AAF}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} + \ldots = \\ = \frac{(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\ldots)\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} = \frac{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} = \frac{\mathbf{LF}}{\mathbf{LF}} \tag{B.6}$$

Clearly, $\mathbf{C}_{(\mathbf{x}_D)fps}$ is a KN×K matrix of ones which is true for all measures of the weighted average number of final production stages. The sum of the weighted average number of intermediate production stages and the weighted average number of final production stages is then equal to:

$$\mathbf{C}_{(\mathbf{x}_D)ps} = \mathbf{C}_{(\mathbf{x}_D)ips} + \mathbf{C}_{(\mathbf{x}_D)fps} = \frac{(\mathbf{I}-\mathbf{A})^{-1}\left((\mathbf{I}-\mathbf{A})^{-1}-\mathbf{I}\right)\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} + \frac{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} = \\ = \frac{(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}}{(\mathbf{I}-\mathbf{A})^{-1}\mathbf{F}} = \frac{\mathbf{LLF}}{\mathbf{LF}} \tag{B.7}$$

Wang et al. (2016) developed a technique of the additive decomposition of production length in cases where value added flows are described by a product of local and/or global Leontief inverse matrices. To apply this technique here, we revert to the second line of equation (B.3) where the eight terms are re-arranged in a more compact form:

$$\mathbf{X}_D = \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}} + \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}} + \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}} + \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}} \tag{B.8}$$

Now, we will count the number of production stages with respect to each of the four terms as outputs move along the value chain from producer to final user. The decomposition above allows us to classify production stages into domestic production stages in the country of origin, domestic production stages in partner countries and cross-border production stages. We will also be able to simultaneously distinguish intermediate production stages and final production stages as explained above.

B.3.1 Production for domestic final use, $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}$

The first term in equation (B.8) calculates industry output that does not leave the domestic economy and is delivered to final users at home in whatever form:

$$\underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{at origin}}} \underbrace{\widehat{\mathbf{F}}}_{\substack{\text{production/delivery}\\ \text{to final demand}\\ \text{at origin}}} = \widehat{\mathbf{F}} + \widehat{\mathbf{A}}\widehat{\mathbf{F}} + \widehat{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}} + \ldots$$

We will count the average weighted number of intermediate production stages and the weighted average number of final production stages.

Number of intermediate production stages The weights are defined by the power series decomposition above, and the count of production stages starts from zero:

$$\mathbf{C}_{(\mathbf{x}_D 1)ips.o} = 0 \times \frac{\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} + 1 \times \frac{\widehat{\mathbf{A}}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} + 2 \times \frac{\widehat{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} + \ldots =$$

$$= \frac{(\widehat{\mathbf{A}} + 2\widehat{\mathbf{A}}^2 + \ldots)\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} - \mathbf{I}\right)\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} \tag{B.9}$$

Number of final production stages Production and delivery to final demand are given by the last transaction at each production round and may be weighted in the same way:

$$\mathbf{C}_{(\mathbf{x}_D 1)fps.o} = 1 \times \frac{\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} + 1 \times \frac{\widehat{\mathbf{A}}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} + 1 \times \frac{\widehat{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} + \ldots =$$

$$= \frac{(\mathbf{I} + \widehat{\mathbf{A}} + \widehat{\mathbf{A}}^2 + \ldots)\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} \tag{B.10}$$

B.3.2 Production for partner eventual domestic use, $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}$

This part of total output necessarily includes intermediates that may first undergo transformation in the country of origin, are then exported and may undergo subsequent transformation in partner countries, and eventually end up in final products for domestic use in partners (but may also return home after a series of transactions). Accordingly, this output involves intermediate production stages at origin, intermediate production stages in partners, final production stages in partners and cross-border intermediate production stages:

$$\begin{aligned}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}} = & \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{at origin}}} \underbrace{\breve{\mathbf{A}}}_{\substack{\text{production and}\\ \text{cross-border}\\ \text{delivery}\\ \text{of intermediates}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{in direct partner}\\ \text{(at destination)}}} \underbrace{\widehat{\mathbf{F}}}_{\substack{\text{production/delivery}\\ \text{to final demand}\\ \text{in direct partner}\\ \text{(at destination)}}} + \\ + & \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{at origin}}} \underbrace{\breve{\mathbf{A}}}_{\substack{\text{production and}\\ \text{cross-border}\\ \text{delivery}\\ \text{of intermediates}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{in direct partner}}} \underbrace{\breve{\mathbf{A}}}_{\substack{\text{production and}\\ \text{cross-border}\\ \text{delivery}\\ \text{of intermediates}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{in 2nd tier partner}\\ \text{(at destination)}}} \underbrace{\widehat{\mathbf{F}}}_{\substack{\text{production/delivery}\\ \text{to final demand}\\ \text{in 2nd tier partner}\\ \text{(at destination)}}} + \ldots\end{aligned} \tag{B.11}$$

Number of intermediate production stages at origin The first term in $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}$ is the matrix of local Leontief inverses that is responsible for the intermediate production stages in the country of origin. Counting the weighted average number of intermediate production stages at origin requires the power series expansion of this matrix:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}} = (\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}} + \widehat{\mathbf{A}}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}} + \widehat{\mathbf{A}}\widehat{\mathbf{A}}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}} + \ldots$$

The first term in this power series does not involve production and delivery of intermediates at home, and the count starts from zero:

$$\mathbf{C}_{(\mathbf{x}_D 2)ips.o} = 0 \times \frac{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} + 1 \times \frac{\widehat{\mathbf{A}}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} + 2 \times \frac{\widehat{\mathbf{A}}\widehat{\mathbf{A}}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} + \\ + \ldots = \frac{(\widehat{\mathbf{A}} + 2\widehat{\mathbf{A}}^2 + \ldots)(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} \tag{B.12}$$

Number of intermediate production stages in partners The second term in $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}$ is responsible for the international part of the production chain.

The power series of $\mathbf{H}$ in equation (B.11) reveal local Leontief inverses that correspond to domestic intermediate production stages in the country of destination and in all other countries between the origin and destination. For example, the first term may be interpreted as follows:

$$\underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery} \\ \text{of intermediates} \\ \text{at origin}}} \underbrace{\breve{\mathbf{A}}}_{\substack{\text{production and} \\ \text{cross-border} \\ \text{delivery} \\ \text{of intermediates}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery} \\ \text{of intermediates} \\ \text{in direct partner} \\ \text{(at destination)}}} \underbrace{\widehat{\mathbf{F}}}_{\substack{\text{production/delivery} \\ \text{to final demand} \\ \text{in direct partner} \\ \text{(at destination)}}}$$

The local Leontief inverses indicate that there are nested power series within the power series of $\mathbf{H}$. Nested power series expansion models the sequence of intermediate production stages in direct partner and gives appropriate weights:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}} = \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{F}} + \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}} + \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}} + \ldots$$

As usual, the count of intermediate production stages starts from zero:

$$\mathbf{C}_{(\mathbf{x}_D 2)ips.p1} = 0 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} + 1 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} + \\ + 2 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} + \ldots = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\widehat{\mathbf{A}} + 2\widehat{\mathbf{A}}^2 + \ldots\right)\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} = \\ = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}$$

The interpretation of the second term in the power series of $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}$ is as follows:

$$\underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{at origin}}}\underbrace{\breve{\mathbf{A}}}_{\substack{\text{production and}\\ \text{cross-border}\\ \text{delivery}\\ \text{of intermediates}}}\underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{in direct partner}}}\underbrace{\breve{\mathbf{A}}}_{\substack{\text{production and}\\ \text{cross-border}\\ \text{delivery}\\ \text{of intermediates}}}\underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{in 2nd tier partner}\\ \text{(at destination)}}}\underbrace{\widehat{\mathbf{F}}}_{\substack{\text{production/delivery}\\ \text{to final demand}\\ \text{in 2nd tier partner}\\ \text{(at destination)}}}$$

Counting the average number of intermediate production stages is, however, more complex than in the previous case. There are two nested power series that denote production stages in direct partner and those in indirect partner:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}=$$
$$=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}+\widehat{\mathbf{A}}+\widehat{\mathbf{A}}^2+\ldots\right)\breve{\mathbf{A}}\left(\mathbf{I}+\widehat{\mathbf{A}}+\widehat{\mathbf{A}}^2+\ldots\right)\widehat{\mathbf{F}}$$

The counting and weighting procedure applies to each of the two power series, and the result after summation (with the intermediate calculation steps dropped) is:

$$\mathbf{C}_{(\mathbf{x}_D2)ips.p2}=\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}+$$
$$+\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}$$

In general, the u^{th} term in the power series of $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}$ involves u nested expansions of local Leontief inverses and a sum of u numbers of domestic intermediate production stages in u partners. Finally, the weighted average of all domestic intermediate production stages in partners converges to:

$$\mathbf{C}_{(\mathbf{x}_D2)ips.p}=\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}\tag{B.13}$$

Number of final production stages Production and delivery to final demand in $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}$ is a domestic transaction in direct or indirect partner countries after a series of transformations of initially exported intermediates.

Recall the first term in equation (B.11) where the nested power series expansion is:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}}+\ldots$$

We count one final production stage involving delivery to final demand per each term in the above expression:

$$\mathbf{C}_{(\mathbf{x}_D 2) fps.p1} = 1 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \widehat{\mathbf{F}}} + 1 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \widehat{\mathbf{F}}} +$$
$$+ 1 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \widehat{\mathbf{F}}} + \ldots = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}+\widehat{\mathbf{A}}+\widehat{\mathbf{A}}^2+\ldots\right) \widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \widehat{\mathbf{F}}} =$$
$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \widehat{\mathbf{F}}}$$

With respect to the second term in equation (B.11), final production stage takes place in the country of destination, and the respective nested power series expansion is:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \widehat{\mathbf{F}} = \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}}\widehat{\mathbf{F}} +$$
$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}} + \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}} + \ldots$$

Counting one final production stage per each term results in the following expression:

$$\mathbf{C}_{(\mathbf{x}_D 2) fps.p2} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \widehat{\mathbf{F}}}$$

In case of final production stages, the u^{th} term in the power series of $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}$ will involve one nested expansion of local Leontief inverses and a sum of one number of production stages in u partners. Finally, the weighted average of all final production stages in partners converges to:

$$\mathbf{C}_{(\mathbf{x}_D 2) fps.p} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} \tag{B.14}$$

Number of cross-border intermediate production stages Consider, again, the power series expansion in equation (B.11) to see that initial outputs only cross borders when one industry in one country sells its intermediate products to another industry in another country. The block-off-diagonal matrix of technical coefficients $\breve{\mathbf{A}}$ is responsible for the cross-border intermediate production stages. We therefore start counting these stages from one:

$$\mathbf{C}_{(\mathbf{x}_D 2)ips.cb} = 1 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} + 2 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} +$$

$$+ 3 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} + \ldots =$$

$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} + 2\left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^2 + 3\left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^3 + \ldots\right)\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} =$$

$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} \tag{B.15}$$

B.3.3 Production for partner final use, $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}$

The third term in equation (B.8) calculates industry output that is exported in the form of final products to direct partners. It may or may not undergo transformation in the home economy before leaving it:

$$\underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{at origin}}} \underbrace{\breve{\mathbf{F}}}_{\substack{\text{production and}\\ \text{cross-border delivery}\\ \text{to final demand}\\ \text{in direct partner}}} = \breve{\mathbf{F}} + \widehat{\mathbf{A}}\breve{\mathbf{F}} + \widehat{\mathbf{A}}\widehat{\mathbf{A}}\breve{\mathbf{F}} + \ldots$$

We will count the average weighted number of intermediate production stages at origin and the weighted average number of cross-border final production stages.

Number of intermediate production stages at origin The first term in the power series above does not involve production and exchange of intermediates among domestic industries, so the count of intermediate production stages starts from zero:

$$\mathbf{C}_{(\mathbf{x}_D 3)ips.o} = 0 \times \frac{\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}} + 1 \times \frac{\widehat{\mathbf{A}}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}} + 2 \times \frac{\widehat{\mathbf{A}}\widehat{\mathbf{A}}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}} + \ldots =$$

$$= \frac{(\widehat{\mathbf{A}} + 2\widehat{\mathbf{A}}^2 + \ldots)\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} - \mathbf{I}\right)\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}} \tag{B.16}$$

Number of cross-border final production stages As usual, the last transaction at each production round is production and delivery to final demand, but it spans across borders in this case:

$$\mathbf{C}_{(\mathbf{x}_D3)fps.cb} = 1 \times \frac{\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}} + 1 \times \frac{\widehat{\mathbf{A}}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}} + 1 \times \frac{\widehat{\mathbf{A}}\widehat{\mathbf{A}}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}} + \ldots =$$

$$= \frac{(\mathbf{I}+\widehat{\mathbf{A}}+\widehat{\mathbf{A}}^2+\ldots)\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}} \tag{B.17}$$

B.3.4 Production for partner eventual exports of final products, $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}$

This part of total output necessarily includes intermediates that may first undergo transformation in the country of origin, are then exported and may undergo subsequent transformation in partner countries, and eventually end up in exports of final products to the last partner in the value chain. This output involves intermediate production stages at origin, intermediate production stages in partners, cross-border intermediate production stages and cross-border final production stages:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}} = \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{at origin}}} \underbrace{\breve{\mathbf{A}}}_{\substack{\text{production and}\\ \text{cross-border}\\ \text{delivery}\\ \text{of intermediates}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{in direct partner}}} \underbrace{\breve{\mathbf{F}}}_{\substack{\text{production and}\\ \text{cross-border delivery}\\ \text{to final demand}\\ \text{in 2nd tier partner}\\ \text{(at destination)}}} +$$

$$+ \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{at origin}}} \underbrace{\breve{\mathbf{A}}}_{\substack{\text{production and}\\ \text{cross-border}\\ \text{delivery}\\ \text{of intermediates}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{in direct partner}}} \underbrace{\breve{\mathbf{A}}}_{\substack{\text{production and}\\ \text{cross-border}\\ \text{delivery}\\ \text{of intermediates}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}}_{\substack{\text{production/delivery}\\ \text{of intermediates}\\ \text{in 2nd tier partner}}} \underbrace{\breve{\mathbf{F}}}_{\substack{\text{production and}\\ \text{cross-border delivery}\\ \text{to final demand}\\ \text{in 3nd tier partner}\\ \text{(at destination)}}} + \ldots \tag{B.18}$$

Number of intermediate production stages at origin Similarly to subsection B.3.2, we identify intermediate production stages in the country of origin in a power series expansion of the matrix of local Leontief inverses in $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}$:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}} = (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}} + \widehat{\mathbf{A}}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}} + \widehat{\mathbf{A}}\widehat{\mathbf{A}}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}} + \ldots$$

Accordingly, the weighted average number of intermediate production stages at origin is:

$$\mathbf{C}_{(\mathbf{x}_D4)ips.o} = 0 \times \frac{(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} + 1 \times \frac{\widehat{\mathbf{A}}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} + 2 \times \frac{\widehat{\mathbf{A}}\widehat{\mathbf{A}}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} +$$

$$+ \ldots = \frac{(\widehat{\mathbf{A}}+2\widehat{\mathbf{A}}^2+\ldots)(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} \tag{B.19}$$

Number of intermediate production stages in partners Here we only have to replicate the decomposition into intermediate production stages in partners as in subsection B.3.2. The nested power series of local Leontief inverses in equation (B.18) model domestic production in partners between the origin and destination. With respect to the first term, this nested power series expansion is as follows:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\breve{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{A}}\breve{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{A}}\breve{\mathbf{F}}+\dots$$

The count of intermediate production stages starts from zero:

$$\begin{aligned}\mathbf{C}_{(\mathbf{X}_D4)ips.p1}&=0\times\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}+1\times\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{A}}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}+\\&+2\times\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{A}}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}+\dots=\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\widehat{\mathbf{A}}+2\widehat{\mathbf{A}}^2+\dots\right)\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}=\\&=\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}\end{aligned}$$

In the second term in equation (B.18), two nested power series denote intermediate production stages in direct partner and those in indirect partner. The counting and weighting procedure followed by summation yields:

$$\begin{aligned}\mathbf{C}_{(\mathbf{X}_D4)ips.p2}=&\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}+\\&+\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}\end{aligned}$$

In general, the u^{th} term in the power series of $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}$ involves u nested expansions of local Leontief inverses and a sum of u numbers of domestic intermediate production stages in u partners. Finally, the weighted average of all domestic intermediate production stages in partners converges to:

$$\mathbf{C}_{(\mathbf{X}_D4)ips.p}=\frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}\tag{B.20}$$

Number of cross-border intermediate production stages Counting cross-border intermediate production stages here is similar to that in subsection B.3.2. The count starts from one and the weights are given by the power series expansion in equation (B.18):

$$\mathbf{C}_{(\mathbf{x}_{D}4)ips.cb} = 1 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} + 2 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} +$$

$$+ 3 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} + \ldots =$$

$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \left(\breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} + \left(\breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^{2} + \left(\breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^{3} + \ldots + \right) \breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} =$$

$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \mathbf{H}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} \tag{B.21}$$

Number of cross-border final production stages In contrast to the second term $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}$ in equation (B.8), the fourth term $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}$ in that equation involves one more border crossing when, in the end of value chain, final product leaves the last country of transformation for the country of final use. Accordingly, we count one more transaction with respect to each term in the power series expansion of the international part of $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}$:

$$\mathbf{C}_{(\mathbf{x}_{D}4)fps.cb} = 1 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} + 1 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} +$$

$$+ 1 \times \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} + \ldots =$$

$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \left(\breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} + \left(\breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^{2} + \left(\breve{\mathbf{A}} \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^{3} + \ldots \right) \breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} =$$

$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} \tag{B.22}$$

B.3.5 From production to final demand: summary numbers

The decomposition of the number of transactions along the downstream value chain in subsection B.3 resulted in twelve indicators. We will now consider various options to summarize these using the respective shares in total output as weights.

Total number of intermediate production stages at origin This requires an aggregation of the weighted average number of intermediate production stages linking producers in the country of origin of output, irrespective of its eventual destination: $\mathbf{C}_{(\mathbf{X}_D 1) ips.o}$, $\mathbf{C}_{(\mathbf{X}_D 2) ips.o}$, $\mathbf{C}_{(\mathbf{X}_D 3) ips.o}$ and $\mathbf{C}_{(\mathbf{X}_D 4) ips.o}$. As in Wang et al. (2016), each number is weighted by the share of the respective component in total output as per equation (B.8):

$$\begin{aligned}
\mathbf{C}_{(\mathbf{X}_D) ips.o} = & \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\mathbf{X}_D} + \\
& + \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\mathbf{X}_D} + \\
& + \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\check{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\check{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\check{\mathbf{F}}}{\mathbf{X}_D} + \\
& + \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)(\mathbf{H}-\mathbf{I})\check{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\check{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\check{\mathbf{F}}}{\mathbf{X}_D} = \\
& = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\widehat{\mathbf{F}} + \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\check{\mathbf{F}}}{\mathbf{X}_D} = \\
& = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\mathbf{F}}{\mathbf{X}_D}
\end{aligned} \tag{B.23}$$

where $\circ$ signifies element-by-element multiplication.

Total number of intermediate production stages in partners Next, we aggregate the number of intermediate production stages that link producers within the borders of partner economies after exports leave the country of origin. Only the second and fourth components in equation (B.8) undergo transformation in partner countries, and only two respective numbers are subject to summation and weighting, $\mathbf{C}_{(\mathbf{X}_D 2) ips.p}$ and $\mathbf{C}_{(\mathbf{X}_D 4) ips.p}$:[11]

[11] Here we also implicitly sum and weigh the average numbers of production stages that correspond to the first and third terms in equation (B.8) and equal zero to ensure that the sum of all weights is a matrix of ones.

$$\mathbf{C}_{(\mathbf{X}_D)ips.p} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\mathbf{X}_D} +$$

$$+ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\mathbf{X}_D} =$$

$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\widehat{\mathbf{F}} + \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\breve{\mathbf{F}}}{\mathbf{X}_D} =$$

$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\mathbf{F}}{\mathbf{X}_D} \tag{B.24}$$

Total number of cross-border intermediate production stages In a similar way, we aggregate the number of intermediate production stages that link producers across national borders throughout entire value chain, $\mathbf{C}_{(\mathbf{X}_D 2)ips.cb}$ and $\mathbf{C}_{(\mathbf{X}_D 4)ips.cb}$:

$$\mathbf{C}_{(\mathbf{X}_D)ips.cb} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\mathbf{X}_D} +$$

$$+ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\mathbf{X}_D} =$$

$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}} + \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\mathbf{X}_D} =$$

$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}(\mathbf{H}-\mathbf{I})\mathbf{F}}{\mathbf{X}_D} \tag{B.25}$$

Equation (B.25) requires careful interpretation. As the denominator is total bilateral output, the result quantifies the number of cross-border intermediate production stages that total output (not only exports!) of one industry in the country of origin has to undergo along the entire value chain until it ends up in final demand of the partner country.

Number of GVC-related production stages Intermediate production stages that link producers in partner countries and across borders can be reasonably classified as production stages relevant to global value chains. Then the weighted average number of GVC-related production stages along the forward value chain is:

$$\mathbf{C}_{(\mathbf{X}_D)ips.GVC} = \mathbf{C}_{(\mathbf{X}_D)ips.p} + \mathbf{C}_{(\mathbf{X}_D)ips.cb} =$$
$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{HF}}{\mathbf{X}_D} + \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}(\mathbf{H}-\mathbf{I})\mathbf{F}}{\mathbf{X}_D} =$$
$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{HF}}{\mathbf{X}_D} \tag{B.26}$$

Total number of all intermediate production stages Finally, we will add up all indicators of the weighted average number of intermediate production stages:

$$\mathbf{C}_{(\mathbf{X}_D)ips} = \mathbf{C}_{(\mathbf{X}_D)ips.o} + \mathbf{C}_{(\mathbf{X}_D)ips.GVC} =$$
$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{HF}}{\mathbf{X}_D} + \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{HF}}{\mathbf{X}_D} =$$
$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\mathbf{H}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{HF}}{\mathbf{X}_D} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{HF}-\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{HF}}{\mathbf{X}_D} =$$
$$= \frac{\mathbf{LLF}-\mathbf{LF}}{\mathbf{X}_D} = \frac{\mathbf{L}(\mathbf{L}-\mathbf{I})\mathbf{F}}{\mathbf{X}_D} \tag{B.27}$$

The result in equation (B.27) quantifies the weighted average number of all intermediate production stages a product has to undergo before it is absorbed in final demand. It is equivalent to the result of the direct aggregate measurement of the number of intermediate production stages using the Leontief inverse in equation (B.5).

Total number of final production stages There are four indicators of the weighted average number of final production stages corresponding to four components in equation (B.8). Those corresponding to the first and third components entail domestic deliveries to final demand and the other two correspond to cross-border deliveries. All four indicators are matrices with KN×K elements equal to 1. It is worth combining all of those into a single measure of the weighted average total number of final production stages, also equal to a matrix of ones:

$$\mathbf{C}_{(\mathbf{X}_D)fps} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\mathbf{X}_D} + \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\mathbf{X}_D} +$$
$$+ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\check{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\check{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\check{\mathbf{F}}}{\mathbf{X}_D} + \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\check{\mathbf{F}}}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\check{\mathbf{F}}} \circ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\check{\mathbf{F}}}{\mathbf{X}_D} =$$
$$= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\check{\mathbf{F}}+\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\check{\mathbf{F}}}{\mathbf{X}_D} = \frac{\mathbf{X}_D}{\mathbf{X}_D} \tag{B.28}$$

The result in equation (B.28) is equal to that in equation (B.6), derived from the global Leontief inverse, and the sum of the measures in (B.27) and (B.28) is equal to the total number of all transactions from production forwards to final use in equation (B.7).

B.3.6 Number of production stages for exports

All summary indicators derived in the previous subsection count the number of production stages that total output undergoes along the downstream value chain. For analytical purposes, it may be useful to count the average number of production stages with respect to only a part of total output, for example, exports. This requires similar aggregation of various indicators of the weighted average number of production stages, but different weights.

Recall that the third, fifth and seventh terms in equation (B.3) add up to cumulative exports $\mathbf{E}_{cum}$ as shown in equation (B.4). Cumulative exports may be treated as direct bilateral exports reallocated to the final destination, i.e., after all production stages. To count those production stages, we will employ the following representation:

$$\mathbf{E}_{cum} = (\mathbf{H} - \mathbf{I})\widehat{\mathbf{F}} + \mathbf{H}\breve{\mathbf{F}}$$

The power series expansion of $\mathbf{H}$ leads to the following expressions:

$$\begin{aligned}(\mathbf{H} - \mathbf{I})\widehat{\mathbf{F}} = {} & \breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}} + \breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}} + \\ & + \breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}} + \ldots \\ \mathbf{H}\breve{\mathbf{F}} = {} & \breve{\mathbf{F}} + \breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}} + \breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}} + \ldots\end{aligned}$$

We will count the weighted average numbers of intermediate production stages in partners (nested power series of local Leontief inverses $(\mathbf{I} - \widehat{\mathbf{A}})^{-1}$), final production stages in partners ($\widehat{\mathbf{F}}$), intermediate cross-border production stages ($\breve{\mathbf{A}}$) and final cross-border production stages ($\breve{\mathbf{F}}$). The counting routines are identical to those described in subsections B.3.2 and B.3.4, but do not cover the intermediate production stages at origin. In the indexing of variables below, the first component of cumulative exports $(\mathbf{H} - \mathbf{I})\widehat{\mathbf{F}}$ will be briefly referred to as $\mathbf{E}_{cum}1$ and the second component $\mathbf{H}\breve{\mathbf{F}}$ as $\mathbf{E}_{cum}2$.

Number of intermediate production stages in partners First, we expand the nested power series of local Leontief inverses in each term of the power series of $(\mathbf{H} - \mathbf{I})\widehat{\mathbf{F}}$. The power series of the first term appears as follows:

$$\breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}} = \breve{\mathbf{A}}\widehat{\mathbf{F}} + \breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}} + \breve{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{A}}\widehat{\mathbf{F}} + \ldots$$

The count starts from zero and the result is:

$$\mathbf{C}_{(\mathbf{E}_{cum}1)ips.p1} = \frac{\breve{\mathbf{A}}\left(\widehat{\mathbf{A}} + 2\widehat{\mathbf{A}}^2 + \ldots\right)\widehat{\mathbf{F}}}{\breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} = \frac{\breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} - \mathbf{I}\right)\widehat{\mathbf{F}}}{\breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}$$

In the second term, there are two nested power series that denote production stages in direct partner and those in indirect partner:

$$\breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}} = \breve{\mathbf{A}}\left(\mathbf{I} + \widehat{\mathbf{A}} + \widehat{\mathbf{A}}^2 + \ldots\right)\breve{\mathbf{A}}\left(\mathbf{I} + \widehat{\mathbf{A}} + \widehat{\mathbf{A}}^2 + \ldots\right)\widehat{\mathbf{F}}$$

The counting and weighting procedure applies to each of the two power series, and the result after summation is:

$$\mathbf{C}_{(\mathbf{E}_{cum}1)ips.p2} = \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} + \\ + \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\widehat{\mathbf{F}}}{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}$$

Parallel to equation (B.13), the sum of all domestic intermediate production stages in all partners results in:

$$\mathbf{C}_{(\mathbf{E}_{cum}1)ips.p} = \frac{(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\widehat{\mathbf{F}}}{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}$$

Next, we expand the nested power series of local Leontief inverses in each term of the power series of $\mathbf{H}\breve{\mathbf{F}}$. This procedure is identical to that described above, and, skipping the derivation, we write the result:

$$\mathbf{C}_{(\mathbf{E}_{cum}2)ips.p} = \frac{(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}}$$

Sum $\mathbf{C}_{(\mathbf{E}_{cum}1)ips.p}$ and $\mathbf{C}_{(\mathbf{E}_{cum}2)ips.p}$ and weigh by the respective shares in cumulative exports to obtain a single weighted average number of intermediate production stages in partners:

$$\mathbf{C}_{(\mathbf{E}_{cum})ips.p} = \frac{(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\widehat{\mathbf{F}}}{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} \circ \frac{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\mathbf{E}_{cum}} + \\ + \frac{(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}} \circ \frac{\mathbf{H}\breve{\mathbf{F}}}{\mathbf{E}_{cum}} = \\ = \frac{(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\mathbf{F}}{\mathbf{E}_{cum}}$$

Number of final production stages in partners We revert to the power series of $(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}$ and count one final production stage per each nested domestic production round in the first term:

$$\mathbf{C}_{(\mathbf{E}_{cum}1)fps.p1} = \frac{\breve{\mathbf{A}}\left(\mathbf{I}+\widehat{\mathbf{A}}+\widehat{\mathbf{A}}^2+\dots\right)\widehat{\mathbf{F}}}{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} = \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}$$

In the second term, final production stage takes place in the country of destination:

$$\mathbf{C}_{(\mathbf{E}_{cum}1)fps.p2} = \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}+\widehat{\mathbf{A}}+\widehat{\mathbf{A}}^2+\ldots\right)\widehat{\mathbf{F}}}{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}} = \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}$$

Finally, the weighted average of all final production stages in partners results in:

$$\mathbf{C}_{(\mathbf{E}_{cum}1)fps.p} = \frac{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}$$

Since the other part of cumulative exports $\mathbf{H}\breve{\mathbf{F}}$ does not involve final production stages in partners, the above result can be weighted by its share in cumulative exports:

$$\mathbf{C}_{(\mathbf{E}_{cum})fps.p} = \frac{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} \circ \frac{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\mathbf{E}_{cum}} + 0 \circ \frac{\mathbf{H}\breve{\mathbf{F}}}{\mathbf{E}_{cum}} = \frac{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\mathbf{E}_{cum}}$$

Number of cross-border intermediate production stages In the power series expansion of $(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}$, we count the number of times $\breve{\mathbf{A}}$ appears in each term signaling about cross-border delivery of intermediates:

$$\begin{aligned}
\mathbf{C}_{(\mathbf{E}_{cum}1)ips.cb} &= 1 \times \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} + 2 \times \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} + \\
&+ 3 \times \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{F}}}{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} + \ldots = \\
&= \frac{\left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} + 2\left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^2 + 3\left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^3 + \ldots\right)\widehat{\mathbf{F}}}{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} = \\
&= \frac{\mathbf{H}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}
\end{aligned}$$

The same applies to the power series of $\mathbf{H}\breve{\mathbf{F}}$, and yields the following:

$$\mathbf{C}_{(\mathbf{E}_{cum}2)ips.cb} = \frac{\mathbf{H}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}}$$

Sum $\mathbf{C}_{(\mathbf{E}_{cum}1)ips.cb}$ and $\mathbf{C}_{(\mathbf{E}_{cum}2)ips.cb}$ and weigh by the respective shares in cumulative exports to obtain a single weighted average number of cross-border intermediate production stages:

$$\mathbf{C}_{(\mathbf{E}_{cum})ips.cb} = \frac{\mathbf{H}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}} \circ \frac{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\mathbf{E}_{cum}} + \frac{\mathbf{H}(\mathbf{H}-\mathbf{I})\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}} \circ \frac{\mathbf{H}\breve{\mathbf{F}}}{\mathbf{E}_{cum}} = \frac{\mathbf{H}(\mathbf{H}-\mathbf{I})\mathbf{F}}{\mathbf{E}_{cum}}$$

Number of cross-border final production stages The second part of cumulative exports $\mathbf{H}\breve{\mathbf{F}}$ also involves final cross-border production stages. Accordingly, we count one more transaction with respect to each term in the power series expansion of $\mathbf{H}\breve{\mathbf{F}}$:

$$\mathbf{C}_{(\mathbf{E}_{cum}2)fps.cb} = 1 \times \frac{\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}} + 1 \times \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}} + 1 \times \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}} +$$
$$+ 1 \times \frac{\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}} + \ldots =$$
$$= \frac{\left(\mathbf{I} + \breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} + \left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^2 + \left(\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)^3 + \ldots\right)\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}} = \frac{\mathbf{H}\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}}$$

The weighted average number of cross-border final production stages that face exports may be calculated as:

$$\mathbf{C}_{(\mathbf{E}_{cum})fps.cb} = 0 \circ \frac{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\mathbf{E}_{cum}} + \frac{\mathbf{H}\breve{\mathbf{F}}}{\mathbf{H}\breve{\mathbf{F}}} \circ \frac{\mathbf{H}\breve{\mathbf{F}}}{\mathbf{E}_{cum}} = \frac{\mathbf{H}\breve{\mathbf{F}}}{\mathbf{E}_{cum}}$$

Total number of border crossings The sum of the weighted average number of cross-border intermediate production stages and cross-border final production stages leads to the weighted average number of border crossings for exports:

$$\mathbf{C}_{(\mathbf{E}_{cum})ps.cb} = \frac{\mathbf{H}(\mathbf{H}-\mathbf{I})\mathbf{F}}{\mathbf{E}_{cum}} + \frac{\mathbf{H}\breve{\mathbf{F}}}{\mathbf{E}_{cum}} = \frac{\mathbf{H}^2\mathbf{F} - \mathbf{H}\widehat{\mathbf{F}}}{\mathbf{E}_{cum}} \tag{B.29}$$

The result is a KN×K matrix where each element may be interpreted as the weighted average number of border crossings along the path of a product of industry i from country r to its final user in country s. This matrix was described in Muradov (2016).

Total number of all production stages Finally, we sum all (domestic and cross-border) intermediate production stages for exports:

$$\mathbf{C}_{(\mathbf{E}_{cum})ips} = \frac{(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} - \mathbf{I}\right)\mathbf{H}\mathbf{F}}{\mathbf{E}_{cum}} + \frac{\mathbf{H}(\mathbf{H}-\mathbf{I})\mathbf{F}}{\mathbf{E}_{cum}} = \frac{(\mathbf{H}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}\mathbf{F}}{\mathbf{E}_{cum}} \tag{B.30}$$

Add final production stages to obtain a complete number of all production stages for exports:

$$\mathbf{C}_{(\mathbf{E}_{cum})ps} = \frac{(\mathbf{H}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}\mathbf{F}}{\mathbf{E}_{cum}} + \frac{(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}}{\mathbf{E}_{cum}} + \frac{\mathbf{H}\breve{\mathbf{F}}}{\mathbf{E}_{cum}} = \frac{(\mathbf{H}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}\mathbf{F} + \mathbf{H}\mathbf{F} - \widehat{\mathbf{F}}}{\mathbf{E}_{cum}} \tag{B.31}$$

B.3.7 A note on value added coefficients

In the literature on the length of global production networks, measurements with respect to the value added flows (see Ye et al., 2015; Wang et al., 2016) are more common. It is shown below that adopting the generation of value added as a starting point does not affect the measurement of the weighted average number of transactions or production stages in value chain at the disaggregate country-industry level.

The relative amount of value added generated by industry j in country s per unit of its total output is recorded in the respective value added coefficient $v_{c,s}^{j} = \frac{v_s^j}{x_s^j}$. In matrix notation, $\hat{\mathbf{v}}_c = \hat{\mathbf{v}}\hat{\mathbf{x}}^{-1}$ is a KN×KN block-diagonal matrix of the value added coefficients. Pre-multiplication of outputs by the value added coefficients provides a measurement of a part of output that is directly contributed by the producer because of employing factors of production, in contrast to using intermediates purchased from other producers. For example, the application of value added coefficients to the matrix of total output reallocated to final destination yields:

$$\hat{\mathbf{v}}_c \mathbf{X}_D = \hat{\mathbf{v}}_c \mathbf{L}\mathbf{F}$$

$\hat{\mathbf{v}}_c\mathbf{LF}$ is a KN×K matrix where a typical entry describes the value added of industry i in country r required to satisfy final demand in country s directly or via production chain. This matrix was described by Johnson and Noguera (2012) and has been extensively utilized for estimates of trade in value added. The power series of this matrix can be interpreted in the same way as that of total bilateral output:

$$\hat{\mathbf{v}}_c\mathbf{LF} = \hat{\mathbf{v}}_c\mathbf{F} + \hat{\mathbf{v}}_c\mathbf{AF} + \hat{\mathbf{v}}_c\mathbf{AAF} + \hat{\mathbf{v}}_c\mathbf{AAAF} + \ldots = \hat{\mathbf{v}}_c\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \ldots\right)\mathbf{F}$$

The first term $\hat{\mathbf{v}}_c\mathbf{F}$ (and $\mathbf{I}$ in the brackets) signifies that part of the value of final products is contributed directly by the producer. This value added reaches the final user in one production/delivery stage. The second term $\hat{\mathbf{v}}_c\mathbf{AF}$ (and, respectively, $\mathbf{A}$) signifies that the production of intermediates that will be used in the production of final products requires a contribution of value added in the same proportion. This value added reaches the final user in two stages, of which one is intermediate production/delivery stage and another one is final stage.

A pre-multiplication of equations (B.3), (B.8) by the value added coefficients converts the calculated flows of the monetary value of products into the flows of value added therein. Obviously, the latter are strictly proportional to the former because the matrix $\hat{\mathbf{v}}_c$ is block-diagonal.

In the same way, the value added coefficients may apply to all measurements of the weighted average number of intermediate and final production stages in subsection B.3. As both numerator and denominator are multiplied row-wise by the same number, this does not affect the results. For example, the total number of transactions (production stages) as per equation (B.7) will appear as follows:

$$\mathbf{C}_{(\mathbf{v})ps} = \frac{\hat{\mathbf{v}}_c\mathbf{LLF}}{\hat{\mathbf{v}}_c\mathbf{LF}} = \frac{\mathbf{LLF}}{\mathbf{LF}} = \mathbf{C}_{(\mathbf{X}_D)ps}$$

The length of value chain or the weighted average number of production stages from producer to final user is therefore equivalent with respect to total output and value added flows. This is true for the disaggregate country-industry level (KN×K dimension), including an aggregation across partner countries (KN×1 dimension).

However, an aggregation across producing industries or countries (K×K, K×1, N×K and N×1 dimensions) is not equivalent in the two cases discussed here. In case the numbers of production stages are calculated with respect to value added flows, the aggregation implicitly involves additional weighting by the value added coefficients:

$$\mathbf{C}_{(\mathbf{v})ps,K\times K} = \frac{\mathbf{S}'_N\hat{\mathbf{v}}_c\mathbf{LLF}}{\mathbf{S}'_N\hat{\mathbf{v}}_c\mathbf{LF}} \neq \frac{\mathbf{S}'_N\mathbf{LLF}}{\mathbf{S}'_N\mathbf{LF}} = \mathbf{C}_{(\mathbf{X}_D)ps,K\times K}$$

C Origin of inputs

C.1 Factorization of the Ghosh inverse

As observed in subsection 2.2, the Ghosh inverse **G** is a KN×KN matrix that attributes total output to primary inputs embodied therein. To apply the Ghosh inverse in the inter-country case, we have first to resize the vector of value added and the valuation layers (margins, net taxes) so that they conform with the dimension of output matrices:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}'_1 & 0 & \cdots & 0 \\ 0 & \mathbf{v}'_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{v}'_K \end{bmatrix} \quad \text{where a block element} \quad \mathbf{v}'_s = \begin{bmatrix} v_s^1 & v_s^2 & \cdots & v_s^N \end{bmatrix}$$

V is a counterpart of **F** in the Leontief model, but, unlike final products, value added is not directly traded across borders, and **V** is a K×KN block-diagonal matrix. Each element v_s^j denotes the amount of value added contributed directly by industry j in country s to its own output.

Similarly, the g^{th} valuation layer is resized as follows:

$$\mathbf{M_Z}(g)_{K\times KN} = \begin{bmatrix} \mathbf{m}'_\mathbf{Z}(g)_{11} & \mathbf{m}'_\mathbf{Z}(g)_{12} & \cdots & \mathbf{m}'_\mathbf{Z}(g)_{1K} \\ \mathbf{m}'_\mathbf{Z}(g)_{21} & \mathbf{m}'_\mathbf{Z}(g)_{22} & \cdots & \mathbf{m}'_\mathbf{Z}(g)_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{m}'_\mathbf{Z}(g)_{K1} & \mathbf{m}'_\mathbf{Z}(g)_{K2} & \cdots & \mathbf{m}'_\mathbf{Z}(g)_{KK} \end{bmatrix}$$

$$\text{where a block element} \quad \mathbf{m}'_\mathbf{Z}(g)_{rs} = \begin{bmatrix} m_\mathbf{Z}(g)_{rs}^1 & m_\mathbf{Z}(g)_{rs}^2 & \cdots & m_\mathbf{Z}(g)_{rs}^N \end{bmatrix}$$

In K×KN matrices $\mathbf{M_Z}(g)_{K\times KN}$, each element $m_\mathbf{Z}(g)_{rs}^j$ describes the amount of g^{th} margin (tax paid, subsidy received or trade/transport cost) paid on intermediate inputs purchased by industry j in country s from all industries in country r. Valuation layers are an intrinsic feature of the Ghosh model and need to be accounted for to ensure the identity between total outputs and total inputs.

Now, we define the K×KN matrix of total output reallocated to the origin of the primary inputs $\mathbf{X}_O$:

$$\mathbf{X}_O = \sum_{g=1}^{G} \mathbf{M_Z}(g)_{K\times KN}\mathbf{G} + \mathbf{VG}$$

Zoom in the **VG** matrix:

$$\mathbf{VG} = \begin{bmatrix} \mathbf{v}'_1\mathbf{G}_{11} & \mathbf{v}'_1\mathbf{G}_{12} & \cdots & \mathbf{v}'_1\mathbf{G}_{1K} \\ \mathbf{v}'_2\mathbf{G}_{21} & \mathbf{v}'_2\mathbf{G}_{22} & \cdots & \mathbf{v}'_2\mathbf{G}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}'_K\mathbf{G}_{K1} & \mathbf{v}'_K\mathbf{G}_{K2} & \cdots & \mathbf{v}'_K\mathbf{G}_{KK} \end{bmatrix}$$

$$\text{where a block element} \quad [\mathbf{VG}]_{rs} = \begin{bmatrix} \sum_{j=1}^{N} v_{rs}^j g_{rs}^{j1} & \sum_{j=1}^{N} v_{rs}^j g_{rs}^{j2} & \cdots & \sum_{j=1}^{N} v_{rs}^j g_{rs}^{jN} \end{bmatrix}$$

Whereas the **LF** matrix identifies the final destination of output of an industry, the **VG** matrix traces that output to the origin of primary inputs contributed directly by the producer or indirectly at previous stages of the production chain. $\mathbf{M_Z}(g)_{K\times KN}\mathbf{G}$ matrices identify the initial origin of products subject to margins or taxes. The sum of primary inputs contributed by all countries, or the column sum of $\mathbf{X}_O$, equals total output: $\mathbf{i}'_K\mathbf{X}_O = \mathbf{i}'_K \sum_{g=1}^{G} \mathbf{M_Z}(g)_{K\times KN}\mathbf{G} + \mathbf{i}'_K\mathbf{VG} = \mathbf{x}'$. A typical entry in $\mathbf{X}_O$ describes the inputs of industry j in country s that were ultimately sourced by factors of production in country i or are associated with payments of margins/taxes by industry j in country s on products of source country i.

Next step is the decomposition of the Ghosh inverse that mirrors that of the Leontief inverse in subsection B.1. Given that $\mathbf{G} = (\mathbf{I}-\mathbf{B})^{-1}$ and $\mathbf{B} = \widehat{\mathbf{B}} + \breve{\mathbf{B}}$, the Ghosh inverse is rewritten as a product of two matrices:

$$\mathbf{G} = (\mathbf{I}-\mathbf{B})^{-1} = \left(\mathbf{I}-\widehat{\mathbf{B}}-\breve{\mathbf{B}}\right)^{-1} = \left(\mathbf{I}-\widehat{\mathbf{B}}-\left(\mathbf{I}-\widehat{\mathbf{B}}\right)\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\right)^{-1} =$$
$$= \left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)\left(\mathbf{I}-\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\right)\right)^{-1} = \left(\mathbf{I}-\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\right)^{-1}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$$

The second factor $\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$ is equal to a block-diagonal matrix of local Ghosh inverses:

$$\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} = \begin{bmatrix} \mathbf{I}-\mathbf{B}_{11} & 0 & \cdots & 0 \\ 0 & \mathbf{I}-\mathbf{B}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{I}-\mathbf{B}_{KK} \end{bmatrix}^{-1} =$$
$$= \begin{bmatrix} (\mathbf{I}-\mathbf{B}_{11})^{-1} & 0 & \cdots & 0 \\ 0 & (\mathbf{I}-\mathbf{B}_{22})^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\mathbf{I}-\mathbf{B}_{KK})^{-1} \end{bmatrix}$$

The power series of this matrix describes the production chain confined to the domestic economy where domestic industries sell intermediates to each other:

$$\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} = \mathbf{I} + \widehat{\mathbf{B}} + \widehat{\mathbf{B}}\widehat{\mathbf{B}} + \widehat{\mathbf{B}}\widehat{\mathbf{B}}\widehat{\mathbf{B}} + \ldots$$

Each term $\widehat{\mathbf{B}}^t$ corresponds to the domestic transactions among producers at t^{th} round of production.

The first factor links the domestic economy to the global value chain via international trade in intermediates. The power series of this matrix is as follows:

$$\left(\mathbf{I}-\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\right)^{-1} = \mathbf{I} + \left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}} + \left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}} +$$
$$+ \left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}} + \ldots$$

In the above expression, each term $\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\right)^u$ corresponds to a sequential border crossing at u^{th} tier of cross-border supply. Within each term, the local Ghosh inverses $\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$ capture the domestic transactions in u^{th} tier partner upstream the value chain.

It will become evident below that power series of the multiplier matrices based on the allocation coefficients **B** model the sequence of production stages in a direction opposite to that associated with the technical coefficients **A**.

For brevity, the first factor will henceforth be denoted as **Q**. Then the factorization of the Ghosh inverse can be written as:

$$\mathbf{G} = \left(\mathbf{I} - \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} \check{\mathbf{B}}\right)^{-1} \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} = \mathbf{Q} \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} \tag{C.1}$$

C.2 Backward decomposition of output: from production to primary origin

With the Ghosh inverse decomposed, the equation of bilateral output reallocated to the primary origins can now be rewritten as:

$$\mathbf{X}_O = \sum_{g=1}^{G} \mathbf{M}_{\mathbf{Z}}(g)_{K \times KN} \mathbf{G} + \mathbf{VG} = \sum_{g=1}^{G} \mathbf{M}_{\mathbf{Z}}(g)_{K \times KN} \mathbf{Q} \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} + \mathbf{VQ} \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} \tag{C.2}$$

As we are less interested at this point in tracing margins and net taxes to their initial origin, we will focus on the second term in equation (C.2) that explains the sources of inputs in terms of the generation of value added at the beginning of production chain. Unlike **F**, **V** is block-diagonal and does not need to be split. Then value added embodied in all inputs can be decomposed as follows:

$$\begin{aligned} \mathbf{VQ} \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} &= \mathbf{V} \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} + \mathbf{V}(\mathbf{Q} - \mathbf{I}) \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} = \\ &= \mathbf{V} + \mathbf{V} \left(\left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} - \mathbf{I}\right) + \mathbf{V}(\mathbf{Q} - \mathbf{I}) + \mathbf{V}(\mathbf{Q} - \mathbf{I}) \left(\left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} - \mathbf{I}\right) \end{aligned} \tag{C.3}$$

In contrast to a similar decomposition of $\mathbf{X}_D$ in subsection B.2, there are only four terms in equation (C.3) that may be interpreted as follows.

1. $\mathbf{V}$ – value generated by domestic producers and directly included in their output. This does not undergo intermediate production stages nor crosses any borders.[12]

2. $\mathbf{V} \left(\left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} - \mathbf{I}\right) = \mathbf{V} \left(\hat{\mathbf{B}} + \hat{\mathbf{B}}\hat{\mathbf{B}} + \ldots\right)$ – value generated by domestic producers and indirectly included in output via supply of intermediate products. This involves at least one intermediate production stage, does not cross borders.

3. $\mathbf{V}(\mathbf{Q} - \mathbf{I}) = \mathbf{V} \left(\left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} \check{\mathbf{B}} + \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} \check{\mathbf{B}} \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} \check{\mathbf{B}} + \ldots\right)$ – value in imported intermediate products that is initially generated by foreign producers in direct and indirect partner countries. These outputs undergo at least one intermediate production stage and cross at least one border.

4. $\mathbf{V}(\mathbf{Q} - \mathbf{I}) \left(\left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} - \mathbf{I}\right) =$
$= \mathbf{V} \left(\left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} \check{\mathbf{B}} + \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} \check{\mathbf{B}} \left(\mathbf{I} - \hat{\mathbf{B}}\right)^{-1} \check{\mathbf{B}} + \ldots\right) \left(\hat{\mathbf{B}} + \hat{\mathbf{B}}\hat{\mathbf{B}} + \ldots\right)$ – value in intermediate products that undergo transformation in both domestic economy and earlier

[12] Production stages and border crossings for the purpose of this interpretation are not weighted.

in partner countries and is generated by foreign producers. These inputs involve at least two intermediate production stages, cross at least one border.

Third and fourth terms link the domestic economy to the upstream global value chain and are therefore of specific analytical interest (see Fig. C.1 for a graphical interpretation).

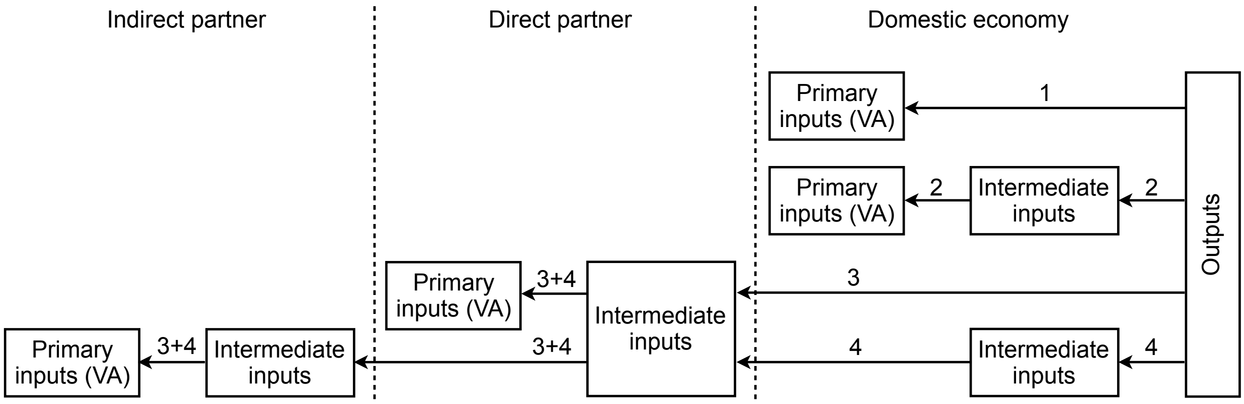

Figure C.1: Backward decomposition of output: a simplified outline of the four terms in equation (C.3)
Note: VA is value added; indirect partner may coincide with the domestic economy.

C.3 Number of production stages: from production to value added

The power series of the "global" Ghosh inverse reveal the history of the generation of value added in industry output (we now put aside margins and net taxes on products):

$$\mathbf{VG} = \mathbf{V} + \mathbf{VB} + \mathbf{VBB} + \mathbf{VBBB} + \ldots = \mathbf{V}\left(\mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \ldots\right)$$

For the interpretation of the terms in this power series, recall that production is an activity where enterprises (institutional units) use inputs of labour, capital, goods and services to produce their outputs. The use of labour and capital, or primary inputs, generates value added for the producer and is described by the first term $\mathbf{V}$. The use of goods and services, or intermediate inputs, generates value added for other producers and is described by all other terms in the power series. Therefore, $\mathbf{V}$ may be treated as the "in-house" generation of value added and $\mathbf{VB}^t$ as the "outsourced" generation of value added at t stages back in the production chain.

Each term in the power series above corresponds to the number of production stages between producers and initial suppliers, or those who generate value added because of employing labour and capital. The "in-house" generation of value added is always in the beginning of the production chain and appears in each term. The total unweighted number of these "in-house" stages is equal to the total number of terms in the power series. The number of $\mathbf{B}$ in each term indicates how many stages back the value was generated by employing the primary inputs. For example, $\mathbf{VBBB}$ signifies that a chain of transactions from the producer belonging to industry j in country s back to the initial supplier in country r includes three transactions in intermediates and one stage of the "in-house" generation of value added. The unweighted sum of all $\mathbf{B}$ leads to an infinite series $1 + 2 + 3 \ldots$

We will henceforth refer to the transactions that involve the use of intermediate inputs as "intermediate production stages" and to the transactions that involve the "in-house" use of primary inputs as "primary production stage".

To obtain a sensible estimate of the number of production stages backwards to the origin of inputs, we will weigh the number of production stages those inputs passed at each production round by the decreasing shares of inputs in total output (less margins and net taxes) at each round. This procedure mirrors that described in subsection B.3 with respect to the forward decomposition of output and also yields a measure conceptually similar to the average propagation length proposed by Dietzenbacher et al. (2005).

The first step is to define weights that will apply to the sequence 1 + 2 + 3 ...:

$$\frac{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} = \frac{\mathbf{V}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} + \frac{\mathbf{VB}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} + \frac{\mathbf{VBB}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} + \ldots$$

Each term in the expression above describes a decreasing share of industry output that was initially created because of the use of primary inputs at the respective production stage. The longer the value chain, the smaller share of output originates in primary inputs.

We will apply the weights separately to intermediate and primary production stages. To count intermediate production stages, we apply the weights to the sequence of numbers starting from zero that conform to the number of **B** in the power series:

$$\begin{aligned}\mathbf{C}_{(\mathbf{x}_O)ips} &= 0 \times \frac{\mathbf{V}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} + 1 \times \frac{\mathbf{VB}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} + 2 \times \frac{\mathbf{VBB}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} + \ldots = \\ &= \frac{\mathbf{V}(\mathbf{B}+2\mathbf{B}^2+\ldots)}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} = \frac{\mathbf{V}\left(\left(\mathbf{I}-\mathbf{B}\right)^{-1}-\mathbf{I}\right)\left(\mathbf{I}-\mathbf{B}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} = \frac{\mathbf{V}(\mathbf{G}-\mathbf{I})\mathbf{G}}{\mathbf{VG}}\end{aligned} \tag{C.4}$$

Equation (C.4) is the Ghosh-based counterpart to equation (B.5), and it counts the average number of intermediate production stages in reverse direction, i.e., backwards to origin of inputs. To count primary production stages, we apply the weights to one additional transaction at each round:

$$\begin{aligned}\mathbf{C}_{(\mathbf{x}_O)pps} &= 1 \times \frac{\mathbf{V}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} + 1 \times \frac{\mathbf{VB}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} + 1 \times \frac{\mathbf{VBB}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} + \ldots = \\ &= \frac{\mathbf{V}(\mathbf{I}+\mathbf{B}+\mathbf{B}^2+\ldots)}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} = \frac{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} = \frac{\mathbf{VG}}{\mathbf{VG}}\end{aligned} \tag{C.5}$$

$\mathbf{C}_{(\mathbf{x}_O)pps}$ is a KN×N matrix of ones which signifies that each production stage is ultimately associated with one primary stage where producers generate value added at the beginning of the production chain. The sum of the weighted average number of intermediate production stages and the weighted average number of primary production stages is then equal to:

$$\begin{aligned}\mathbf{C}_{(\mathbf{x}_O)ps} &= \mathbf{C}_{(\mathbf{x}_O)ips} + \mathbf{C}_{(\mathbf{x}_O)pps} = \frac{\mathbf{V}\left(\left(\mathbf{I}-\mathbf{B}\right)^{-1}-\mathbf{I}\right)\left(\mathbf{I}-\mathbf{B}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} + \frac{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} = \\ &= \frac{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}\left(\mathbf{I}-\mathbf{B}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\mathbf{B}\right)^{-1}} = \frac{\mathbf{VGG}}{\mathbf{VG}}\end{aligned} \tag{C.6}$$

Now, we will decompose the total average weighted number of production stages into various components using the technique of Wang et al. (2016). Equation (C.3) provides the underlying decomposition of total industry outputs (less margins and net taxes) into four input components, and the first line of that equation shows how these may be re-grouped into two components, optimizing the counting routines:

$$\mathbf{VG} = \mathbf{V}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1} + \mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1} \tag{C.7}$$

We will be able to distinguish between domestic production stages in the country of destination (i.e., country that produces output), domestic production stages in partner countries (i.e., upstream producers in the value chain) and cross-border production stages. We will also be able to simultaneously identify intermediate production stages and primary production stages that correspond, respectively, to the use of intermediate and primary inputs. By and large, the counting routines in this subsection are similar to those in subsection B.3, therefore many intermediate calculation steps will be skipped.

It is also worth noting that we will now focus on the origin of primary inputs disregarding the inputs related to payments of margins and net taxes.

C.3.1 Production that originates in domestic primary inputs, $\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$

The first term in equation (C.7) calculates domestic primary inputs in industry output that do not leave the domestic economy in the production process. In other words, this is domestic value added that goes through domestic value chain and is embodied in industry output:

$$\underbrace{\mathbf{V}}_{\substack{\text{“in-house” use}\\ \text{of primary inputs}\\ \text{at destination}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}_{\substack{\text{use of}\\ \text{intermediates}\\ \text{at destination}}} = \mathbf{V} + \mathbf{V}\widehat{\mathbf{B}} + \mathbf{V}\widehat{\mathbf{B}}\widehat{\mathbf{B}} + \ldots$$

We will count the average weighted number of intermediate production stages and the weighted average number of primary production stages.

Number of intermediate production stages The power series above gives the necessary weights, and the number of $\widehat{\mathbf{B}}$ corresponds to the number of intermediate production stages. The count starts from zero:

$$\mathbf{C}_{(\mathbf{x}_O 1)ips.d} = 0 \times \frac{\mathbf{V}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + 1 \times \frac{\mathbf{V}\widehat{\mathbf{B}}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + 2 \times \frac{\mathbf{V}\widehat{\mathbf{B}}\widehat{\mathbf{B}}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + \ldots =$$
$$= \frac{\mathbf{V}\left(\widehat{\mathbf{B}} + 2\widehat{\mathbf{B}}^2 + \ldots\right)}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} = \frac{\mathbf{V}\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} - \mathbf{I}\right)\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} \tag{C.8}$$

Number of primary production stages The “in-house” generation of value added is always the first transaction at each production round:

$$\mathbf{C}_{(\mathbf{x}_O 1)pps.d} = 1 \times \frac{\mathbf{V}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + 1 \times \frac{\mathbf{V}\widehat{\mathbf{B}}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + 1 \times \frac{\mathbf{V}\widehat{\mathbf{B}}\widehat{\mathbf{B}}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + \ldots =$$
$$= \frac{\mathbf{V}\left(\mathbf{I} + \widehat{\mathbf{B}} + \widehat{\mathbf{B}}^2 + \ldots\right)}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} = \frac{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} \tag{C.9}$$

C.3.2 Production that originates in partner primary inputs, $\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$

This part of total output includes intermediates that embody primary inputs from partner countries. The relevant production chain may include transformation in the country of destination, necessarily includes importation of intermediates from at least the direct partner, and may include transformation in the country of origin. In other words, this term calculates value added from partner countries that may undergo some production process before being used in the form of intermediate inputs at destination. In fact, it may also include domestic value added that returns to home economy after a series of transactions (in cases where second or higher-tier partner back in the value chain coincides with the country of destination).

Accordingly, this output involves intermediate production stages at destination, intermediate production stages in partners, primary production stages in partners and cross-border intermediate production stages:

$$\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} = \underbrace{\mathbf{V}}_{\substack{\text{“in-house” use}\\ \text{of primary inputs}\\ \text{in direct partner}\\ \text{(at origin)}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}_{\substack{\text{use of}\\ \text{intermediates}\\ \text{at origin}}} \underbrace{\breve{\mathbf{B}}}_{\substack{\text{cross-border}\\ \text{supply and use}\\ \text{of intermediates}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}_{\substack{\text{use of}\\ \text{intermediates}\\ \text{at destination}}} +$$

$$+ \underbrace{\mathbf{V}}_{\substack{\text{“in-house” use}\\ \text{of primary inputs}\\ \text{in 2nd tier partner}\\ \text{(at origin)}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}_{\substack{\text{use of}\\ \text{intermediates}\\ \text{at origin}}} \underbrace{\breve{\mathbf{B}}}_{\substack{\text{cross-border}\\ \text{supply and use}\\ \text{of intermediates}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}_{\substack{\text{use of}\\ \text{intermediates}\\ \text{in direct partner}}} \underbrace{\breve{\mathbf{B}}}_{\substack{\text{cross-border}\\ \text{supply and use}\\ \text{of intermediates}}} \underbrace{\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}_{\substack{\text{use of}\\ \text{intermediates}\\ \text{at destination}}} + \dots \quad \text{(C.10)}$$

Number of intermediate production stages at destination The second term in $\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$ is the matrix of local Ghosh inverses that is responsible for the intermediate production stages in the country of destination. Counting the weighted average number of intermediate production stages at destination builds on the power series expansion of this matrix:

$$\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} = \mathbf{V}(\mathbf{Q}-\mathbf{I}) + \mathbf{V}(\mathbf{Q}-\mathbf{I})\widehat{\mathbf{B}} + \mathbf{V}(\mathbf{Q}-\mathbf{I})\widehat{\mathbf{B}}\widehat{\mathbf{B}} + \dots$$

The first term in this power series does not involve any transactions in intermediates at home, and the count starts from zero:

$$\mathbf{C}_{(\mathbf{x}_O2)ips.d} = 0 \times \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + 1 \times \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\widehat{\mathbf{B}}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + 2 \times \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\widehat{\mathbf{B}}\widehat{\mathbf{B}}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} +$$

$$+ \dots = \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\widehat{\mathbf{B}} + 2\widehat{\mathbf{B}}^2 + \dots\right)}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} = \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} - \mathbf{I}\right)\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} \quad \text{(C.11)}$$

Number of intermediate production stages in partners The first term in $\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$ is responsible for the international part of the production chain.

The power series of $\mathbf{Q}$ in equation (C.10) bring forward the sequences of local Ghosh inverses that correspond to domestic intermediate production stages in the country of origin and in all other countries between origin and destination. The first term in equation (C.10) may be further expanded as follows:

$$\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} = \mathbf{V}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} + \mathbf{V}\widehat{\mathbf{B}}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} + \mathbf{V}\widehat{\mathbf{B}}\widehat{\mathbf{B}}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} + \dots$$

As in the case of the forward decomposition in subsection B.3.2, there are nested power series of local Ghosh inverses within the power series of $\mathbf{Q}$. Nested power series expansion models the sequence of intermediate production stages in direct partner and gives appropriate weights. The count of intermediate production stages starts from zero (skipped here) and the result is:

$$\mathbf{C}_{(\mathbf{x}_O 2)ips.p1} = \frac{\mathbf{V}\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}$$

In the second term in equation (C.10) involves two nested power series that correspond to production stages in direct partner and those in indirect partner:

$$\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} =$$
$$= \mathbf{V}\left(\mathbf{I}+\widehat{\mathbf{B}}+\widehat{\mathbf{B}}^2+\dots\right)\breve{\mathbf{B}}\left(\mathbf{I}+\widehat{\mathbf{B}}+\widehat{\mathbf{B}}^2+\dots\right)\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$$

The counting and weighting procedure applies to each of the two power series, and the result after summation is:

$$\begin{aligned}\mathbf{C}_{(\mathbf{x}_O 2)ips.p2} = {} & \frac{\mathbf{V}\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}+ \\ & + \frac{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}\end{aligned} \tag{C.12}$$

In general, the u^{th} term in the power series of $\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$ involves u nested expansions of local Ghosh inverses and a sum of u numbers of domestic intermediate production stages in u partners. Finally, the weighted average of all domestic intermediate production stages in partners converges to:

$$\mathbf{C}_{(\mathbf{x}_O 2)ips.p} = \frac{\mathbf{VQ}\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} \tag{C.13}$$

Number of primary production stages Primary production stage, or the "in-house" generation of value added in $\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$, is a domestic transaction in direct or indirect partner countries prior to a series of transformations of imported intermediates.

There is one primary production stage per each term in the nested power series expansion of the local Ghosh inverse in the first term of equation (C.10). The weighted sum is:

$$\mathbf{C}_{(\mathbf{x}_O 2)pps.p1} = \frac{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}$$

In the second term of equation (C.10), there are two nested power series, but primary production stage is always the first transaction in the country of origin. We count one primary production stage per each term in the nested power series closer to the origin, and the result is:

$$\mathbf{C}_{(\mathbf{x}_O 2)pps.p2} = \frac{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}$$

In case of primary production stages, the u^{th} term in the power series of $\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}$ will involve one nested expansion of local Ghosh inverses and a sum of one number of production stages in u partners. Finally, the weighted average of all primary production stages in partners converges to:

$$\mathbf{C}_{(\mathbf{x}_O 2)pps.p} = \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} \tag{C.14}$$

Number of cross-border intermediate production stages The power series expansion in equation (C.10) shows that, from the backward perspective, inputs received at destination only cross borders when one industry in one country purchases intermediate products from another industry in another country. The block-off-diagonal matrix of allocation coefficients $\breve{\mathbf{B}}$ is therefore responsible for the cross-border intermediate production stages. The count of border crossings starts from one:

$$\begin{aligned}
\mathbf{C}_{(\mathbf{x}_O 2)ips.cb} &= 1\times\frac{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + 2\times\frac{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + \\
&+ 3\times\frac{\mathbf{V}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + \ldots = \\
&= \frac{\mathbf{V}\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}} + 2\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\right)^{2} + 3\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\right)^{3} + \ldots\right)\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} = \\
&= \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\mathbf{Q}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}
\end{aligned} \tag{C.15}$$

C.3.3 From production to value added: summary numbers

The decomposition of the number of transactions along the upstream value chain in subsection C.3 resulted in six indicators. There are various options available to summarize these using the respective shares in total output (less margins and net taxes) as weights.

Total number of intermediate production stages at destination This requires an aggregation of the weighted average number of intermediate production stages linking producers in the country of destination where output is produced, irrespective of where it originates: $\mathbf{C}_{(\mathbf{x}_O 1)ips.d}$ and $\mathbf{C}_{(\mathbf{x}_O 2)ips.d}$. Each number is weighted by the share of the respective component in total output (exclusive of margins and net taxes) as per equation (C.7):

$$\begin{aligned}\mathbf{C}_{(\mathbf{x}_O)ips.d} &= \frac{\mathbf{V}\left(\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}} \circ \frac{\mathbf{V}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} + \\ &+ \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}} \circ \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} = \\ &= \frac{\mathbf{VQ}\left(\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}}\end{aligned} \tag{C.16}$$

where $\circ$ signifies element-by-element multiplication.

Total number of intermediate production stages in partners There is no need to calculate the total number of intermediate production stages in partner countries because it is given by $\mathbf{C}_{(\mathbf{x}_O 2)ips.p}$, but it is worth normalising it with respect to total output for the use in other summary indicators:

$$\mathbf{C}_{(\mathbf{x}_O)ips.p} = \mathbf{C}_{(\mathbf{x}_O 2)ips.p} \circ \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} = \frac{\mathbf{VQ}\left(\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} \tag{C.17}$$

Number of GVC-related production stages Add up the total number of intermediate production stages in partner countries and across borders to obtain the weighted average number of GVC-related production stages along the backward value chain:

$$\begin{aligned}\mathbf{C}_{(\mathbf{x}_O)ips.GVC} &= \mathbf{C}_{(\mathbf{x}_O)ips.p} + \mathbf{C}_{(\mathbf{x}_O)ips.cb} = \\ &= \frac{\mathbf{VQ}\left(\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} + \mathbf{C}_{(\mathbf{x}_O 2)ips.cb} \circ \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} = \\ &= \frac{\mathbf{VQ}\left(\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} + \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\mathbf{Q}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} = \\ &= \frac{\mathbf{VQ}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}}\end{aligned} \tag{C.18}$$

Total number of all intermediate production stages Add up the weighted average number of domestic and cross-border intermediate production stages:

$$
\begin{aligned}
\mathbf{C}_{(\mathbf{X}_O)ips} &= \mathbf{C}_{(\mathbf{X}_O)ips.d} + \mathbf{C}_{(\mathbf{X}_O)ips.GVC} = \\
&= \frac{\mathbf{VQ}\left(\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1} - \mathbf{I}\right)\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} + \frac{\mathbf{VQ}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} = \\
&= \frac{\mathbf{VQ}\left(\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}\mathbf{Q} - \mathbf{I}\right)\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} = \frac{\mathbf{VQ}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}\mathbf{Q}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1} - \mathbf{VQ}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} = \\
&= \frac{\mathbf{VGG}-\mathbf{VG}}{\mathbf{VG}} = \frac{\mathbf{V}(\mathbf{G}-\mathbf{I})\mathbf{G}}{\mathbf{VG}} \qquad \text{(C.19)}
\end{aligned}
$$

The result in equation (C.19) quantifies the weighted average number of all intermediate production stages inputs undergo from the stage when value is generated therein. It is also equivalent to the direct measurement of the number of intermediate production stages using the global Ghosh inverse in equation (C.4).

Total number of primary production stages Normalise and combine $\mathbf{C}_{(\mathbf{X}_{O}1)pps.d}$ and $\mathbf{C}_{(\mathbf{X}_{O}2)pps.p}$ into a single measure of the weighted average total number of primary production stages:

$$
\begin{aligned}
\mathbf{C}_{(\mathbf{X}_O)pps} &= \frac{\mathbf{V}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{V}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}} \circ \frac{\mathbf{V}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} + \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}} \circ \frac{\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} = \\
&= \frac{\mathbf{V}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1} + \mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} = \frac{\mathbf{VQ}\left(\mathbf{I}-\hat{\mathbf{B}}\right)^{-1}}{\mathbf{VG}} = \frac{\mathbf{VG}}{\mathbf{VG}} \qquad \text{(C.20)}
\end{aligned}
$$

The result in equation (C.20) is also a K×KN matrix of ones and is equal to that in equation (C.5), derived from the global Ghosh inverse. The sum of the measures in (C.19) and (C.20) is then equal to the total number of all transactions from production backwards to the generation of value added in equation (C.6).

C.3.4 A note on final demand coefficients

Total industry output reallocated to the initial origin of inputs $\mathbf{X}_O$ may be converted into final products with the application of the final demand coefficients. These coefficients $f_{c,r}^i = \frac{f_r^i}{x_r^i}$ describe a part of total industry output that ends up in final demand and are parallel to the widely employed value added coefficients. In matrix notation, $\hat{\mathbf{f}}_c = \hat{\mathbf{x}}^{-1}\hat{\mathbf{f}}$ is a KN×KN block-diagonal matrix of the final demand coefficients, and $\mathbf{f}$ is a column vector of final demand ($\mathbf{f} = \mathbf{F}\mathbf{i}_K$).

The application of final demand coefficients to the matrix of total output reallocated to initial origin yields:

$$
\mathbf{X}_O\hat{\mathbf{f}}_c = \sum_{g=1}^{G} \mathbf{M}_{\mathbf{Z}}(g)_{K\times KN}\mathbf{G}\hat{\mathbf{f}}_c + \mathbf{VG}\hat{\mathbf{f}}_c
$$

$\mathbf{VG}\hat{\mathbf{f}}_c$ is a K×KN matrix where a typical entry describes the amount of final products of industry j in country s that may be entirely attributed to value added of country r, generated directly or via the production chain. The power series of this matrix can be interpreted in the same way as that of total bilateral output:

$$\mathbf{VG}\hat{\mathbf{f}}_c = \mathbf{V}\hat{\mathbf{f}}_c + \mathbf{VB}\hat{\mathbf{f}}_c + \mathbf{VBB}\hat{\mathbf{f}}_c + \mathbf{VBBB}\hat{\mathbf{f}}_c + \ldots = \mathbf{V}\left(\mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \ldots\right)\hat{\mathbf{f}}_c$$

Each term describes the path from final products backwards in the production process to their primary inputs, or value added. For example, the first term $\mathbf{V}\hat{\mathbf{f}}_c$ (and $\mathbf{I}$ in the brackets) signifies that value is generated "in-house" and goes directly into final products. All subsequent terms signify that final products are made from intermediates and generation of their value is "outsourced" to the initial supplier. As usual, the number of $\mathbf{B}$ corresponds to the number of these intermediate production stages.

An important note is that the expression above models the production sequence that ends in the final production or completion stage not consumption or accumulation. The reason is that $\hat{\mathbf{f}}_c$ informs us that the output is for final use but does not tell us anything about its final user. Unlike the $\mathbf{F}$ matrix, $\hat{\mathbf{f}}_c$ suppresses the information on the country distribution of final products.

Final demand coefficients may apply to all measurements of the weighted average number of intermediate and primary production stages in subsection C.3. As both numerator and denominator are multiplied column-wise by the same number (because the matrix $\hat{\mathbf{f}}_c$ is block-diagonal), this does not affect the results. For example, the total number of production stages as per equation (C.6) will appear as follows:

$$\mathbf{C}_{(\mathbf{F})ps} = \frac{\mathbf{VGG}\hat{\mathbf{f}}_c}{\mathbf{VG}\hat{\mathbf{f}}_c} = \frac{\mathbf{VGG}}{\mathbf{VG}}$$

The length of value chain or the weighted average number of production stages from final producer to supplier of primary inputs does not depend on whether it is measured with respect to total output or final products only. This holds at the country-industry level (K×KN, 1×KN dimensions) but not at the aggregate country or industry levels (K×K, 1×K, K×N, 1×N dimensions).

C.3.5 A note on the convergence of forward and backward decomposition of output: from primary origin to final destination

In only one case do the measurements of the number of production stages from both forward and backward perspectives give the same result: if the count starts from the industry of origin where primary inputs are used and ends in the industry of completion where final products are created. This requires calculation in the full KN×KN matrix dimension.

First, we observe that value added reallocated to its final product destination equals final products reallocated to the origin of value added therein:

$$\hat{\mathbf{v}}_c\mathbf{L}\hat{\mathbf{f}} = \hat{\mathbf{v}}_c\hat{\mathbf{x}}\mathbf{G}\hat{\mathbf{x}}^{-1}\hat{\mathbf{f}} = \hat{\mathbf{v}}\mathbf{G}\hat{\mathbf{f}}_c$$

Then it is clear that the total weighted average number of production stages from primary inputs to final products is equal to that from final products to primary inputs:

$$\frac{\hat{\mathbf{v}}_c\mathbf{LL}\hat{\mathbf{f}}}{\hat{\mathbf{v}}_c\mathbf{L}\hat{\mathbf{f}}} = \frac{\mathbf{LL}}{\mathbf{L}} = \frac{\hat{\mathbf{x}}\mathbf{G}\hat{\mathbf{x}}^{-1}\hat{\mathbf{x}}\mathbf{G}\hat{\mathbf{x}}^{-1}}{\hat{\mathbf{x}}\mathbf{G}\hat{\mathbf{x}}^{-1}} = \frac{\mathbf{GG}}{\mathbf{G}} = \frac{\hat{\mathbf{v}}\mathbf{GG}\hat{\mathbf{f}}_c}{\hat{\mathbf{v}}\mathbf{G}\hat{\mathbf{f}}_c}$$

The additive decomposition of the total weighted average number of production stages expressed in terms of Leontief or Ghosh global inverses is fully symmetric. This means that, for example, the average number of intermediate production stages at destination from the backward perspective is equal to the average number of intermediate production stages at origin from the forward perspective. This is true for all other indicators in subsections B.3 and C.3, including the average numbers of intermediate production stages in partners, cross-border intermediate production stages, primary and final production stages.

This is a well-known result, observed by the proponents of the average propagation length measurements (see Dietzenbacher et al., 2005). A change of perspective on value chain in the full KN×KN matrix dimension does not affect the calculated length of production process: the number of production stages from industry i in country r to industry j in country s equals that from industry j in country s to industry i in country r. This is not the case for the indicators in subsections B.3 and C.3 because they involve implicit aggregation of industries that deliver final output at destination or contribute primary inputs at origin.

The average weighted number of domestic production stages is also the same in both directions, between industry i and industry j in country r or s. Hence, even when the change of the underlying model reverses the starting point in the counting procedure, i.e., from destination to origin or vice versa, the length of production chain remains unchanged because the industry dimension is not altered. One may naturally expect this result: irrespective of whether the economy in question is origin or destination of output, products have to undergo the same chain of inter-industry transactions.

D Relative length of and position in global value chains

D.1 Participation in global value chains

Decomposition of outputs according to their paths forward to the final destination (as in section B) or backward to the primary origin (as in section C) is a precursor to developing various summary measures of the importance and length of global value chains. Some of these measures answer the question: how big is the part of total output (or value added, or final products) that is destined for or originates in multi-stage production beyond the home economy? Other measures address questions: is this country or industry closer to the final user or primary producer in production chain? It may also be of great analytical interest to see how the importance of and position in the cross-country production chains evolve over time. This section discusses these measures, building on and further developing the ideas of Wang et al. (2016).

Wang et al. (2016) define a "GVC participation index based on forward industrial linkage" as a share of domestic value added generated from GVC-related production activities to total industry value added. At the country-industry level, this index may be defined in terms of output (see the note on value added coefficients in subsection B.3.7) giving the same result. One may observe that the output used in GVC-related production activities corresponds to the second and fourth terms in equation (B.8). Then the GVC forward participation index in the notation of this paper is:

$$\mathbf{p}_{(\mathbf{x}_D)GVC} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}\mathbf{i}_K + \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\check{\mathbf{F}}\mathbf{i}_K}{\mathbf{LFi}_K} = \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\mathbf{f}}{\mathbf{Lf}} =$$

$$= \frac{\left(\mathbf{L}-\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}}{\mathbf{Lf}} \tag{D.1}$$

The result in equation (D.1) is a KN×1 vector where each element is the GVC forward participation index for industry i in country r. Please see Appendix E for the explanation of the algebraic manipulations used in equation (D.1) and subsequent equations. To ensure consistency with Wang et al. (2016), an aggregation of $\mathbf{p}_{(\mathbf{X}_D)GVC}$ to country (K×1) or industry (N×1) level still requires pre-multiplication by the value added coefficients.

Similarly, the "GVC participation index based on backward industrial linkage" of Wang et al. (2016) can be written using the second term from equation (C.7):

$$\mathbf{p}'_{(\mathbf{X}_O)GVC} = \frac{\mathbf{i}'_K \mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{i}'_K \mathbf{V}\mathbf{G}} = \frac{\mathbf{v}'\left(\mathbf{G}-\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\right)}{\mathbf{v}'\mathbf{G}} \tag{D.2}$$

The result in equation (D.2) is a 1×KN vector where each element is the GVC backward participation index for industry j in country s.[13] It also needs noting that an aggregation of $\mathbf{p}'_{(\mathbf{X}_O)GVC}$ to country (1×K) or industry (1×N) level in line with Wang et al. (2016) requires post-multiplication by the final demand coefficients.

D.2 Orientation towards global value chains

As discussed in section B, production stages that outputs undergo on the way to their final destination may be classified into intermediate production stages at origin, intermediate production stages in partners, intermediate cross-border production stages and final production stages:

$$\mathbf{C}_{(\mathbf{X}_D)ps} = \mathbf{C}_{(\mathbf{X}_D)ips.o} + \mathbf{C}_{(\mathbf{X}_D)ips.p} + \mathbf{C}_{(\mathbf{X}_D)ips.cb} + \mathbf{C}_{(\mathbf{X}_D)fps} \tag{D.3}$$

Each term on the right side of equation (D.3) is the average number of production stages normalized with respect to total output (see subsection B.3.5). The underlying weighting scheme has two important advantages. First, it allows us to express the average number of all production stages as a simple sum of components. Second, it implicitly combines measures of the length and importance of production chain because it requires multiplying the average number of production stages by the share of total output that is relevant to that production chain.

We may gauge the relative importance of or orientation towards global value chains from the forward-looking perspective by computing a ratio of the weighted average number of GVC-related production stages to the weighted average number of purely domestic production stages:

$$\mathbf{p}_{(\mathbf{C},\mathbf{X}_D)GVC} = \frac{\mathbf{c}_{(\mathbf{X}_D)ips.p} + \mathbf{c}_{(\mathbf{X}_D)ips.cb}}{\mathbf{c}_{(\mathbf{X}_D)ips.o}} \tag{D.4}$$

Measuring the participation in or orientation towards global value chains is more sensible with respect to all partner countries. The aggregated indicators may be understood as weighted averages across all partners (see Appendix G for a detailed exposition of the aggregation options). Accordingly, $\mathbf{p}_{(\mathbf{C},\mathbf{X}_D)GVC}$ is a KN×1 vector of indices that compare the length and importance of global value chain and those of domestic value chain.

Formulation of this index from the backward-looking perspective mirrors the above. First, we recall the decomposition of the weighted average number of all production stages back to the origin of inputs:

$$\mathbf{C}_{(\mathbf{X}_O)ps} = \mathbf{C}_{(\mathbf{X}_O)ips.d} + \mathbf{C}_{(\mathbf{X}_O)ips.p} + \mathbf{C}_{(\mathbf{X}_O)ips.cb} + \mathbf{C}_{(\mathbf{X}_O)pps} \tag{D.5}$$

[13] The GVC forward participation index $\mathbf{p}_{(\mathbf{X}_D)GVC}$ is denoted in Wang et al. (2016) as $GVCPt\_f$ and the GVC backward participation index $\mathbf{p}'_{(\mathbf{X}_O)GVC}$ as $GVCPt\_b$.

Next, we define a 1×KN vector of indices of orientation towards upstream global value chain:

$$\mathbf{p}'_{(\mathbf{C},\mathbf{x}_O)GVC} = \frac{\mathbf{c}'_{(\mathbf{x}_O)ips.p} + \mathbf{c}'_{(\mathbf{x}_O)ips.cb}}{\mathbf{c}'_{(\mathbf{x}_O)ips.d}} \tag{D.6}$$

D.3 Position in global value chains

Wang et al. (2016) define the average production length in forward GVC as a ratio of GVC-related domestic value added and its induced gross output. In the notation of this paper, it should be equal to the sum of $\mathbf{C}_{(\mathbf{x}_D 2)ips.o}$ (see equation B.12), $\mathbf{C}_{(\mathbf{x}_D 2)ips.p}$ (equation B.13), $\mathbf{C}_{(\mathbf{x}_D 2)ips.cb}$ (equation B.15), $\mathbf{C}_{(\mathbf{x}_D 4)ips.o}$ (equation B.19), $\mathbf{C}_{(\mathbf{x}_D 4)ips.p}$ (equation B.20) and $\mathbf{C}_{(\mathbf{x}_D 4)ips.cb}$ (equation B.21), weighted by the share of the respective outputs in the GVC-related output (not in total output!) and aggregated across partner countries:

$$\begin{aligned}
\mathbf{c}_{(\mathbf{x}_D)GVC} &= \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}\mathbf{i}_K}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\mathbf{F}\mathbf{i}_K} + \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\widehat{\mathbf{F}}\mathbf{i}_K}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\mathbf{F}\mathbf{i}_K} + \\
&+ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}\mathbf{i}_K}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\mathbf{F}\mathbf{i}_K} + \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)(\mathbf{H}-\mathbf{I})\check{\mathbf{F}}\mathbf{i}_K}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\mathbf{F}\mathbf{i}_K} + \\
&+ \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{H}\check{\mathbf{F}}\mathbf{i}_K}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\mathbf{F}\mathbf{i}_K} + \frac{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}(\mathbf{H}-\mathbf{I})\check{\mathbf{F}}\mathbf{i}_K}{\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{H}-\mathbf{I})\mathbf{F}\mathbf{i}_K} = \\
&= \frac{\mathbf{L}\left(\mathbf{L}-\mathbf{I}\right)\mathbf{f}-\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{I}\right)\mathbf{f}}{\mathbf{L}\mathbf{f}-\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{f}} = \frac{\left(\mathbf{L}\mathbf{L}-\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}}{\left(\mathbf{L}-\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}} - \mathbf{i}_{KN}
\end{aligned} \tag{D.7}$$

In a similar way, the average production length backward in Wang et al. (2016) is the ratio of GVC-related foreign value added and its induced gross output. Using the Ghosh-type decomposition in this paper, it can be expressed as a sum of $\mathbf{C}_{(\mathbf{x}_O 2)ips.d}$ (see equation C.11), $\mathbf{C}_{(\mathbf{x}_O 2)ips.p}$ (equation C.13) and $\mathbf{C}_{(\mathbf{x}_O 2)ips.cb}$ (equation C.15) and does not require weighting because the denominator is the GVC-related inputs:

$$\begin{aligned}
\mathbf{c}'_{(\mathbf{x}_O)GVC} &= \frac{\mathbf{i}'_K\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{i}'_K\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + \\
&+ \frac{\mathbf{i}'_K\mathbf{V}\mathbf{Q}\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{i}'_K\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} + \frac{\mathbf{i}'_K\mathbf{V}(\mathbf{Q}-\mathbf{I})\mathbf{Q}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{i}'_K\mathbf{V}(\mathbf{Q}-\mathbf{I})\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} = \\
&= \frac{\mathbf{v}'\left(\mathbf{G}-\mathbf{I}\right)\mathbf{G}-\mathbf{v}'\left(\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}-\mathbf{I}\right)\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}}{\mathbf{v}'\mathbf{G}-\mathbf{v}'\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}} = \frac{\mathbf{v}'\left(\mathbf{G}\mathbf{G}-\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\right)}{\mathbf{v}'\left(\mathbf{G}-\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\right)} - \mathbf{i}'_{KN}
\end{aligned} \tag{D.8}$$

There is a subtle difference in the formulation of these two indicators in Wang et al. (2016) that is reviewed in more detail in Appendix F.

Finally, Wang et al. (2016) define the "average production line position in the global value chain" as the ratio of the two production lengths:

$$\mathbf{p}_{(\mathbf{C}_{D/O})GVC} = \frac{\mathbf{c}_{(\mathbf{X}_D)GVC}}{\mathbf{c}_{(\mathbf{X}_O)GVC}} \tag{D.9}$$

This paper proposes a modification of the average production line position index by removing the intermediate production stages at home and accounting for the importance of the GVC-related outputs and inputs:

$$\mathbf{p}_{(\mathbf{C},\mathbf{X}_{D/O})GVC} = \frac{\mathbf{c}_{(\mathbf{X}_D)ips.p} + \mathbf{c}_{(\mathbf{X}_D)ips.cb}}{\mathbf{c}_{(\mathbf{X}_O)ips.p} + \mathbf{c}_{(\mathbf{X}_O)ips.cb}} \tag{D.10}$$

Each term on the right side of equation (D.10) implies a normalization with respect to total output that goes through the respective production chain. $\mathbf{p}_{(\mathbf{C},\mathbf{X}_{D/O})GVC}$ therefore is not prone to a possible overemphasis in cases where the production length is significant but the share of output (value added, final products) that is relevant to this production chain does not really matter for the whole industry or economy.

Appendix F discusses the differences between the GVC-related analytical indices of Wang et al. (2016) and those in this paper.

E Some useful algebraic properties of the H and Q matrices

Here we review some algebraic manipulations with the **H** and **Q** matrices that are useful to explore the relationship among the measures proposed in this and earlier papers, including Wang et al. (2016).

E.1 Relationships with local and global Leontief (Ghosh) inverse matrices

The factorization of the global Leontief inverse in equation (B.1) allows us to express the **H** matrix as follows:

$$\mathbf{H} = \left(\mathbf{I} - \widehat{\mathbf{A}}\right)\mathbf{L} = \left(\mathbf{I} - \mathbf{A} + \breve{\mathbf{A}}\right)\mathbf{L} = \left(\mathbf{I} - \mathbf{A}\right)\mathbf{L} + \breve{\mathbf{A}}\mathbf{L} = \mathbf{I} + \breve{\mathbf{A}}\mathbf{L} \tag{E.1}$$

Similarly, the **Q** matrix can be rewritten using the factorization of the global Ghosh inverse in equation (C.1):

$$\mathbf{Q} = \mathbf{G}\left(\mathbf{I} - \widehat{\mathbf{B}}\right) = \mathbf{G}\left(\mathbf{I} - \mathbf{B} + \breve{\mathbf{B}}\right) = \mathbf{G}\left(\mathbf{I} - \mathbf{B}\right) + \mathbf{G}\breve{\mathbf{B}} = \mathbf{I} + \mathbf{G}\breve{\mathbf{B}} \tag{E.2}$$

Equations (E.1) and (E.2) point out at easier ways to compute the **H** and **Q** matrices.

The next exercise will explore the difference between global and local Leontief inverses. Using that $\mathbf{A} = \widehat{\mathbf{A}} + \breve{\mathbf{A}}$, start with the following expression:

$$\breve{\mathbf{A}} = \left(\mathbf{I} - \widehat{\mathbf{A}}\right) - \left(\mathbf{I} - \mathbf{A}\right)$$

Multiply both sides of the above equation by local Leontief inverse on the left and global Leontief inverse on the right:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\mathbf{L}=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)\mathbf{L}-\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}(\mathbf{I}-\mathbf{A})\,\mathbf{L}=\mathbf{L}-\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}$$

Now, multiply the same by global Leontief inverse on the left and local Leontief inverse on the right:

$$\mathbf{L}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}=\mathbf{L}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}-\mathbf{L}\,(\mathbf{I}-\mathbf{A})\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}=\mathbf{L}-\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}$$

Then it follows that:

$$\mathbf{L}-\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\breve{\mathbf{A}}\mathbf{L}=\mathbf{L}\breve{\mathbf{A}}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \tag{E.3}$$

Wang et al. (2016) are perhaps the first to observe this property and use it in numerous formulations.

Using that $\mathbf{B}=\widehat{\mathbf{B}}+\breve{\mathbf{B}}$ and skipping the intermediate derivation steps, we can confirm the same property for the global and local Ghosh matrices:

$$\mathbf{G}-\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}=\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\breve{\mathbf{B}}\mathbf{G}=\mathbf{G}\breve{\mathbf{B}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1} \tag{E.4}$$

E.2 Relationship between H and Q

It is well known that the matrices of technical coefficients $\mathbf{A}$ and allocation coefficients $\mathbf{B}$ are similar:[14]

$$\mathbf{A}=\hat{\mathbf{x}}\mathbf{B}\hat{\mathbf{x}}^{-1}\qquad\text{and}\qquad\mathbf{B}=\hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}}$$

The Leontief and Ghosh inverses, either global or local, are also similar:

$$\mathbf{L}=\hat{\mathbf{x}}\mathbf{G}\hat{\mathbf{x}}^{-1}\qquad\text{and}\qquad\mathbf{G}=\hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}},$$
$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}=\hat{\mathbf{x}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\hat{\mathbf{x}}^{-1}\qquad\text{and}\qquad\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}=\hat{\mathbf{x}}^{-1}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\hat{\mathbf{x}}$$

Using the above properties, we explore below the relationship between $\mathbf{H}$ and $\mathbf{Q}$. Inserting the factorized global Leontief and Ghosh inverses into $\mathbf{L}=\hat{\mathbf{x}}\mathbf{G}\hat{\mathbf{x}}^{-1}$ produces the following expression:

$$\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\mathbf{H}=\hat{\mathbf{x}}\mathbf{Q}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\hat{\mathbf{x}}^{-1}$$

And we are able to express $\mathbf{H}$ in terms of $\mathbf{Q}$:

$$\mathbf{H}=\left(\mathbf{I}-\widehat{\mathbf{A}}\right)\hat{\mathbf{x}}\mathbf{Q}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\hat{\mathbf{x}}^{-1}$$

Recalling that $\left(\mathbf{I}-\widehat{\mathbf{A}}\right)=\hat{\mathbf{x}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)\hat{\mathbf{x}}^{-1}$, we proceed to the following expression:

$$\mathbf{H}=\hat{\mathbf{x}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)\hat{\mathbf{x}}\hat{\mathbf{x}}^{-1}\mathbf{Q}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\hat{\mathbf{x}}^{-1}=\hat{\mathbf{x}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)\mathbf{Q}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\hat{\mathbf{x}}^{-1}=$$
$$=\hat{\mathbf{x}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)\mathbf{Q}\left(\hat{\mathbf{x}}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)\right)^{-1} \tag{E.5}$$

Matrix $\mathbf{H}$ is therefore similar to matrix $\mathbf{Q}$. And, vice versa:

$$\mathbf{Q}=\left(\left(\mathbf{I}-\widehat{\mathbf{A}}\right)\hat{\mathbf{x}}\right)^{-1}\mathbf{H}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)\hat{\mathbf{x}} \tag{E.6}$$

[14] Two matrices, $\mathbf{P}$ and $\mathbf{R}$, are said to be similar if the following relation holds: $\mathbf{P}=\mathbf{M}\mathbf{R}\mathbf{M}^{-1}$.

F GVC position indices in Wang et al. (2016) and this paper

A GVC position index identifies whether forward or backward global value chain is longer for each industry in each country.

Wang et al. (2016) define the production line position index as the ratio of the average production length forward to the average production length backward.[15] Their average production length forward results from the count of the intermediate production stages within and beyond the home economy with respect to the GVC-related value added. In the notation of this paper, it can be expressed as follows:

$$\mathbf{c}_{(\mathbf{x}_D)GVC^*} = \frac{\hat{\mathbf{v}}_c \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}}\mathbf{LF}\mathbf{i}_K + \hat{\mathbf{v}}_c \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}}\mathbf{LLF}\mathbf{i}_K}{\hat{\mathbf{v}}_c \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \breve{\mathbf{A}}\mathbf{LF}\mathbf{i}_K} =$$

$$= \frac{\hat{\mathbf{v}}_c\mathbf{LLf} - \hat{\mathbf{v}}_c \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \mathbf{f}}{\hat{\mathbf{v}}_c\mathbf{Lf} - \hat{\mathbf{v}}_c \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \mathbf{f}} = \frac{\left(\mathbf{LL} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right) \mathbf{f}}{\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right) \mathbf{f}}$$

In the same way, the average production length backward sums the number of intermediate production stages within and beyond the home economy with respect to the GVC-related value of final products:

$$\mathbf{c}'_{(\mathbf{x}_O)GVC^*} = \frac{\mathbf{v}'_c\mathbf{L}\breve{\mathbf{A}} \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \hat{\mathbf{f}} + \mathbf{v}'_c\mathbf{LL}\breve{\mathbf{A}} \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \hat{\mathbf{f}}}{\mathbf{v}'_c\mathbf{L}\breve{\mathbf{A}} \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \hat{\mathbf{f}}} =$$

$$= \frac{\mathbf{v}'_c\mathbf{LL}\hat{\mathbf{f}} - \mathbf{v}'_c \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \hat{\mathbf{f}}}{\mathbf{v}'_c\mathbf{L}\hat{\mathbf{f}} - \mathbf{v}'_c \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \hat{\mathbf{f}}} = \frac{\mathbf{v}'_c \left(\mathbf{LL} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)}{\mathbf{v}'_c \left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)}$$

Note that $\mathbf{c}'_{(\mathbf{x}_O)GVC^*}$ can also be defined in terms of the Ghosh global and local inverses:

$$\mathbf{c}'_{(\mathbf{x}_O)GVC^*} = \frac{\mathbf{v}' \left(\mathbf{GG} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1} \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)}{\mathbf{v}' \left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)}$$

Finally, the GVC production line position index of Wang et al. (2016) is the ratio of the two production length indicators:

[15]The production line position index is denoted in Wang et al. (2016) as $GVCPs$, the average production length forward in GVCs as PLv_GVC and the average production length backward in GVCs as PLy_GVC.

$$\mathbf{p}_{(\mathbf{C}_{D/O})GVC^*} = \frac{\mathbf{c}_{(\mathbf{X}_D)GVC^*}}{\mathbf{c}_{(\mathbf{X}_O)GVC^*}} =$$

$$= \frac{\left(\mathbf{LL} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}}{\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}} \oslash \left[\frac{\mathbf{v}'\left(\mathbf{GG} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)}{\mathbf{v}'\left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)}\right]' \tag{F.1}$$

where $\oslash$ signifies the element-by-element division, as do fractions. One may clearly see that $\mathbf{p}_{(\mathbf{C}_{D/O})GVC^*}$ differs from $\mathbf{p}_{(\mathbf{C}_{D/O})GVC}$ in equation (D.9). The difference arises because, when measuring the length of domestic production forwards and backwards in GVCs, Wang et al. (2016) implicitly count final production stages within and between partners and primary production stages in partners that must not be treated as intermediate production stages and therefore must not enter this calculation. Equation (D.9), based on equations (D.7) and (D.8), provides the "true" GVC production line position index where one final production stage is subtracted from both length indicators:

$$\mathbf{p}_{(\mathbf{C}_{D/O})GVC} = \frac{\mathbf{c}_{(\mathbf{X}_D)GVC}}{\mathbf{c}_{(\mathbf{X}_O)GVC}} =$$

$$= \left[\frac{\left(\mathbf{LL} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}}{\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}} - \mathbf{i}_{KN}\right] \oslash \left[\frac{\mathbf{v}'\left(\mathbf{GG} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)}{\mathbf{v}'\left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)} - \mathbf{i}'_{KN}\right]' \tag{F.2}$$

In the above formulation, the position index is a function of only GVC-related production length forward and backward, not normalized with respect to total output (or value added, final demand).

In this paper, the modified production line position index is a function of GVC-related production length forward and backward that do not include the intermediate segment of production at home and are normalized with respect to total output. In other words, it is a function of both length and relative size of GVC-related production activities (see equation D.10):

$$\mathbf{p}_{(\mathbf{C},\mathbf{X}_{D/O})GVC} = \frac{\mathbf{c}_{(\mathbf{X}_D)ips.p} + \mathbf{c}_{(\mathbf{X}_D)ips.cb}}{\mathbf{c}'_{(\mathbf{X}_O)ips.p} + \mathbf{c}'_{(\mathbf{X}_O)ips.cb}} =$$

$$= \frac{\dfrac{\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}(\mathbf{H} - \mathbf{I})\left(\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} - \mathbf{I}\right)\mathbf{Hf}}{\mathbf{X}_D\mathbf{i}_K} + \dfrac{\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\mathbf{H}(\mathbf{H} - \mathbf{I})\mathbf{f}}{\mathbf{X}_D\mathbf{i}_K}}{\left[\dfrac{\mathbf{v}'\mathbf{Q}\left(\left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1} - \mathbf{I}\right)(\mathbf{Q} - \mathbf{I})\left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}}{\mathbf{v}'\mathbf{G}}\right]' + \left[\dfrac{\mathbf{v}'(\mathbf{Q} - \mathbf{I})\mathbf{Q}\left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}}{\mathbf{v}'\mathbf{G}}\right]'} =$$

$$= \frac{\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{Lf}}{\left[\mathbf{v}'\mathbf{G}\left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)\right]'} \oslash \frac{\mathbf{Lf}}{[\mathbf{v}'\mathbf{G}]'} = \frac{\left(\mathbf{L} - \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{Lf}}{\left(\mathbf{G} - \left(\mathbf{I} - \widehat{\mathbf{B}}\right)^{-1}\right)'\mathbf{G}'\mathbf{v}} \oslash \frac{\mathbf{Lf}}{\mathbf{G}'\mathbf{v}} \tag{F.3}$$

If value added $\mathbf{v}'$ includes net taxes on products, or net taxes on products are zero, the last term in the equation above would equal a vector of ones and might be neglected. Both indices are distributed around 1, but the variance of the new index is usually greater as the following tables reveal.

The difference in formulation produces notable differences in results as Tables F.1-F.2 show. Removing one final production stage from the original formulation of Wang et al. (2016) does not significantly alter the results, see third and fourth columns in Tables F.1 and F.2. This ensures that the GVC production line position index only counts intermediate production stages.

However, disregarding the chain of intermediate production stages within the home economy and normalizing the length components with respect to total outputs does reshuffle the entire country and industry rankings. Country ranking in Table F.1 based on Wang et al. (2016) places such oil and gas exporters as Russia and Norway far from top positions, respectively, 37^{th} (not seen in Table F.1) and 51^{st}. Greece and New Zealand rank unusually high.

The modified index provides more intuitive results: Norway and Russia now appear among the top upstream countries next to other oil and gas exporters. Greece and New Zealand move down in the list, with Greece positioned downstream. Seven top positions upstream now exclusively belong to countries with significant contribution of mining to their economies: Brunei, Saudi Arabia, Norway, Russia, Chile, Australia and South Africa. At the other extreme, there are developing economies such as Cambodia, Turkey, India, Tunisia, China, Viet Nam, and some smaller EU members Croatia, Cyprus, Malta. Somewhat surprising results are Colombia's rank as 11^{th} most upstream economy and that of the U.S. as 10^{th} most downstream economy.

Conventional wisdom suggests that industries supplying their products entirely or predominantly to final demand should be at the end of value chain, either domestic or global. However, the index of Wang et al. (2016) places such industries as hotels and restaurants (C55), health and social work (C85), construction (C45), public administration and defence (C75) and education (C80) much closer to the beginning of global value chain. Wang et al. (2016) explain this by the indirect involvement of those industries in the international production sharing via other domestic industries. For example, construction industry may deliver its intermediate products to another domestic industry that would use those for its GVC-related exports. Then such indirect connection to the forward value chain may be relatively lengthy. Yet these results seem counterintuitive and may also be influenced by marginal values of the GVC-induced output (or value added) that is used in the denominator in the equations of the average production length in GVCs.

The modified index addresses this issue by normalizing all length indicators with respect to total output, as opposed to the GVC-induced output. As Table F.2 shows, all five industries mentioned above are now ranked lowest, i.e., most downstream in global value chain. Overall, transition to the new index at the aggregate industry level reshuffles the ranking. Mining and quarrying (C10T14) unequivocally prevails as the most upstream industry, i.e., a industry that uses little intermediate inputs but supplies a lot of those to global value chain. Coke, refined petroleum products and nuclear fuel (C23) and electricity, gas and water supply (C40T41) change their position from top upstream to downstream where they receive more intermediate inputs from than they supply to global value chain.

The new index shows that there are 33 out of 61 countries positioned upstream in 2011, compared to 44 countries in the version of Wang et al. (2016). Out of 33 industries, only 11 are positioned upstream, while there would be 19 such industries according to Wang et al.

Table F.1: GVC production line position index of Wang et al. (2016) vs. modified GVC position index: country rankings, 2011

Wang et al. (2016): $\mathbf{p}_{(\mathbf{C}_{D/O})GVC*}$		Wang et al. (2016), corrected: $\mathbf{p}_{(\mathbf{C}_{D/O})GVC}$		This paper: $\mathbf{p}_{(\mathbf{C},\mathbf{X}_{D/O})GVC}$	
ZAF	1.1226	LUX	1.1679	BRN	7.8730
BRN	1.1191	ZAF	1.1648	SAU	5.0610
LUX	1.1181	BRN	1.1642	NOR	1.9522
AUS	1.0820	AUS	1.1071	RUS	1.8618
GRC	1.0730	GRC	1.1011	CHL	1.8596
JPN	1.0719	JPN	1.0949	AUS	1.7020
NZL	1.0685	NZL	1.0914	ZAF	1.6122
ISL	1.0676	ISL	1.0904	HKG	1.5344
BRA	1.0611	ISR	1.0815	CHE	1.4662
ISR	1.0589	BRA	1.0803	ISL	1.3549
NLD	1.0575	NLD	1.0767	COL	1.3546
SAU	1.0557	SAU	1.0765	NLD	1.3324
DNK	1.0528	DNK	1.0714	SWE	1.2441
SGP	1.0494	SGP	1.0665	SGP	1.2398
IRL	1.0468	IRL	1.0656	BRA	1.2155
CHL	1.0452	CHL	1.0603	DEU	1.1722
FRA	1.0370	FRA	1.0492	CAN	1.1665
FIN	1.0366	FIN	1.0485	IDN	1.1640
ITA	1.0343	ITA	1.0453	LUX	1.1415
BEL	1.0341	BEL	1.0452	NZL	1.1128
...	...	...	...	...	...
CAN	1.0033	CAN	1.0044	GRC	0.8870
THA	1.0032	THA	1.0042	BGR	0.8789
ESP	1.0011	ESP	1.0015	ITA	0.8699
HUN	0.9997	HUN	0.9996	POL	0.8304
MLT	0.9963	MLT	0.9951	ESP	0.8252
USA	0.9886	USA	0.9849	HUN	0.8183
GBR	0.9823	GBR	0.9764	ROU	0.8088
HRV	0.9814	POL	0.9754	CZE	0.8000
POL	0.9812	HRV	0.9753	THA	0.7789
NOR	0.9762	NOR	0.9685	PRT	0.7729
CHN	0.9624	CHN	0.9529	USA	0.7577
MEX	0.9545	MEX	0.9386	MLT	0.7514
ROU	0.9532	ROU	0.9386	CYP	0.7474
VNM	0.9528	VNM	0.9383	VNM	0.7211
ARG	0.9481	ARG	0.9315	CHN	0.6924
TUN	0.9449	TUR	0.9277	TUN	0.6878
COL	0.9445	COL	0.9261	IND	0.6765
TUR	0.9441	TUN	0.9255	HRV	0.5472
CYP	0.9195	CYP	0.8912	TUR	0.4763
KHM	0.8279	KHM	0.7763	KHM	0.3872

Note: In the index labelling system of Wang et al. (2016), the GVC production line position index $\mathbf{p}_{(\mathbf{C}_{D/O})GVC*}$ is denoted as $GVCPs$. The results are not shown for the rest of world.

Source: OECD ICIO tables, author's calculations.

Table F.2: GVC production line position index of Wang et al. (2016) vs. modified GVC position index: industry rankings, 2011

Wang et al. (2016): $\mathbf{p}_{(\mathbf{C}_{D/O})GVC*}$		Wang et al. (2016), corrected: $\mathbf{p}_{(\mathbf{C}_{D/O})GVC}$		This paper: $\mathbf{p}_{(\mathbf{C},\mathbf{X}_{D/O})GVC}$	
C40T41_EGW	1.4311	C23_PET	1.6136	C10T14_MIN	7.5552
C23_PET	1.4021	C40T41_EGW	1.6005	C71_RMQ	2.7516
C55_HTR	1.2265	C55_HTR	1.2965	C73T74_BZS	2.1084
C85_HTH	1.1718	C85_HTH	1.2258	C50T52_WRT	2.0857
C45_CON	1.1604	C45_CON	1.2084	C65T67_FIN	1.9679
C70_REA	1.1391	C70_REA	1.1786	C27_MET	1.4926
C75_GOV	1.1294	C75_GOV	1.1696	C72_ITS	1.4550
C80_EDU	1.1122	C80_EDU	1.1460	C60T63_TRN	1.3642
C10T14_MIN	1.1018	C10T14_MIN	1.1390	C24_CHM	1.2993
C90T93_OTS	1.1002	C90T93_OTS	1.1318	C21T22_PAP	1.1047
C65T67_FIN	1.0943	C65T67_FIN	1.1249	C25_RBP	1.0034
C64_PTL	1.0880	C64_PTL	1.1149	C20_WOD	0.9920
C27_MET	1.0688	C27_MET	1.0922	C28_FBM	0.9021
C72_ITS	1.0424	C72_ITS	1.0562	C64_PTL	0.8919
C21T22_PAP	1.0365	C21T22_PAP	1.0483	C23_PET	0.8469
C60T63_TRN	1.0284	C60T63_TRN	1.0377	C01T05_AGR	0.8321
C24_CHM	1.0094	C24_CHM	1.0127	C70_REA	0.8294
C71_RMQ	1.0044	C71_RMQ	1.0058	C30.32.33_CEQ	0.8292
C73T74_BZS	1.0032	C73T74_BZS	1.0042	C40T41_EGW	0.8263
C26_NMM	0.9938	C26_NMM	0.9916	C29_MEQ	0.7755
C20_WOD	0.9653	C20_WOD	0.9547	C31_ELQ	0.7185
C28_FBM	0.9640	C50T52_WRT	0.9526	C36T37_OTM	0.6744
C50T52_WRT	0.9638	C28_FBM	0.9525	C26_NMM	0.6684
C01T05_AGR	0.9564	C01T05_AGR	0.9426	C35_TRQ	0.6517
C25_RBP	0.9451	C25_RBP	0.9275	C90T93_OTS	0.5467
C36T37_OTM	0.9417	C36T37_OTM	0.9230	C17T19_TEX	0.5363
C17T19_TEX	0.8807	C17T19_TEX	0.8433	C34_MTR	0.4451
C30.32.33_CEQ	0.8759	C30.32.33_CEQ	0.8360	C15T16_FOD	0.3486
C15T16_FOD	0.8713	C15T16_FOD	0.8312	C55_HTR	0.3027
C31_ELQ	0.8552	C31_ELQ	0.8117	C80_EDU	0.1916
C29_MEQ	0.8363	C29_MEQ	0.7881	C75_GOV	0.1179
C34_MTR	0.8246	C34_MTR	0.7735	C45_CON	0.1077
C35_TRQ	0.7812	C35_TRQ	0.7143	C85_HTH	0.0407

Note: In the index labelling system of Wang et al. (2016), the GVC production line position index $\mathbf{p}_{(\mathbf{C}_{D/O})GVC*}$ is denoted as $GVCPs$. "Private households with employed persons" (C95) industry is dropped from the table because of zero output in the majority of countries.
Source: OECD ICIO tables, author's calculations.

(2016).[16] Overall, the modification of the index proposed in this paper increases its range and variance which is useful for quantitative analysis.

G Aggregation options

The default dimension of the results in sections B and C is, respectively, country-industry by country, or KN×K, and country by country-industry, K×KN. For analysis and visualization, the dimension of the results need to be reduced to K×K, KN×1, K×1 and N×1. This requires two aggregation matrices and an appropriately sized summation vector. The aggregation procedure necessarily implies weighting and averaging the results of higher dimension.

The industry-wise aggregation matrix $\mathbf{S}_N$ is constructed from the N×1 summation vectors $\mathbf{i}_N$:

$$\mathbf{S}_N = \begin{bmatrix} \mathbf{i}_N & 0 & \cdots & 0 \\ 0 & \mathbf{i}_N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{i}_N \end{bmatrix}$$

The dimension of $\mathbf{S}_N$ is KN×K. Pre-multiplying a KN×K matrix by $\mathbf{S}'_N$ compresses it to the K×K (country by country) dimension.

The country-wise aggregation matrix $\mathbf{S}_K$ requires N×N identity matrices $\mathbf{I}_N$:

$$\mathbf{S}_K = \begin{bmatrix} \mathbf{I}_N & \mathbf{I}_N & \cdots & \mathbf{I}_N \end{bmatrix}$$

The dimension of $\mathbf{S}_K$ is N×KN. Pre-multiplying a KN×K matrix by $\mathbf{S}_K$ compresses it to the N×K, industry by country dimension.

Summation vectors aggregate results row-wise or column-wise across all partners and/or industries.

For example, consider the aggregation of the number of cross-border intermediate production stages from equation (B.15) from KN×K to K×K dimension:

$$\mathbf{C}_{(\mathbf{X}_D 2)ips.cb,K\times K} = \frac{\mathbf{S}'_N \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \mathbf{H}(\mathbf{H} - \mathbf{I})\widehat{\mathbf{F}}}{\mathbf{S}'_N \left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} (\mathbf{H} - \mathbf{I})\widehat{\mathbf{F}}}$$

Each entry in the K×K matrix above counts the average number of cross-border production stages from producing country r to partner country s.

The aggregation of the same matrix from KN×K to KN×1 dimension requires K×1 summation vector:

$$\mathbf{c}_{(\mathbf{X}_D 2)ips.cb,KN\times 1} = \frac{\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} \mathbf{H}(\mathbf{H} - \mathbf{I})\widehat{\mathbf{F}}\mathbf{i}_K}{\left(\mathbf{I} - \widehat{\mathbf{A}}\right)^{-1} (\mathbf{H} - \mathbf{I})\widehat{\mathbf{F}}\mathbf{i}_K}$$

Now, the result quantifies the average number of cross-border production stages from industry i in country r to all partner countries. This type of aggregation underlies the derivation of the analytical indices in Appendix D.

The aggregation to the K×1 dimension is as follows:

[16] The "Rest of world" region and "Private households with employed persons" (C95) industry are dropped from the ranking results.

$$\mathbf{c}_{(\mathbf{X}_D 2) ips.cb, K\times 1} = \frac{\mathbf{S}'_N \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \mathbf{H}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}\mathbf{i}_K}{\mathbf{S}'_N \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}\mathbf{i}_K}$$

And, finally, the aggregation to the N×1 dimension is:

$$\mathbf{c}_{(\mathbf{X}_D 2) ips.cb, N\times 1} = \frac{\mathbf{S}_K \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} \mathbf{H}(\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}\mathbf{i}_K}{\mathbf{S}_K \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1} (\mathbf{H}-\mathbf{I})\widehat{\mathbf{F}}\mathbf{i}_K}$$

The aggregation of the GVC production line position index of Wang et al. (2016) requires particular care: the diagonalised vectors of value added coefficients or final demand may be dropped in the KN×1 dimension, but must be kept in the K×1 or N×1 dimensions. As their GVC index system is designed to handle value added rather than output flows, the aggregation implicitly requires weighting by the value added coefficients or final demand (final demand coefficients if written in Ghosh terms). Therefore, the index in equation (F.2) in the K×1 dimension must be defined as follows:

$$\mathbf{p}_{(\mathbf{C}_{D/O}) GVC, K\times 1} = \frac{\mathbf{c}_{(\mathbf{X}_D) GVC, K\times 1}}{\mathbf{c}_{(\mathbf{X}_O) GVC, K\times 1}} =$$

$$= \left[\frac{\mathbf{S}'_N \hat{\mathbf{v}}_c \left(\mathbf{LL} - \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}}{\mathbf{S}'_N \hat{\mathbf{v}}_c \left(\mathbf{L} - \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}} - \mathbf{i}_K \right] \oslash \left[\frac{\mathbf{v}' \left(\mathbf{GG} - \left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\right)\hat{\mathbf{f}}_c \mathbf{S}_N}{\mathbf{v}' \left(\mathbf{G} - \left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\right)\hat{\mathbf{f}}_c \mathbf{S}_N} - \mathbf{i}'_K \right]'$$

Similarly, GVC production line position index of Wang et al. (2016) in N×1 dimension is:

$$\mathbf{p}_{(\mathbf{C}_{D/O}) GVC, N\times 1} = \frac{\mathbf{c}_{(\mathbf{X}_D) GVC, N\times 1}}{\mathbf{c}_{(\mathbf{X}_O) GVC, N\times 1}} =$$

$$= \left[\frac{\mathbf{S}_K \hat{\mathbf{v}}_c \left(\mathbf{LL} - \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}}{\mathbf{S}_K \hat{\mathbf{v}}_c \left(\mathbf{L} - \left(\mathbf{I}-\widehat{\mathbf{A}}\right)^{-1}\right)\mathbf{f}} - \mathbf{i}_N \right] \oslash \left[\frac{\mathbf{v}' \left(\mathbf{GG} - \left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\right)\hat{\mathbf{f}}_c \mathbf{S}'_K}{\mathbf{v}' \left(\mathbf{G} - \left(\mathbf{I}-\widehat{\mathbf{B}}\right)^{-1}\right)\hat{\mathbf{f}}_c \mathbf{S}'_K} - \mathbf{i}'_N \right]'$$