Energy and environmental studies: when to use which method of decomposition?

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Abstract

In many energy and environmental studies an aggregate change in a variable is either multiplicatively or additively decomposed into a certain number of factors. For each case this paper considers three widely used methods, all six sharing the properties of time and factor reversal. On the basis of theoretical and empirical considerations (consistency-in aggregation; change-in-sign robustness; completeness; simplicity of implementation) we provide an answer to the question of when using which method. In an example of the decomposition of sectoral carbon dioxide emissions into five factors we apply all methods and compare the outcomes.

Keywords and phrases: Index number theory, Decomposition analysis, Consistency-in aggregation, Change-in-sign robustness, Completeness.

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1. Introduction

In many energy and environmental studies an aggregate change in a variable V is decomposed into a certain number of factors. It takes on two different forms: a *multiplicative* one (V^1/V^0), where the superscript 1 denotes the comparison period and the superscript 0 the base period, and an *additive* one ($V^1 - V^0$).

This paper considers six widely used methods, all of them sharing the property of 'time reversal' and being 'ideal', i.e. they satisfy the requirement of 'factor reversal'. Five of them were already known in the field of the theory of indices and indicators[2]. The following table summarizes the names used in both fields.

Table 1.

Multiplicative		Additive	
Decomposition	Index	Decomposition	Indicator
SDA or generalized Fisher	Fisher	SDA or Sun-Shapley	Bennet
LMDI- I	Montgomery-Vartia	LMDI- I	Montgomery
LMDI- II	Sato-Vartia (S-V)	LMDI- II	Additive S-V

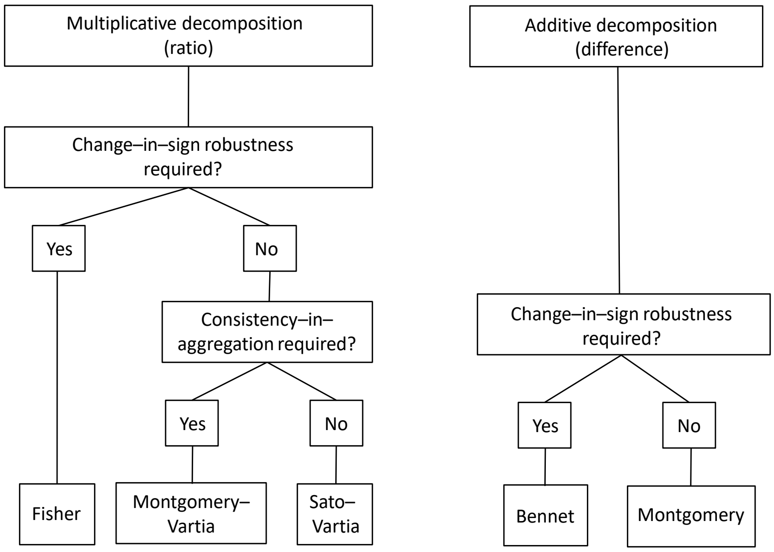
In a previous paper De Boer (2018) dealt with the multiplicative and additive SDA decomposition; in terminology of the theory of indices and indicators with 'Fisher' and 'Bennet'. He used the generic formula of Siegel (1945) that generalizes the index of Fisher (1922), originally designed for the decomposition of total consumption expenditure into two factors (price and quantity), to the general case of n factors. By collecting duplicates the computation of the unweighted average of $n!$ *permutations* ('elementary decompositions') is reduced to the computation of the weighted average of 2^{n-1} *combinations*. Since both decompositions use the same combinations and weights one Matlab program, given in the paper, suffices to deal with both of them. In the empirical example he dealt with a decomposition of carbon dioxide emissions in the Netherlands into five factors (emission coefficients, production techniques, final demand mix, demand structure and scale effects) so that the computation of 120 elementary decompositions is reduced to the computation of 16 combinations.

In this paper we apply the four LMDI methods to the very same example. We give one Matlab program that deals with these four methods at the same time. As expected, the methods for the multiplicative decomposition (Fisher, Montgomery-Vartia and Sato-Vartia) and the additive one (Bennet, Montgomery and Additive Sato-Vartia) yield similar results.

Based on theoretical and empirical arguments, we propose in Picture 1 below the following answer to the question when to use which method.

[2] The Additive Sato-Vartia method is not known in the field of the theory of indicators, but, as we will see below, it is a 1-1 transformation of the Sato-Vartia index.

Picture 1:



The paper is structured in the following way. Section 2 is devoted to a theoretical treatment of ideal decomposition methods. In the historical overview (section 2.1) we first pay attention to the theory of indices and indicators. We argue that if in a multiplicative decomposition ‘consistency-in-aggregation’ is required we have to apply Montgomery-Vartia rather than Sato-Vartia and take the non-fulfilment of ‘proportionality’ for granted. Next, we pay attention to the correspondence with structural decomposition analysis (SDA) and with index decomposition analysis (IDA). On empirical grounds we reject the use of the Additive Sato-Vartia (additive LMDI-II) decomposition. On industry level this decomposition turns out to be complete (‘no residual term’), but not at sector level, whereas the decompositions according to Bennet and Montgomery are complete at both levels. In section 2.2 we give the general form of the n-factor decomposition to which the methods can be applied and in section 2.3 the generic formula of Siegel from which the n-factor decompositions according to Fisher and Bennet can be derived. Section 2.4 contains the mathematics of the logarithmic mean which is defined for two positive numbers. In practice, we can replace zeros by epsilon small values so that it is ‘zero value robust’, but it is not ‘change-in-sign-robust’. The decompositions according to Fisher and Bennet can handle zero values and changes in sign so that in case ‘change-in-sign robustness’ is required we can only apply these methods and not one based on the logarithmic mean. In the final two subsections we give the mathematics of the n-factor decompositions according to Montgomery and Montgomery-Vartia (LMDI-I) on one hand and according to Sato-Vartia and Additive Sato-Vartia (LMDI-II) on the other. We show that there is a one-to-one relationship between the additive decomposition and the multiplicative one which is advantageously used in the computer program given in Appendix A. In Section 3 we describe our example of the decomposition of sectoral carbon dioxide emissions into n = 5 factors and give the specific formula for all decomposition methods. The empirical outcomes are given in Section 4. In our dataset there are five final demand categories (consumption, government consumption, investments, change in stocks, and exports) of which one, ‘change in stocks’, exhibits changes in sign. The only methods that can directly handle this situation are ‘Fisher’ for the multiplicative case and ‘Bennet’ for the additive one. However, as argued by de Boer (2008) it is not a genuine final demand category and we follow his proposal of splitting it over the other items of a row according to the pertinent shares in total output. Then, we can apply the four methods based on the logarithmic mean. This procedure entails that the number of final demand categories is reduced from five to four. For the multiplicative case it turns out that the results of the three decompositions (Fisher, Montgomery-Vartia, and Sato-Vartia) are very close to each other. This means that the split of ‘change in stocks’ over the other items has no effect and that the non-fulfilment of proportionality of the Montgomery-Vartia decomposition can be taken for granted. Unsurprisingly, the results for the additive case of the three decompositions (Bennet, Montgomery, and Additive Sato-Vartia) turn out to be very close to each other, as well. Section 5, finally, concludes.

2. Ideal decomposition methods

2.1. Historical overview

Theory of indices and indicators

In the traditional theory of indices and indicators an aggregate value change, expressed as ratio (index) or difference (indicator), is decomposed into two factors, price and quantity. In the 19th century two famous price and quantity indices were due to Laspeyres (1871) and Paasche (1874). These indices do not satisfy time reversal and factor reversal. In order to overcome these deficiencies Fisher (1922) derived his ‘ideal index’ by taking the geometric mean of the Laspeyres and Paasche indices. Balk (2003) remarked that the additive counterpart of the Fisher index is the indicator of Bennet (1920).

Balk (2003) gave simple derivations of the Montgomery-Vartia and Sato-Vartia indices, and of the Montgomery indicator. In Balk (2008) it is proven that the indices of Fisher and of Sato-Vartia (Sato, 1976; Vartia, 1974, 1976) obey time and factor reversal, they satisfy the axiom of ‘proportionality’[3], but are not ‘consistent-in-aggregation’[4]. The index of Montgomery-Vartia (Montgomery, 1929, 1937; Vartia, 1976) is shown to obey time and factor reversal, but it does not satisfy ‘proportionality’. On the other hand it is proven to be ‘consistent-in-aggregation’[5]. That is the reason why in Picture 1, on theoretical grounds, we advise to apply the Montgomery-Vartia index only in case ‘consistency-in-aggregation’ is required, but if it is not required, to apply either the Fisher index or the Sato-Vartia one. In Balk (2008) it is proven that the indicators of Bennet and of Montgomery (1929, 1937) obey time and factor reversal, and are ‘consistent-in-aggregation’. That is the reason why we have no preference on theoretical grounds. The Additive Sato-Vartia indicator, as we named it, is a ‘by-product’ of the Sato-Vartia decomposition. It is, however, unknown in the theory of indices and indicators so that we have no theoretical ground to exclude it.

[3] If all the price relatives are the same, the price index is equal to these relatives.

[4] Commodities used in the construction of the consumer price index are frequently grouped in four or five levels of sub-aggregates. At the lowest level prices are combined into price indices, whereas at higher levels, these price indices are combined into higher level indices. If at each stage the same functional form is used, the price index is called ‘consistent-in-aggregation’.

[5] In equation (3.168) Balk (2008) gives the formalization he proposed in earlier work (1995, 1996).

Correspondence with Structural Decomposition Analysis (SDA)

De Boer (2008) showed the correspondence between the theory of indices and indicators and applied the Montgomery indicator to the additive SDA decomposition of the example[6] of Dietzenbacher and Los (1998). They decompose the change in labor cost of 214 sectors in the Netherlands between 1986 and 1992 into four components: the effects of a change in the labor cost per unit of output; the effects of technical change; the effects of changes in the final demand mix; and the effects of the changes in the final demand levels. De Boer replicated their results and showed that the Montgomery decompositions were very close to the arithmetic mean of all elementary decompositions. In de Boer (2018) it is shown that this arithmetic mean is equivalent to the generalization of Siegel (1945) of the Bennet indicator. De Boer (2009a) applied the index of Sato-Vartia to the SDA multiplicative decomposition in the framework of the same example (Dietzenbacher, Hoen and Los, 2000). He replicated their results and showed that the Sato-Vartia decompositions were very close to the geometric averages of all elementary decompositions. De Boer (2009b) showed that this geometric average is equivalent to the generalization of Siegel of the index of Fisher to n factors.

Correspondence with Index Decomposition Analysis (IDA)

Boyd et al (1987) introduced their so-called 'Divisia index approach'. It is assumed that all variables are continuous and each is given as a function of time. The resulting equation is differentiated with respect to t, integrated over the time interval 0 to T, and the integral path is approximated using the arithmetic mean weight function. It results in the so-called AMDI method ('Artihmetic Mean Divisia Index'.) In the theory of indices and indicators this method is known under the name of 'Törnqvist index' (Törnqvist and Törnqvist, 1937) which is defined as the geometric mean of the Geometric Laspeyres and Geometric Paasche indices (Balk, 2008, p.72.) Since it is not an ideal index we do not consider it in this paper. Ang and Choi (1997) introduced 'A refined Divisia index method' by replacing the arithmetic mean by the logarithmic mean weight scheme proposed by Sato (1976). It is renamed to LMDI- II method by Ang and Liu (2001). Obviously, it is equivalent to the Sato-Vartia index in the theory of indices and indicators. Ang, Zhang and Choi (1998) proposed 'a refined Divisia index method based on decomposition of a differential quantity'. This method is nothing but the Montgomery indicator. Ang and Liu (2001) propose the so-called LMDI-I method which is equivalent to the Montgomery-Vartia index. They rename the method proposed by Ang et al (1998) to (additive) LMDI-I. The mathematical derivations of the multiplicative LMDI-I and LMDI-II methods and of the additive LMDI-I method (or Montgomery-Vartia, Sato-Vartia and Montgomery, respectively) are mathematically involved. As said before, Balk (2003) supplied much simpler derivations. The additive LMDI-II method is introduced in Appendix B of Ang, Liu and Chew (2003). As said before, this method in unknown in the theory of indices and indicators. We do not recommend to use it because, as pointed out by Ang, Huang and Mu (2009), it has one serious drawback. They show in their example that on industry level the decomposition is complete ('no residual term'), but that at sector level the decomposition is not complete since the residuals are unequal to zero. It can be shown that the decompositions according to Bennet and Montgomery are not only complete at industry level, but also at sector level so that use of one of these methods is recommended.

[6] The author is indebted to Bart Los for putting the data at his disposal.

2.2. Applicability of methods

We can only apply the methods given in this section if the n-factor decomposition can be written in the following form:

$$y = \sum_{\{I\}} \prod_{f=1}^{n} x_f \qquad (1)$$

where:

y: aggregate to be decomposed;

$\{I\}$: set of summation indices;

x_f : factor f $(= 1, \dots, n)$.

As a consequence, decompositions where in the Leontief inverse the matrix of technical coefficients is written as a Hadamard product of two matrices, like for instance in Xu and Dietzenbacher (2014), or more than two, like for instance in Zhang and Lahr (2014), cannot be decomposed according to the methods given in this section.

2.3 Fisher and Bennet

Siegel's generic formula for the geometric mean, A_n, reads:

$$A_n = \left[\prod_{r=1;j=1}^{r=n;j={}_{n-1}C_{r-1}} \left\{A\left(rt_{1,j}\right)_n\right\}^{(r-1)!(n-r)!}\right]^{1/n!} \qquad (2)$$

where

n: number of factors;

$n!$: number of elementary decompositions;

$r = 1, \dots, n$: number of '1' in the numerator (including the '1' that is always present);

$j = 1, \dots, {}_{n-1}C_{r-1}$: number of components, i.e. $\binom{n-1}{r-1}$;

$(r-1)!\,(n-r)!$: number of duplicates;

$A(rt_1, j)_n$: one of the components with exponent: $\frac{(r-1)!(n-r)!}{n!}$, $j = 1, \dots, \binom{n-1}{r-1}$.

The Bennet decomposition is the additive counterpart to the Fisher decomposition (Balk, 2003). It is the weighted arithmetic mean with the same components, the weights being the same as the exponents of the Fisher decomposition.

2.4 The logarithmic mean

The logarithmic mean for two *positive* numbers a and b is defined as:

$$L(a,b) = \frac{a-b}{ln(^a/_b)} \text{ and } L(a,a) = a \qquad (3)$$

For our purposes, the most important property is symmetry in its arguments[7], i.e.:

$L(a,b) = L(b,a)$

as can easily be verified from (3).

The logarithmic mean is very convenient when switching from a ratio to a difference and vice versa (Balk, 2003). It follows straightforwardly from (3) that:

$$a/b = \exp\{(\mathrm{a}-\mathrm{b})/\mathrm{L}(\mathrm{a},\mathrm{b})\} \tag{4}$$

and

$$(a-b) = L(a,b)\ln(a/b) \tag{5}$$

The logarithmic mean is '*zero value robust*': in practice we can replace zeros by epsilon small positive numbers (Ang and Liu, 2007a). If a and b are both negative, it can still be used. However, if there is a change-in-sign, that is to say when a is positive (negative) and b is negative (positive), the logarithmic mean (3) is not defined so that it is not '*change-in-sign robust'*. According to Ang and Liu (2007b) the logarithmic mean might handle changes in sign using the so-called 'Analytical Limit Strategy'. Their procedure has to be applied to each change in sign individually. This is so cumbersome that we advise to use the decompositions according to Fisher or Bennet which are change-in-sign, as well as zero value robust.

2.5 Montgomery (additive LMDI- I) and Montgomery- Vartia (multiplicative LMDI-I)

Define (cf. (1)):

$$v = \prod_{f=1}^{n} x_f \tag{6}$$

In decomposition according to Montgomery the weight of all factors $x_f \ \ (f=1,\dots,n)$ is equal to:

$$w^M = L(v^1, v^0) \tag{7}$$

The effect of factor x_f reads:

$$D_{x_f}^M = \sum_{\{I\}} w^M ln\left(\frac{x_f^1}{x_f^0}\right) (f=1,\dots,n); \tag{8}$$

and the Montgomery decomposition is equal to:

$$Dy = (y^1 - y^0) = \sum_{f=1}^{n} D_{x_f}^M$$

The multiplicative decomposition (i.e. Montgomery-Vartia) can be derived from the Montgomery decomposition using the transformation given in (4):

$$R_{x_f}^{MV} = exp\left[D_{x_f}^M / L(y^1, y^0)\right] (f=1,\dots,n) \tag{9}$$

The weight for of all factors $x_f \ \ (f=1,\dots,n)$ is equal to:

[7] Other properties are: $\min(a,b) \le L(a,b) \le \max(a,b)$; $L(a,b)$ is continuous; $L(\lambda a, \lambda b) = \lambda L(a,b)$; $\sqrt{ab} \le L(a,b) \le (a+b)/2$. For more details, we refer to section 3.11, Appendix 1 of Balk (2008).

$$w^{MV} = \frac{w^M}{L(y^1, y^0)} \tag{10}$$

Using Jensen's Inequality Balk (2003) proves that

$$\sum_{\{I\}} w^{MV} \leq 1 \tag{11}$$

implying that the Montgomery-Vartia decomposition does not satisfy the requirement of 'proportionality'.

The effect of factor x_f reads:

$$R_{x_f}^{MV} = \prod_{\{I\}} \left(\frac{x_f^1}{x_f^0}\right)^{w^{MV}} \quad (f = 1, \dots, n)$$

and the Montgomery-Vartia decomposition is equal to:

$$Ry = \frac{y^1}{y^0} = \prod_{j=1}^{n} R_{x_f}^{MV}$$

2.6 Sato-Vartia (multiplicative LMDI-II) and Additive Sato-Vartia (additive LMDI-II)

Define (cf. (1)):

$$s = \frac{\prod_{f=1}^{n} x_f}{y} = \frac{v}{y} \tag{12}$$

The weight for all factors x_f $((f = 1, \dots, n)$ is equal to:

$$w^{SV} = \frac{L(s^1, s^0)}{\sum_{\{I\}} L(s^1, s^0)} \tag{13}$$

It follows straightforwardly that

$$\sum_{\{I\}} w^{SV} = 1 \tag{14}$$

implying that the Sato-Vartia decomposition satisfies the requirement of 'proportionality'.

The effect of factor x_f reads:

$$R_{x_f}^{SV} = \prod_{\{I\}} \left(\frac{x_f^1}{x_f^0}\right)^{w^{SV}} \quad (f = 1, \dots, n); \tag{15}$$

and the Sato-Vartia decomposition is equal to:

$$Ry = \frac{y^1}{y^0} = \prod_{j=1}^{n} R_{x_f}^{SV}$$

The Additive Sato-Vartia decomposition can be derived from the Sato-Vartia decomposition using the transformation given in (5):

$$D_{x_f}^{ASV} = L(y^1, y^0) ln \left[R_{x_f}^{SV}\right] \; (f = 1, \dots, n) \tag{16}$$

The weight for of all factors $x_f \quad (f = 1, \ldots, n)$ is equal to:

$$w^{ASV} = L(y^1, y^0) w^{SV} \tag{17}$$

The effect of factor x_f reads:

$$D_{x_f}^{ASV} = \sum_{\{I\}} w^{ASV} \, ln\left(\frac{x_f^1}{x_f^0}\right) (f = 1, \ldots, n);$$

and the Additive Sato-Vartia decomposition is equal to:

$$Dy = (y^1 - y^0) = \sum_{f=1}^{n} D_{x_f}^{M}$$

3. Example

3.1. The model and the decompositions

De Boer (2018) disposed of two input-output tables and of the sectoral carbon dioxide emissions. The number of sectors is denoted by ‘s’ and the number of final demand categories by ‘m’.

Defining the following vectors and matrices:

$co2$: $s \times 1$ vector of sectoral emissions of carbon dioxide;

x: $s \times 1$ vector of sectoral outputs;

f: $s \times 1$vector of sectoral emissions per unit of output;

$\hat{f}$: $s \times s$ diagonal diagonal matrix with f on the main diagonal;

A : $s \times s$ matrix of input-output coefficients a_{ij} measuring the input from sector *i* in sector *j*, per unit of sector *j*’s output;

B : $n \times m$ matrix of bridge coefficients b_{jk}measuring the fraction of final demand in category *k* that is spent on products from sector *j*;

u : $m \times 1$ vector of shares u_k of final demand category *k* in total final demand; and

y: total final demand.

he considers the model:

$$co2 = \hat{f}x$$

$$x = Ax + Buy$$

of which the solution is:

$$co2 = \hat{f}DBuy \tag{18}$$

where: $D = (I - A)^{-1}$ is the Leontief inverse.

In sum notation (18) reads:

$$co2_i = \sum_{j=1}^{s} \sum_{k=1}^{m} f_i d_{ij} b_{jk} u_k \, y \tag{19}$$

Consequently, the aggregate to be decomposed, y in (1), is 'carbon dioxide emissions' $(co2_i)$, the set of summation indices $\{I\} = j, k$; and the factors are: x_1 ('emission coefficients' f_i); x_2 ('production techniques' d_{ij}); x_3 ('final demand mix' b_{jk}); x_4 ('demand structure' u_k); and x_5 ('size of economy' y), respectively. We want to decompose the change in carbon dioxide emissions from the base period, denoted by the superscript '0', to the comparison period, denoted by the superscript '1', into the changes of these five factors.

Multiplicative (ratio) decomposition

The ratio change in carbon dioxide emissions of sector *i* is defined to be:

$$RCO2_i = co2_i^1/co2_i^0$$

From (19) we obtain

$$RCO2_i = \frac{\sum_{j=1}^{s}\sum_{k=1}^{m} f_i^1 d_{ij}^1 b_{jk}^1 u_k^1 y^1}{\sum_{j=1}^{s}\sum_{k=1}^{m} f_i^0 d_{ij}^0 b_{jk}^0 u_k^0 y^0} \tag{20}$$

We want to decompose (20) into the ratio changes in emission coefficients (RF_i), production techniques (RD_i), final demand mix(RB_i), demand structure (RU_i) and size of the economy (RY_i), i.e.:

$$RCO2_i = RF_i \times RD_i \times RB_i \times RU_i \times RY_i.$$

Additive (difference) decomposition

The difference change in carbon dioxide emissions of sector *i* is defined to be:

$$DCO2_i = CO2_i^1 - CO2_i^0$$

From (19) we obtain:

$$DCO2_i = \sum_{j=1}^{n}\sum_{k=1}^{m}\left(f_i^1 d_{ij}^1 b_{jk}^1 u_k^1 y^1 - f_i^0 d_{ij}^0 b_{jk}^0 u_k^0 y^0\right) \tag{21}$$

We want to decompose (21) into the difference changes in emission coefficients (DF_i), production techniques (DD_i), final demand mix(DB_i), demand structure (DU_i), and size of the economy (DY_i), i.e.:

$$DCO2_i = DF_i + DD_i + DB_i + DU_i + DY_i$$

3.2 The Fisher and Bennet decompositions

Without proof de Boer (2009b) supplied a table with the combinations, the number of duplicates and the weights for the case of the decomposition of a variable into five factors.

Table 1. Summary for the case of five factors

Appendix A Equation:	Number of ones	Combinations				Number of duplicates	Weight
A.1	0	**{0,0,0,0}**				24	1/5
A.2	1	**{1,0,0,0}**	{0,1,0,0}	{0,0,1,0}	**{0,0,0,1}**	6	1/20
A.3	2	**{1,1,0,0}**	{1,0,1,0}	{1,0,0,1}		4	1/30
A.3	2	{0,0,1,1}	{0,1,0,1}	**{0,1,1,0}**		4	1/30
A.4	3	**{0,1,1,1}**	{1,0,1,1}	{1,1,0,1}	**{1,1,1,0}**	6	1/20
A.5	4	**{1,1,1,1}**				24	1/5

From this table we can derive the decomposition formulas for each of the five factors. For factor 1[8], 'ratio change in emission coefficients', it results in:

$$\left[\frac{\sum x_1^1 x_2^{\mathbf{0}} x_3^{\mathbf{0}} x_4^{\mathbf{0}} x_5^{\mathbf{0}}}{\sum x_1^0 x_2^{\mathbf{0}} x_3^{\mathbf{0}} x_4^{\mathbf{0}} x_5^{\mathbf{0}}}\right]^{1/5} \times \left[\frac{\sum x_1^1 x_2^{\mathbf{1}} x_3^{\mathbf{0}} x_4^{\mathbf{0}} x_5^{\mathbf{0}}}{\sum x_1^0 x_2^{\mathbf{1}} x_3^{\mathbf{0}} x_4^{\mathbf{0}} x_5^{\mathbf{0}}}\right]^{1/20} \times \ldots \times \left[\frac{\sum x_1^1 x_2^{\mathbf{0}} x_3^{\mathbf{0}} x_4^{\mathbf{0}} x_5^{\mathbf{1}}}{\sum x_1^0 x_2^{\mathbf{0}} x_3^{\mathbf{0}} x_4^{\mathbf{0}} x_5^{\mathbf{1}}}\right]^{1/20} \times \left[\frac{\sum x_1^1 x_2^{\mathbf{1}} x_3^{\mathbf{1}} x_4^{\mathbf{0}} x_5^{\mathbf{0}}}{\sum x_1^0 x_2^{\mathbf{1}} x_3^{\mathbf{1}} x_4^{\mathbf{0}} x_5^{\mathbf{0}}}\right]^{1/30} \times$$

(22)

$$\times \ldots \times \left[\frac{\sum x_1^1 x_2^{\mathbf{0}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{0}}}{\sum x_1^0 x_2^{\mathbf{0}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{0}}}\right]^{1/30} \times \left[\frac{\sum x_1^1 x_2^{\mathbf{0}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{1}}}{\sum x_1^0 x_2^{\mathbf{0}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{1}}}\right]^{1/20} \times \ldots \times \left[\frac{\sum x_1^1 x_2^{\mathbf{1}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{0}}}{\sum x_1^0 x_2^{\mathbf{1}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{0}}}\right]^{1/20} \times \left[\frac{\sum x_1^1 x_2^{\mathbf{1}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{1}}}{\sum x_1^0 x_2^{\mathbf{1}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{1}}}\right]^{1/5}$$

In the very same way, we use Table 1 for the Bennet decomposition. For factor 1, 'difference change in emission coefficients' we obtain:

$$\frac{1}{5}\left[\sum(\Delta x_1) x_2^{\mathbf{0}} x_3^{\mathbf{0}} x_4^{\mathbf{0}} x_5^{\mathbf{0}}\right] + \frac{1}{20}\left[\sum\left(\Delta x_1 x_2^{\mathbf{1}} x_3^{\mathbf{0}} x_4^{\mathbf{0}} x_5^{\mathbf{0}}\right)\right] + \cdots + \frac{1}{20}\left[\sum(\Delta x_1) x_2^{\mathbf{0}} x_3^{\mathbf{0}} x_4^{\mathbf{0}} x_5^{\mathbf{1}}\right] +$$

(23)

$$+ \frac{1}{30}\left[\sum(\Delta x_1)\, x_2^{\mathbf{1}} x_3^{\mathbf{1}} x_4^{\mathbf{0}} x_5^{\mathbf{0}}\right] + \cdots + \frac{1}{30}\left[\sum(\Delta x_1)\, x_2^{\mathbf{0}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{0}}\right] + \frac{1}{20}\left[\sum(\Delta x_1)\, x_2^{\mathbf{0}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{1}}\right] +$$

$$+ \cdots + \frac{1}{20}\left[\left[\sum(\Delta x_1)\, x_2^{\mathbf{1}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{0}}\right]\right] + \frac{1}{5}\left[\sum\left(\Delta x_1 x_2^{\mathbf{1}} x_3^{\mathbf{1}} x_4^{\mathbf{1}} x_5^{\mathbf{1}}\right)\right]$$

De Boer (2018) gives the derivation of Table 1 from Siegel's formula in his Appendix A, whereas in his Appendix B one Matlab program is given that performs the Siegel and Bennet decompositions at the same time. His results of the Siegel decomposition are mentioned in Table 3 below and are compared to those of the Sato-Vartia and Montgomery-Vartia decompositions. The results of the Bennet decomposition are mentioned in Table 4 below and are compared to those of the Montgomery and Additive Sato-Vartia decompositions.

3.4 The Montgomery and Montgomery-Vartia decompositions

Montgomery

According to equations (6) and (19) we have

$v_{ijk}^1 = f_i^1 d_{ij}^1 b_{jk}^1 u_k^1 y^1$ and $v_{ijk}^0 = f_i^0 d_{ij}^0 b_{jk}^0 u_k^0 y^0$

so that equations (7) and (8) imply:

$$w_{ijk}^M = L\left(v_{ijk}^1, v_{ijk}^0\right) \qquad (24)$$

$$DF_i^M = \sum_{j=1}^{s} \sum_{k=1}^{m} w_{ijk}^M \, ln\left(\frac{f_i^1}{f_i^0}\right) \qquad DD_i^M = \sum_{j=1}^{s} \sum_{k=1}^{m} w_{ijk}^M \, ln\left(\frac{d_{ij}^1}{d_{ij}^0}\right)$$

$$DB_i^M = \sum_{j=1}^{s} \sum_{k=1}^{m} w_{ijk}^M \, ln\left(\frac{b_{jk}^1}{b_{jk}^0}\right) \qquad DU_i^M = \sum_{j=1}^{s} \sum_{k=1}^{m} w_{ijk}^M \, ln\left(\frac{u_k^1}{u_k^0}\right) \qquad (25)$$

$$DY_i^M = \sum_{j=1}^{s} \sum_{k=1}^{m} w_{ijk}^M \, ln\left(\frac{y^1}{y^0}\right)$$

[8] The same formulas are used for the other factors.

Montgomery-Vartia

Using the transformation (9), i.c.

$R?^{MV} = exp\left[D?^{M}/L\left(co2_i^1, co2_i^0\right)\right]$ (? = F_i, D_i, B_i, U_i, Y_i) (26)

and (10), we arrive at:

$w_{ijk}^{MV} = w_{ijk}^{M}/ L\left(co2_i^1, co2_i^0\right)$ (27)

and:

$RF_i^{MV} = \prod_{j=1}^{s}\prod_{k=1}^{m}\left[\frac{f_i^1}{f_i^0}\right]^{w_{ijk}^{MV}} (\leq \frac{f_i^1}{f_i^0})$ because of $\sum_{j=1}^{s}\sum_{k=1}^{m} w_{ijk}^{MV} \leq 1$ (cf. (11))

$RD_i^{SV} = \prod_{j=1}^{s}\prod_{k=1}^{m}\left[\frac{d_{ij}^1}{d_{ij}^0}\right]^{w_{ijk}^{MV}}$ $RB_i^{SV} = \prod_{j=1}^{s}\prod_{k=1}^{m}\left[\frac{b_{jk}^1}{b_{jk}^0}\right]^{w_{ijk}^{MV}}$ $RU_i^{SV} = \prod_{j=1}^{s}\prod_{k=1}^{m}\left[\frac{u_k^1}{u_k^0}\right]^{w_{ijk}^{MV}}$

$RY_i^{MV} = \prod_{j=1}^{s}\prod_{k=1}^{m}\left[\frac{y^1}{y^0}\right]^{w_{ijk}^{MV}} (\leq \frac{y^1}{y^0})$ (28)

Sato-Vartia and Additive Sato-Vartia decompositions

Sato-Vartia

Equation (12) implies:

$s_{ijk}^1 = v_{ijk}^1/co2_i^1$ and $s_{ijk}^0 = v_{ijk}^0/co2_i^0$

Consequently, the equation (13) becomes:

$w_{ijk}^{SV} = \frac{L\left[s_{ijk}^1, s_{ijk}^0\right]}{\sum_{j'=1}^{s}\sum_{k'=1}^{m} L\left[s_{ij'k'}^1, s_{ij'k'}^0\right]}$ (29)

Obviously: $\sum_{j=1}^{s}\sum_{k=1}^{m} w_{ijk}^{SV} = 1$ (fulfillment of 'proportionality').

According to equation (15), we have:

$RF_i^{SV} = \prod_{j=1}^{s}\prod_{k=1}^{m}\left[\frac{f_i^1}{f_i^0}\right]^{w_{ijk}^{SV}} (= \frac{f_i^1}{f_i^0})$

$RD_i^{SV} = \prod_{j=1}^{s}\prod_{k=1}^{m}\left[\frac{d_{ij}^1}{d_{ij}^0}\right]^{w_{ijk}^{SV}}$ $RB_i^{SV} = \prod_{j=1}^{s}\prod_{k=1}^{m}\left[\frac{b_{jk}^1}{b_{jk}^0}\right]^{w_{ijk}^{SV}}$ (30)

$RU_i^{SV} = \prod_{j=1}^{s}\prod_{k=1}^{m}\left[\frac{u_k^1}{u_k^0}\right]^{w_{ijk}^{SV}}$ $RY_i^{SV} = \prod_{j=1}^{s}\prod_{k=1}^{m}\left[\frac{y^1}{y^0}\right]^{w_{ijk}^{SV}} (= \frac{y^1}{y^0})$

Additive Sato-Vartia

Using the transformation (16):

$D?^{ASV} = L\left(co2_i^1, co2_i^0\right) ln[R?^{MV}]$ (? = F_i, D_i, B_i, U_i, Y_i) (31)

we arrive at the weighting factors:

$$w_{ijk}^{ASV} = L\left(co2_i^1, co2_i^0\right) w_{ijk}^{SV} \tag{32}$$

so that:

$$DF_i^{ASV} = \sum_{j=1}^{s} w_{ijk}^{ASV} \, ln\left(\frac{f_i^1}{f_i^0}\right) \qquad DD_i^{ASV} = \sum_{j=1}^{s} \sum_{k=1}^{m} w_{ijk}^{ASV} \, ln\left(\frac{d_{ij}^1}{d_{ij}^0}\right)$$

$$DB_i^{ASV} = \sum_{j=1}^{s} \sum_{k=1}^{m} w_{ijk}^{ASV} \, ln\left(\frac{b_{jk}^1}{b_{jk}^0}\right) \qquad DU_i^{ASV} = \sum_{j=1}^{s} \sum_{k=1}^{m} w_{ijk}^{ASV} \, ln\left(\frac{u_k^1}{u_k^0}\right)$$

$$DY_i^{ASV} = \sum_{j=1}^{s} \sum_{k=1}^{m} w_{ijk}^{ASV} \, ln\left(\frac{y^1}{y^0}\right)$$

4. Results

4.1. Dataset[9]

The dataset consists of two Excel files. In ‘Base period’ the emissions of carbon dioxide (in million kg) are given for 60 sectors of the Dutch economy in 2004, together with the 60 x 60 matrix of intermediate deliveries (in million €) and the 60 x 5 matrix of final deliveries (consumption, government consumption, investments, change in stocks, and exports.) In ‘Comparison period’ the same data are given for 2005, the matrices of intermediate and final deliveries are recorded in prices 2004.

In the last row of Table 2 below we give the percentages of the ten largest emitters in the total economy. Together they count for slightly more than 80% of the emissions while their share in total final demand is 18.5%. Between brackets we give the percentages in the total economy of the largest emitter. Not unsurprisingly it is sector number 25 ‘Electricity and gas supply’. It counts for about one third of total emissions whereas its share in total final demand is only 1.3%. From the column ‘DCO2’ we gather that the largest emitter accounts for 51.8% of the reduction of carbon dioxide emissions from 2004 to 2005. The next nine largest sectors account for 1% only. Consequently, the share in the reduction by the remaining 50 sectors is equal to 47.2%

In de Boer (2018) this dataset was used for the decomposition, according to Fisher and Bennet, of the change of carbon dioxide emissions into the five factors: emission coefficients, production techniques, final demand mix, demand structure, and size of the economy. The results are given in the Tables 3 and 4 below.

[9] The author is indebted to Sjoerd Schenau of Statistics Netherlands for putting these two tables at his disposal.

Table 2. Carbon dioxide emissions (million kg), ratio and difference change, and total final demand (million €) for the ten largest emitters and for the total economy.

#	Sector	CO2 2004	CO2 2005	RCO2	DCO2	Final 2004	Final 2005
25	Electricity and gas supply	56,538	55,076	0.974	-1,462	7,689	7,657
13	Chemicals; man-made fibres	15,149	15,215	1.004	66	19,978	20,605
12	Petroleum products, cokes, etc.	12,941	12,826	0.991	-115	13,217	13,097
36	Air transport	12,425	12,940	1.041	515	5,910	6,332
34	Land transport	8,821	8,478	0.961	-343	8,121	8,195
35	Water transport	7,409	7,709	1.041	300	4,980	5,265
16	Basic metals	7,280	6,957	0.956	-323	5,306	5,108
55	Sewage; refuse disposal	7,234	7,268	1.005	34	1,213	1,166
2	Horticulture	6,846	6,821	0.996	-25	7,030	6,963
8	Food, beverages, Tobacco	4,439	4,305	0.970	-134	34,232	34,962
	Ten largest emitters, 2004	139,082	137,594	0.989	-1,488	107,676	109,350
	Total economy	171,419	168,599	0.984	-2,820	580,936	592,633
	% ten largest in total economy	81.1 (33.0)	81.6 (32.7)		52.8 (51.8)	18.5 (1.3)	18.5 (1.3)

4.2 Empirical implementation of the decompositions based on the logarithmic mean

As stated before, the logarithmic mean is 'zero value robust' that is to say: in practice we can replace zeros by epsilon small positive numbers (Ang and Liu, 2007a). In the Matlab program (Appendix A) 10^{-14} is used. However, it is not 'change-in-sign robust'. We cannot apply the decompositions on our data set because of the presence of the final demand category 'change in stocks'. However, as argued by De Boer (2008), this is not a genuine final demand category. A correct treatment is the following: in the final demand matrix a column should be included with the (non-negative) 'addition to stocks' and in the input-output table a row with the (non-negative) 'depletion of stocks'. Due to problems of data collection, national account statisticians only include the balancing item 'change in stocks'. De Boer (2008) solved the problem of changes-in-sign for stocks by splitting them over the other items of a row according to the pertinent shares in total output. The column sums are not any longer equal to total output so that he added a row (which plays no role in decompositions) in which he recorded the adjustment for the stocks. In the example this procedure is applied. As a consequence, the number of final demand categories is reduced from 5 to 4.

4.3 Matlab program for the methods based on the logarithmic mean

In the first part of the Matlab program (Appendix A) the original data are read; the change in stocks are split over the other items of a row according to the pertinent shares in total output;

and transformed to the data for the five factors: emission coefficients, production techniques, final demand mix, demand structure*,* and size of the economy.

In the second part we program the methods based on the logarithmic mean. Since the weight for the Montgomery decomposition (cf. (24)) is easier to program than the one for Montgomery- Vartia (cf. (27)) we choose performing the Montgomery decomposition and to use transformation (26) for Montgomery- Vartia. The weight for the Sato- Vartia decomposition (cf. (29)) is easier to program than the one for the Additive Sato- Vartia (cf. (32)). Therefore we choose to perform the Sato-Vartia decomposition and use the transformation (31) to obtain its additive counterpart.

4.4 Results for the multiplicative decompositions

Table 3 Results of the three multiplicative decompositions

Sector	Method	RF	RD	RB	RU	RY*
25	Fisher	0.9093	1.0670	0.9839	1.0010	
	Montgomery-Vartia	0.9102	1.0646	0.9854	1.0003	1.011939
	Sato-Vartia	0.9093	1.0655	0.9853	1.0003	
13	Fisher	0.9850	0.9899	1.0002	1.0088	
	Montgomery-Vartia	0.9857	0.9899	1.0005	1.0084	1.021028
	Sato- Vartia	0.9857	0.9899	1.0005	1.0084	
12	Fisher	0.9960	0.9962	0.9732	1.0073	
	Montgomery-Vartia	0.9949	0.9962	0.9734	1.0071	1.020125
	Sato-Vartia	0.9949	0.9962	0.9734	1.0071	
36	Fisher	0.9734	1.0070	1.0354	1.0065	
	Montgomery-Vartia	0.9734	1.0064	1.0356	1.0063	1.020129
	Sato-Vartia	0.9734	1.0064	1.0356	1.0063	
34	Fisher	0.9449	0.9920	0.9980	1.0024	
	Montgomery-Vartia	0.9449	0.9920	0.9974	1.0024	1.020130
	Sato-Vartia	0.9499	0.9920	0.9974	1.0024	
35	Fisher	0.9912	0.9950	1.0257	1.0090	
	Montgomery-Vartia	0.9912	0.9950	1.0260	1.0081	1.020134
	Sato-Vartia	0.9912	0.9950	1.0260	1.0081	
16	Fisher	0.9982	0.9990	0.9469	1.0088	
	Montgomery-Vartia	0.9982	0.9983	0.9473	1.0084	1.020125
	Sato-Vartia	0.9982	0.9984	0.9473	1.0084	
55	Fisher	1.0402	0.9960	0.9949	0.9860	
	Montgomery-Vartia	1.0402	0.9651	0.9949	0.9860	1.020125
	Sato-Vartia	1.0402	0.9651	0.9949	0.9860	
2	Fisher	1.0039	1.0011	0.9635	1.0088	
	Montgomery-Vartia	1.0039	1.0010	0.9638	1.0084	1.020130
	Sato-Vartia	1.0039	1.0010	0.9638	1.0084	
8	Fisher	0.9516	0.9979	0.9962	1.0050	
	Montgomery-Vartia	0.9516	0.9979	0.9963	1.0049	1.020132
	Sato-Vartia	0.9516	0.9979	0.9963	1.0049	

* For Fisher and Sato-Vartia all figures are equal to 1.020135

As we conclude from Table 3 all decompositions are very close to each other. From an empirical point of view the split of 'change in stocks' over the other items of the pertinent row has no effect. If 'change-in-sign robustness' is required we need to apply the Fisher decomposition, but if it is not required, like in this example, we advise to use either the Montgomery- Vartia or the Sato-Vartia decomposition since the latter two are easier to program than Fisher's. The decompositions according to Fisher (cf. equation (22)) and Sato-Vartia (cf. equation (14)) satisfy 'proportionality' which implies that for all sectors the effect of the factor 'size of the economy' (y^1/y^0) is equal to 1.020135. Because $RY \leq y^1/y^0$ (equation (28)) Montgomery-Vartia does not satisfy 'proportionality'. As we conclude from Table 3, all reported effects of 'size of the economy' of Montgomery-Vartia are very close to the ones of Fisher and Sato-Vartia. For all 60 sectors the minimum effect is equal to 1.019438; the maximum to 1.020134; while the mean effect is equal to 1.020098 with a standard deviation of 0.000119. If in this example we desire to have consistency-in-aggregation, we can easily take the non-fulfilment of 'proportionality' for granted and apply Montgomery- Vartia. If not, we advise to use Sato-Vartia since it satisfies proportionality.

4.5 Results for the additive decompositions

Table 4 Results for the additive decompositions

Sector	Method	DF	DD	DB	DU	DY
25	Bennet	-5,314	3,662	-912	17	1,115
	Montgomery	-5,253	3,492	-821	18	1,102
	Additive S-V	-5,304	3,540	-829	18	1,112
13	Bennet	-219	-154	3	132	303
	Montgomery	-218	-153	8	128	303
	Additive S-V	-219	-153	8	128	303
12	Bennet	-66	-49	-351	94	257
	Montgomery	-66	-49	-347	91	257
	Additive S-V	-66	-49	-348	91	257
36	Bennet	-342	81	441	82	253
	Montgomery	-341	81	443	80	253
	Additive S-V	-342	81	443	80	253
34	Bennet	-445	-69	-22	20	172
	Montgomery	-445	-70	-22	21	172
	Additive S-V	-445	-70	-22	21	172
35	Bennet	-67	-38	192	63	151
	Montgomery	-67	-38	194	61	151
	Additive S-V	-67	-38	194	61	151
16	Bennet	-128	-12	-388	62	142
	Montgomery	-128	-12	-385	60	142
	Additive S-V	-128	-12	-386	60	142
55	Bennet	286	-257	-37	-103	145
	Montgomery	286	-258	-37	-102	144
	Additive S-V	286	-258	-37	-102	145
2	Bennet	26	7	-254	60	136
	Montgomery	26	7	-252	57	136
	Additive S-V	26	7	-252	57	136
8	Bennet	-217	-9	-17	22	87
	Montgomery	-217	-9	-16	21	87
	Additive S-V	-217	-9	-16	21	87

Obviously, the results for the additive decompositions show the same picture as those of the multiplicative decompositions with the same conclusion that the three methods yield the same results so that the split of 'change in stocks' over the other items of a row has no effect. If 'change-in-sign robustness' is required we need to apply the Bennet decomposition, but if it is not required, like in this example, we advise to use the Montgomery decomposition because it easier to program than Bennet's decomposition.

5. Concluding remarks

In this paper we paid attention to six widely used decomposition methods which all share the properties of time reversal and of being ideal, i.e. satisfying factor reversal. On the basis of theoretical and empirical considerations we gave an answer to the question of when using which method.

Multiplicative decomposition

We considered:

1. Fisher (SDA) which is zero value and change-in-sign robust, satisfies proportionality, but is not consistent-in-aggregation;

2. Montgomery- Vartia (multiplicative LMDI-I) which is zero value robust, but not change-in-sign robust, is consistent-in-aggregation, but does not satisfy proportionality; and

3. Sato-Vartia (multiplicative LMDI-II) which is zero value robust, but not change-in-sign robust, satisfies proportionality, but is not consistent-in-aggregation.

If there are changes in sign in the data set which cannot be resolved the only method that can be applied is ‘Fisher’. If the data set is ‘change-in-sign robust’ we can apply all three methods. If we wish to apply a method which is ‘consistent-in-aggregation’, we have to apply the Montgomery-Vartia decomposition and take the non-fulfilment of proportionality for granted. If we are not interested in consistency in aggregation we can either apply Sato-Vartia or Fisher which both satisfy proportionality. Since the first method is simpler to implement than the latter we recommend to use Sato-Vartia.

Additive decomposition

We considered:

4. Bennet (SDA) which is zero value and change-in-sign robust, and complete at industry and sector level;

5. Montgomery (additive LMDI-I) which is zero value robust, but not change-in-sign robust, and is complete at industry and sector level; and

6. Additive Sato-Vartia (additive LMDI-II) which is zero value robust, but not change-in-sign robust, and is complete at industry level, but not at sector level. That is the reason why we do not recommend the use of this method.

If there are changes in sign in the data set which cannot be resolved, only Bennet can be applied. Otherwise, we can use either Montgomery or Bennet, but since the first method is simpler to implement we recommend the use of Montgomery.

Example

We applied all methods to an example in which the change from 2004 to 2005 in sectoral carbon dioxide emissions of the Netherlands are decomposed into five factors: emission coefficients, production techniques, final demand mix, demand structure and size of the economy. The data set is not change-in-sign-robust because of the presence ‘change in stocks’ which, as argued by de Boer (2008), is not a genuine final demand category. It was resolved by spreading the change in stocks over the other items of the pertinent row in the input-output table. We applied the methods of Fisher and Bennet to the full data set, i.e. including the change in stocks, and the other methods which are based on the logarithmic mean to the data set where the number of final demand categories is reduced to four.

Multiplicative decomposition

From Table 3 it followed that all decompositions are very close to each other so that the split of 'change in stocks' over the other items of the pertinent row has no effect. We advise to use either the Montgomery- Vartia or the Sato-Vartia decomposition since the latter two are easier to program than Fisher's. The effect of the factor 'size of the economy' for the Montgomery-Vartia decomposition turned out to be so close to the effect according to the decompositions of Fisher and Sato- Vartia which both satisfy 'proportionality'. Consequently, we can take its non-fulfilment for granted. If one wishes to adopt a method that is consistent-in-aggregation Montgomery-Vartia needs to be applied, if not, Sato-Vartia is recommended since it satisfies 'proportionality'.

Additive decomposition

Again, the split of 'change in stocks' over the other items of a row has no effect. We advise to use the Montgomery decomposition because it easier to program than Bennet's.

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Appendix A Matlab code for the decompositions of Montgomery (MO), Montgomery-Vartia (MV), Sato-Vartia (SV) and Additive Sato-Vartia (ASV).

```
clc; clear;

% Number of factors
n = 5;
% Number of final demand categories
m = 4;
% Number of sectors
s = 60;

%% Base Period

% Read the original data
CO20 = xlsread('Base Period','C2:BJ2');
CO20(CO20==0) = 10^-14;
data = xlsread('Base Period','C3:BQ62');

% Splitting stocks
temp1 = data(1:60,[61:63 65]);
stocks = data(:,64);

temp2 = temp1./repmat(sum(temp1,2),1,4);
temp2(60,:) = 0;

temp1 = temp1 + temp2.*repmat(stocks,1,4);
temp = [data(1:60,1:60) temp1];

transaction = temp(1:60,1:60);
x = data(1:60,67);
z = temp(1:60,61:64);

% Transformation of original data to data for the five factors
```

```
% Carbon dioxide emissions per unit of output
f0 = diag(CO20./x');

% Leontief inverse
A = transaction./repmat(x',60,1);
D0 = inv(eye(60)-A);
D0(D0==0) = 10^-14;

% Total final demand
y0 = sum(sum(z,1));

% Shares of final demand categories in Total final demand
u0 = transpose(sum(z,1)/y0);

% Bridge matrix
B0 = z./repmat(transpose(u0*y0),s,1);
B0(B0==0) = 10^-14;

%% Comparison period

% Read the original data
CO21 = xlsread('Comparison Period','C2:BJ2');
CO21(CO21==0) = 10^-14;
data = xlsread('Comparison Period','C3:BQ62');

% Spreading stocks over other final demand categories
temp1 = data(1:60,[61:63 65]);
stocks = data(:,64);

temp2 = temp1./repmat(sum(temp1,2),1,4);
temp2(60,:) = 0;

temp1 = temp1 + temp2.*repmat(stocks,1,4);
temp = [data(1:60,1:60) temp1];

transaction = temp(1:60,1:60);
x = data(1:60,67);
z = temp(1:60,61:64);

% Transformation of original data to data for the five factors

% Carbon dioxide emissions per unit of output
f1 = diag(CO21./x');

% Leontief inverse
A = transaction./repmat(x',60,1);
D1 = inv(eye(60)-A);
D1(D1==0) = 10^-14;

% Total final demand
y1 = sum(sum(z,1));

% Shares of final demand categories in Total final demand
u1 = transpose(sum(z,1)/y1);
```

```
% Bridge matrix
B1 = z./repmat(transpose(u1*y1),s,1);
B1(B1==0) = 10^-14;

%% Decompositions

% Sato-Vartia (SV) and Additive Sato-Vartia (ASV)
[sv,asv] = SV(f0,f1,D0,D1,B0,B1,u0,u1,y0,y1,CO20,CO21,n,m,s);

% Montgomery (MO) and Montgomery-Vartia (MV)
[mo,mv] = MO(f0,f1,D0,D1,B0,B1,u0,u1,y0,y1,CO20,CO21,n,m,s);
```

```
% This function computes the logarithmic mean of two positive numbers

function l = L(a,b)

% Adjust for the case when a/b equals 1, which leads to a division by
% zero.
if a/b == 1 || b == 0
   l = (a-b)/log(1+10^-14);
else
   l = (a-b)/log(a/b);
end

end
```

```
% This function performs the Montgomery (MO) and Montgomery-Vartia decompositions

function [mo,mv] = MO(f0,f1,D0,D1,B0,B1,u0,u1,y0,y1,CO20,CO21,n,m,s)
%% Montgomery (MO)

mo = zeros(s,n);
for i = 1:s
   for j = 1:s
      for k = 1:m
         weight = L(f1(i,i)*D1(i,j)*B1(j,k)*u1(k)*y1,f0(i,i)*D0(i,j)*B0(j,k)*u0(k)*y0);
         mo(i,1) = mo(i,1) + weight * log(f1(i,i)/f0(i,i));
         mo(i,2) = mo(i,2) + weight * log(D1(i,j)/D0(i,j));
         mo(i,3) = mo(i,3) + weight * log(B1(j,k)/B0(j,k));
         mo(i,4) = mo(i,4) + weight * log(u1(k)/u0(k));
         mo(i,5) = mo(i,5) + weight * log(y1/y0);
      end
   end
end

%% Montgomery-Vartia (MV)
mv = zeros(s,n);
for i = 1:s
   if CO20(i)==10^-14 || CO21(i)==10^-14
```

```
        mv(i,:) = 1;
    else
        mv(i,:) = exp(mo(i,:)./L(CO20(i),CO21(i)));
    end
end

end
```


```
% This function performs the Sato-Vartia (SV) and Additive Sato-Vartia decompositions

function [sv,asv] = SV(f0,f1,D0,D1,B0,B1,u0,u1,y0,y1,CO20,CO21,n,m,s)
%% Sato Vartia (SV)

sv = ones(s,n);

for i = 1:s
    sum = 0;
    for j1 = 1:s
        for k1 = 1:m
            sum = sum + 
L(f1(i,i)*D1(i,j1)*B1(j1,k1)*u1(k1)*y1/CO21(i),f0(i,i)*D0(i,j1)*B0(j1,k1)*u0(k1)*y0/CO20(i));
        end
    end

    for j = 1:s
        for k = 1:m
            weight = 
(L(f1(i,i)*D1(i,j)*B1(j,k)*u1(k)*y1/CO21(i),f0(i,i)*D0(i,j)*B0(j,k)*u0(k)*y0/CO20(i))/sum);
            sv(i,1) = sv(i,1) * ((f1(i,i)/f0(i,i)) ^ weight);
            sv(i,2) = sv(i,2) * ((D1(i,j)/D0(i,j)) ^ weight);
            sv(i,3) = sv(i,3) * ((B1(j,k)/B0(j,k)) ^ weight);
            sv(i,4) = sv(i,4) * ((u1(k)/u0(k)) ^ weight);
            sv(i,5) = sv(i,5) * ((y1/y0) ^ weight);
        end
    end
end


%% Additive Sato-Vartia (ASV)
asv = zeros(s,n);
for i = 1:s
    asv(i,:) = L(CO21(i),CO20(i)).*log(sv(i,:));
end

end
```
