Transforming a Rectangular Input-Output Model into the Coordinates with Respect to Eigenbasis

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A rectangular input-output table with N products and M industries is defined by production matrix $\mathbf{X}$ and intermediate consumption matrix $\mathbf{Z}$ with N rows and M columns both. The square matrices $\mathbf{FF}'=(\mathbf{X}-\mathbf{Z})(\mathbf{X}-\mathbf{Z})'$ of order N and $\mathbf{F}'\mathbf{F}=(\mathbf{X}-\mathbf{Z})'(\mathbf{X}-\mathbf{Z})$ of order M are symmetric and have the same spectrum of nonzero real eigenvalues.

The eigenvectors of matrix $\mathbf{FF}'$ form an orthonormal basis of N-dimensional vector space that could be considered as eigenbasis for rectangular input-output model at $N>M$ (i.e., the number of products exceeds the number of industries as it often happens in statistical practice). Being transformed with respect to the eigenbasis, matrices $\mathbf{X}$ and $\mathbf{Z}$ have $N-M$ lower rows coincided between each other (with zero final demand for the last $N-M$ products). This property allows employing rectangular input-output table written in the coordinates with respect to the eigenvectors of matrix $\mathbf{FF}'$ as operational demand-driven input-output model in which M components of final demand are exogenous variables and the other $N-M$ components are set to zero.

In turn, the eigenvectors of matrix $\mathbf{F}'\mathbf{F}$ constitute an orthonormal basis of M-dimensional vector space that could serve as eigenbasis for rectangular input-output model at $M>N$ (i.e., the number of industries exceeds the number of products). Being transformed with respect to this basis, matrices $\mathbf{X}$ and $\mathbf{Z}$ have $M-N$ right columns coincided between each other (with zero value added in the last $M-N$ industries). Thus, rectangular input-output table written in the coordinates with respect to the eigenvectors of matrix $\mathbf{F}'\mathbf{F}$ can be used as operational supply-driven input-output model in which N components of value added are exogenous variables and the other $M-N$ components are set to zero.

The analytical opportunities of practical applying the proposed models are slightly limited because of explicit shortage of exogenous variables. Nevertheless, it is shown that the models appear to be a useful additional toolbox to regular computational schemes of input-output analysis. Their main advantage is direct handling the initial rectangular input-output table without obvious data distortion being entailed by transformations the table to symmetric format under various assumptions.

Keywords: rectangular input-output table, exogenous changes in final demand and value added, eigenvalues and eigenvectors, demand-driven and supply-driven models, eigenbasis for rectangular input-output model

JEL Classification: C67; D57

1. Linear input-output model: a general formulation

The general linear input-output model of an economy with N products (commodities) and M industries (sectors) for the certain time period leans on a pair of rectangular matrices, namely supply (production) matrix **X** and use for intermediates (intermediate consumption) matrix **Z** of the same dimension $N\times M$ both. In mathematical notation, the model includes the vector equation for material balance of products' intermediate and final uses, i.e.,

$$\mathbf{X}\mathbf{e}_M = \mathbf{Z}\mathbf{e}_M + \mathbf{y}, \tag{1}$$

and the following vector equation for financial balance of industries' intermediate and primary (combined into value added) inputs:

$$\mathbf{e}'_N\mathbf{X} = \mathbf{e}'_N\mathbf{Z} + \mathbf{v}' \tag{2}$$

where $\mathbf{e}_N$ and $\mathbf{e}_M$ are $N\times 1$ and $M\times 1$ summation column vectors with unit elements, **y** is a column vector of net final demand with dimensions $N\times 1$, and **v** is a column vector of value added with dimensions $M\times 1$. Here putting a prime after vector's (matrix's) symbol denotes a transpose of this vector (matrix).

The balance model (1), (2) contains N+M linear equations with $2NM + N$+M scalar variables. Hence, for exact identifiability of the model it is required to include in it 2NM additional independent equations describing established, presumed and exogenous linkages between the variables.

"One of the major uses of the information in an input-output model is to assess the effect on an economy of changes in elements that are exogenous to the model of that economy" (Miller and Blair, 2009, p. 243). To measure the changes mentioned above, in most practical cases there usually is the supply and use table for economy under consideration for some time period (say, period 0) compiled from available statistical data. This table includes the production matrix X_0 and intermediate consumption matrix $\mathbf{Z}_0$ with dimensions $N\times M$, ($N\times 1$)-dimensional column vector of net final demand $\mathbf{y}_0$, and ($M\times 1$)-dimensional column vector of value added $\mathbf{v}_0$ (see Eurostat, 2008). Note that the equations (1) and (2) are exactly met for the initial supply and use table components. The structure of initial supply and use table serves as an informational framework for constructing the additional linkage equations in general linear input-output model (1), (2).

With accordance to the quotation above, one of the main aims of constructing input-output models is to assess an impact of the exogenous changes (either absolute or relative) in net final demand and, by virtue of evident symmetry in the balance equations under consideration, an impact of the exogenous changes in gross value added on simultaneous behavior of the economy as a whole and its industries.

2. The price and quantity transformations of the model variables

In principle, any finite variations in exogenous elements of the input-output model (1), (2) lead to the changes of price and quantity proportions in the resulting (i.e., disturbed) supply and use table. The most general way to describe an impact of these changes on matrices $\mathbf{X}$ and $\mathbf{Z}$ is as follows:

$$\mathbf{X} = \mathbf{P_X} \circ \mathbf{Q_X} \circ \mathbf{X}_0, \qquad \mathbf{Z} = \mathbf{P_Z} \circ \mathbf{Q_Z} \circ \mathbf{Z}_0$$

where $\mathbf{P_X}$ and $\mathbf{P_Z}$ are $N\times M$-dimensional matrices of the relative price indices for products, $\mathbf{Q_X}$ and $\mathbf{Q_Z}$ are $N\times M$ matrices of the relative quantity (physical volume) indices for industries of the economy, and the character " $\circ$ " denotes the Hadamard's (element-wise) product of two matrices with the same dimensions.

Following Motorin (2017), one can assume that in market economy $\mathbf{P_X} = \mathbf{P_Z} = \mathbf{P}$, and $\mathbf{Q_X} = \mathbf{Q_Z} = \mathbf{Q}$ on the current level of production. Besides, it is quite natural to propose also that the price on certain product does not vary along the row of producing-and-consuming industries, i.e., $p_{nm} = p_n$ for all $m = 1\div M$ at $n = 1\div N$ where the character " $\div$ " between the lower and upper bounds of index's changing range means that the index sequentially runs all integer values in the specified range, and, moreover, that the production quantity index for the certain industry's output and intermediate consumption is keeping invariable through all products produced and consumed, namely, $q_{nm} = q_m$ for all $n = 1\div N$ at $m = 1\div M$.

Thus, matrices $\mathbf{P}$ and $\mathbf{Q}$ can be represented respectively as $\mathbf{P} = \mathbf{p} \otimes \mathbf{e}'_M$ and $\mathbf{Q} = \mathbf{e}_N \otimes \mathbf{q}'$ where $\mathbf{p}$ is a column vector of the relative price indices on products with dimensions $N\times 1$, $\mathbf{q}$ is a column vector of the relative quantity indices for industries with dimensions $M\times 1$, and the character " $\otimes$ " denotes the Kronecker product for two matrices.

Transforming the above statements into regular matrix notation gives two nonlinear multiplicative patterns

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0\hat{\mathbf{q}} \tag{3}$$

where putting a "hat" over vector's symbol (or angled bracketing around it) denotes a diagonal matrix with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697). The patterns (3) provide the combined price and quantity description of an economy response to exogenous changes in the input-output model's variables, inter alia, in net final demand and in gross value added.

The nonlinear multiplicative patterns (3) generate a nonlinear problem of input-output analysis as follows:

$$\hat{\mathbf{p}}\mathbf{X}_0\mathbf{q} = \hat{\mathbf{p}}\mathbf{Z}_0\mathbf{q} + \mathbf{y}, \qquad \mathbf{p}'\mathbf{X}_0\hat{\mathbf{q}} = \mathbf{p}'\mathbf{Z}_0\hat{\mathbf{q}} + \mathbf{v}'.$$

Note that here the unknown vectors $\mathbf{p}$ and $\mathbf{q}$ cannot be estimated unambiguously as the functions of exogenous vectors $\mathbf{y}$ and $\mathbf{v}$ because the multiplicative patterns (3) are hyperbolically homogeneous, since $\mathbf{X} = \mathbf{pq}' \circ \mathbf{X}_0$, $\mathbf{Z} = \mathbf{pq}' \circ \mathbf{Z}_0$, and $\mathbf{pq}' = c\mathbf{p} \cdot \mathbf{q}'/c$ for any nonzero scalar c.

Nevertheless, evaluating of input-output model (1), (2) in terms of the production quantity changing at constant prices on the products and/or in terms of price changing at constant level of production in the industries is of great theoretical and practical interest.

3. The general linear input-output models at constant prices and at constant production level

In a case of constant prices on products we have $\hat{\mathbf{p}} = \mathbf{E}_N$ and $\mathbf{p} = \mathbf{e}_N$ where $\mathbf{E}_N$ is identity matrix of order N, and $\mathbf{e}_N$ is $N\times 1$ summation column vector, as earlier, so the nonlinear multiplicative patterns (3) can be rewritten in linear form, namely

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}. \tag{4}$$

Substituting multiplicative patterns (4) in the equations of input-output model (1), (2), we obtain

$$\left(\mathbf{X}_0 - \mathbf{Z}_0\right)\mathbf{q} = \mathbf{y}, \tag{5}$$

$$\left\langle \mathbf{e}'_N\left(\mathbf{X}_0 - \mathbf{Z}_0\right)\right\rangle \mathbf{q} = \mathbf{v} \tag{6}$$

respectively.

Assessing the input-output model (1), (2) at constant level of production in the industries (at $\hat{\mathbf{q}} = \mathbf{E}_M$ and $\mathbf{q} = \mathbf{e}_M$ where $\mathbf{E}_M$ is identity matrix of order M, and $\mathbf{e}_M$ is $M\times 1$ summation column vector, as earlier) leads to following linear patterns

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0. \tag{7}$$

Finally, substituting multiplicative patterns (7) in the equations of input-output model (1), (2), we have

$$\left\langle \left(\mathbf{X}_0 - \mathbf{Z}_0\right)\mathbf{e}_M \right\rangle \mathbf{p} = \mathbf{y}, \tag{8}$$

$$\left(\mathbf{X}'_0 - \mathbf{Z}'_0\right)\mathbf{p} = \mathbf{v} \tag{9}$$

respectively.

4. Exploring a general linear input-output model at constant prices

Consider some operational opportunities in obtaining solutions for the input-output model (5), (6) in the cases of evaluating a response of the economy to exogenous changes in the net final demand vector $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ with dimensions $N\times 1$ or in the value added vector $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ with dimensions $M\times 1$ at constant prices. Here it is assumed that "disturbed" vectors $\mathbf{y}_*$ and $\mathbf{v}_*$ do not have any zero components.

The material balance model (5) contains N linear equations with M unknown scalar variables $\mathbf{q}$, whereas the financial balance model (6) includes *M* linear equations with the same *M* unknowns. Hence, in most general case *N* > = < *M* one can assess a response of the economy only to exogenous change in the value added vector $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ by resolving the equation (6) written as $\langle \mathbf{e}'_N (\mathbf{X}_0 - \mathbf{Z}_0) \rangle \mathbf{q} = \hat{\mathbf{v}}_0 \mathbf{q} = \mathbf{v}_*$ with respect to the column vector of the relative quantity indices for industries, namely

$$\mathbf{q} = \hat{\mathbf{v}}_0^{-1} \mathbf{v}_* . \tag{10}$$

It should be noted that the solution (10) is valid at any numbers of products and industries in the economy. Nevertheless, this common solution is trivial because a response of input-output model (5), (6) to the disturbance $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ comes to the alternate multiplying the columns of production and intermediate consumption matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ on the growth indices of value added through all industries at constant prices on the products.

However, at *N* = *M* = *J* the choice of alternative exogenous condition is also feasible in finding a supplementary solution for the model (5), (6). Under the exogenous final demand condition $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$, the equation (5) written as $(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{q} = \mathbf{y}_*$ can be resolved with respect to the column vector of the relative quantity indices for industries, namely

$$\mathbf{q} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}_* , \tag{11}$$

of course, provided that inverse of the square matrix $\mathbf{X}_0 - \mathbf{Z}_0$ of order *K* exists as it is expected to be. (Note that initial production matrix $\mathbf{X}_0$ usually has the dominant main diagonal, so one can assume it nondegenerate.) The supplementary solution (11) is valid only if the values of *N* and *M* coincide, but it is not trivial in contrast to common solution (10).

The model (5), (6) with its supplementary solution (11) at *N* = *M* = *J* describes an impact of exogenous changes in final demand in terms of a production quantity changing at constant prices on the products and can be considered as a generalized version of well-known Leontief demand-driven model (see Miller and Blair, 2009, Section 2.2.2). In accordance with the multiplicative patterns (4), the total requirements matrix, which links the vector of product outputs with the final demand vector, can be derived as follows:

$$\mathbf{X}\mathbf{e}_J = \mathbf{X}_0 \mathbf{q} = \mathbf{X}_0 (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}_* = \left[(\mathbf{X}_0 - \mathbf{Z}_0) \mathbf{X}_0^{-1} \right]^{-1} \mathbf{y}_* = \left(\mathbf{E}_J - \mathbf{Z}_0 \mathbf{X}_0^{-1} \right)^{-1} \mathbf{y}_* . \tag{12}$$

Note that generalized form of Leontief technical coefficients $\mathbf{Z}_0 \mathbf{X}_0^{-1}$ have been explored by Jansen and ten Raa (1990) and other authors in the context of constructing symmetric input-output tables; this form of technical coefficients is known as commodity technology model. Thus, any transformation of the square input-output model at constant prices implies an input-output data changing within a product technology paradigm (see Model A in Eurostat, 2008; see also Motorin, 2019).

5. Exploring a general linear input-output model at constant production level

In its turn, consider operational opportunities in getting solutions for the input-output model (8), (9) in the cases of evaluating a response of the economy to exogenous changes in the final demand vector $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ or in the value added vector $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ at constant level of production.

The material balance model (8) contains N linear equations with N unknown scalar variables $\mathbf{p}$, whereas the financial balance model (9) includes M linear equations with the same N unknowns. Hence, in a general case $N >=< M$ one can evaluate a response of the economy only to exogenous change in the final demand vector $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ by resolving the equation (8) written as $\langle (\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_M \rangle \mathbf{p} = \hat{\mathbf{y}}_0 \mathbf{p} = \mathbf{y}_*$ with respect to the column vector of the relative price indices on products, namely

$$\mathbf{p} = \hat{\mathbf{y}}_0^{-1} \mathbf{y}_* . \tag{13}$$

The solution (13) is valid at any numbers of products and industries in the economy. Nevertheless, this common solution is trivial because a response of input-output model (8), (9) to the disturbance $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ comes to the alternate multiplying the rows of production and intermediate consumption matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ on the value indices of final demand through all products at constant level of production in the industries.

However, at $N = M = J$ the choice of alternative exogenous condition is also feasible in finding a supplementary solution for the model (8), (9). Under the exogenous value added condition $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$, the equation (9) written as $(\mathbf{X}'_0 - \mathbf{Z}'_0)\mathbf{p} = \mathbf{v}_*$ can be resolved with respect to the column vector of the relative price indices on products, namely

$$\mathbf{p} = (\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1} \mathbf{v}_* , \tag{14}$$

of course, provided that inverse of the square (at $N = M = J$) matrix $\mathbf{X}'_0 - \mathbf{Z}'_0$ exists. (Recall that initial production matrix $\mathbf{X}_0$ usually has the dominant main diagonal.) The supplementary solution (14) is valid only if the values of N and M coincide, but in contrast to the common solution (13), it is not trivial.

The model (8), (9) with its supplementary solution (14) describes an impact of exogenous changes in value added in terms of a price changing at constant level of production in the industries and can be classified as a generalized version of Ghosh supply-driven model (see Miller and Blair, 2009, Section 12.1). According to the multiplicative patterns (4), Ghosh analogue of total requirements matrix, which links the vector of industry outputs with the value added vector, can be derived as follows:

$$\mathbf{X}'\mathbf{e}_J = \mathbf{X}_0'\mathbf{p} = \mathbf{X}_0'(\mathbf{X}_0' - \mathbf{Z}_0')^{-1}\mathbf{v}_* = \left[(\mathbf{X}_0' - \mathbf{Z}_0')(\mathbf{X}_0')^{-1}\right]^{-1}\mathbf{v}_* = \left[\mathbf{E}_J - \mathbf{Z}_0'(\mathbf{X}_0')^{-1}\right]^{-1}\mathbf{v}_* . \quad (15)$$

This generalized form of Ghosh allocation coefficients is dual to commodity technology model. Besides, it is interesting here to pay attention to the fact that models (5), (6) and (8), (9) do demonstrate remarkable duality properties in pairwise comparison of the common solutions (10) and (13) at any values N and M as well as the supplementary solution (11) with total requirements matrix (12) and the supplementary solution (14) with total requirements matrix (15) at $N = M = J$, respectively.

6. Further opportunities for analytical employing the general linear input-output models

It is shown above that general input-output models (5), (6) and (8), (9) have trivial common solutions and nontrivial solutions in particular case $N = M = J$. Then the quite natural question arises: is it possible to provide an operational use of these models for the purposes of input-output analysis in any other cases and how to do it?

Note that the key equations (5) and (9) of the general input-output models at constant prices and at constant production level can be written in unified form, namely

$$\mathbf{F}\mathbf{q} = \mathbf{y}_* , \qquad \mathbf{F}'\mathbf{p} = \mathbf{v}_*$$

where $\mathbf{F} = \mathbf{X}_0 - \mathbf{Z}_0$ is rectangular matrix with dimensions $N\times M$. Let us introduce into consideration the square matrix $\mathbf{F}\mathbf{F}' = (\mathbf{X}_0 - \mathbf{Z}_0)(\mathbf{X}_0 - \mathbf{Z}_0)'$ of order N and the square matrix $\mathbf{F}'\mathbf{F} = (\mathbf{X}_0 - \mathbf{Z}_0)'(\mathbf{X}_0 - \mathbf{Z}_0)$ of order M. It can be easily shown that they both are symmetric and have the same spectrum of nonzero real eigenvalues.

Let $\mathbf{D}_K$ and $\mathbf{D}_L$ be the square matrices of orders $K = \max\{N, M\}$ and $L = \min\{N, M\}$, respectively, defined as follows:

$$\mathbf{D}_K = \begin{cases} \mathbf{F}\mathbf{F}', & \text{if } K = N; \\ \mathbf{F}'\mathbf{F}, & \text{if } K = M; \end{cases} \qquad \mathbf{D}_L = \begin{cases} \mathbf{F}'\mathbf{F}, & \text{if } L = M; \\ \mathbf{F}\mathbf{F}', & \text{if } L = N. \end{cases} \quad (16)$$

It is easy to see that both matrices are of the same rank $L < K$, so matrix $\mathbf{D}_K$ of higher order K is degenerate.

Symmetric matrix $\mathbf{D}_L$ of order $L < K$ has L nonzero real eigenvalues that could be assembled into $(L\times 1)$-dimensional column vector $\boldsymbol{\lambda}_L$. In turn, symmetric matrix $\mathbf{D}_K$ has the same L nonzero real eigenvalues and also $K - L$ zero eigenvalues. Thus, $(1\times K)$-dimensional row vector of $\mathbf{D}_K$'s eigenvalues is equal to $\boldsymbol{\lambda}_K' = (\boldsymbol{\lambda}_L', \mathbf{0}_{K-L}')$ where $\mathbf{0}_{K-L}'$ is the null row vector with dimensions $1\times(K-L)$.

The eigenvectors of symmetric matrix $\mathbf{D}_K$ form an orthonormal basis of K-dimensional vector space. Let $\mathbf{S}_K$ be a square matrix of order K whose columns coincide with the eigenvectors of matrix $\mathbf{D}_K$. They are pairwise orthogonal to each other, i.e. $\mathbf{S}'_K\mathbf{S}_K = \mathbf{E}_K$. Hence, the inverse of $\mathbf{S}_K$ can be derived as $\mathbf{S}_K^{-1} = \mathbf{S}'_K$ (in linear algebra such matrices are called orthogonal). By definition of an eigenvector we have $\mathbf{D}_K\mathbf{S}_K = \mathbf{S}_K\hat{\boldsymbol{\lambda}}_K$ from which

$$\mathbf{S}'_K\mathbf{D}_K\mathbf{S}_K = \hat{\boldsymbol{\lambda}}_K . \tag{17}$$

If the number of products exceeds the number of industries as it often happens in statistical practice (i.e., at $K = N > M$), it follows from (16) that $\mathbf{D}_K = \mathbf{F}\mathbf{F}' = (\mathbf{X}_0 - \mathbf{Z}_0)(\mathbf{X}_0 - \mathbf{Z}_0)'$. The eigenvectors of matrix **FF**' form an orthonormal basis of N-dimensional vector space that could be considered as eigenbasis for rectangular input-output model at constant prices. Transformed into the coordinates with respect to this eigenbasis, matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ become $\mathbf{S}'_K\mathbf{X}_0$ and $\mathbf{S}'_K\mathbf{Z}_0$ and according to (17) have N–M lower rows coincided between each other (with zero final demand for the last N–M products). This property allows employing rectangular input-output table written in the coordinates with respect to the eigenvectors of matrix **FF**' as operational demand-driven input-output model in which M components of final demand are exogenous variables and the other N–M components are set to zero.

In turn, if the number of industries exceeds the number of products (i.e., at $K = M > N$), formula (16) gives $\mathbf{D}_K = \mathbf{F}'\mathbf{F} = (\mathbf{X}_0 - \mathbf{Z}_0)'(\mathbf{X}_0 - \mathbf{Z}_0)$. The eigenvectors of matrix **F'F** constitute an orthonormal basis of M-dimensional vector space that could serve as eigenbasis for rectangular input-output model at constant level of production. Transformed into the coordinates with respect to this eigenbasis, matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ become $\mathbf{X}_0\mathbf{S}_K$ and $\mathbf{Z}_0\mathbf{S}_K$ and according to (17) have M–N right columns coincided between each other (with zero value added for the last M–N industries). Thus, rectangular input-output table written in the coordinates with respect to the eigenvectors of matrix **F'F** can be used as operational supply-driven input-output model in which N components of value added are exogenous variables and the other M–N components are set to zero.

7. Numerical example for the linear input-output model at constant prices

To illustrate a proposed approach to analytical employing the general linear input-output model at constant prices, consider the following initial supply and use table for the economy with $N = 5$ products and $M = 3$ industries (at $K = N > M$):

	$\mathbf{Z}_0$			$\mathbf{Z}_0\mathbf{e}_5$		$\mathbf{X}_0$			$\mathbf{X}_0\mathbf{e}_3$	$\mathbf{y}_0$
	20	0	10	30		60	0	0	60	**30**
	34	152	72	258		80	230	0	310	**52**
	0	50	0	50		0	60	30	90	**40**
	36	188	98	322		0	190	210	400	**78**
	10	15	0	25		0	10	30	40	**15**
$\mathbf{e}_5'\mathbf{Z}_0$	100	405	180	**685**	$\mathbf{e}_5'\mathbf{X}_0$	140	490	270	**900**	**215**
					$\mathbf{v}_0'$	**40**	**85**	**90**	**215**	

Here $K = \max\{5, 3\} = 5$, $L = \min\{5, 3\} = 3$ and

$$\mathbf{D}_{K=5} = \mathbf{FF}' = (\mathbf{X}_0 - \mathbf{Z}_0)(\mathbf{X}_0 - \mathbf{Z}_0)' = \begin{bmatrix} 1700 & 2560 & -300 & -2560 & -700 \\ 2560 & 13384 & -1380 & -9564 & -3010 \\ -300 & -1380 & 1000 & 3380 & 850 \\ -2560 & -9564 & 3380 & 13384 & 3710 \\ -700 & -3010 & 850 & 3710 & 1025 \end{bmatrix},$$

$$\mathbf{D}_{L=3} = \mathbf{F}'\mathbf{F} = (\mathbf{X}_0 - \mathbf{Z}_0)'(\mathbf{X}_0 - \mathbf{Z}_0) = \begin{bmatrix} 5112 & 3566 & -8044 \\ 3566 & 6213 & -5242 \\ -8044 & -5242 & 19628 \end{bmatrix}, \qquad \boldsymbol{\lambda}_{L=3} = \begin{bmatrix} 25254.218 \\ 4549.455 \\ 1149.327 \end{bmatrix},$$

$$\boldsymbol{\lambda}_{K=5} = \begin{bmatrix} 25254.218 \\ 4549.455 \\ 1149.327 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{S}_{K=5} = \left[\begin{array}{c|c|c|c|c} -0.154795 & 0.048728 & 0.971194 & 0.161103 & 0.067017 \\ -0.659364 & 0.722569 & -0.159281 & 0.126081 & -0.043208 \\ 0.143082 & 0.314440 & 0.169888 & -0.903579 & -0.187983 \\ 0.694066 & 0.607323 & 0.016691 & 0.266171 & 0.279832 \\ 0.197681 & 0.088354 & 0.047654 & 0.266171 & -0.938082 \end{array}\right]$$

It is easy to check that equation (17) is met, namely, $\mathbf{S}'_{K=5}\mathbf{D}_{K=5} = \hat{\boldsymbol{\lambda}}_{K=5}\mathbf{S}'_{K=5}$. Transforming initial supply and use table into the coordinates with respect to eigenbasis (constituted by columns of matrix $\mathbf{S}_{K=5}$) gives the following supply and use table:

	$\mathbf{S}'_{K=5}\mathbf{Z}_0$			$\mathbf{S}'_{K=5}\mathbf{Z}_0\mathbf{e}_3$		$\mathbf{S}'_{K=5}\mathbf{X}_0$			$\mathbf{S}'_{K=5}\mathbf{X}_0\mathbf{e}_3$	$(\mathbf{y}_0)_{\text{EB}}$
	1.45	***40.38***	***19.00***	60.83		***-62.04***	***-9.22***	***155.98***	84.72	***23.89***
	48.29	***241.05***	***112.03***	401.37		***60.73***	***301.33***	***139.62***	501.68	***100.31***
	15.09	***-11.86***	***-0.12***	3.10		***45.53***	***-22.79***	***10.03***	32.77	***29.67***
	19.75	28.02	36.77	84.54		19.75	28.02	36.77	84.54	**0.00**
	0.56	22.57	24.98	48.12		0.56	22.57	24.98	48.12	**0.00**
$\mathbf{e}_5'\mathbf{S}'_{K=5}\mathbf{Z}_0$	85.14	320.16	192.66	**597.96**	$\mathbf{e}_5'\mathbf{S}'_{K=5}\mathbf{X}_0$	64.54	319.91	367.39	**751.83**	**153.87**
					$(\mathbf{v}_0)'_{\text{EB}}$	**-20.60**	**-0.25**	**174.72**	**153.87**	

Indeed, matrices $\mathbf{S}'_{K-5}\mathbf{X}_0$ and $\mathbf{S}'_{K=5}\mathbf{Z}_0$ have N–M=5–3=2 lower rows coincided between each other (with zero final demand for the last N–M=2 products); in the table these rows are slightly darkened. (Besides, subscript "EB" in $(\mathbf{y}_0)_{\text{EB}}$ and $(\mathbf{v}_0)'_{\text{EB}}$ means the initial final demand and value added vectors $\mathbf{y}_0$ and $\mathbf{v}_0'$ written in the coordinates with respect to eigenbasis.)

Let $(\Delta\mathbf{y})_{\text{EB}} = (1,1,1)'$ be the column vector of the unit changes of final demand on product

1, 2 and 3 in the new classification (blue box in the latter table) so that

$$(\mathbf{y}_*)_{\mathrm{EB}} = (\mathbf{y}_0)_{\mathrm{EB}} + (\Delta\mathbf{y})_{\mathrm{EB}} = \begin{bmatrix} 23.89 \\ 100.31 \\ 29.67 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 24.89 \\ 101.31 \\ 30.67 \end{bmatrix}$$

Applying supplementary solution (11) for 3-dimensional square demand-driven input-output model (5), (6) at constant prices (see green boxes in the latter table) we obtain the quantity index vector

$$\mathbf{q}_{\mathrm{EB}} = \left(\begin{bmatrix} -62.04 & -9.22 & 155.98 \\ 60.73 & 301.33 & 139.62 \\ 45.53 & -22.79 & 10.03 \end{bmatrix} - \begin{bmatrix} 1.45 & 40.38 & 19.00 \\ 48.29 & 241.05 & 112.03 \\ 15.09 & -11.86 & -0.12 \end{bmatrix} \right)^{-1} \begin{bmatrix} 24.89 \\ 101.31 \\ 30.67 \end{bmatrix} = \begin{bmatrix} 1.026709 \\ 1.001776 \\ 1.020322 \end{bmatrix}$$

and disturbed supply and use table (written over eigenbasis)

	$\mathbf{S}'_{K=5}\mathbf{Z}_0\hat{\mathbf{q}}_{\mathrm{EB}}$					$\mathbf{S}'_{K=5}\mathbf{X}_0\hat{\mathbf{q}}_{\mathrm{EB}}$				$(\mathbf{y}_*)_{\mathrm{EB}}$
	1.488	40.452	19.382	61.322		-63.694	-9.236	159.146	86.217	**24.895**
	49.579	241.482	114.307	405.368		62.351	301.867	142.459	506.677	**101.310**
	15.489	-11.885	-0.123	3.481		46.745	-22.834	10.235	34.146	**30.665**
	20.280	28.068	37.521	85.869		20.280	28.068	37.521	85.869	**0.000**
	0.579	22.610	25.490	48.680		0.579	22.610	25.490	48.680	**0.000**
	87.415	320.728	196.577	**604.720**		66.262	320.475	374.852	**761.590**	**156.870**
					$(\mathbf{v})'_{\mathrm{EB}}$	**-21.152**	**-0.253**	**178.275**	**156.870**	

Disturbed supply and use table (written over natural basis) is

	$\mathbf{Z}_0\hat{\mathbf{q}}_{\mathrm{EB}}$			$\mathbf{Z}_0\mathbf{q}_{\mathrm{EB}}$		$\mathbf{X}_0\hat{\mathbf{q}}_{\mathrm{EB}}$			$\mathbf{X}_0\mathbf{q}_{\mathrm{EB}}$	$\mathbf{y}$
	20.534	0.000	10.203	30.737		61.603	0.000	0.000	61.603	**30.865**
	34.908	152.270	73.463	260.641		82.137	230.408	0.000	312.545	**51.904**
	0.000	50.089	0.000	50.089		0.000	60.107	30.610	90.716	**40.627**
	36.962	188.334	99.992	325.287		0.000	190.337	214.268	404.605	**79.318**
	10.267	15.027	0.000	25.294		0.000	10.018	30.610	40.627	**15.334**
$\mathbf{e}'_N\mathbf{Z}_0\hat{\mathbf{q}}_{\mathrm{EB}}$	102.671	405.719	183.658	**692.048**	$\mathbf{e}'_N\mathbf{X}_0\hat{\mathbf{q}}_{\mathrm{EB}}$	143.739	490.870	275.487	**910.096**	**218.048**
					$\mathbf{v}'$	**41.068**	**85.151**	**91.829**	**218.048**	

8. Numerical example for the linear input-output model at constant production level

To illustrate a proposed approach to analytical employing the general linear input-output model at constant level of production, consider the following initial supply and use table for the economy with $N = 3$ products and $M = 5$ industries (at $K = M > N$):

$\mathbf{Z}_0$					$\mathbf{Z}_0\mathbf{e}_5$	$\mathbf{X}_0$					$\mathbf{X}_0\mathbf{e}_5$	$\mathbf{y}_0$
20	34	0	36	10	100	60	80	0	0	0	140	**40**
0	152	50	188	15	405	0	230	60	190	10	490	**85**
10	72	0	98	0	180	0	0	30	210	30	270	**90**
30	258	50	322	25	**685**	60	310	90	400	40	**900**	**215**
					$\mathbf{v}'_0$	**30**	**52**	**40**	**78**	**15**	**215**	

Here $K = \max\{3, 5\} = 5$, $L = \min\{3, 5\} = 3$ and

$$\mathbf{D}_{K=5}=\mathbf{F}'\mathbf{F}=(\mathbf{X}_0-\mathbf{Z}_0)'(\mathbf{X}_0-\mathbf{Z}_0)=\begin{bmatrix} 1700 & 2560 & -300 & -2560 & -700 \\ 2560 & 13384 & -1380 & -9564 & -3010 \\ -300 & -1380 & 1000 & 3380 & 850 \\ -2560 & -9564 & 3380 & 13384 & 3710 \\ -700 & -3010 & 850 & 3710 & 1025 \end{bmatrix},$$

$$\mathbf{D}_{L=3}=\mathbf{F}\mathbf{F}'=(\mathbf{X}_0-\mathbf{Z}_0)(\mathbf{X}_0-\mathbf{Z}_0)'=\begin{bmatrix} 5112 & 3566 & -8044 \\ 3566 & 6213 & -5242 \\ -8044 & -5242 & 19628 \end{bmatrix}, \qquad \boldsymbol{\lambda}_{L=3}=\begin{bmatrix} 25254.218 \\ 4549.455 \\ 1149.327 \end{bmatrix},$$

$$\boldsymbol{\lambda}_{K=5}=\begin{bmatrix} 25254.218 \\ 4549.455 \\ 1149.327 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{S}_{K=5}=\left[\begin{array}{c|c|c|c|c} -0.154795 & 0.048728 & 0.971194 & 0.161103 & 0.067017 \\ -0.659364 & 0.722569 & -0.159281 & 0.126081 & -0.043208 \\ 0.143082 & 0.314440 & 0.169888 & -0.903579 & -0.187983 \\ 0.694066 & 0.607323 & 0.016691 & 0.266171 & 0.279832 \\ 0.197681 & 0.088354 & 0.047654 & 0.266171 & -0.938082 \end{array}\right]$$

It is easy to check that equation (17) is met, namely, $\mathbf{D}_{K=5}\mathbf{S}_{K=5}=\mathbf{S}_{K=5}\hat{\boldsymbol{\lambda}}_{K=5}$ nearly resembling as it was done in previous example. Transforming initial supply and use table into the coordinates with respect to eigenbasis (constituted by columns of matrix $\mathbf{S}_{K=5}$) gives the following supply and use table:

$\mathbf{Z}_0\mathbf{S}_{K=5}$					$\mathbf{Z}_0\mathbf{S}_{K=5}\mathbf{e}_5$	$\mathbf{X}_0\mathbf{S}_{K=5}$					$\mathbf{X}_0\mathbf{S}_{K=5}\mathbf{e}_5$	$(\mathbf{y}_0)_{\mathrm{EB}}$
1.45	*48.29*	*15.09*	19.75	0.56	85.14	*-62.04*	*60.73*	*45.53*	19.75	0.56	64.54	**-20.60**
40.38	*241.05*	*-11.86*	28.02	22.57	320.16	*-9.22*	*301.33*	*-22.79*	28.02	22.57	319.91	**-0.25**
19.00	*112.03*	*-0.12*	36.77	24.98	192.66	*155.98*	*139.62*	*10.03*	36.77	24.98	367.39	**174.72**
60.83	401.37	3.10	84.54	48.12	**597.96**	84.72	501.68	32.77	84.54	48.12	**751.83**	**153.87**
					$(\mathbf{v}_0)'_{\mathrm{EB}}$	***23.89***	***100.31***	***29.67***	**0.00**	**0.00**	**153.87**	

Indeed, matrices $\mathbf{X}_0\mathbf{S}_{K-5}$ and $\mathbf{Z}_0\mathbf{S}_{K=5}$ have *M*–*N*=5–3=2 right columns coincided between each other (with zero value added for the last *M*–*N* =2 industries); in the table these columns are slightly darkened. (As earlier, subscript "EB" in $(\mathbf{y}_0)_{\mathrm{EB}}$ and $(\mathbf{v}_0)'_{\mathrm{EB}}$ means the initial final demand and value added vectors $\mathbf{y}_0$ and $\mathbf{v}'_0$ written in the coordinates with respect to eigenbasis.)

Let $(\Delta\mathbf{v})'_{\mathrm{EB}}=(1,1,1)$ be the row vector of the unit changes of value added in industry 1, 2 and 3 in the new classification (blue box in the latter table) so that

$$(\mathbf{v}_*)_{\mathrm{EB}}=(\mathbf{v}_0)_{\mathrm{EB}}+(\Delta\mathbf{v})_{\mathrm{EB}}=\begin{bmatrix} 23.89 \\ 100.31 \\ 29.67 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 24.89 \\ 101.31 \\ 30.67 \end{bmatrix}$$

Applying supplementary solution (14) for 3-dimensional square supply-driven input-output model (8), (9) at constant level of production (see green boxes in the latter table) we obtain the price index vector

$$\mathbf{p}_{\mathrm{EB}} = \left(\begin{bmatrix} -62.04 & 60.73 & 45.53 \\ -9.22 & 301.33 & -22.79 \\ 155.98 & 139.62 & 10.03 \end{bmatrix}' - \begin{bmatrix} 1.45 & 48.29 & 15.09 \\ 40.38 & 241.05 & -11.86 \\ 19.00 & 112.03 & -0.12 \end{bmatrix}' \right)^{-1} \begin{bmatrix} 24.89 \\ 101.31 \\ 30.67 \end{bmatrix} = \begin{bmatrix} 1.026709 \\ 1.001776 \\ 1.020322 \end{bmatrix}$$

and disturbed supply and use table (written over eigenbasis)

$\hat{\mathbf{p}}_{\mathrm{EB}}\mathbf{Z}_0\mathbf{S}_{K=5}$						$\hat{\mathbf{p}}_{\mathrm{EB}}\mathbf{X}_0\mathbf{S}_{K=5}$						$(\mathbf{y})_{\mathrm{EB}}$
1.488	49.579	15.489	20.280	0.579	87.415	-63.694	62.351	46.745	20.280	0.579	66.262	**-21.152**
40.452	241.482	-11.885	28.068	22.610	320.728	-9.236	301.867	-22.834	28.068	22.610	320.475	**-0.253**
19.382	114.307	-0.123	37.521	25.490	196.577	159.146	142.459	10.235	37.521	25.490	374.852	**178.275**
61.322	405.368	3.481	85.869	48.680	**604.720**	86.217	506.677	34.146	85.869	48.680	**761.590**	**156.870**
					$(\mathbf{v}_*)'_{\mathrm{EB}}$	**24.895**	**101.310**	**30.665**	**0.000**	**0.000**	**156.870**	

Disturbed supply and use table (written over natural basis) is

$\hat{\mathbf{p}}_{\mathrm{EB}}\mathbf{Z}_0$						$\hat{\mathbf{p}}_{\mathrm{EB}}\mathbf{X}_0$						$\mathbf{y}$
20.534	34.908	0.000	36.962	10.267	102.671	61.603	82.137	0.000	0.000	0.000	143.739	**41.068**
0.000	152.270	50.089	188.334	15.027	405.719	0.000	230.408	60.107	190.337	10.018	490.870	**85.151**
10.203	73.463	0.000	99.992	0.000	183.658	0.000	0.000	30.610	214.268	30.610	275.487	**91.829**
30.737	260.641	50.089	325.287	25.294	**692.048**	61.603	312.545	90.716	404.605	40.627	**910.096**	**218.048**
					$\mathbf{v}'$	**30.865**	**51.904**	**40.627**	**79.318**	**15.334**	**218.048**	

9. Concluding remarks

The opportunities of practical applying the proposed approach to analytical employing the general linear input-output models at constant prices and at constant level of production are slightly limited because of explicit shortage of exogenous variables. Nevertheless, it is shown that the models appear to be a useful additional toolbox to regular computational schemes of input-output analysis. Their main advantage is direct handling the initial rectangular input-output table without obvious data distortion being entailed by transformations the table to symmetric format under various assumptions.

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