Thirlwall's Law and uneven development under Global Value Chains: a multi-country input-output approach

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Abstract
A notable macroeconomic explanation of uneven development, with particular relevance to developing countries, has been the problem of balance of payments constraints, as captured by Thirlwall's Law: where relative growth rates are explained by differences between income elasticities for exports and imports. Araujo and Lima have developed a one-country multisectoral disaggregation of this hypothesis using a vertically integrated input-output framework, which is extended here in two main ways. First, international trade in intermediate inputs – the basis for Global Value Chains – is introduced; second, the model is extended to multiple countries. The main outcome of the paper is the development of a new multisectoral method for modelling balance of payments constraints: a Multi Country Sectoral Thirlwall Law (MCSTL) under which key sector relationships are nested in intercountry trading relationships that encompass both intermediate and final goods. The identification of this input-output structure is developed in analytical stages, moving from a one-country vertically integrated system, to two, three and finally multi-country systems. In addition to its theoretical contribution to understanding the industrial structure of trade, an implication of this multi-country/multi-sector approach is that it can also be tested in future empirical work using the recently available World Input-Output Database of national tables.

Key words
Uneven development; Thirlwall's Law; Global Value Chains; Input-output analysis; Vertical integration; Trade in Value Added

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1. INTRODUCTION

A notable explanation for uneven development between countries has been the problem of balance of payments constraints, as advanced by Thirlwall (1979). For developing countries in particular, this can undermine growth when low income elasticities for exports combine with high income elasticities for imports, as captured by Thirlwall's Law, for which the vast majority of studies have provided empirical support (see Thirlwall, 2013, Table 5.2). The suggested policy prescription under this balance of payments-constrained growth model is for structural change to promote exports in favour of imports.

This structural change approach has recently been developed with the specification of income elasticities for industrial sectors under the Multisectoral Thirlwall Law (MSTL), first developed by Araujo and Lima (2007), with its starting point the vertically integrated closed input-output model devised by Pasinetti (1993). Pasinetti shows how under vertical integration different stages of production can be succinctly represented in a model of final consumption: a 'genuinely' macroeconomic approach since relationships hold regardless of the degree of disaggregation (*ibid*, p. 20). This somewhat abstract theoretical insight has promulgated an emerging empirical literature on Thirlwall's Law in which import and export elasticities have been estimated using econometric functions at the sectoral level (e.g.: Romero and McCombie 2016; Blecker and Ibarra 2013; Romero, Silveira and Jayme Jr. 2011; Govea and Lima 2010). Valuable insights have been provided into the structure of sectoral elasticities, of particular note being evidence to suggest that some countries should specialise in the production of exports from high tech industrial sectors.

One problem that has emerged in this literature, however, is the role of Global Value Chains (GVCs) in the specification of sectoral income elasticities. Blecker and Ibarra (2013), for example, show in a study of Mexico that income elasticities are biased if the role of intermediate inputs is not taken into account. Since in a global production network

intermediate inputs can be traded multiple times between countries, gross trade flows can fail to locate the original source of value added and overestimate its scale. UNCTAD (2013, x) estimate that such double counting accounts for 28% of global trade. The problem of double counting has been systematically addressed by a recent literature on the use of input-output analysis – a key technique for analysing national accounts under GVCs – to account for Trade in Value Added (see Los *et al*, 2015, Koopman *et al*, 2014; Johnson and Noguera, 2012; Stehrer, 2012, Daudin *et al*, 2001; Hummels *et al*, 2001). By decomposing the input-output accounts, multiple global flows of value added can be traced back to their country of origin. This importance of input-output analysis to modelling GVCs suggests a re-consideration of how Thirlwall'a Law can be modelled using the original input-output approach devised by Araujo and Lima (2007).

Drawing on the largely empirical input-output literature developing the notion of Trade in Value Added, the theoretical contribution of this paper is to generalize the Multisectoral Thirlwall Law in order to take into account the structure of Global Value Chains. A new Multi Country Sectoral Thirlwall Law (MCSTL) is proposed, by extending the one-country Araujo-Lima model to a multi-country input-output framework in which trade in intermediate inputs is fully taken into account. This is a type of multi-regional input-output model, originally pioneered by Isard (1951) and Moses (1955), in which each region is a country.

There is, in addition, a problem that Pasinetti's closed input-output model generates unit income elasticities, which are not suitable for capturing uneven development. To address this issue an open input-output system is established that incorporates autonomous final consumer expenditures.

In order to delineate the structural characteristics and complex patterns of trade associated with Global Value Chains, the analysis here builds up in incremental steps, from

the basic one-country macroeconomic model to two, three and multi-country models. In Section 2, Thirlwall's Law is explained in its basic macroeconomic form, followed in Section 3 by a consideration of its multisectoral foundations in the Araujo-Lima adaptation of Pasinetti's closed input-output model. In Section 4 a two-country extension of Araujo-Lima is developed, incorporating intermediate inputs and autonomous expenditures. This model is used in Section 5 to derive the Multisectoral Thirlwall Law, providing new insights into its structure. Building on these foundations, Section 6 turns to the specification of a three-country system, followed in Section 7 by a derivation of a new Multi Country Sectoral Thirlwall Law. Some concluding remarks are provided in Section 8.

2. THIRLWALL'S LAW

Thirlwall's Law can be introduced for the case of a home economy. The aggregate income (Y) of this home economy consists of consumption (C) and exports (X):

$$Y = C + X \tag{1}$$

By letting imports (M) depend on income such that

$$M = mY \tag{2}$$

and by assuming a balance of payments constraint that requires a balance of trade $(X = M)$ it follows that

$$Y = \frac{1}{m}X \tag{3}$$

Demand for exports is the key driver for income according to $1/m$, the Harrod trade multiplier.

This demand-side approach can be further elaborated using income elasticities. By differentiating (3) with respect to *X* and (2) with respect to *Y*:

$$\frac{dY}{dX} = \frac{1}{m} = \frac{1}{\frac{dM}{dY}} \tag{4}$$

Since $X = M$, the right hand side of (4) can be multiplied by M/Y and the left hand side by X/Y so that

$$dY\frac{X}{Y} = \frac{dX}{\frac{dM}{dY}}\frac{M}{Y} \tag{5}$$

or

$$\frac{dY}{Y} = \frac{dX}{X}\Big/\left(\frac{dM}{M}\Big/\frac{dY}{Y}\right) \tag{6}$$

(see Thirlwall, 2013, p. 121). If Y_f is the income of the foreign country then it also follows that

$$\frac{dY}{Y}\Big/\frac{dY_f}{Y_f} = \left(\frac{dX}{X}\Big/\frac{dY_f}{Y_f}\right)\Big/\left(\frac{dM}{M}\Big/\frac{dY}{Y}\right) \tag{7}$$

or

$$\frac{g_h}{g_f} = \frac{\varepsilon}{\pi} \tag{8}$$

This is Thirlwall's hypothesis, which has come to be known as Thirlwall's Law, under which, with a balance of trade constraint, the relative growth rates of the home country (g_h) and the foreign country (g_f) are governed by the ratio of the income elasticities for exports (ε) and imports (π) (see Thrilwall 2013, p. 85).

3. THE ARAUJO-LIMA MODEL

In an important contribution to understanding how trade and structural change are related, Araujo and Lima (2007) explore the multisectoral foundations of Thirlwall's Law by extending the model of pure labour developed by Pasinetti (1993). This builds on the earlier

work of Araujo and Teixeira (2004), with for simplicity exchange rates not considered for this abstract starting point.

In this Pasinetti model, a key abstraction is that labour is the sole factor of production and provides the only source of demand for its output. Although intermediate inputs can be incorporated into this framework (see Section 4), they are not explicitly modelled in the first instance in order to focus on the structural relationship between output and demand. This stripped down closed input-output model is shown by Pasinetti to allow a clear focus on the relationship between structural change and demand, before introducing further complexities at a more concrete level of analysis.

Core to the Pasinetti model are per capital consumption coefficients (c_i) defined for each industry i for the home country, which has a total employed population of N. Modification of the model to incorporate trade requires the specification of per capita consumption coefficients for exports of each commodity (c_i^e), which depend on employment in the foreign country (N_f). The key innovation is use of an employment ratio, $\xi = N_f/N$, which allows home employment to be transformed into employment in the foreign country. Hence, the physical output (Q_i) of each commodity is directed in part to exports by combining these coefficients such that $\xi c_i^e N = c_i^e N_f$, as shown in the quantity equation

$$Q_i = (c_i + \xi c_i^e)N \tag{9}$$

With l_i representing labour coefficients, for S single-commodity producing industries the level of employment in the home country is specified as

$$N = \sum_{i=1}^{S} l_i Q_i \tag{10}$$

These two equations show the interdependence between production and consumption. In (9), output of each commodity depends on the macroeconomic consumption expenditure of

workers, whose employment depends on sectoral output, as captured in (2). Collecting these elements into $S \times 1$ column vectors of outputs $(\mathbf{Q})$, domestic consumption coefficients $(\mathbf{c})$ and export related consumption coefficients $(\boldsymbol{c}^e)$, together with a $1 \times S$ row vector of labour coefficients $(\mathbf{l})$, the quantity system takes the form:

$$\begin{bmatrix} \mathbf{I} & -(\mathbf{c} + \xi \mathbf{c}^e \\ -\mathbf{l} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{Q} \\ N \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \tag{11}$$

Dual to this quantity system, the Araujo-Lima price system can be established with a row vector of money prices $(\mathbf{p})$ and a money wage rate (w). With $\mathbf{c}^m$ defined as the column vector of consumption coefficients for imports, the price system can be shown in matrix form as

$$[\mathbf{p} \quad w] \begin{bmatrix} \mathbf{I} & -(\mathbf{c} + \mathbf{c}^m \\ -\mathbf{l} & 1 \end{bmatrix} = [\mathbf{0} \quad 0] \tag{12}$$

Comparison of (11) and (12) shows that under duality between price and quantity systems, requiring the same coefficient matrix constituent to each system,

$$\xi \mathbf{c}^e = \mathbf{c}^m \tag{13}$$

Multiplying by N and the vector of money prices:

$$\mathbf{p}\mathbf{c}^e N_f = \mathbf{p}\mathbf{c}^m N \tag{14}$$

This shows that the total money value of exports is equal to the total money value of imports. Under duality, the balance of trade is built into the extended Pasinetti system.

The Araujo-Lima system is an inventive introduction of trade into a Pasinetti based multisectoral framework, but with two main limitations. First, the model focuses mainly on the production activity of the home country, with limited treatment of the production system operating in the foreign country. It is ostensibly a one-country model. This is somewhat restrictive given the increasingly global structure of production networks between industries across multiple countries.

A second problem is that as a closed input-output model it generates unit income elasticities. Consider the money value of total imports:

$$M = \mathbf{p}\mathbf{c}^m N \tag{15}$$

Since from (12) $\mathbf{p} = w\mathbf{l}$, it follows that

$$M = \mathbf{l}\mathbf{c}^m wN \tag{16}$$

In the pure labour economy total income is constituted by wage income, $Y = wN$, such that

$$M = \mathbf{l}\mathbf{c}^m Y \tag{17}$$

Differentiating with respect to income:

$$\frac{dM}{dY} = \mathbf{l}\mathbf{c}^m \tag{18}$$

Hence, under (17) and (18), the income elasticity for imports is

$$\pi = \frac{dM}{dY}\frac{Y}{M} = \frac{\mathbf{l}\mathbf{c}^m Y}{\mathbf{l}\mathbf{c}^m Y} = 1 \tag{19}$$

The income elasticity of imports is unitary. With no autonomous components in this closed input-output system, Engel curves go through the origin, generating unit income elasticities. The income elasticity for exports is also unitary since

$$X = \mathbf{l}\mathbf{c}^e Y_f \tag{20}$$

and

$$\frac{dX}{dY_f} = \mathbf{l}\mathbf{c}^e \tag{21}$$

which yields

$$\varepsilon = \frac{dX}{dY_f}\frac{Y_f}{X} = \frac{\mathbf{l}\mathbf{c}^e Y_f}{\mathbf{l}\mathbf{c}^e Y_f} = 1 \tag{22}$$

Under a balance of trade in the Araujo-Lima model, from (8) Thirlwall's Law generates equal growth rates for the home and foreign countries:

$$\frac{g_h}{g_f} = \frac{\varepsilon}{\pi} = 1 \qquad (23)$$

There is no possibility for uneven growth between the two countries under this closed input-output system. An autonomous component is required in order to specify Engel curves that go through the origin with non-unitary income elasticities. This autonomous component has been introduced in studies inspired by Araujo and Lima (2007) to estimate elasticities using econometric functions at the sectoral level (e.g.: Romero and McCombie 2016), but the objective here is to introduce this autonomous component as an extension of their original input-output formulation.

4. A TWO-COUNTRY EXTENSION

In order to address these issues, the one-country Araujo-Lima model can be extended into an explicitly two-country framework in which countries *A* and *B* each have their own production systems. Each country produces the same *S* commodities in different physical quantities, collected in $S \times 1$ column vectors of gross outputs, $\mathbf{X}^A$ and $\mathbf{X}^B$. These outputs are produced with volumes of labour employed in each country of $N^A$and N^B. Column vectors of per capita consumption coefficients are intra-country, reflecting consumption of domestic commodities that depends on domestic employment ($\mathbf{c}^{AA}$ and $\mathbf{c}^{BB}$) and inter-country, consumed by labour in the other country ($\mathbf{c}^{AB}$ and $\mathbf{c}^{BA}$).

Autonomous expenditure on imports can be introduced in order to transform the Araujo-Lima system from a closed to an open input-output system. The column vector $\bar{\mathbf{Q}}^{AB}$represents autonomous imports of final consumption goods by country *B* from country *A*; the column vector $\bar{\mathbf{Q}}^{BA}$ represents autonomous imports from *B* into *A*. These flows are exogenous to the system, independent of economic activity in the two countries.

An additional extension of the Araujo-Lima system, to make it suitable for the study of Global Value Chains, is in the treatment of technology. With labour coefficients collected in row vectors $\mathbf{l}^A$ and $\mathbf{l}^B$, interindustry technical coefficient matrices, $\mathbf{A}^{AA}$ and $\mathbf{A}^{BB}$, can be defined to capture intermediate input requirements that are produced in each country. Inter-country trade, on the other hand, includes flows of intermediate goods between countries, an increasingly important phenomenon in global production networks. To model this, $\mathbf{A}^{AB}$ is a matrix capturing intermediate inputs imported from country *A* by country *B*; and $\mathbf{A}^{BA}$ is a matrix of intermediate inputs imported from country *B* by country *A*. This use of foreign inputs represents the most general type of Global Value Chain relationship, referred to by Baldwin and Lopez-Gonzalez (2015, 1686) as 'importing to produce'.

We are now able to set up an open two-country input-output model:

$$\begin{bmatrix} \mathbf{I}-\mathbf{A}^{AA} & -\mathbf{A}^{AB} & -\mathbf{c}^{AA} & -\boldsymbol{c}^{AB} \\ -\mathbf{A}^{BA} & \mathbf{I}-\mathbf{A}^{BB} & -\boldsymbol{c}^{BA} & -\boldsymbol{c}^{BB} \\ -\mathbf{l}^A & \mathbf{0} & 1 & 0 \\ \mathbf{0} & -\mathbf{l}^B & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}^A \\ \mathbf{X}^B \\ N^A \\ N^B \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{Q}}^{AB} \\ \bar{\mathbf{Q}}^{BA} \\ 0 \\ 0 \end{bmatrix} \qquad (24)$$

Following the approach developed in Pasinetti's (1981) one-country system, (24) can now be re-configured in vertically integrated form. Column vectors of final outputs, $\mathbf{Q}^A$ and $\mathbf{Q}^B$, are defined net of intermediate inputs for each country:

$$\begin{bmatrix} \mathbf{I}-\mathbf{A}^{AA} & -\mathbf{A}^{AB} \\ -\mathbf{A}^{BA} & \mathbf{I}-\mathbf{A}^{BB} \end{bmatrix} \begin{bmatrix} \mathbf{X}^A \\ \mathbf{X}^B \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^A \\ \mathbf{Q}^B \end{bmatrix} \qquad (25)$$

We can also define a matrix of vertically integrated labour coefficients, which capture the direct and indirect labour required across the two countries:

$$\begin{bmatrix} \mathbf{n}^{AA} & \mathbf{n}^{AB} \\ \mathbf{n}^{BA} & \mathbf{n}^{BB} \end{bmatrix} = \begin{bmatrix} \mathbf{l}^A & \mathbf{0} \\ \mathbf{0} & \mathbf{l}^B \end{bmatrix} \begin{bmatrix} \mathbf{I}-\mathbf{A}^{AA} & -\mathbf{A}^{AB} \\ -\mathbf{A}^{BA} & \mathbf{I}-\mathbf{A}^{BB} \end{bmatrix}^{-1} \qquad (26)$$

The row vector $\mathbf{n}^{AB}$, for example, captures the (direct and indirect) labour embodied in intermediate goods produced in country *A*, per unit of final consumption in country *B*. Using

(25) and (26) the quantity system for the two-country input output model can be shown in vertically integrated form as

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & -\mathbf{c}^{AA} & -\boldsymbol{c}^{AB} \\ \mathbf{0} & \mathbf{I} & -\boldsymbol{c}^{BA} & -\boldsymbol{c}^{BB} \\ -\mathbf{n}^{AA} & -\mathbf{n}^{AB} & 1 & 0 \\ -\mathbf{n}^{BA} & -\mathbf{n}^{BB} & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{Q}^{A} \\ \mathbf{Q}^{B} \\ N^{A} \\ N^{B} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{Q}}^{AB} \\ \bar{\mathbf{Q}}^{BA} \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

which may be written as

$$(\mathbf{I} - \mathbf{A}^{+})\mathbf{Q} = \mathbf{F} \tag{28}$$

And, providing matrix $(\mathbf{I} - \mathbf{A}^{+})$ is non-singular, the system can be solved using a Type II Leontief inverse, $(\mathbf{I} - \mathbf{A}^{+})^{-1}$, closed in part with respect to household consumption, such that

$$\mathbf{Q} = (\mathbf{I} - \mathbf{A}^{+})^{-1}\mathbf{F} \tag{29}$$

The quantities $(\mathbf{Q}^{A}, \mathbf{Q}^{B}, N^{A}, N^{B})$ are determined, via the Leontief inverse, by the autonomous consumption flows contained in **F.**

Dual to this quantity system a price system can be defined. With given quantities of labour (N^{A} and N^{B}) determined by autonomous imports ($\bar{\mathbf{Q}}^{AB}$and $\bar{\mathbf{Q}}^{BA}$) set up in the quantity system, we can define autonomous consumption coefficients:

$$\bar{\mathbf{c}}^{AB} = \frac{\bar{\mathbf{Q}}^{AB}}{N^{B}}$$

$$\bar{\mathbf{c}}^{BA} = \frac{\bar{\mathbf{Q}}^{BA}}{N^{A}} \tag{30}$$

In order to establish an equilibrium solution to this open input-output system, the consumption coefficients in (30) can be interpreted as given parameters in a price system that is dual to the quantity system:

$$[\mathbf{p}^{A} \quad \mathbf{p}^{B} \quad w^{A} \quad w^{B}] \begin{bmatrix} \mathbf{I} & \mathbf{0} & -\mathbf{c}^{AA} & -(\boldsymbol{c}^{AB} + \bar{\mathbf{c}}^{AB}) \\ \mathbf{0} & \mathbf{I} & -(\boldsymbol{c}^{BA} + \bar{\mathbf{c}}^{BA}) & -\boldsymbol{c}^{BB} \\ -\mathbf{n}^{AA} & -\mathbf{n}^{AB} & 1 & 0 \\ -\mathbf{n}^{BA} & -\mathbf{n}^{BB} & 0 & 1 \end{bmatrix} = [\mathbf{0} \quad \mathbf{0} \quad 0 \quad 0]$$

(31)

Row vectors of money prices ($\mathbf{p}^A$ and $\mathbf{p}^B$) represent value added, which in this system are wage costs captured by scalar money wage rates (w^A and w^A). For simplicity, in the analysis that follows a uniform wage rate $w = w^A = w^B$ is assumed across the two countries, a restriction that could in principle be a relaxed at a later stage of analysis.

In this input-output framework the balance of trade between the two countries can be examined, as required under Thirlwall's Law. In Global Value Chains it is necessary to capture flows of value added required to produce imports and exports: referred to as Trade in Value Added (TiVA) (see Johnson and Noguera, 2012; Stehrer, 2012). This can be first identified for the case of imports to country *A*. From the quantity system (27) we have expressions for employment and output:

$$N^B = \boldsymbol{n}^{BA}\boldsymbol{Q}^A + \boldsymbol{n}^{BB}\boldsymbol{Q}^B \tag{32}$$

$$\boldsymbol{Q}^A = \boldsymbol{c}^{AA}N^A + \boldsymbol{c}^{AB}N^B + \overline{\boldsymbol{Q}}^{AB} \tag{33}$$

$$\boldsymbol{Q}^B = \boldsymbol{c}^{BA}N^A + \boldsymbol{c}^{BB}N^B + \overline{\boldsymbol{Q}}^{BA} \tag{34}$$

Substituting (33) and (34) into (32):

$$N^B = \boldsymbol{n}^{BA}\boldsymbol{c}^{AA}N^A + \boldsymbol{n}^{BA}\boldsymbol{c}^{AB}N^B + \boldsymbol{n}^{BA}\overline{\boldsymbol{Q}}^{AB} + \boldsymbol{n}^{BB}\boldsymbol{c}^{BA}N^A + \boldsymbol{n}^{BB}\boldsymbol{c}^{BB}N^B + \boldsymbol{n}^{BB}\overline{\boldsymbol{Q}}^{BA} \tag{35}$$

Re-arranging, and multiplying throughout by the wage rate (w), the income ($Y^B = wN^B$) for country *B* has the structure

$$Y^B = w\boldsymbol{n}^{BB}(\overline{\boldsymbol{Q}}^{BA} + \boldsymbol{c}^{BA}N^A) + w\boldsymbol{n}^{BA}\boldsymbol{c}^{AA}N^A + w\boldsymbol{n}^{BA}(\overline{\boldsymbol{Q}}^{AB} + \boldsymbol{c}^{AB}N^B) + w\boldsymbol{n}^{BB}\boldsymbol{c}^{BB}N^B \tag{36}$$

Elements of this expression can be considered each in turn. The first term ($w\boldsymbol{n}^{BB}(\overline{\boldsymbol{Q}}^{BA} + \boldsymbol{c}^{BA}N^{A})$ captures, via $w\boldsymbol{n}^{BB}$, the value added by country *B* in its production of final goods (autonomous and endogenous) exported to country *A*. The second term, ($w\boldsymbol{n}^{BA}\boldsymbol{c}^{AA}N^{A}$) captures country *B*'s value added embodied in intermediate goods, via $\boldsymbol{n}^{BA}$, that are exported to country *A* for use in its domestically produced final good consumption. This trade in final and intermediate goods makes up the TiVA of imports by country *A* from Country *B* (to be used in the next section to derive the Multisectoral Thirlwall Law):

$$M^{BA} = w\mathbf{n}^{BB}(\overline{\mathbf{Q}}^{BA} + \mathbf{c}^{BA}\mathrm{N}^{A}) + w\mathbf{n}^{BA}\boldsymbol{c}^{AA}N^{A} \qquad (37)$$

It may be noted for completion that the remaining terms to consider in (36) capture value added created and extinguished in country *B*. The third term, $w\boldsymbol{n}^{BA}(\overline{\boldsymbol{Q}}^{AB} + \boldsymbol{c}^{AB}N^{B})$, captures value added in country *B* in the production of intermediate goods that are exported to country *A* for subsequent use in the production of exports of final goods (autonomous and endogenous) back to country *B*. This is a supply chain sequence, referred to by Baldwin and Lopez-Gonzalez (2015, 1686) as 're-importing'. These flows are *not* included in TiVA, but accounted for as part of country B's domestically produced value added. The final term, $w\boldsymbol{n}^{BB}\boldsymbol{c}^{BB}N^{B}$, captures country B's domestically produced value added embodied in domestic consumption

A symmetrical TiVA expression for exports from country *A* into country *B* takes the form

$$X^{AB} = w\mathbf{n}^{AA}(\overline{\mathbf{Q}}^{AB} + \mathbf{c}^{AB}\mathrm{N}^{B}) + w\mathbf{n}^{AB}\boldsymbol{c}^{BB}N^{B} \qquad (38)$$

Again the first term captures valued added flows in final goods, with the second term representing flows in intermediate goods. In this input-output framework the balance of trade,

measured in Trade in Value Added, requires that $M^{BA} = X^{AB}$. This constraint will be imposed under Thirlwall's Law in the analysis that follows.

5. THE MULTISECTORAL THIRLWALL LAW

Using this input-output system, it is possible to consider the structure of sectoral income elasticities. To illustrate how this works it can be assumed that each country has two single-commodity producing industries ($S = 2$). A constituent element of the column vector of consumption coefficients ($\mathbf{c}^{BA}$), for example, is c_1^{BA}, which represents the consumption of Country *B*'s output of commodity 1 per unit of employment in Country *A*.

Using (37), the starting point is to focus on the money value of each of the two commodities imported by Country *A* from Country *B*:

$$M_1^{BA} = wn_1^{BB}\bar{Q}_1^{BA} + wn_1^{BA}c_1^{AA}N^A + wn_1^{BB}c_1^{BA}N^A$$
$$M_2^{BA} = wn_2^{BB}\bar{Q}_2^{BA} + wn_2^{BA}c_2^{AA}N^A + wn_2^{BB}c_2^{BA}N^A \quad (39)$$

Now with income constituted by wages ($Y^A = wN^A$), equation (39) can be re-expressed as

$$M_1^{BA} = wn_1^{BB}\bar{Q}_1^{BA} + (n_1^{BA}c_1^{AA} + n_1^{BB}c_1^{BA})Y^A$$
$$M_2^{BA} = wn_2^{BB}\bar{Q}_2^{BA} + (n_2^{BA}c_2^{AA} + n_2^{BB}c_2^{BA})Y^A \quad (40)$$

Hence the total money value of imports into country *A* from *B* is

$$M^{BA} = w(n_1^{BB}\bar{Q}_1^{BA} + n_2^{BB}\bar{Q}_2^{BA}) + (n_1^{BA}c_1^{AA} + n_1^{BB}c_1^{BA} + n_2^{BA}c_2^{AA} + n_2^{BB}c_2^{BA})Y^A \quad (41)$$

Derivatives of these import quantities with respect to income can then be specified as

$$\frac{dM_1^{BA}}{dY^A} = n_1^{BA}c_1^{AA} + n_1^{BB}c_1^{BA}$$

$$\frac{dM_2^{BA}}{dY^A} = n_2^{BA}c_2^{AA} + n_2^{BB}c_2^{BA}$$

$$\frac{dM^{BA}}{dY^A} = n_1^{BA}c_1^{AA} + n_1^{BB}c_1^{BA} + n_2^{BA}c_2^{AA} + n_2^{BB}c_2^{BA} \tag{42}$$

Using these derivatives, the multisectoral structure of income elasticities can be established. Examination of (42) shows that the derivative of total imports with respect to income is decomposed into a summation of the two derivatives for each industry:

$$\frac{dM^{BA}}{dY^A} = \frac{dM_1^{BA}}{dY^A} + \frac{dM_2^{BA}}{dY^A} \tag{43}$$

These derivatives can be manipulated to give an expression for the income elasticity for imports of

$$\frac{dM^{BA}}{dY^A}\frac{Y^A}{M^{BA}} = \frac{dM_1^{BA}}{dY^A}\frac{Y^A}{M_1^{BA}}\frac{M_1^{BA}}{M^{BA}} + \frac{dM_2^{BA}}{dY^A}\frac{Y^A}{M_2^{BA}}\frac{M_2^{BA}}{M^{BA}} \tag{44}$$

With the income elasticity for imports of each commodity expressed as π_i^{BA} and by defining weightings, $\omega_i^{BA} = M_i^{BA}/M^{BA}$, the income elasticity can be written as

$$\pi^{BA} = \pi_1^{BA}\omega_1^{BA} + \pi_2^{BA}\omega_2^{BA} \tag{45}$$

Without showing all of the same steps, symmetrical to the derivation presented from (39) to (45), the income elasticity for imports into *B* from *A* takes the form:

$$\pi^{AB} = \pi_1^{AB}\omega_1^{AB} + \pi_2^{AB}\omega_2^{AB} \tag{46}$$

It can be noted that the income elasticity of imports into *B* from *A* is also the income elasticity for exports from *A* to *B*: $\pi^{AB} = \varepsilon^{AB}$. Hence, using (8) together with (45) and (46), relative growth rates are expressed as the ratio of income elasticities:

$$\frac{g_A}{g_B} = \frac{\varepsilon^{AB}}{\pi^{BA}} = \frac{\pi_1^{AB}\omega_1^{AB} + \pi_2^{AB}\omega_2^{AB}}{\pi_1^{BA}\omega_1^{BA} + \pi_2^{BA}\omega_2^{BA}} \tag{47}$$

This is the Multisectoral Thirlwall Law as devised by Araujo and Lima (2007), for the illustrative case of two sectors. As mentioned earlier, despite its original evolution from the work of Pasinetti, subsequent research on this multisectoral variant of the Law has moved away from its input-output foundations. Its derivation here provides a succinct interface between the problem of uneven development, as captured by the ratio of growth rates, and input-output technology, as embodied in multisectoral income elasticities for imports and exports.

Lying behind these income elasticities is a complex value chain of interrelationships between sectors across countries. To give some insight into how this structure governs uneven development, using (42) for imports, and the same approach for exports, (47) can be re-expressed as

$$\frac{g_A}{g_B} = \frac{(n_1^{AB}c_1^{BB} + n_1^{AA}c_1^{AB})\frac{Y^B}{M_1^{AB}}\frac{M_1^{AB}}{M^{AB}} + (n_2^{AB}c_2^{BB} + n_2^{AA}c_2^{AB})\frac{Y^B}{M_2^{AB}}\frac{M_2^{AB}}{M^{AB}}}{(n_1^{BA}c_1^{AA} + n_1^{BB}c_1^{BA})\frac{Y^A}{M_1^{BA}}\frac{M_1^{BA}}{M^{BA}} + (n_2^{BA}c_2^{AA} + n_2^{BB}c_2^{BA})\frac{Y^A}{M_2^{BA}}\frac{M_2^{BA}}{M^{BA}}} \tag{48}$$

Through cancellation of terms, and since under Thirlwall's Law there is a balance of trade equality ($M^{AB} = M^{BA}$):

$$\frac{g_A}{g_B} = \frac{Y^B}{Y^A}\frac{(n_1^{AB}c_1^{BB} + n_1^{AA}c_1^{AB} + n_2^{AB}c_2^{BB} + n_2^{AA}c_2^{AB})}{(n_1^{BA}c_1^{AA} + n_1^{BB}c_1^{BA} + n_2^{BA}c_2^{AA} + n_2^{BB}c_2^{BA})} \tag{49}$$

Relative growth rates therefore have two theoretical explanations. The first involves the ratio of income levels in each country. Since $Y^A = wN^A$ and $Y^B = wN^B$ it follows that

$$\frac{Y^B}{Y^A} = \frac{N^B}{N^A} = \xi \tag{50}$$

This is the ratio of employment levels between the two countries (ξ), shown in Section 3 to be central to Araujo and Lima's one-country modelling of exports using the Pasinetti closed input-output model (see equation 11). This employment ratio is shown here to retain a core role in understanding relative growth rates in a two-country open input-output framework:

$$\frac{g_A}{g_B} = \xi \frac{(n_1^{AB}c_1^{BB} + n_1^{AA}c_1^{AB} + n_2^{AB}c_2^{BB} + n_2^{AA}c_2^{AB})}{(n_1^{BA}c_1^{AA} + n_1^{BB}c_1^{BA} + n_2^{BA}c_2^{AA} + n_2^{BB}c_2^{BA})} \qquad (51)$$

If, for example, country *A* was underdeveloped (compared to *B*), then its growth rate would benefit the higher is the size of the other economy, country *B*. Larger, richer economies have a role to play in fostering the development of poorer economies.

The second theoretical explanation operates at the sectoral level. Consider, for example, the two elements in the numerator of (51) for sector 1, each of which is a linear combination of an employment multiplier and consumption coefficient. In the first element, $n_1^{AB}c_1^{BB}$, the employment multiplier, n_1^{AB}, captures the labour embodied in intermediate goods produced in *A* required to produce consumption goods, represented by the consumption coefficient, c_1^{BB}, produced and consumed in *B*. These are global supply chain purchases of intermediate inputs by country *B* from country *A*. The second expression, $n_1^{AA}c_1^{AB}$, consists of a domestic employment multiplier, n_1^{AA}, for intermediate inputs produced in country *A* for use in country A's production of consumption goods, represented by c_1^{AB}, exported to country B: more traditional trade in final consumption goods. This provides a more in depth insight into the identification of key sectors than under the Multisectoral Thirlwall Law. Whereas the numerator of (47) suggests an examination of the income elasticity for exports, and its proportionate share of trade weighted for sector 1, $\pi_1^{AB}\omega_1^{AB}$, a more in depth disaggregated analysis is provided by distinguishing between supply chain trade of intermediate goods and the more traditional trade in final goods. This distinction can be drawn for each of the sectors in an input-output system.

This has implications for key sector analysis. A sector may have a high income elasticity for exports, and a high weighting, but this may disguise an underlying imbalance; the sector may, for example, have weak trade linkages for final goods alongside strong linkages for intermediate goods, or vice versa. By deriving the Multisectoral Thirlwall, under an input-output system, key sectors are no longer chosen just by identifying sectors with high income elasticities that make up a significant proportion of trade. The export potential of sectors can also be explored by distinguishing between intermediate and final good trade linkages.

6. A THREE-COUNTRY EXTENSION

More interesting value chain linkages emerge once a third country is introduced, this being key to discovering the structure of the global multi-country relationships in Section 7. In addition to countries *A* and *B*, a new third country, *C*, can be considered.

Extending equation (27), the quantity system for three countries, each with *S* sectors, has the structure

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & -\boldsymbol{c}^{AA} & -\boldsymbol{c}^{AB} & -\boldsymbol{c}^{AC} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\boldsymbol{c}^{BA} & -\boldsymbol{c}^{BB} & -\boldsymbol{c}^{BC} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & -\boldsymbol{c}^{CA} & -\boldsymbol{c}^{CB} & -\boldsymbol{c}^{CC} \\ -\boldsymbol{n}^{AA} & -\boldsymbol{n}^{AB} & -\boldsymbol{n}^{AC} & 1 & 0 & 0 \\ -\boldsymbol{n}^{BA} & -\boldsymbol{n}^{BB} & -\boldsymbol{n}^{BC} & 0 & 1 & 0 \\ -\boldsymbol{n}^{CA} & -\boldsymbol{n}^{CB} & -\boldsymbol{n}^{CC} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{Q}^A \\ \mathbf{Q}^B \\ \mathbf{Q}^C \\ N^A \\ N^B \\ N^C \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{Q}}^{AB} + \overline{\boldsymbol{Q}}^{AC} \\ \overline{\boldsymbol{Q}}^{BA} + \overline{\boldsymbol{Q}}^{BC} \\ \overline{\boldsymbol{Q}}^{CA} + \overline{\boldsymbol{Q}}^{CB} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{52}$$

In order to derive Thirlwall's Law for a three-country system, the analysis will focus on imports into country *A* from countries *B* and *C*, each in turn. Consider first the expression for employment relating to country *B*

$$N^B = \boldsymbol{n}^{BA}\boldsymbol{Q}^A + \boldsymbol{n}^{BB}\boldsymbol{Q}^B + \boldsymbol{n}^{BC}\boldsymbol{Q}^C \tag{53}$$

The expressions relating to output for all three countries are:

$$\boldsymbol{Q}^{A} = \boldsymbol{c}^{AA}N^{A} + \boldsymbol{c}^{AB}N^{B} + \boldsymbol{c}^{AC}N^{C} + \overline{\boldsymbol{Q}}^{AB} + \overline{\boldsymbol{Q}}^{AC} \quad (54)$$

$$\boldsymbol{Q}^{B} = \boldsymbol{c}^{BA}N^{A} + \boldsymbol{c}^{BB}N^{B} + \boldsymbol{c}^{BC}N^{C} + \overline{\boldsymbol{Q}}^{BA} + \overline{\boldsymbol{Q}}^{BC} \quad (55)$$

$$\boldsymbol{Q}^{C} = \boldsymbol{c}^{CA}N^{A} + \boldsymbol{c}^{CB}N^{B} + \boldsymbol{c}^{CC}N^{C} + \overline{\boldsymbol{Q}}^{CA} + \overline{\boldsymbol{Q}}^{CB} \quad (56)$$

Substituting (54), (55) and (56) into (53) yields

$$\begin{aligned} N^{B} = {} & \boldsymbol{n}^{BA}\boldsymbol{c}^{AA}N^{A} + \boldsymbol{n}^{BA}\boldsymbol{c}^{AB}N^{B} + \boldsymbol{n}^{BA}\boldsymbol{c}^{AC}N^{C} + \boldsymbol{n}^{BA}\overline{\boldsymbol{Q}}^{AB} + \boldsymbol{n}^{BA}\overline{\boldsymbol{Q}}^{AC} \\ & + \boldsymbol{n}^{BB}\boldsymbol{c}^{BA}N^{A} + \boldsymbol{n}^{BB}\boldsymbol{c}^{BB}N^{B} + \boldsymbol{n}^{BB}\boldsymbol{c}^{BC}N^{C} + \boldsymbol{n}^{BB}\overline{\boldsymbol{Q}}^{BA} + \boldsymbol{n}^{BB}\overline{\boldsymbol{Q}}^{BC} \\ & + \boldsymbol{n}^{BC}\boldsymbol{c}^{CA}N^{A} + \boldsymbol{n}^{BC}\boldsymbol{c}^{CB}N^{B} + \boldsymbol{n}^{BC}\boldsymbol{c}^{CC}N^{C} + \boldsymbol{n}^{BC}\overline{\boldsymbol{Q}}^{CA} + \boldsymbol{n}^{BC}\overline{\boldsymbol{Q}}^{CB} \end{aligned} \quad (57)$$

Following the same derivation as in equations (32) to (37), from (57) a new expression, that includes linkages with the third country *C*, is derived for imports into country *A* from country *B*:

$$M^{BA} = w(\mathbf{n}^{BB}\overline{\boldsymbol{Q}}^{BA} + \mathbf{n}^{BC}\overline{\boldsymbol{Q}}^{CA}) + w(\mathbf{n}^{BA}\boldsymbol{c}^{AA}N^{A} + \boldsymbol{n}^{BB}\boldsymbol{c}^{BA}N^{A} + \boldsymbol{n}^{BC}\boldsymbol{c}^{CA}N^{A}) \quad (58)$$

Compared to the import expression for the two-country model (37), there are two new third party elements. First, $w\mathbf{n}^{BC}\overline{\boldsymbol{Q}}^{CA}$ shows the value added that flows from country *B* to country *C*, embodied in intermediate inputs, to produce final goods that are autonomously demanded from country *C* by country *A*. Second, $w\boldsymbol{n}^{BC}\boldsymbol{c}^{CA}N^{A}$ shows value added, embodied in intermediate inputs, from *B* to *C*, for final goods endogenously demanded by country *A*. This use of foreign inputs has been referred to in the literature as 'importing to export' (Baldwin and Lopez-Gonzalez, 2015, 1686); Hummels *et al* (2001) call it 'vertical specialization'. It represents a subset of the more general 'importing to produce' category, focusing specifically on the importing of intermediate goods for use in the production of exports.

A second expression for imports from country *C* into country *A* can be derived using the employment equation from (52):

$$\boldsymbol{N^C = n^{CA}Q^A + n^{CB}Q^B + n^{CC}Q^C} \quad \textbf{(59)}$$

Substituting (54), (55) and (56) into (59) gives

$$N^C = \boldsymbol{n^{CA}c^{AA}}N^A + \boldsymbol{n^{CA}c^{AB}}N^B + \boldsymbol{n^{CA}c^{AC}}N^C + \boldsymbol{n^{CA}\overline{Q}^{AB}} + \boldsymbol{n^{CA}\overline{Q}^{AC}}$$
$$+ \boldsymbol{n^{CB}c^{BA}}N^A + \boldsymbol{n^{CB}c^{BB}}N^B + \boldsymbol{n^{CB}c^{BC}}N^C + \boldsymbol{n^{CB}\overline{Q}^{BA}} + \boldsymbol{n^{CB}\overline{Q}^{BC}}$$
$$+ \boldsymbol{n^{CC}c^{CA}}N^A + \boldsymbol{n^{CC}c^{CB}}N^B + \boldsymbol{n^{CC}c^{CC}}N^C + \boldsymbol{n^{CC}\overline{Q}^{CA}} + \boldsymbol{n^{CC}\overline{Q}^{CB}} \quad (60)$$

Using the same steps as before,

$$M^{CA} = w(\mathbf{n}^{CC}\overline{\boldsymbol{Q}}^{CA} + \mathbf{n}^{CB}\overline{\boldsymbol{Q}}^{BA}) + w(\boldsymbol{n^{CA}c^{AA}}N^A + \mathbf{n}^{CB}\boldsymbol{c^{BA}}N^A + \boldsymbol{n^{CC}c^{CA}}N^A) \quad (61)$$

Hence, total imports of value added into country *A* are $M^{*A} = M^{BA} + M^{CA}$, which from (58) and (61), with $Y^A = wN^A$, is

$$M^{*A} = w(\mathbf{n}^{BB}\overline{\boldsymbol{Q}}^{BA} + \mathbf{n}^{BC}\overline{\boldsymbol{Q}}^{CA} + \mathbf{n}^{CC}\overline{\boldsymbol{Q}}^{CA} + \mathbf{n}^{CB}\overline{\boldsymbol{Q}}^{BA})$$
$$+ (\mathbf{n}^{BA}\boldsymbol{c^{AA}} + \boldsymbol{n^{BB}c^{BA}} + \boldsymbol{n^{BC}c^{CA}} + \boldsymbol{n^{CA}c^{AA}} + \mathbf{n}^{CB}\boldsymbol{c^{BA}} + \boldsymbol{n^{CC}c^{CA}})Y^A \quad (62)$$

Hence the derivate of imports against income is

$$\frac{dM^{*A}}{dY^A} = \mathbf{n}^{BA}\boldsymbol{c^{AA}} + \boldsymbol{n^{BB}c^{BA}} + \boldsymbol{n^{BC}c^{CA}} + \boldsymbol{n^{CA}c^{AA}} + \mathbf{n}^{CB}\boldsymbol{c^{BA}} + \boldsymbol{n^{CC}c^{CA}} \quad (63)$$

Finally, turning to exports, the employment equation, from (52), for country *A* can be considered:

$$N^A = \boldsymbol{n^{AA}Q^A + n^{AB}Q^B + n^{AC}Q^C} \quad (64)$$

Again by substituting (54), (55) and (56) into the employment equation (64),

$$N^A = \boldsymbol{n}^{AA}\boldsymbol{c}^{AA}N^A + \boldsymbol{n}^{AA}\boldsymbol{c}^{AB}N^B + \boldsymbol{n}^{AA}\boldsymbol{c}^{AC}N^C + \boldsymbol{n}^{AA}\overline{\boldsymbol{Q}}^{AB} + \boldsymbol{n}^{AA}\overline{\boldsymbol{Q}}^{AC}$$
$$+ \boldsymbol{n}^{AB}\boldsymbol{c}^{BA}N^A + \boldsymbol{n}^{AB}\boldsymbol{c}^{BB}N^B + \boldsymbol{n}^{AB}\boldsymbol{c}^{BC}N^C + \boldsymbol{n}^{AB}\overline{\boldsymbol{Q}}^{BA} + \boldsymbol{n}^{AB}\overline{\boldsymbol{Q}}^{BC}$$
$$+ \boldsymbol{n}^{AC}\boldsymbol{c}^{CA}N^A + \boldsymbol{n}^{AC}\boldsymbol{c}^{CB}N^B + \boldsymbol{n}^{AC}\boldsymbol{c}^{CC}N^C + \boldsymbol{n}^{AC}\overline{\boldsymbol{Q}}^{CA} + \boldsymbol{n}^{AC}\overline{\boldsymbol{Q}}^{CB} \tag{65}$$

Selecting out the export-related terms, total exports of country *A*, to both of the other countries, are

$$X^{A*} = w(\mathbf{n}^{AA}\overline{\boldsymbol{Q}}^{AB} + \mathbf{n}^{AC}\overline{\boldsymbol{Q}}^{CB} + \mathbf{n}^{AA}\overline{\boldsymbol{Q}}^{AC} + \mathbf{n}^{AB}\overline{\boldsymbol{Q}}^{BC})$$
$$+ w(\boldsymbol{n}^{AA}\boldsymbol{c}^{AC}N^{\boldsymbol{C}} + \mathbf{n}^{AB}\boldsymbol{c}^{BC}N^{\boldsymbol{C}} + \boldsymbol{n}^{AC}\boldsymbol{c}^{CC}N^{\boldsymbol{C}})$$
$$+(\boldsymbol{n}^{AA}\boldsymbol{c}^{AB} + \mathbf{n}^{AB}\boldsymbol{c}^{BB} + \boldsymbol{n}^{AC}\boldsymbol{c}^{CB})Y^B \tag{66}$$

where $Y^B = wN^B$. The derivative of country *A*'s exports with respect to income in country *B* is hence

$$\frac{dX^{A*}}{dY^B} = \boldsymbol{n}^{AA}\boldsymbol{c}^{AB} + \mathbf{n}^{AB}\boldsymbol{c}^{BB} + \boldsymbol{n}^{AC}\boldsymbol{c}^{CB} \tag{67}$$

Adapting (51), a three-country structure for Thirlwall's Law can therefore be established, with relative growth rates defined using (63) and (67):

$$\frac{g_A}{g_B} = \xi\frac{\frac{dX^{A*}}{dY^B}}{\frac{dM^{*A}}{dY^A}} = \xi\frac{\boldsymbol{n}^{AA}\boldsymbol{c}^{AB} + \mathbf{n}^{AB}\boldsymbol{c}^{BB} + \boldsymbol{n}^{AC}\boldsymbol{c}^{CB}}{\mathbf{n}^{BA}\boldsymbol{c}^{AA} + \boldsymbol{n}^{BB}\boldsymbol{c}^{BA} + \boldsymbol{n}^{BC}\boldsymbol{c}^{CA} + \boldsymbol{n}^{CA}\boldsymbol{c}^{AA} + \mathbf{n}^{CB}\boldsymbol{c}^{BA} + \boldsymbol{n}^{CC}\boldsymbol{c}^{CA}} \tag{68}$$

As in the two-country model, Thirlwall's Law has a multisectoral structure. Each term in (68) is an aggregate of sectoral components. In the *S* sector case, the first term in the numerator, for example, has the structure.

$$\boldsymbol{n}^{AA}\boldsymbol{c}^{AB} = [n_1^{AA} \quad n_2^{AA} \quad \cdots \quad n_S^{AA}]\begin{bmatrix} c_1^{AB} \\ c_2^{AB} \\ \vdots \\ c_S^{AB} \end{bmatrix} = \sum_{i=1}^{s} n_i^{AA}\, c_i^{AB} \tag{69}$$

As we have seen, the three-country structure also introduces new third-party linkages, incorporated here into Thirlwall's Law. The term, $\boldsymbol{n}^{AC}\boldsymbol{c}^{CB}$, in the numerator, captures the labour embodied in exports of intermediate inputs from country *A* to *C* that are 'imported to export', embodied in exports to country *B*. These third party linkages are also present in the denominator, with $\boldsymbol{n}^{BC}\boldsymbol{c}^{CA}$ representing the labour embodied by country *B* (via intermediate inputs) in exports of final goods from country *C* to *A*. Finally, $\mathbf{n}^{CB}\boldsymbol{c}^{BA}$ represents the labour embodied by country *C* in exports of final goods from *B* to *A*. All three countries in this model are linked together in an interdependent global production network.

7. THE MULTI COUNTRY THIRLWALL LAW

A final simple step is to generalise this multisectoral disaggregation of Thirlwall's Law to a multi-country setting. Preliminary to this, the three country version can be further examined, with equation (68) written as

$$\frac{g_A}{g_B} = \xi \frac{\boldsymbol{n}_A\boldsymbol{c}^B}{\boldsymbol{n}_B\boldsymbol{c}^A + \boldsymbol{n}_C\boldsymbol{c}^A} \tag{70}$$

where terms are collected in vectors:

$\boldsymbol{n}_A = [\boldsymbol{n}^{AA} \quad \boldsymbol{n}^{AB} \quad \boldsymbol{n}^{AC}]$,

$\boldsymbol{n}_B = [\boldsymbol{n}^{BA} \quad \boldsymbol{n}^{BB} \quad \boldsymbol{n}^{BC}]$,

$\boldsymbol{n}_C = [\boldsymbol{n}^{CA} \quad \boldsymbol{n}^{CB} \quad \boldsymbol{n}^{CC}]$,

$$\boldsymbol{c}^{A} = \begin{bmatrix} \boldsymbol{c}^{AA} \\ \boldsymbol{c}^{BA} \\ \boldsymbol{c}^{CA} \end{bmatrix}, \boldsymbol{c}^{B} = \begin{bmatrix} \boldsymbol{c}^{AB} \\ \boldsymbol{c}^{BB} \\ \boldsymbol{c}^{CB} \end{bmatrix}$$

Thirlwall's Law is based here on the ratio of labour embodied in exports from country *A* to *B* to the labour embodied in all imports into country *A*, an insight that will now be built upon for the multi-country case.

With numbered superscripts (1 for country 1, 2 for country 2, etc.) the quantity system for *R* countries takes the form

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{c}^{11} & -\mathbf{c}^{12} & \cdots & -\mathbf{c}^{1R} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & -\mathbf{c}^{21} & -\mathbf{c}^{22} & \cdots & -\mathbf{c}^{2R} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & -\mathbf{c}^{R1} & -\mathbf{c}^{R2} & \cdots & -\mathbf{c}^{RR} \\ -\mathbf{n}^{11} & -\mathbf{n}^{12} & \cdots & -\mathbf{n}^{1R} & 1 & 0 & \cdots & 0 \\ -\mathbf{n}^{21} & -\mathbf{n}^{22} & \cdots & -\mathbf{n}^{2R} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mathbf{n}^{R1} & -\mathbf{n}^{R2} & \cdots & -\mathbf{n}^{RR} & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \mathbf{Q}^{1} \\ \mathbf{Q}^{2} \\ \vdots \\ \mathbf{Q}^{R} \\ \mathrm{N}^{1} \\ \mathrm{N}^{2} \\ \vdots \\ \mathrm{N}^{R} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{Q}}^{1} \\ \overline{\mathbf{Q}}^{2} \\ \vdots \\ \overline{\mathbf{Q}}^{R} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{71}$$

where the multisectoral structure of each component is

$$\boldsymbol{n}^{ij} = \begin{bmatrix} n_1^{ij} & n_2^{ij} & \cdots & n_S^{ij} \end{bmatrix},$$

$$\boldsymbol{c}^{ij} = \begin{bmatrix} c_1^{ij} \\ c_2^{ij} \\ \vdots \\ c_S^{ij} \end{bmatrix}, \boldsymbol{Q}^{i} = \begin{bmatrix} Q_1^{i} \\ Q_2^{i} \\ \vdots \\ Q_S^{i} \end{bmatrix}, \text{ and } \overline{\boldsymbol{Q}}^{i} = \begin{bmatrix} \bar{Q}_1^{i} \\ \bar{Q}_2^{i} \\ \vdots \\ \bar{Q}_S^{i} \end{bmatrix}.$$

Each vector in (71) is a collection of *S* sectoral terms, with superscripts indicating inter-country relationships. Multiple sectoral and inter-country value chain linkages are captured by each (vertically integrated) vector of labour coefficients $\boldsymbol{n}^{ij}$, representing the (direct and indirect) labour embodied in intermediate goods exported for each sector from country *i* to country *j*. The element n_2^{34}, for example, shows the labour embodied in intermediate goods

produced in sector 2 in country 3 that are exported to country 4. This trade in intermediate goods, as characterised by Global Value Chains, is captured for *R* countries.

Trade in final goods is captured by the consumption coefficients, $\mathbf{c}^{ij}$, representing the consumption of goods by country *j* that are produced by each sector in country *i*. For example, the element c_2^{13}captures country 3's consumption of good 2, as produced in sector 2 of country 1.

Using this multi-country/multi-sector model, and replicating the pattern identified for Thirlwall's Law in (70), for any two countries *i* and *j*, we have

$$\frac{g_i}{g_j} = \xi \frac{\boldsymbol{n}_i \boldsymbol{c}^j}{(\sum_{k=1}^{R} \boldsymbol{n}_k - \boldsymbol{n}_i)\boldsymbol{c}^i} \tag{72}$$

where $\xi = N^j/N^i$ is the employment ratio, and the remaining terms consist of vectors:

$\boldsymbol{n}_i = [\boldsymbol{n}^{i1} \quad \boldsymbol{n}^{i2} \quad \cdots \quad \boldsymbol{n}^{iR}]$,

$\boldsymbol{n}_k = [\boldsymbol{n}^{k1} \quad \boldsymbol{n}^{k2} \quad \cdots \quad \boldsymbol{n}^{kR}]$,

$\boldsymbol{c}^j = \begin{bmatrix} \boldsymbol{c}^{1j} \\ \boldsymbol{c}^{2j} \\ \vdots \\ \boldsymbol{c}^{Rj} \end{bmatrix}$, and

$\boldsymbol{c}^i = \begin{bmatrix} \boldsymbol{c}^{1i} \\ \boldsymbol{c}^{2i} \\ \vdots \\ \boldsymbol{c}^{Ri} \end{bmatrix}$.

A new generalized version of Thirlwall's Law is suggested by (72), which may be referred to as the Multi Country Sectoral Thirlwall Law (MCSTL). Despite the high degree of disaggregation, it has a simple structure. In addition to the employment ratio, the relative growth rates of any two countries *i* and *j* depends the ratio of the labour embodied in sectoral exports $\boldsymbol{n}_i \boldsymbol{c}^j$ from *i* to *j* to the labour embodied in total sectoral imports $(\sum_{k=1}^{R} \boldsymbol{n}_k - \boldsymbol{n}_i)\boldsymbol{c}^i$ into country *i*. This is a genuinely macroeconomic relationships that holds regardless of the degree of disaggregation across sectors and countries.

The model incorporates trade in both final and intermediate goods. Consider first the trade in just final goods. For example, when *i*=2 (denoting country 2) and *j*=3 (denoting country 3) the term in the numerator, $\boldsymbol{n}^{i2}\boldsymbol{c}^{2j}$ is equal to $\boldsymbol{n}^{22}\boldsymbol{c}^{23}$. This shows the labour carried out in country 2 embodied in exports of final goods from country 2 to 3. The MCSTL models all of these final good linkages across *S* sectors and *R* countries.

Second, consider $\boldsymbol{n}^{i2}\boldsymbol{c}^{2j}$ when *i*=1 (denoting country 1) and *j*=4 (denoting country 4) such that it equals $\boldsymbol{n}^{12}\boldsymbol{c}^{24}$. This captures the labour embodied in intermediate goods produced in country 1 for use in country 2 for the production of final consumption goods exported to country 4. In this example, trade in both intermediate and final goods is captured: one of the multiple 'imported to export' relationships that are incorporated in this multiple country model. If this linear combination of employment multiplier $(\boldsymbol{n}^{12})$ and consumption coefficient $(\boldsymbol{c}^{24})$ is strong, a key inter-country relationship is indicated. Further nested within this inter-country relationship are possible key sector relationships; in this example we have $\boldsymbol{n}^{12}\boldsymbol{c}^{24} = n_1^{12}c_1^{24} + n_2^{12}c_2^{24} + \cdots + n_S^{12}c_S^{24}$, from which a strong linear combination can be identified for any key sector. This general system allows for the identification of key inter-country relationships, and within these key inter-sector relationships, incorporating trade in intermediate and final goods, for all *S* sectors across *R* countries in a global production network.

8. CONCLUDING REMARKS

A new multi-country/multi-sectoral model of Thirlwall's Law is developed here that extends the one-country input-output model of Araujo and Lima (2007) that is based on Pasinetti's system of vertical integration (Pasinetti, 1993). In addition to its generalization to *R* countries, this model allows for non-unitary income elasticities by accounting for autonomous

expenditure in an open input-output system. Furthermore, it introduces tradeable intermediate inputs in Global Value Chains, building on the recent empirical input-output literature (e.g. Los *et al*, 2015; Johnson and Noguera, 2012). Global linkages identified in the latter include the importing of intermediate inputs for export production, accounted for as Trade in Value Added. A demonstration is provided here of how such complex value chain linkages can be succinctly captured using Pasinetti's framework of vertical integration: an alternative perspective and contribution to the input-output literature.

Thirlwall's Law is re-interpreted in two main ways. First, the Multisectoral Thirlwall Law (MSTL) is disaggregated to take into account two types of trade, in intermediate and final goods. Instead of examining income elasticities for exports and imports, a more detailed disaggregation is proposed in which key sector analysis distinguishes between intermediate and final good linkages. Second, this new disaggregated version is generalized to a Multi Country and Sectoral Thirlwall Law (MCSTL). Key inter-country relationships can be identified, of which key sector analysis is a constituent part. Uneven development is shown to depend on a global production network of intermediate inputs and final goods traded between multiple countries.

This multi-country/multi-sector approach to modelling Thirlwall's Law has the additional merit that there is on hand a relatively new dataset available for its possible testing and application: the World Input-Output Database, made up of national tables with interlinking trade flows (see Dietzenbacher et al, 2013). The theoretical model developed here is some way from being concrete enough to be modelled empirically since, for example, there is as yet no category for value added other than wages, and no variation in wage rates, assumptions that will be relaxed in future research.

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