Constructing Demand-Driven Input–Output Models by Direct Introducing Quantity and Price Parameters into the Product Balance Identity

Vladimir Motorin

National Research University Higher School of Economics, Moscow, Russian Federation

motoriny@gmail.com

The study starts with the product balance identity according to logical scheme [output 0 = intermediate input 0 + final demand 0] for the base year 0. Written in free variables it generates a product balance equation, or a generalized demand-driven input–output model. Analytical properties and peculiarities of vector and matrix calibration of the generalized model are considered.

Formal introducing unknown matrices of quantity and price indices leads to generalized nonlinear input–output model with exogenous final demand. This model comprises an excessive number of unknown quantity and price parameters and is not identifiable itself.

Nevertheless, under simplifying assumption about diagonal form of unknown parameters matrices one can get still nonlinear demand-driven input–output model with exogenous final demand but written with usual matrix operations and so much more operational. In general, this model could be linearize in four ways: set the matrix of price indices equal to identity matrix (constant prices), allow the quantity parameters matrix being identity matrix (constant levels of production by industries), and, finally, use two variants of combined including price and quantity parameters into the model.

Main attention in the study is paid to examining the analytical properties of four groups of linearized input–output models with various sets of price and quantity parameters. All the models turn out to be strictly identifiable under not so cumbersome technical assumptions and satisfy the vector and matrix calibration conditions. In particular, it is shown that the linear demand-driven price model could be appreciate as almost trivial whereas the linear demand-driven quantity model is in accordance with the formal pattern of product technology assumption widely known in input–output analysis.

Other two groups of linear demand-driven models with combined using price and quantity parameters seem to be out of economic sense, mostly not mentioned in the special literature but some of their features are of theoretical interest and deserves further exploration. In particular, one of them is in accordance with the formal pattern of industry technology assumption widely known in input–output analysis.

Keywords: product balance identity, generalized product balance equation, vector and matrix calibration of the model, linear vector-valued cost function, nonlinear demand-driven input–output model, relative price and quantity indices, demand-driven price and quantity models, demand-driven price/quantity and quantity/price models

JEL Classification: C67; D57

1. Introduction: identity and equation of product balance

The information for constructing input–output models in accordance with the main methodological recommendations of the UN Handbook on Supply, Use and Input–Output Tables with Extensions and Applications (United Nations, 2018, Ch.12) and the Eurostat Manual of Supply, Use and Input–Output Tables (Eurostat, 2008, Ch.11) includes the following components of supply and use tables:

- supply table at basic prices,
- use table at basic prices,
- domestic use table at basic prices,
- import use table at basic prices.

The initial informational and analytical premise for constructing input–output models with exogenous final demand is a product (material) balance identity in basic prices formed on the basis of the data from supply and use tables for the base year 0:

$$\mathbf{X}_0\mathbf{e}_M = \mathbf{Z}_0\mathbf{e}_M + \mathbf{y}_0 \tag{1}$$

where $\mathbf{X}_0$ and $\mathbf{Z}_0$ are supply (production) matrix and use (intermediate consumption) matrix of the same dimension $N \times M$ for the base year 0, N and M are the numbers of products and industries in the economy, respectively, $\mathbf{e}_M$ is M-dimensional summation column vector with unit elements, and $\mathbf{y}_0$ is N-dimensional column vector of final demand in base year 0. Vector formula (1) describes the system of N scalar identities with production and intermediate consumption matrices given.

In order to transform the ***product balance identity*** (1) into a mathematical model, one can formally write it in free variables $\mathbf{X}$, $\mathbf{Z}$ and $\mathbf{y}$. As a result, we obtain a generalized ***product balance equation***, or a generalized demand-driven input–output model

$$\mathbf{X}\mathbf{e}_M = \mathbf{Z}\mathbf{e}_M + \mathbf{y} . \tag{2}$$

It is easy to see that this generalized model contains N linear equations with $2NM$+N scalar variables – elements of the production and intermediate consumption matrices $\mathbf{X}$, $\mathbf{Z}$ and components of the final demand vector $\mathbf{y}$ (or with $3N$ scalar variables – components of the vectors $\mathbf{X}\mathbf{e}_M$, $\mathbf{Z}\mathbf{e}_M$ and $\mathbf{y}$). Thus, the model (2) is unidentifiable at choosing the final demand vector as an exogenous variable (Motorin, 2017).

2. The properties of a generalized product balance equation

The linear input–output model with exogenous final demand (2) is written in a very common form, but nevertheless has a number of analytical properties that should be taken into account when choosing or constructing procedures for model identification.

Property 1. Let a pair of matrices $\mathbf{X}^*(\mathbf{y})$ and $\mathbf{Z}^*(\mathbf{y})$ be some solution to the generalized product balance equation for a given final demand vector $\mathbf{y}$. Then the diagonal matrix $\langle \mathbf{X}^*(\mathbf{y})\mathbf{e}_M \rangle$ of order N with a vector $\mathbf{X}^*(\mathbf{y})\mathbf{e}_M$ on the main diagonal and the diagonal matrix $\langle \mathbf{Z}^*(\mathbf{y})\mathbf{e}_M \rangle$ of order N with a vector $\mathbf{Z}^*(\mathbf{y})\mathbf{e}_M$ on the main diagonal also satisfy equation (2), since

$$\langle \mathbf{X}^*(\mathbf{y})\mathbf{e}_M \rangle \mathbf{e}_M - \langle \mathbf{Z}^*(\mathbf{y})\mathbf{e}_M \rangle \mathbf{e}_M = \mathbf{X}^*(\mathbf{y})\mathbf{e}_M - \mathbf{Z}^*(\mathbf{y})\mathbf{e}_M = \mathbf{y} . \tag{3}$$

Property 2. Next, let $\mathbf{S}$ be an arbitrary matrix of dimensions $N \times M$. It is easy to verify that the matrices $\mathbf{X}^*(\mathbf{y}) - \mathbf{S}$ and $\mathbf{Z}^*(\mathbf{y}) - \mathbf{S}$ also form a solution to the generalized product balance equation (2) since

$$\left[\mathbf{X}^*(\mathbf{y})-\mathbf{S}\right]\mathbf{e}_M-\left[\mathbf{Z}^*(\mathbf{y})-\mathbf{S}\right]\mathbf{e}_M=\mathbf{X}^*(\mathbf{y})\mathbf{e}_M-\mathbf{Z}^*(\mathbf{y})\mathbf{e}_M=\mathbf{y}\,. \tag{4}$$

However, it is clear that the practical worth of such additive solution, true for any choice of matrix **S**, is relatively small because formula (4) exhibits obvious feature of mathematical tautology.

Property 3. Finally, let $\mathbf{S}_\mathbf{X}$ and $\mathbf{S}_\mathbf{Z}$ be a pair of square right stochastic matrices of order M such that $\mathbf{S}_\mathbf{X}\mathbf{e}_M=\mathbf{S}_\mathbf{Z}\mathbf{e}_M=\mathbf{e}_M$. Then the matrices $\mathbf{X}^*(\mathbf{y})\mathbf{S}_\mathbf{X}$ and $\mathbf{Z}^*(\mathbf{y})\mathbf{S}_\mathbf{Z}$ satisfy equation (2) for a given vector **y** since

$$\mathbf{X}^*(\mathbf{y})\mathbf{S}_\mathbf{X}\mathbf{e}_M-\mathbf{Z}^*(\mathbf{y})\mathbf{S}_\mathbf{Z}\mathbf{e}_M=\mathbf{X}^*(\mathbf{y})\mathbf{e}_M-\mathbf{Z}^*(\mathbf{y})\mathbf{e}_M=\mathbf{y}\,. \tag{5}$$

It should be emphasized that the practical worth of such multiplicative solution is limited because in the general case it is quite difficult to give an economic meaning to a pair of arbitrary right stochastic matrices. Nevertheless, as it will be shown below, Property 3 plays an important technical role in transforming the unidentifiable generalized linear input–output model with exogenous final demand (2) into its various operational forms.

The established analytical properties (3), (4) and (5) of the generalized product balance equation prove useful in further studying the conditions of exact identifiability of model (2) and its correct calibration at $\mathbf{y}=\mathbf{y}_0$.

3. Vector and matrix calibrating a generalized demand-driven input–output model

The endogenous variables in model (2) are either a pair of matrices and with $2NM$ unknown elements, or a pair of vectors and with $2N$ unknown components. The noted ambiguity of the choice of endogenous variables in the system of N linear equations (2) generates two different conditions for model calibrating with respect to the initial product balance identity (1).

Vector calibration condition. At $\mathbf{y}=\mathbf{y}_0=(\mathbf{X}_0-\mathbf{Z}_0)\mathbf{e}_M$, the vector of product outputs by all industries $\mathbf{X}\mathbf{e}_M(\mathbf{y})$ and the vector of product intermediate consumption $\mathbf{Z}\mathbf{e}_M(\mathbf{y})$ satisfy the requirements $\mathbf{X}\mathbf{e}_M(\mathbf{y}_0)=\mathbf{X}_0\mathbf{e}_M$ and $\mathbf{Z}\mathbf{e}_M(\mathbf{y}_0)=\mathbf{Z}_0\mathbf{e}_M$, respectively.

Matrix calibration condition. At $\mathbf{y}=\mathbf{y}_0=(\mathbf{X}_0-\mathbf{Z}_0)\mathbf{e}_M$, the production matrix $\mathbf{X}(\mathbf{y})$ and the intermediate consumption matrix $\mathbf{Z}(\mathbf{y})$ satisfy the requirements $\mathbf{X}(\mathbf{y}_0)=\mathbf{X}_0$ and $\mathbf{Z}(\mathbf{y}_0)=\mathbf{Z}_0$, respectively.

It is easy to see that fulfillment of the matrix calibration condition always entails the fulfillment of the vector calibration condition but the converse statement is obviously not true. Thus, the vector calibration condition is essentially a weakened version of model’s matrix calibration condition.

Within the formal framework of vector calibrating, the generalized input–output model with exogenous final demand (2) contains N linear equations with $2N$ unknowns. To make this model operational, it is sufficient to supplement it with an equation for the relationship between the output and intermediate consumption vectors, for example, in the form of a linear vector-valued cost function

$$\mathbf{Ze}_M = \mathbf{C}_0\mathbf{Xe}_M \,, \tag{6}$$

where $\mathbf{C}_0$ is N-dimensional square matrix of so-called input–output coefficients, determined a priori on the basis of supply and use tables for the base year 0. Having formally resolved the system (2), (6) with respect to $\mathbf{Xe}_M$, we obtain the following equation for the dependence of the product output vector on the final demand vector:

$$\mathbf{Xe}_M = (\mathbf{E}_N - \mathbf{C}_0)^{-1}\mathbf{y} \tag{7}$$

where $\mathbf{E}_N$ is the identity matrix of order N, and the matrix in parentheses is assumed to be non-singular. This equation not only demonstrates a simple technique for reducing the number of unknowns in (2), but is also very useful analytically: if the spectral radius of matrix $\mathbf{C}_0$ is less than one, the inverse matrix in (7) can be expanded into a convergent power series that can serve as an effective analytical tool for a detailed study of inter-sectoral interactions in the economy (see, e.g., Miller and Blair, 2009).

Relation (7) is widely used in input–output analysis as a universal way of transforming the product balance equation into various models with exogenous final demand for constructing product-by-product input–output tables. In authoritative international reference handbooks United Nations (2018) and Eurostat (2008) it is recommended using three types of transformations resembling (7) in the practice of input–output modeling, namely, based on the product technology assumption, the industry technology assumption, and the hybrid (mixed) technology assumption.

3.1. Calibrating the product technology model

The product technology pattern (or product technology model) in the accepted notation has the following form:

$$\mathbf{C}_0 = \mathbf{Z}_0\mathbf{X}_0^{-1} \tag{8}$$

– see, e.g., Kop Jansen and ten Raa (1990); ten Raa and Rueda-Cantuche (2003), (2007); Rueda-Cantuche (2011).

This formula implicitly assumes that in the economy under consideration the number of products N coincides with the number of industries M, that is, $M = N = K$, and the square production matrix of order K is non-singular. It should be emphasized that in accordance with the product technology assumption each product is produced in its own specific way, irrespective of the industry where it is produced (United Nations, 2018; Eurostat, 2008).

Substituting the product technology pattern (8) into (7) leads to the equation

$$\mathbf{X}\mathbf{e}_K = \left(\mathbf{E}_K - \mathbf{Z}_0\mathbf{X}_0^{-1}\right)^{-1}\mathbf{y} = \mathbf{X}_0\left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y} \tag{9}$$

provided that the square matrices of order K in parentheses are non-singular. In turn, substituting (8) and (9) into the formula for the linear vector-valued cost function (6) yields

$$\mathbf{Z}\mathbf{e}_K = \mathbf{Z}_0\mathbf{X}_0^{-1}\mathbf{X}\mathbf{e}_K = \mathbf{Z}_0\mathbf{X}_0^{-1}\mathbf{X}_0\left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y} = \mathbf{Z}_0\left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y}\,. \tag{10}$$

It is easy to see that at $\mathbf{y} = \mathbf{y}_0 = \left(\mathbf{X}_0 - \mathbf{Z}_0\right)\mathbf{e}_K$ equations (9) and (10) entails the statements $\mathbf{X}\mathbf{e}_K(\mathbf{y}_0) = \mathbf{X}_0\mathbf{e}_K$ и $\mathbf{Z}\mathbf{e}_K(\mathbf{y}_0) = \mathbf{Z}_0\mathbf{e}_K$. Thus, the product technology pattern guarantees the fulfillment of the vector calibration condition formulated above. At the same time, it is clear that equations (9) and (10) do not allow us to uniquely identify the output and intermediate consumption matrices as functions of the exogenous vector of final demand, since, by virtue of Properties 1 and 3 of the generalized product balance equation, they have an infinite set of solutions with respect to **X** and **Z**. Consequently, the product technology assumption, generally speaking, does not ensure the fulfillment of the matrix calibration condition without involving any additional a priori information.

3.2. Calibrating the industry technology model

The industry technology pattern (or industry technology model) in the accepted notation has the following form:

$$\mathbf{C}_0 = \mathbf{Z}_0\left\langle \mathbf{e}_N'\mathbf{X}_0\right\rangle^{-1}\mathbf{X}_0'\left\langle \mathbf{X}_0\mathbf{e}_M\right\rangle^{-1} \tag{11}$$

– see, e.g., Kop Jansen and ten Raa (1990); ten Raa and Rueda-Cantuche (2003), (2007); Rueda-Cantuche (2011).

It is clear that chosen form of the input–output coefficients matrix can be used for any combination of the number of products *N* and the number of industries *M* in the economy under consideration. It should be emphasized that according to the industry technology assumption each industry has its own specific way of production, irrespective of its product mix (United Nations, 2018; Eurostat, 2008).

Substituting the industry technology pattern (11) into (7) gives the equation

$$\mathbf{X}\mathbf{e}_M = \left(\mathbf{E}_N - \mathbf{Z}_0\left\langle \mathbf{e}_N'\mathbf{X}_0\right\rangle^{-1}\mathbf{X}_0'\left\langle \mathbf{X}_0\mathbf{e}_M\right\rangle^{-1}\right)^{-1}\mathbf{y} \tag{12}$$

provided that the square matrix of order *N* in parentheses is non-singular. In turn, substituting (11) and (12) into the formula for the linear vector-valued cost function (6) yields

$$\mathbf{Z}\mathbf{e}_M = \mathbf{C}_0\mathbf{X}\mathbf{e}_M = \mathbf{Z}_0\langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{X}'_0\langle\mathbf{X}_0\mathbf{e}_M\rangle^{-1}\left(\mathbf{E}_N - \mathbf{Z}_0\langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{X}'_0\langle\mathbf{X}_0\mathbf{e}_M\rangle^{-1}\right)^{-1}\mathbf{y}\,. \tag{13}$$

Let us prove that equation (12) generates the vector calibrating formula $\mathbf{X}\mathbf{e}_M(\mathbf{y}_0) = \mathbf{X}_0\mathbf{e}_M$ at $\mathbf{y} = \mathbf{y}_0 = (\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_M$. Indeed, the equality

$$\left(\mathbf{E}_N - \mathbf{Z}_0\langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{X}'_0\langle\mathbf{X}_0\mathbf{e}_M\rangle^{-1}\right)^{-1}(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_M = \mathbf{X}_0\mathbf{e}_M$$

can be transformed into

$$(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_M = \left(\mathbf{E}_N - \mathbf{Z}_0\langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{X}'_0\langle\mathbf{X}_0\mathbf{e}_M\rangle^{-1}\right)\mathbf{X}_0\mathbf{e}_M$$

from where after opening all circle brackets we get the identity

$$\mathbf{X}_0\mathbf{e}_M - \mathbf{Z}_0\mathbf{e}_M = \mathbf{X}_0\mathbf{e}_M - \mathbf{Z}_0\langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{X}'_0\langle\mathbf{X}_0\mathbf{e}_M\rangle^{-1}\mathbf{X}_0\mathbf{e}_M = \mathbf{X}_0\mathbf{e}_M - \mathbf{Z}_0\mathbf{e}_M\,.$$

This means that first requirement of the vector calibration condition is satisfied under the industry technology assumption. Moreover, as it follows from equation (13), fulfillment of first requirement entails the fulfillment of the second requirement since

$$\mathbf{Z}\mathbf{e}_M(\mathbf{y}_0) = \mathbf{Z}_0\langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{X}'_0\langle\mathbf{X}_0\mathbf{e}_M\rangle^{-1}\mathbf{X}_0\mathbf{e}_M = \mathbf{Z}_0\langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{X}'_0\mathbf{e}_N = \mathbf{Z}_0\mathbf{e}_M\,.$$

It is important to emphasize that equation (12) and (13) have an infinite set of solutions with respect to **X** and **Z** as well as in previous case of product technology assumption. Therefore, the industry technology assumption, generally speaking, does not ensure the fulfillment of the matrix calibration condition without involving any additional a priori information.

3.3. Calibrating the hybrid technology model

The hybrid technology pattern (or hybrid technology model) in the accepted notation has the following form:

$$\mathbf{C}_0 = (\mathbf{Z}_0 - \mathbf{X}_{02})\,\mathbf{X}_{01}^{-1} \tag{14}$$

– see, e.g., Kop Jansen and ten Raa (1990). Here $\mathbf{X}_{01}$ and $\mathbf{X}_{02}$ are the matrix terms of exogenous additive decomposition of the initial production matrix $\mathbf{X}_0 = \mathbf{X}_{01} + \mathbf{X}_{02}$.

The hybrid technology pattern could be considered as a combination of the product and industry technology models – see, e.g., Eurostat (2008). As in the case of the product technology assumption, formula (14) implicitly assumes that in the economy under consideration the number of products N coincides with the number of industries M, that is, $M = N = K$, and the square matrix $\mathbf{X}_{01}$ of order K is non-singular.

Substituting the hybrid technology pattern (14) into (7) leads to the equation

$$\mathbf{X}\mathbf{e}_K = \left[\mathbf{E}_K - (\mathbf{Z}_0 - \mathbf{X}_{02})\mathbf{X}_{01}^{-1}\right]^{-1}\mathbf{y} = \mathbf{X}_{01}(\mathbf{X}_{01} - \mathbf{Z}_0 + \mathbf{X}_{02})^{-1}\mathbf{y} = \mathbf{X}_{01}(\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y} \tag{15}$$

provided that the square matrices of order K in parentheses are non-singular. In turn, substituting

(14) and (15) into the formula for the linear vector-valued cost function (6) yields

$$\mathbf{Z}\mathbf{e}_K = (\mathbf{Z}_0 - \mathbf{X}_{02})\,\mathbf{X}_{01}^{-1}\mathbf{X}\mathbf{e}_K = (\mathbf{Z}_0 - \mathbf{X}_{02})(\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}\,. \tag{16}$$

At $\mathbf{y} = \mathbf{y}_0 = (\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_M$, the equations (15) and (16) can be transforms as follows:

$$\mathbf{X}\mathbf{e}_K(\mathbf{y}_0) = \mathbf{X}_{01}(\mathbf{X}_0 - \mathbf{Z}_0)^{-1}(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_K = \mathbf{X}_{01}\mathbf{e}_K = \mathbf{X}_0\mathbf{e}_K - \mathbf{X}_{02}\mathbf{e}_K \neq \mathbf{X}_0\mathbf{e}_K\,,$$

$$\mathbf{Z}\mathbf{e}_K(\mathbf{y}_0) = (\mathbf{Z}_0 - \mathbf{X}_{02})(\mathbf{X}_0 - \mathbf{Z}_0)^{-1}(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_K = \mathbf{Z}_0\mathbf{e}_K - \mathbf{X}_{02}\mathbf{e}_K \neq \mathbf{Z}_0\mathbf{e}_K\,.$$

The expressions obtained serve as explicit manifestation of tautological Property 2 of the generalized product balance equation (2). Thus, the hybrid technology assumption, generally speaking, does not ensure the fulfillment of neither matrix calibration condition nor even vector calibration condition without involving any additional a priori information.

4. Introducing price and quantity parameters into the product balance equation

Purposeful varying the exogenous elements of final demand in the generalized demand-driven input–output model (2) changes the price and quantity proportions in the resulting (disturbed) supply-use table. The most general (and widely used in macroeconomic statistics) way of describing the influence of exogenous changes on production and intermediate consumption matrices **X** and **Z** is apparently using the following multiplicative patterns

$$\mathbf{X} = \mathbf{P}_\mathbf{X} \circ \mathbf{Q}_\mathbf{X} \circ \mathbf{X}_0\,, \qquad\qquad \mathbf{Z} = \mathbf{P}_\mathbf{Z} \circ \mathbf{Q}_\mathbf{Z} \circ \mathbf{Z}_0\,, \tag{17}$$

where $\mathbf{P}_\mathbf{X}$ and $\mathbf{P}_\mathbf{Z}$ are $N\times M$-dimensional matrices of the relative price indices for products in output and intermediate consumption, $\mathbf{Q}_\mathbf{X}$ and $\mathbf{Q}_\mathbf{Z}$ are $N\times M$ matrices of the relative quantity (physical volume) indices for industries (producers and consumers), and the character " $\circ$ " denotes the Hadamard (element-wise) product of two matrices with the same dimensions. However, this method of transforming the production and intermediate consumption matrices cannot be considered as operational because it leads to increasing the number of variables in the generalized linear model (2) from $N(2M+1)$ to $N(4M+1)$.

To make the patterns (17) more operational it is necessary to simplify the situation by introducing some additional restrictions.

Simplifying assumption 1. Letting $\mathbf{P}_\mathbf{X} = \mathbf{P}_\mathbf{Z} = \mathbf{P}$ and $\mathbf{Q}_\mathbf{X} = \mathbf{Q}_\mathbf{Z} = \mathbf{Q}$.

Simplifying assumption 2. Letting $\mathbf{P} = \mathbf{p} \otimes \mathbf{e}'_M$, $\mathbf{Q} = \mathbf{e}_N \otimes \mathbf{q}'$ where **p** is a column vector of the relative price indices on products with dimensions $N\times 1$, **q** is a column vector of the relative quantity indices for industries with dimensions $M\times 1$, and the character " $\otimes$ " denotes the Kronecker product for two matrices.

Under Simplifying assumptions 1 and 2, multiplicative patterns (17) in the usual matrix notation take the following form:

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0\hat{\mathbf{q}}, \tag{18}$$

where putting a "hat" over vector's symbol (or angled bracketing around it, as earlier in Section 2) denotes a diagonal matrix with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697).

Nonlinear multiplicative patterns (18) together with the generalized product balance equation (2) provide a combined quantity'n'price description of economy's response to changes of the final demand components in a generalized nonlinear demand-driven input–output model

$$\hat{\mathbf{p}}\mathbf{X}_0\mathbf{q} = \hat{\mathbf{p}}\mathbf{Z}_0\mathbf{q} + \mathbf{y}. \tag{19}$$

Note that the obvious properties of diagonal matrices $\hat{\mathbf{q}}\mathbf{e}_M = \mathbf{q}$ and $\hat{\mathbf{p}}\mathbf{e}_N = \mathbf{p}$ were used while transiting from (18) to (19).

It should be emphasized that in the nonlinear model (19) the vectors $\mathbf{p}$ and $\mathbf{q}$ cannot be identified unambiguously because the formulas (18) define hyperbolically homogeneous functions $p_n q_m$ since

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0\hat{\mathbf{q}} = \mathbf{p}\mathbf{q}' \circ \mathbf{X}_0, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0\hat{\mathbf{q}} = \mathbf{p}\mathbf{q}' \circ \mathbf{Z}_0$$

and $\mathbf{p}\mathbf{q}' = c\mathbf{p} \cdot \mathbf{q}'/c$ where c is an arbitrary nonzero scalar. At the same time, the generalized nonlinear model (19) can be easily linearized in four ways that seem to be formally feasible and operational.

5. Classification of the linear demand-driven models with price and quantity parameters

First simple way to convert the nonlinear demand-driven input–output model (19) into a linear form is setting the quantity parameters matrix equal to identity matrix of order *M*, i.e., $\hat{\mathbf{q}} = \mathbf{E}_M$. As a result, nonlinear multiplicative patterns (18) become

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0, \tag{20}$$

and the following linear equation with vector of price parameters $\mathbf{p}$ arises:

$$\hat{\mathbf{p}}\mathbf{X}_0\mathbf{e}_M = \hat{\mathbf{p}}\mathbf{Z}_0\mathbf{e}_M + \mathbf{y}. \tag{21}$$

It is quite natural to classify equation (21) as a linear demand-driven price model at constant production levels. (It is important to note that main difference between this model and the Leontief price model, which is well known in input–output analysis (see, e.g., Miller and Blair, 2009), is manifested in their exogenous factors – final demand in the first case and value added coefficients in the second one.)

Another simple way to transform the nonlinear demand-driven input–output model (19) into a linear form is setting the price parameters matrix equal to identity matrix of order *N*, i.e.,

$\hat{\mathbf{p}} = \mathbf{E}_N$. As a consequence, nonlinear multiplicative patterns (18) become

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}, \tag{22}$$

and the following linear equation with vector of quantity parameters **q** appears:

$$\mathbf{X}_0\mathbf{q} = \mathbf{Z}_0\mathbf{q} + \mathbf{y}. \tag{23}$$

One can consider equation (23) as a linear demand-driven quantity (or volume) model at constant prices on products (because $\mathbf{p} = \mathbf{e}_N$).

The variety of methods for transforming the generalized nonlinear demand-driven input–output model (19) to one or another linear form is not limited to constructing the price model (21), in which the role of unknown parameters is played by relative price indices, and the quantity model (23) with relative physical volume parameters for output and intermediate consumption. From formal viewpoint, there are two alternative variants of linearizing the generalized nonlinear input–output model (19) with a combined inclusion of price and volume parameters: the linear demand-driven model with ***price changes*** in the production matrix and ***volume changes*** in the intermediate consumption matrix

$$\hat{\mathbf{p}}\mathbf{X}_0\mathbf{e}_M = \mathbf{Z}_0\mathbf{q} + \mathbf{y} \tag{24}$$

based on linear multiplicative patterns

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0, \qquad \mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}, \tag{25}$$

and the linear demand-driven model with, vice versa, ***volume changes*** in the output matrix and ***price changes*** in the intermediate consumption matrix

$$\mathbf{X}_0\mathbf{q} = \hat{\mathbf{p}}\mathbf{Z}_0\mathbf{e}_M + \mathbf{y}, \tag{26}$$

based on linear multiplicative patterns

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0. \tag{27}$$

The listed above variants of linearizing the generalized nonlinear model (19) can be classified jointly as the linear demand-driven models with combined set of price and quantity parameters. However, the eclecticism of the economic interpretation of these two models is quite obvious, and both of them seem to be out of economic sense.

Nevertheless, the linear models (24) and (26) as formal objects represent a certain theoretical interest and deserve further study. Based on the order of appearing the parameters **p** and **q** in the corresponding formulas, we will henceforth mention (24) with the patterns (25) as the linear combined price/quantity demand-driven model and (26) with the patterns (27) as the linear combined quantity/price demand-driven model.

6. The linear demand-driven price model

As noted above, at constant levels of production and intermediate consumption in economy's industries one can set $\hat{\mathbf{q}} = \mathbf{E}_M$ and $\mathbf{q} = \mathbf{e}_M$; then nonlinear multiplicative patterns (18) are transformed into linear form (20) and besides the generalized nonlinear demand-driven input–output model (19) becomes linear and can be written as (21).

Formally, the model (21) contains *N* linear equations with *N* scalar unknowns **p** and, as one might expect, should be strictly identifiable with final demand as exogenous vector of dimensions *N*×1. However, to make the situation more operational, it is necessary to move in equation (21) from the unknown diagonal matrix $\hat{\mathbf{p}}$ to the unknown vector of price parameters **p** and write it as follows:

$$\langle \mathbf{X}_0 \mathbf{e}_M \rangle \mathbf{p} = \langle \mathbf{Z}_0 \mathbf{e}_M \rangle \mathbf{p} + \mathbf{y} \tag{28}$$

where one has used an obvious property $\hat{\mathbf{a}}\mathbf{b} = \hat{\mathbf{b}}\mathbf{a}$ that is satisfied for any pair of column vectors **a** and **b** of the same dimension and their diagonalizations.

It should be emphasized again that model (21) and its transformed version (28) provide a description of economy's response to changes in the components of final demand exclusively in terms of relative prices on products. That is the reason why it can be called a linear demand-driven price model, as it was done in a title of Section 6.

The solution of equation (28) with respect to unknown vector **p** provides the following analytical representation for the vector of relative prices on products produced and consumed in the economy:

$$\mathbf{p}_* = \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \langle \mathbf{Z}_0 \mathbf{e}_M \rangle\right)^{-1} \mathbf{y} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{P}_*^{-1} \mathbf{y} \tag{29}$$

where $\mathbf{P}_* = \mathbf{E}_N - \langle \mathbf{Z}_0 \mathbf{e}_M \rangle \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1}$ is the non-singular square matrix of order *N*. Note that its inverse matrix is equal to $\mathbf{P}_*^{-1} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle \hat{\mathbf{y}}_0^{-1}$ since $\mathbf{P}_* \langle \mathbf{X}_0 \mathbf{e}_M \rangle = \langle \mathbf{X}_0 \mathbf{e}_M \rangle - \langle \mathbf{Z}_0 \mathbf{e}_M \rangle = \hat{\mathbf{y}}_0$.

It is not difficult to show that analytical representation (29) and linear multiplicative patterns (20) yields the following equations of linkages between vectors and matrices of output and intermediate consumption and the final demand vector:

$$\mathbf{X}\mathbf{e}_M = \hat{\mathbf{p}}_* \mathbf{X}_0 \mathbf{e}_M = \langle \mathbf{X}_0 \mathbf{e}_M \rangle \mathbf{p}_* = \mathbf{P}_*^{-1} \mathbf{y}, \qquad \mathbf{Z}\mathbf{e}_M = \langle \mathbf{Z}_0 \mathbf{e}_M \rangle \mathbf{p}_* = \langle \mathbf{Z}_0 \mathbf{e}_M \rangle \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{P}_*^{-1} \mathbf{y};$$

$$\mathbf{X} = \hat{\mathbf{p}}_* \mathbf{X}_0 = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \langle \mathbf{P}_*^{-1} \mathbf{y} \rangle \mathbf{X}_0, \qquad \mathbf{Z} = \hat{\mathbf{p}}_* \mathbf{Z}_0 = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \langle \mathbf{P}_*^{-1} \mathbf{y} \rangle \mathbf{Z}_0.$$

These expressions indicate that linear vector-valued cost function (6) for demand-driven price model (21) is

$$\mathbf{Z}\mathbf{e}_M = \langle \mathbf{Z}_0 \mathbf{e}_M \rangle \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}\mathbf{e}_M$$

that is interesting to compare with correspondent formulas for the cases of commodity technology (8), industry technology (11) and hybrid technology (14).

It is easy to see that at $\mathbf{y} = \mathbf{y}_0$ the column vector of relative prices on products (29) becomes equal to

$$\mathbf{p}_*(\mathbf{y}_0) = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{P}_*^{-1} \mathbf{y}_0 = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \langle \mathbf{X}_0 \mathbf{e}_M \rangle \hat{\mathbf{y}}_0^{-1} \mathbf{y}_0 = \mathbf{e}_N ,$$

from which it immediately follows that

$$\mathbf{X}\mathbf{e}_M(\mathbf{y}_0) = \hat{\mathbf{p}}_*(\mathbf{y}_0)\mathbf{X}_0\mathbf{e}_M = \mathbf{X}_0\mathbf{e}_M , \qquad \mathbf{Z}\mathbf{e}_M(\mathbf{y}_0) = \langle \mathbf{Z}_0\mathbf{e}_M \rangle \mathbf{p}_*(\mathbf{y}_0) = \mathbf{Z}_0\mathbf{e}_M ;$$

$$\mathbf{X}(\mathbf{y}_0) = \hat{\mathbf{p}}_*(\mathbf{y}_0)\mathbf{X}_0 = \mathbf{X}_0 , \qquad \mathbf{Z}(\mathbf{y}_0) = \hat{\mathbf{p}}_*(\mathbf{y}_0)\mathbf{Z}_0 = \mathbf{Z}_0 .$$

Thus, the linear demand-driven price model (21) ensures that vector and matrix calibration conditions are met.

Nevertheless, it can be argued that linear demand-driven price model is almost trivial. In fact, the formula for calculating the N-dimensional vector of relative prices on products (29), as is easy to verify, is equivalent to an N-fold application of the scalar formula

$$p_n = \left(\sum_{m=1}^{M} x_{nm}^0 - \sum_{m=1}^{M} z_{nm}^0 \right)^{-1} y_n = \frac{y_n}{y_n^0} , \qquad n = 1 \div N ,$$

for each of the products produced and consumed in the economy.

7. The linear demand-driven quantity model

At constant prices on products it is natural to set $\hat{\mathbf{p}} = \mathbf{E}_N$ and $\mathbf{p} = \mathbf{e}_N$; then nonlinear multiplicative patterns (18) are transformed into linear form (22) and besides the generalized nonlinear demand-driven input–output model (19) becomes linear and can be written as (23). From a formal viewpoint, system (23) contains N linear equations with M scalar variables $\mathbf{q}$ and have a unique solution only if in the economy under consideration the number of products N coincides with the number of industries M, i.e., if $M = N = K$.

Model (23) provides a description of the reaction of economy's production system to changes in final demand components exclusively in terms of relative physical volumes of production and intermediate consumption in industries. That is the reason why it can be called a linear demand-driven quantity model.

The solution of the vector equation (23) with respect to the vector $\mathbf{q}$ (provided that square matrix $\mathbf{X}_0 - \mathbf{Z}_0$ of order K is invertible) gives the following analytical representation for the vector of the physical volumes of products produced and consumed in the economy:

$$\mathbf{q}_* = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y} = \left[\left(\mathbf{E}_K - \mathbf{Z}_0 \mathbf{X}_0^{-1} \right) \mathbf{X}_0 \right]^{-1} \mathbf{y} = \mathbf{X}_0^{-1} \mathbf{Q}_*^{-1} \mathbf{y} , \tag{30}$$

where $\mathbf{Q}_* = \mathbf{E}_K - \mathbf{Z}_0\mathbf{X}_0^{-1}$ is the non-singular square matrix of order *N*.

It is easy to show that analytical representation (30) and linear multiplicative patterns (22) brings the following equations of linkages between vectors and matrices of output and intermediate consumption and the final demand vector:

$$\mathbf{X}\mathbf{e}_K = \mathbf{X}_0\mathbf{q}_* = \mathbf{Q}_*^{-1}\mathbf{y}, \qquad \mathbf{Z}\mathbf{e}_K = \mathbf{Z}_0\mathbf{q}_* = \mathbf{Z}_0\mathbf{X}_0^{-1}\mathbf{Q}_*^{-1}\mathbf{y};$$

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}_* = \mathbf{X}_0\left\langle \mathbf{X}_0^{-1}\mathbf{Q}_*^{-1}\mathbf{y} \right\rangle, \qquad \mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}_* = \mathbf{Z}_0\left\langle \mathbf{X}_0^{-1}\mathbf{Q}_*^{-1}\mathbf{y} \right\rangle.$$

It should be noted that at $\mathbf{y} = \mathbf{y}_0$ the column vector of relative quantity indices (30) becomes equal to

$$\mathbf{q}_*(\mathbf{y}_0) = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1}(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_K = \mathbf{e}_K,$$

from which it immediately follows that

$$\mathbf{X}\mathbf{e}_K(\mathbf{y}_0) = \mathbf{X}_0\mathbf{q}_*(\mathbf{y}_0) = \mathbf{X}_0\mathbf{e}_K, \qquad \mathbf{Z}\mathbf{e}_K(\mathbf{y}_0) = \mathbf{Z}_0\mathbf{q}_*(\mathbf{y}_0) = \mathbf{Z}_0\mathbf{e}_K;$$

$$\mathbf{X}(\mathbf{y}_0) = \mathbf{X}_0\hat{\mathbf{q}}_*(\mathbf{y}_0) = \mathbf{X}_0, \qquad \mathbf{Z}(\mathbf{y}_0) = \mathbf{Z}_0\hat{\mathbf{q}}_*(\mathbf{y}_0) = \mathbf{Z}_0.$$

Thus, the linear demand-driven quantity model (23) ensures that vector and matrix calibration conditions are met.

The presented above formula for calculating the matrix $\mathbf{Q}_*$ indicates that linear vector-valued cost function (6) for demand-driven quantity model (23) is

$$\mathbf{Z}\mathbf{e}_K = \mathbf{Z}_0\mathbf{X}_0^{-1}\mathbf{X}\mathbf{e}_K,$$

and, moreover, its matrix of input–output coefficients exactly coincides with matrix (8) – a set of input–output coefficients considered within a framework of product technology assumption. As noted above, the linear demand-driven quantity model operates at constant prices, i.e., at $\hat{\mathbf{p}} = \mathbf{E}_N$ and $\mathbf{p} = \mathbf{e}_N$. It is worth to mention here that this conclusion exactly corresponds to price invariance axiom fulfillment established by Kop Jansen and ten Raa (1990) for product technology pattern.

Nevertheless, demand-driven quantity model (23) does fundamentally differ from the input–output model (7), (8) based on the product technology assumption. The formal premises for constructing model (23) are the generalized product balance equation (2) and a pair of nonlinear multiplicative patterns (19) together with the linearizing conditions $\hat{\mathbf{p}} = \mathbf{E}_N$ and $\mathbf{p} = \mathbf{e}_N$, whereas model (7), (8) is generated by the product balance equation (2) and the linear vector-valued cost function (6) with the matrix of input–output coefficients (8). An important consequence of these differences in the formal premises for constructing the models under consideration is the impossibility of unambiguous identifying the matrices of output and intermediate consumption of products as functions of the exogenous final demand vector in model (7), (8) that entails a noncompliance with the matrix calibration condition.

8. The linear combined price/quantity demand-driven model

The formal framework for constructing the linear combined price/quantity model (24) is the multiplicative patterns (25). The model contains N linear equations with N+M scalar components of parameter vectors $\mathbf{p}$ and $\mathbf{q}$ and therefore requires involving some additional information to ensure its strict identifiability. As an instance, one can try to introduce the instrumental linkages between quantity and price parameters either in the form $\mathbf{q} = \mathbf{R}_0\mathbf{p}$ or in the form $\mathbf{p} = \mathbf{S}_0\mathbf{q}$ where $\mathbf{R}_0$ is a matrix of dimension $M\times N$ and $\mathbf{S}_0$ is a matrix of dimension $N\times M$. Note that the matrices $\mathbf{R}_0$ and $\mathbf{S}_0$ should be based on initial data from the product balance identity (1).

Substituting the first instrumental linkage into model (24) leads to the equation

$$\left(\langle \mathbf{X}_0\mathbf{e}_M \rangle - \mathbf{Z}_0\mathbf{R}_0\right)\mathbf{p} = \mathbf{y} . \tag{31}$$

In turn, substituting the second instrumental linkage yields

$$\left(\langle \mathbf{X}_0\mathbf{e}_M \rangle \mathbf{S}_0 - \mathbf{Z}_0\right)\mathbf{q} = \mathbf{y} . \tag{32}$$

It is easy to see that square matrix in parentheses in (31) has order N, whereas the matrix in parentheses in (32) is rectangular and has dimension $N\times M$. Thus, equation (32) cannot be solved with respect to vector $\mathbf{q}$ and becomes operational only if in the economy under consideration the number of products N coincides with the number of industries M, i.e., provided that $M = N = K$.

It is interesting to note that the original price/quantity model (24) characterizes the reaction of economy's production system to varying the final demand components in terms of price changes in the output matrix and volume changes in the intermediate consumption matrix, while the transformed model (31) essentially turns out to be a price model and transformed model (32) is actually a quantity model. The noted metamorphoses seem to be direct consequence of introducing two instrumental linkages between quantity and price parameters into the original model in order to ensure its identifiability.

Clearly, the transformed models (31) and (32) satisfy both calibration conditions if and only if $\mathbf{p}(\mathbf{y}_0) = \mathbf{e}_N$ and $\mathbf{q}(\mathbf{y}_0) = \mathbf{e}_K$. For model (31), we have

$$\mathbf{y}_0 = \mathbf{X}_0\mathbf{e}_M - \mathbf{Z}_0\mathbf{e}_M = \left(\langle \mathbf{X}_0\mathbf{e}_M \rangle - \mathbf{Z}_0\mathbf{R}_0\right)\mathbf{e}_N = \mathbf{X}_0\mathbf{e}_M - \mathbf{Z}_0\mathbf{R}_0\mathbf{e}_N$$

from where one can derive the requirement $\mathbf{R}_0\mathbf{e}_N = \mathbf{e}_M$, i.e., matrix $\mathbf{R}_0$ should be right stochastic. In turn, for model (32) at $M = N = K$ we get

$$\mathbf{y}_0 = \mathbf{X}_0\mathbf{e}_K - \mathbf{Z}_0\mathbf{e}_K = \left(\langle \mathbf{X}_0\mathbf{e}_K \rangle \mathbf{S}_0 - \mathbf{Z}_0\right)\mathbf{e}_K = \langle \mathbf{X}_0\mathbf{e}_K \rangle \mathbf{S}_0\mathbf{e}_K - \mathbf{Z}_0\mathbf{e}_K$$

from where it follows the resembling requirement $\mathbf{S}_0\mathbf{e}_K = \mathbf{e}_K$.

Thus, the linear combined price/quantity model (24) with the instrumental linkages between quantity and price parameters generates two (price and quantity) families of input–output models (31) and (32) on a set of right stochastic matrices $\mathbf{R}_0$ и $\mathbf{S}_0$. A variety of these models is defined by practical possibilities of choosing the linking matrices based on initial data from the product balance identity (1).

Two examples of constructing the linear combined price/quantity model (24) with two specifications of the linking right stochastic matrices are given below.

8.1. The example: $\mathbf{R}_0$ is the transpose of $M\times N$ product mix matrix

Model: the linear combined price/quantity demand-driven model (24), version (31)

Dimensions: $M\times N$

Instrumental linkage specification: the transpose of the product mix matrix $\mathbf{R}_0 = \langle \mathbf{X}_0'\mathbf{e}_N \rangle^{-1}\mathbf{X}_0'$

Derivation of the formulas:

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0, \qquad \mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}, \qquad \mathbf{q} = \mathbf{R}_0\mathbf{p};$$

$$\hat{\mathbf{p}}\mathbf{X}_0\mathbf{e}_M = \mathbf{Z}_0\mathbf{q} + \mathbf{y}, \quad \langle \mathbf{X}_0\mathbf{e}_M \rangle \mathbf{p} - \mathbf{Z}_0\mathbf{R}_0\mathbf{p} = \mathbf{y};$$

$$\langle \mathbf{X}_0\mathbf{e}_M \rangle \mathbf{p} - \mathbf{Z}_0\langle \mathbf{X}_0'\mathbf{e}_N \rangle^{-1}\mathbf{X}_0'\mathbf{p} = \mathbf{y};$$

$$\mathbf{p}_\circ = \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1}\left(\mathbf{E}_N - \mathbf{Z}_0\langle \mathbf{X}_0'\mathbf{e}_N \rangle^{-1}\mathbf{X}_0'\langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1}\right)^{-1}\mathbf{y} = \langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1}\mathbf{P}_\circ^{-1}\mathbf{y};$$

$$\mathbf{q}_\circ = \mathbf{R}_0\mathbf{p}_\circ = \langle \mathbf{X}_0'\mathbf{e}_N \rangle^{-1}\mathbf{X}_0'\langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1}\mathbf{P}_\circ^{-1}\mathbf{y};$$

$$\mathbf{X}\mathbf{e}_M = \hat{\mathbf{p}}_\circ\mathbf{X}_0\mathbf{e}_M = \langle \mathbf{X}_0\mathbf{e}_M \rangle \mathbf{p}_\circ = \mathbf{P}_\circ^{-1}\mathbf{y}, \qquad \mathbf{Z}\mathbf{e}_M = \mathbf{Z}_0\mathbf{q}_\circ = \mathbf{Z}_0\langle \mathbf{X}_0'\mathbf{e}_N \rangle^{-1}\mathbf{X}_0'\langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1}\mathbf{P}_\circ^{-1}\mathbf{y};$$

$$\mathbf{X} = \hat{\mathbf{p}}_\circ\mathbf{X}_0 = \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}\langle \mathbf{P}_\circ^{-1}\mathbf{y} \rangle \mathbf{X}_0, \qquad \mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}_\circ = \mathbf{Z}_0\langle \mathbf{X}_0'\mathbf{e}_N \rangle^{-1}\left\langle \mathbf{X}_0'\langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1}\mathbf{P}_\circ^{-1}\mathbf{y} \right\rangle$$

Calibration: the solution satisfies both vector and matrix conditions

Input–output coefficients: $\mathbf{C}_0^\circ = \mathbf{Z}_0\langle \mathbf{X}_0'\mathbf{e}_N \rangle^{-1}\mathbf{X}_0'\langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1}$

Conclusion: input–output coefficients matrix coincides with the industry technology pattern (11)

8.2. The example: $\mathbf{S}_0$ is the inverse of $K\times K$ market shares matrix

Model: the linear combined price/quantity demand-driven model (24), version (32)

Dimensions: $K\times K$

Instrumental linkage specification: the inverse of the market shares matrix $\mathbf{S}_0 = \mathbf{X}_0^{-1}\langle \mathbf{X}_0\mathbf{e}_K \rangle$

Derivation of the formulas:

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0, \qquad \mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}, \qquad \mathbf{p} = \mathbf{S}_0\mathbf{q};$$

$$\hat{\mathbf{p}}\mathbf{X}_0\mathbf{e}_K = \mathbf{Z}_0\mathbf{q} + \mathbf{y}, \qquad \langle \mathbf{X}_0\mathbf{e}_K \rangle \mathbf{S}_0\mathbf{q} - \mathbf{Z}_0\mathbf{q} = \mathbf{y};$$

$$\langle \mathbf{X}_0\mathbf{e}_K \rangle \mathbf{X}_0^{-1} \langle \mathbf{X}_0\mathbf{e}_K \rangle \mathbf{q} - \mathbf{Z}_0\mathbf{q} = \mathbf{y} \;;$$

$$\mathbf{q}_\circ = \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \mathbf{X}_0 \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \left(\mathbf{E}_K - \mathbf{Z}_0 \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \mathbf{X}_0 \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \right)^{-1} \mathbf{y} = \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \mathbf{X}_0 \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \mathbf{Q}_\circ^{-1}\mathbf{y} \;;$$

$$\mathbf{p}_\circ = \mathbf{S}_0\mathbf{q}_\circ = \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \mathbf{Q}_\circ^{-1}\mathbf{y} \;;$$

$$\mathbf{X}\mathbf{e}_K = \hat{\mathbf{p}}_\circ \mathbf{X}_0\mathbf{e}_K = \langle \mathbf{X}_0\mathbf{e}_K \rangle \mathbf{p}_\circ = \mathbf{Q}_\circ^{-1}\mathbf{y}, \qquad \mathbf{Z}\mathbf{e}_K = \mathbf{Z}_0\mathbf{q}_\circ = \mathbf{Z}_0 \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \mathbf{X}_0 \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \mathbf{Q}_\circ^{-1}\mathbf{y} \;;$$

$$\mathbf{X} = \hat{\mathbf{p}}_\circ \mathbf{X}_0 = \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \langle \mathbf{Q}_\circ^{-1}\mathbf{y} \rangle \mathbf{X}_0, \qquad \mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}_\circ = \mathbf{Z}_0 \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \left\langle \mathbf{X}_0 \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \mathbf{Q}_\circ^{-1}\mathbf{y} \right\rangle$$

Calibration: the solution satisfies both vector and matrix conditions

Input–output coefficients: $\mathbf{C}_0^\circ = \mathbf{Z}_0 \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1} \mathbf{X}_0 \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}$

Conclusion: the model is not mentioned in the special literature apparently

9. The linear combined quantity/price demand-driven model

The formal framework for constructing the linear combined quantity/price model (26) is the multiplicative patterns (27). The model contains N linear equations with M+N scalar components of parameter vectors $\mathbf{q}$ and $\mathbf{p}$ and therefore requires involving some additional information to ensure its strict identifiability. As in previous case, one can try to introduce the instrumental linkages between quantity and price parameters either in the form $\mathbf{q} = \mathbf{R}_0\mathbf{p}$ or in the form $\mathbf{p} = \mathbf{S}_0\mathbf{q}$ where $\mathbf{R}_0$ is a matrix of dimension $M\times N$ and $\mathbf{S}_0$ is a matrix of dimension $N\times M$. Recall that the matrices $\mathbf{R}_0$ and $\mathbf{S}_0$ should be based on initial data from the product balance identity (1).

Substituting the first instrumental linkage into model (26) leads to the equation

$$\left(\mathbf{X}_0\mathbf{R}_0 - \langle \mathbf{Z}_0\mathbf{e}_M \rangle \right) \mathbf{p} = \mathbf{y} \,. \tag{33}$$

In turn, substituting the second instrumental linkage yields

$$\left(\mathbf{X}_0 - \langle \mathbf{Z}_0\mathbf{e}_M \rangle \mathbf{S}_0 \right) \mathbf{q} = \mathbf{y} \,. \tag{34}$$

It is easy to see that square matrix in parentheses in (33) has order N, whereas the matrix in parentheses in (34) is rectangular and has dimension $N\times M$. Thus, equation (34) cannot be solved with respect to vector $\mathbf{q}$ and becomes operational only if in the economy under consideration the number of products N coincides with the number of industries M, i.e., provided that $M = N = K$.

It is interesting to note that the original quantity/price (26) characterizes the reaction of economy's production system to varying the final demand components in terms of volume changes in the output matrix and price changes in the intermediate consumption matrix, while the transformed model (33) actually represent a price model and transformed model (34) is essentially a quantity model. The noted metamorphoses seem to be direct consequence of introducing two instrumental linkages between quantity and price parameters into the original model in order to ensure its identifiability.

Clearly, the transformed models (33) and (34) satisfy both calibration conditions if and only if $\mathbf{p}(\mathbf{y}_0) = \mathbf{e}_N$ and $\mathbf{q}(\mathbf{y}_0) = \mathbf{e}_K$. For model (33), we have

$$\mathbf{y}_0 = \mathbf{X}_0\mathbf{e}_M - \mathbf{Z}_0\mathbf{e}_M = \left(\mathbf{X}_0\mathbf{R}_0 - \langle \mathbf{Z}_0\mathbf{e}_M \rangle\right)\mathbf{e}_N = \mathbf{X}_0\mathbf{R}_0\mathbf{e}_N - \mathbf{Z}_0\mathbf{e}_M$$

from where one can derive the requirement $\mathbf{R}_0\mathbf{e}_N = \mathbf{e}_M$, i.e., matrix $\mathbf{R}_0$ should be right stochastic, as earlier. In turn, for model (34) at $M = N = K$ we get

$$\mathbf{y}_0 = \mathbf{X}_0\mathbf{e}_K - \mathbf{Z}_0\mathbf{e}_K = \left(\mathbf{X}_0 - \langle \mathbf{Z}_0\mathbf{e}_K \rangle \mathbf{S}_0\right)\mathbf{e}_K = \mathbf{X}_0\mathbf{e}_K - \langle \mathbf{Z}_0\mathbf{e}_K \rangle \mathbf{S}_0\mathbf{e}_K$$

from where it follows the resembling requirement $\mathbf{S}_0\mathbf{e}_K = \mathbf{e}_K$.

Thus, the linear combined quantity/price model (26) with the instrumental linkages between quantity and price parameters generates two (price and quantity) families of input–output models (33) and (34) on a set of right stochastic matrices $\mathbf{R}_0$ и $\mathbf{S}_0$. A variety of these models is defined by practical possibilities of choosing the linking matrices based on initial data from the product balance identity (1).

Two examples of constructing the linear combined quantity/price model (26) with two specifications of the linking right stochastic matrices are given below.

9.1. The example: $\mathbf{R}_0$ is the inverse of $K\times K$ market shares matrix

Model: the linear combined quantity/price demand-driven model (26), version (33)

Dimensions: $K\times K$

Instrumental linkage specification: the inverse of the market shares matrix $\mathbf{R}_0 = \mathbf{X}_0^{-1}\langle \mathbf{X}_0\mathbf{e}_K \rangle$

Derivation of the formulas:

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0, \qquad \mathbf{q} = \mathbf{R}_0\mathbf{p};$$

$$\mathbf{X}_0\mathbf{q} = \hat{\mathbf{p}}\mathbf{Z}_0\mathbf{e}_K + \mathbf{y}, \quad \mathbf{X}_0\mathbf{R}_0\mathbf{p} - \langle \mathbf{Z}_0\mathbf{e}_K \rangle\mathbf{p} = \mathbf{y}, \quad \langle \mathbf{X}_0\mathbf{e}_K \rangle\mathbf{p} - \langle \mathbf{Z}_0\mathbf{e}_K \rangle\mathbf{p} = \mathbf{y};$$

$$\mathbf{p}_\bullet = \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}\left(\mathbf{E}_K - \langle \mathbf{Z}_0\mathbf{e}_K \rangle\langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}\right)^{-1}\mathbf{y} = \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}\mathbf{P}_\bullet^{-1}\mathbf{y}, \qquad \mathbf{q}_\bullet = \mathbf{R}_0\mathbf{p}_\bullet = \mathbf{X}_0^{-1}\mathbf{P}_\bullet^{-1}\mathbf{y};$$

$$\mathbf{X}\mathbf{e}_K = \mathbf{X}_0\mathbf{q}_\bullet = \mathbf{X}_0\,\mathbf{X}_0^{-1}\mathbf{P}_\bullet^{-1}\mathbf{y} = \mathbf{P}_\bullet^{-1}\mathbf{y}, \qquad \mathbf{Z}\mathbf{e}_K = \langle \mathbf{Z}_0\mathbf{e}_K \rangle\mathbf{p}_\bullet = \langle \mathbf{Z}_0\mathbf{e}_K \rangle\langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}\mathbf{P}_\bullet^{-1}\mathbf{y};$$

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}_\bullet = \mathbf{X}_0\left\langle \mathbf{X}_0^{-1}\mathbf{P}_\bullet^{-1}\mathbf{y} \right\rangle, \qquad \mathbf{Z} = \hat{\mathbf{p}}_\bullet\mathbf{Z}_0 = \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}\left\langle \mathbf{P}_\bullet^{-1}\mathbf{y} \right\rangle\mathbf{Z}_0$$

Calibration: the solution satisfies both vector and matrix conditions

Input–output coefficients: $\mathbf{C}_0^\circ = \langle \mathbf{Z}_0\mathbf{e}_K \rangle\langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}$

Conclusion: the model has the same input–output coefficients matrix
as the linear demand-driven price model (21)

9.2. The example: $\mathbf{S}_0$ is the inverse & transpose of $K\times K$ product mix matrix

Model: the linear combined quantity/price demand-driven model (26), version (34)

Dimensions: $K\times K$

Instrumental linkage specification: the inverse & transpose of

the product mix matrix $\mathbf{S}_0 = (\mathbf{X}_0')^{-1}\langle \mathbf{X}_0'\mathbf{e}_K\rangle$

Derivation of the formulas:

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0, \qquad \mathbf{p} = \mathbf{S}_0\mathbf{q};$$

$$\mathbf{X}_0\mathbf{q} = \hat{\mathbf{p}}\mathbf{Z}_0\mathbf{e}_K + \mathbf{y}, \quad \mathbf{X}_0\mathbf{q} - \langle \mathbf{Z}_0\mathbf{e}_K\rangle \mathbf{S}_0\mathbf{q} = \mathbf{y}, \quad \mathbf{X}_0\mathbf{q} - \langle \mathbf{Z}_0\mathbf{e}_K\rangle (\mathbf{X}_0')^{-1}\langle \mathbf{X}_0'\mathbf{e}_K\rangle \mathbf{q} = \mathbf{y};$$

$$\mathbf{q}_\bullet = \mathbf{X}_0^{-1}\left(\mathbf{E}_K - \langle \mathbf{Z}_0\mathbf{e}_K\rangle (\mathbf{X}_0')^{-1}\langle \mathbf{X}_0'\mathbf{e}_K\rangle \mathbf{X}_0^{-1}\right)^{-1}\mathbf{y} = \mathbf{X}_0^{-1}\mathbf{Q}_\bullet^{-1}\mathbf{y}, \quad \mathbf{p}_\bullet = \mathbf{S}_0\mathbf{q}_\bullet = (\mathbf{X}_0')^{-1}\langle \mathbf{X}_0'\mathbf{e}_K\rangle \mathbf{X}_0^{-1}\mathbf{Q}_\bullet^{-1}\mathbf{y};$$

$$\mathbf{X}\mathbf{e}_K = \mathbf{X}_0\mathbf{q}_\bullet = \mathbf{X}_0\,\mathbf{X}_0^{-1}\mathbf{Q}_\bullet^{-1}\mathbf{y} = \mathbf{Q}_\bullet^{-1}\mathbf{y}, \quad \mathbf{Z}\mathbf{e}_K = \langle \mathbf{Z}_0\mathbf{e}_K\rangle \mathbf{p}_\bullet = \langle \mathbf{Z}_0\mathbf{e}_K\rangle (\mathbf{X}_0')^{-1}\langle \mathbf{X}_0'\mathbf{e}_K\rangle \mathbf{X}_0^{-1}\mathbf{Q}_\bullet^{-1}\mathbf{y};$$

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}_\bullet = \mathbf{X}_0\left\langle \mathbf{X}_0^{-1}\mathbf{Q}_\bullet^{-1}\mathbf{y}\right\rangle, \qquad \mathbf{Z} = \hat{\mathbf{p}}_\bullet\mathbf{Z}_0 = \left\langle (\mathbf{X}_0')^{-1}\langle \mathbf{X}_0'\mathbf{e}_K\rangle \mathbf{X}_0^{-1}\mathbf{Q}_\bullet^{-1}\mathbf{y}\right\rangle \mathbf{Z}_0$$

Calibration: the solution satisfies both vector and matrix conditions

Input–output coefficients: $\mathbf{C}_0^\bullet = \langle \mathbf{Z}_0\mathbf{e}_K\rangle (\mathbf{X}_0')^{-1}\langle \mathbf{X}_0'\mathbf{e}_K\rangle \mathbf{X}_0^{-1}$

Conclusion: the model is not mentioned in the special literature apparently

10. Concluding remarks

The proposed approach to constructing demand-driven input-output models by direct introducing quantity and price parameters appears to be a useful analytical toolbox for further study of theoretical and applied aspects of input–output modeling. Its main advantage is the universality and flexibility of describing a wide class of input–output models with exogenous final demand including those apparently not mentioned in the special literature.

The idea of introducing the instrumental linkages between quantity and price parameters can be extended to cases of double sets of quantity or price indices that could expand the analytical possibilities of sensitivity analysis and implementing scenario calculations.

References

Euro stat (2008) *Euro stat Manual of Supply, Use and Input–Output Tables*. Methodologies and Working Papers. Luxembourg, Office for Official Publications of the European Communities, KS-RA-07-013-EN-N. Available from: https://ec.europa.eu/eurostat/ documents/3859598/5902113/KS-RA-07-013-EN.PDF/b0b3d71e-3930 - 4442-94be-70b36cea9b39.

Kop Jansen P. and ten Rae T. (1990) The Choice of Model in the Construction of Input–Output Coefficients Matrices. *International Economic Review*, 31(1), 213–227.

Miller R.E. and Blair P.D. (2009) *Input–Output Analysis: Foundations and Extensions*, 2nd ed. Cambridge, Cambridge University Press.

Motoring V. (2017) The Linear Matrix-Valued Cost Functions as a Source of Leontief and Gosh Models. *Aretha Vienna*, 59 (4), 291–301. Available from: https://doi.org/10.21648/arthavij%2F2017%2Fv59%2Fi4%2F170795.

Rueda-Cantuche J.M. (2011) The Choice of Type of Input–Output Table Revisited: Moving Towards the Use of Supply-Use Tables in Impact Analysis. *SORT*, 35 (1), 21-38. Available from: https://raco.cat/index.php/SORT/article/view/242561/325182.

ten Raa T. and Rueda-Cantuche J.M. (2003) The Construction of Input–Output Coefficients Matrices in an Axiomatic Context: Some Further Considerations. *Economic Systems Research*, 15 (4), 439–455. Available from: https://www.tandfonline.com/doi/abs/10.1080/0953531032000152317.

ten Raa T. and Rueda-Cantuche J.M. (2007) A Generalized Expression for the Commodity and the Industry Technology Models in Input–Output Analysis. *Economic Systems Research*, 19 (1), 99–104. Available from: https://www.tandfonline.com/doi/abs/10.1080/09535310601164849.

United Nations (2018) *Handbook on Supply, Use and Input–Output Tables with Extensions and Applications*. Studies in Methods. Handbook of National Accounting. New York: United Nations, Department for Economic and Social Affairs, Statistics Division, ST/ESA/STAT/SER.F/74/Rev.1. Available from: https://unstats.un.org/unsd/national_account/docs/SUT_IOT_HB_Final_Cover.pdf.